Twistor Quantization of String Theory:

A Three-dimensional Prototype.

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Abstract.
The twistor transform of a classical open string defines natural canonical variables for twistor quantization in dimensions where one can solve the reality constraints on the twistor description of the string. The first dimension where this can be done covariantly is three. The twistor formulation of this theory is discussed as a prototype for higher dimensional situations. This type of covariant analysis generalizes to the dimension ten case more readily than to dimension four.

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1. Introduction.

In a flat Minkowskian space-time of any dimension the classical real strings correspond to pairs of null curves. The description of strings in terms of such conformally invariant structures permits the introduction of twistor methods (Penrose 1967, Penrose and MacCallum 1972, Penrose and Rindler 1986). The twistor description of *classical* null curves is now well established in dimensions three (Hitchin 1982, Shaw 1985), four (Shaw 1985, 1986a) and six (Hughston and Shaw 1986a). The results in dimensions three and four describe the geometrical picture underlying the parameterizations of minimal surfaces given by Weierstrass (1866), Montcheuil (1905) and Eisenhart (1911) and work is in progress to extend these ideas to higher dimensions and the supersymmetric regime (see e.g. Fairlie and Manogue 1986, Shaw 1986a,b).

It is important to appreciate that the twistor approach generates complex null curves more readily than real null curves. For example, a holomorphic curve in complex projective 3-space defines a holomorphic null curve in complex four-dimensional Minkowski space, which defines a spacelike minimal surface in the associated real Minkowski space. To obtain timelike strings it is necessary to introduce reality conditions on the twistor curves. The ease with which one can solve the reality conditions depends in a critical way on the dimension and signature of the underlying space-time. This has implications for the quantization in twistor space, where one imagines replacing reality conditions by holomorphic differential constraints on "wave functions" on the loop space of the twistor space. It turns out that this procedure is most straightforward when the space-time is of a dimension and signature where it makes sense to define a real spinor and, by implication, a real twistor. While this is not the case in dimension four (restricting attention to Minkowskian signature) it is possible in three dimensions and also in ten dimensions. In dimension three the twistor realization of the conformal group $SO(2,3)$ is the real symplectic group $Sp(4,R)$, whereas in four dimensions the corresponding groups are $SO(2,4)$ and $SU(2,2)$ respectively. It should be emphasized that at this early stage in the development of twistorial string models these reality considerations should not be regarded as being dimensional constraints at the same level as those found in string theory: the approach to twistor quantization considered here is the most obvious and least subtle line of approach and one cannot yet rule out a more sophisticated approach for dealing with the dimension four case. At the same time it is somewhat curious that the light-cone gauge quantization of a free superstring is Lorentz covariant in both ten and three dimensions, while the corresponding free bosonic quantization is covariant in twenty-six and three dimensions. The covariance in three dimensions (also two) is present for different reasons than in twenty-six or ten, and does not require that the theory contain a tachyon. The three-dimensional case is therefore interesting for a variety of reasons, and is the main subject of this note.

The goal of the analysis is to find a twistorial formulation of classical string theory which is fit for quantization. The approach to quantization is to find a set of twistor variables with respect to which the symplectic structure on the space of strings is diagonal and the reality conditions are at worst first class constraints. The standard twistor quantization may then be invoked. Generically, if one denotes a twistor by Z, then the quantization represents the quantum states as holomorphic functions of Z and the classical quantity $\bar{Z}$ is quantized by

$$\bar{Z} \rightarrow -i\frac{\partial}{\partial Z} \quad .$$

The difficulty with the four dimensional case, as it stands at present, is that the only known choice of canonical variables leads to reality conditions whose Poisson bracket algebra fails to close: the constraints are *second class.* This is also the prevailing situation in six dimensions. However, in the case of three dimensions one can solve the reality conditions before quantization and the only remaining constraints are first class. The classical Poisson algebra of these residual constraints is a twistorial realization of the Lie algebra of the group $Diff(S^1)$. In some sense this represents, at the classical level, a "fermionization" of the bosonic string alluded to by Sparling (1986). Twistors provide a "square root" for space-time geometry; the corresponding twistor loop space provides a square root for bosonic string theory (see also Hughston 1986).

