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CONSERVED CURRENTS OF THE MAXWELL EQUATIONS IN THE PRESENCE OF ELECTRIC AND MAGNETIC SOURCES

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ABSTRACT

Lagrangians are employed which, when varied with respect to field strengths and electric and magnetic sources, lead to the Maxwell equations. In this way, conserved currents are derived for field strengths and electromagnetic sources. The conserved density of the duality transformation is related to the total helicity of the electromagnetic field.

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The Maxwell equations seem to contain an unexpected amount of symmetries and conservation laws[1)-5)]. Here we should like to present a new approach towards finding conservation laws. We construct Lagrangians for which the equations of motion are obtained by varying both field strengths as well as the electric and magnetic sources. In this way, conserved currents can be found which depend on the field strengths as well as on the electric and magnetic sources. Furthermore, using the equations of motion, the electromagnetic sources can be eliminated. In the following, we shall use the notation of Ref. 6).

If magnetic sources are included, the Maxwell equations take the form:

$$\partial_\mu F_{\mu\nu} = -(4\pi/c)\, j^e_\nu , \tag{1a}$$

$$\partial_\mu F^D_{\mu\nu} = -i\,(4\pi/c)\, j^m_\nu , \tag{1b}$$

where j^e_ν are the electric sources and j^m_ν the magnetic ones ($j^m_4 = ic\rho^m$, where ρ^m is the magnetic charge distribution, $j^e_4 = ic\rho^e$). $F_{\mu\nu}$ is the antisymmetric tensor of the electromagnetic field and $F^D_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\lambda\sigma}F_{\lambda\sigma}$ is its dual. In the enlarged version of this letter[7)], we use equations for the self-dual $F_{\mu\nu}+F^D_{\mu\nu}$ and anti-self-dual $F_{\mu\nu}-F^D_{\mu\nu}$ combinations. If we define

$$\psi_\alpha = H_\alpha - i E_\alpha , \quad \alpha = 1,2,3 ; \quad \psi_4 = 0 , \tag{2}$$

$$q_\mu = (4\pi/c)\left(j^e_\mu + i j^m_\mu \right) , \tag{3a}$$

$$k_\mu = q^*_\mu + 2\partial_4 \psi^*_\mu \tag{3b}$$

we can show that the following equations are satisfied:

$$i\,(R^\dagger_\mu)_{\nu\alpha}\, \partial_\mu \psi_\alpha = q_\nu \tag{4a}$$

$$i\,(R^\dagger_\mu)_{\nu\alpha}\, \partial_\mu \psi^*_\alpha = k_\nu , \tag{4b}$$

where the matrices R_μ are given by

$$R_1=\begin{pmatrix} & & & -i\\ & & i & \\ & -i & & \\ i & & & \end{pmatrix},\quad R_2=\begin{pmatrix} & & -i & \\ & & & -i\\ i & & & \\ & i & & \end{pmatrix},\quad R_3=\begin{pmatrix} & i & & \\ -i & & & \\ & & & -i\\ & & i & \end{pmatrix},\quad R_4=\begin{pmatrix} i & & & \\ & -i & & \\ & & -i & \\ & & & -i\end{pmatrix}. \tag{5}$$

Each one of Eqs. (4) is equivalent to Eqs. (1).

One can check that the following relation is satisfied:

$$(-i R_\nu \partial_\nu)(i R^{\dagger}_\mu \partial_\mu) = \partial_\mu \partial_\mu \equiv \Box \tag{6}$$

Applying to Eqs. (4) the operator $(-iR_\nu \partial_\nu)$, we obtain:

$$\Box \psi_a = -i\,(R_\mu)_{a\nu}\,\partial_\mu q_\nu\,, \tag{7a}$$

$$\Box \psi^*_a = -i\,(R_\mu)_{a\nu}\,\partial_\mu k_\nu\,, \tag{7b}$$

We construct the following Lagrangian:

$$\mathcal{L} = -(\partial_\mu \psi^*_a)(\partial_\mu \psi_a) + i\psi^*_a (R_\mu)_{a\nu}\partial_\mu q_\nu + i\psi_a (R_\mu)_{a\nu}\partial_\mu k_\nu - k_\nu q_\nu\,, \tag{8}$$

which, being varied with respect to ψ_a, ψ^*_a, q_ν and k_ν, leads to Eqs. (7b), (7a), (4b) and (4a) respectively, thus reproducing the Maxwell equations or equations consistent with them. As the Lagrangian (8) does not depend explicitly on the co-ordinates, using Noether's theorem, the following tensor is conserved:

$$T_{\mu\nu} = \mathcal{L}\,\delta_{\mu\nu} - \frac{\partial \mathcal{L}}{\partial(\partial_\nu \psi_a)}\partial_\mu \psi_a - \frac{\partial \mathcal{L}}{\partial(\partial_\nu \psi^*_a)}\partial_\mu \psi^*_a - \frac{\partial \mathcal{L}}{\partial(\partial_\nu k_\lambda)}\partial_\mu k_\lambda - \frac{\partial \mathcal{L}}{\partial(\partial_\nu q_\lambda)}\partial_\mu q_\lambda\,, \tag{9}$$

which depends on both field strengths and on the electromagnetic sources. The sources can be eliminated by using the equations of motion (4) and (7). Thus

$$T_{\mu\nu} = \delta_{\mu\nu}\left[-(\partial_\lambda\psi_a^*)(\partial_\lambda\psi_a) + i\psi_a^*(R_\lambda)_{a\sigma}\partial_\lambda q_\sigma + i\psi_a(R_\lambda)_{a\sigma}\partial_\lambda k_\sigma - k_\sigma q_\sigma\right]$$
$$+ (\partial_\nu\psi_a^*)(\partial_\mu\psi_a) + (\partial_\nu\psi_a)(\partial_\mu\psi_a^*) + i\psi_a(R_\nu)_{a\lambda}\partial_\mu k_\lambda$$
$$+ i\psi_a^*(R_\nu)_{a\lambda}\partial_\mu q_\lambda \tag{10}$$

$$= \delta_{\mu\nu}\left[-(\partial_\lambda\psi_a^*)(\partial_\lambda\psi_a) - \psi_a^*\Box\psi_a - \psi_a\Box\psi_a^* + (R_\lambda^\dagger)_{\sigma a}\partial_\lambda\psi_a(R_\eta^\dagger)_{\sigma b}\partial_\eta\psi_b^*\right]$$
$$+ (\partial_\nu\psi_a^*)(\partial_\mu\psi_a) + (\partial_\nu\psi_a)(\partial_\mu\psi_a^*) - \psi_a(R_\nu)_{a\lambda}(R_\eta^\dagger)_{\lambda b}\partial_\mu\partial_\eta\psi_b^*$$
$$- \psi_a^*(R_\nu)_{a\lambda}(R_\eta^\dagger)_{\lambda b}\partial_\mu\partial_\eta\psi_b \tag{11}$$

The Lagrangian (8) is also invariant with respect to the simultaneous duality transformations of the field strengths and sources

$$\psi_a \to e^{i\alpha}\psi_a\,,\quad \psi_a^* \to e^{-i\alpha}\psi_a^*\,,\quad q_\nu \to e^{i\alpha}q_\nu\,,\quad k_\nu \to e^{-i\alpha}k_\nu\,, \tag{12}$$

with the corresponding conserved current J_μ:

$$J_\mu = (i/2)(\psi_a^*\partial_\mu\psi_a - \psi_a\partial_\mu\psi_a^*) - \tfrac{1}{2}\psi_a^*(R_\mu)_{a\nu}q_\nu + \tfrac{1}{2}\psi_a(R_\mu)_{a\nu}k_\nu\,. \tag{13}$$

The conserved tensor (10) or (11) is quite complicated and further insight is needed to understand its meaning. On the other hand, the conserved duality current (13) is less complicated and the conserved density is

$$J_0 = -i\,J_4 = \tfrac{1}{2}\left[\psi_a^*\partial_4\psi_a - \psi_a\partial_4\psi_a^* - \psi_a^*(R_4R_\lambda^\dagger)_{ab}\partial_\lambda\psi_b + \psi_a(R_4R_\lambda^\dagger)_{ab}\partial_\lambda\psi_b^*\right]$$
$$= -\vec{E}\cdot(\nabla\times\vec{E}) - \vec{H}\cdot(\nabla\times\vec{H})\,. \tag{14}$$

This result coincides with a conserved density found by Lipkin[8] for the free electromagnetic field. Most recently it was discussed in Refs. 3), 4) and 5). Calkin[9] has shown that this conserved quantity is proportional to the difference in the number of right and left circulatory polarized photons (or to the total helicity). We found that this quantity is also conserved in the presence of electromagnetic sources.

As the method presented here for finding conserved currents in the presence of the electromagnetic sources seems to be completely new, we concentrate mainly on its presentation, leaving other aspects for future consideration.

An enlarged and detailed version of this letter[7] is being prepared. It contains other Lagrangians, whose conserved currents are related to those presented here.

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