The theory to be described represents a prototype for more general twistorial string theories. The ideas described below generalize to ten dimensions in an obvious way, and also isolate the problems which have to be faced and overcome in dimension four.

The plan of this work is as follows. In section two 3-dimensional twistor geometry is introduced via the twistor equation. The appropriate reality structures are described in section three. In section four the process of twistor quantization for null geodesics is reviewed. This is important since the real null geodesics are the classical ground states of the string. In section five classical string theory is recast in twistor terms. An open string corresponds to a quasi-periodic curve on twistor space and these curves can be associated with the loop space of twistor space. The symplectic structure of the space of open strings turns out to be the loop integral of the symplectic form of twistor space itself. This suggests natural canonical variables, namely the Fourier coefficients in an expansion of the loop. The constraint algebra is then investigated, revealing dimension three as being particularly straightforward. In section six the quantization is considered. Some exact solutions (both massless and massive) of the quantum constraints are exhibited. The quantum constraints yield a twistorial representation of the Virasoro (1970) algebra, but with a curious difference: the central charge of the algebra corresponds to dimension two rather than three.

Throughout this note the string tension $\mathbf{T}$ is set at unity.

2. The twistor equation in three dimensions.

Let $x^a = (t, x, z)$ be coordinates for Minkowskian 3-space $\mathcal{M}_{1,2}$, with metric

$$ds^2 = dx^a dx_a = dt^2 - dx^2 - dz^2 \quad .$$

The points of $\mathcal{M}_{1,2}$ correspond to symmetric 2-index spinors x^{AB} via

$$x^{AB} = \frac{1}{\sqrt{2}} \begin{bmatrix} t-z & x \\ x & t+z \end{bmatrix}$$

and

$$x^a x_a = 2 \det x^{AB} = x^{AB} x^{CD} \varepsilon_{AC} \varepsilon_{BD} \quad .$$

Complex conjugation on tensors is extended to spinors as an *involutory* operation $\dagger$. The inner product on spin space defined by

$$< \alpha, \alpha >= i \alpha^A \alpha^{\dagger B} \varepsilon_{AB}$$

defines the group $SU(1,1)$. The properties of such spinors on Lorentzian 3-spaces are discussed in more detail in section three and elsewhere (Shaw, 1983).

The twistor equation is the condition

$$\nabla_{(CD} \omega_{A)}(x) = 0$$

on a spinor field $\omega^A(x)$, where $\nabla_{CD} = \nabla_{(CD)}$ is the spinor covariant derivative. Equivalently one may write

$$\nabla_{CD} \omega_A(x) = \varepsilon_{A(C} \pi_{D)} \tag{2.1}$$

for some spinor field π_A. Application of the Ricci identity implies that π_A is constant and that consequently the solution to (2.1) is

$$\omega^A(x) = \omega^A + x^{AB} \pi_B \quad ,$$

where ω^A is constant. The *twistor correspondence* is defined by the zeroes of $\omega^A(x)$:

$$\omega^A + x^{AB} \pi_B = 0 \quad . \tag{2.2}$$

Twistor space may be considered as the set of pairs $Z^\alpha = \{\omega^A, \pi_A\}$. It comes equipped with an antisymmetric metric $\varepsilon_{\alpha\beta} = \varepsilon_{[\alpha\beta]}$, defined by

$$Z_1^\alpha Z_2^\beta \varepsilon_{\beta\alpha} = \omega^A p_A - \Omega^A \pi_A \quad ,$$

where

$$Z_1^\alpha = \{\omega^A, \pi_A\} \quad ; \quad Z_2^\alpha = \{\Omega^A, p_A\} \quad .$$

Raising and lowering of indices is defined in the usual way:

$$Z_\alpha = Z^\beta \varepsilon_{\beta\alpha} = \{\pi_A, -\omega^A\}$$

and $Z^\alpha Z_\alpha = 0$.

The incidence relation (2.2) defines the twistor correspondence between space-time and twistor space. Given a space-time point x^{AB} the set of all twistors incident with x is an isotropic 2-plane: If Z_1^α and Z_2^α are any two points incident with x then

$$Z_1^\alpha Z_{2\alpha} = 0 \quad . \tag{2.3}$$

Projectively one obtains a null line in projective twistor space. Any two twistors satisfying (2.3) define a unique point in space-time. Conversely, given a pair $\{\omega^A, \pi_A\}$ the corresponding space-time structure is a γ-plane (in general odd dimension), which in this dimension is a null line:

$$x^{AB} = x_0^{AB} + t\pi^A \pi^B \quad .$$

The structure of these relations can be reduced much further. If one defines

$$F = \omega^A \pi_A$$

and write

$$\pi_A = (\pi_0, \pi_1) = \pi_1(\zeta, 1) \quad ; \quad F = -\eta \pi_1^2 \quad ,$$

then all of the information of the incidence relations is contained in the pair (η, ζ), which may be regarded in a natural way as local coordinates for part of $T(CP^1)$. Much of the theory of twistors in three dimensions can be developed by working only with the space $T(CP^1)$ (see Hitchin, 1982). For example, the points x of the space-time correspond to holomorphic sections of $T(CP^1)$, which are necessarily quadratic in ζ:

$$\eta = t + z + 2x\zeta + \zeta^2(t - z) \quad .$$

Note that the approach of Hitchin involves Euclidean 3-space, so that the complex conjugation conventions used here are different. The variables (η, ζ) play an important role in the contour integral formulae for massless fields which arise from twistor quantization of the null geodesics.

3. Reality Structures.

Suppose that a twistor Z^α is incident with a *real* space-time point:

$$x^{\dagger AB} = x^{AB} \quad .$$

Applying † to (2.2) gives

$$\omega^{\dagger A} + x^{AB}\pi^\dagger_B = 0 \quad ,$$

and so

$$Z^\alpha Z^\dagger_\alpha \equiv \omega^A \pi^\dagger_A - \omega^{\dagger A}\pi_A = 0 \quad . \tag{3.1}$$

This condition *can* be solved by requiring that Z^α is itself real:

$$Z^{\dagger\alpha} = Z^\alpha \quad , \tag{3.2}$$

which, with an appropriate choice of spinor basis, means that all the components of Z^α are real. More generally, Z^α could be real up to a phase:

$$Z^{\dagger\alpha} = \exp(-2i\gamma)Z^\alpha \quad , \tag{3.3}$$

so that the twistor is now $\exp(i\gamma)$ times a real twistor. The most general twistor satisfying (3.1) does not necessarily satisfy (3.3), but the twistors which do satisfy (3.3) are of considerable importance, as will be demonstrated shortly.

For any choice of π_A the vector

$$\pi^{\dagger(A}\pi^{B)}$$

is real and any real vector in $\mathcal{M}_{1,2}$ can be thus expressed. The vector is null if and only if

$$\pi^{\dagger A} \propto \pi^A \quad .$$

If one represents the null geodesic as

$$x^{AB}(t) = x^{AB}_0 + t\pi^{\dagger(A}\pi^{B)}$$

with

$$\pi_A \pi^{\dagger A} = 0 \quad , \tag{3.4}$$

then the null geodesic defines a twistor $Z^\alpha = \{\omega^A, \pi_A\}$ by the incidence condition (2.2). This twistor satisfies (3.4) and hence also (3.3). Note that the condition (3.3) is equivalent to the pair of conditions

$$Z^\alpha Z^\dagger_\alpha = 0 \quad , \quad \pi_A \pi^{\dagger A} = 0 \quad . \tag{3.5}$$

This leads to two essentially equivalent twistorial views of classical real null geodesics: a real picture and a complex picture. If one works with scaled null geodesics, the null tangent vector V^{AB} defines both a real spinor π^A (unique up to $\pm$) and a complex spinor $\hat{\pi}^A$ unique up to a phase satisfying $\hat{\pi}^A\hat{\pi}^\dagger_A = 0$. The incidence relations define a real twistor and a twistor subject to (3.5) respectively. These two views of null geodesics lead to different quantization procedures. One can solve the reality condition at the classical level by making the twistors real, and then quantize. Alternatively one can quantize the complex classical picture and impose the reality conditions at the quantum level. This latter approach is the true "twistor quantization" and will be considered in the next section.

The real picture is interesting for a variety of reasons. The spinorial structures described above give a concrete realization of the geometrical structures which are present. The space of unscaled real null geodesics is precisely the space of real Z^α modulo real rescalings, and is therefore RP^3. The antisymmetric metric defines the symplectic group $Sp(4,R)$, which is the twistor translation of the 3-dimensional conformal group $SO(2,3)$. The points of $\mathcal{M}_{1,2}$ are (non-projectively) the isotropic 2-planes of R^4, i.e. the Lagrangian Grassmannian associated with $Sp(4,R)$. The crucial difference between dimensions three and four is that the real conformal group is realized twistorially as a real symplectic group rather than a complex unitary group, and this results in considerable simplifications in the description of real strings. Otherwise, the symplectic structure plays the same roles as does the unitary structure in one greater dimension. For example, the "future tube" of $\mathcal{M}_{1,2}$ is characterized as the set of complex points $x^{AB} = X^{AB} - iY^{AB}$, where Y^a is timelike and future-pointing. Points on the corresponding lines in twistor space must satisfy

$$-iZ^\alpha Z^\dagger_\alpha > 0 \quad . \tag{3.6}$$

One further remark is pertinent. With the present conventions on complex conjugation and the ε-spinor this spinor is *real*:

$$\varepsilon^\dagger_{AB} = +\varepsilon_{AB} \quad . \tag{3.7}$$

Thus the action of complex conjugation commutes with the raising and lowering of spinor indices. This is the opposite of what is most convenient when one considers spinors on Lorentzian 3-surfaces as embedded in a Lorentzian 4-space (as is done in Shaw 1983). In the present context it simplifies matters considerably to adopt (3.7)

4. Twistor Quantization I: Null Geodesics.

Consider the space N of scaled real null geodesics. In terms of standard coordinates $\{x^a, p_a\}$, with

$$p_a p^a = 0 \quad , x^a \simeq x^a + tp^a \quad ,$$

N is equipped with a 2-form

$$\omega = dp_a \wedge dx^a.$$

Suppose first that one writes

$$p_{AB} = \pi_A \pi_B \quad , \quad \omega^A + x^{AB}\pi_B = 0 \quad ,$$

where $Z^\alpha = \{\omega^A, \pi_A\}$ is *real.* Then an elementary calculation gives

$$\omega = -dx^{AB} \wedge d(\pi_A \pi_B) = 2d\omega^A \wedge d\pi_A = dZ^\alpha \wedge dZ_\alpha \quad . \tag{4.1}$$

Now suppose instead that one writes

$$p_{AB} = \pi_A \pi^\dagger_B \quad , \quad \omega^A + x^{AB}\pi_B = 0 \quad , \tag{4.2}$$

$$\pi^A \pi^\dagger_A = 0 \quad , \quad Z^\alpha Z^\dagger_\alpha = 0 \quad . \tag{4.3}$$

In this case a similar calculation gives

$$\omega = d\omega^A \wedge d\pi^\dagger_A - d\pi_A \wedge d\omega^{\dagger A} = dZ^\alpha \wedge dZ^\dagger_\alpha \quad . \tag{4.4}$$

Note the similarity between this latter result and the usual situation in four dimensions, where one has

$$\omega = dZ^\alpha \wedge d\bar{Z}_\alpha \ , \quad Z^\alpha \bar{Z}_\alpha = 0 \quad .$$

Here the action of complex conjugation on twistors is the standard one appropriate to 4-dimensional Minkowski space. The inner product is invariant under the action of $SU(2,2)$. In three dimensions the constraint that the "norm" of Z^α vanish is supplemented by the proportionality condition $\pi^A \pi^\dagger_A = 0$. The first form of ω is obtained from the second by solving this constraint by the choice

$$\pi^{\dagger A} = \pi^A \quad .$$

The various expressions for the symplectic 2-form suggest a host of quantization procedures. For the present the standard "twistor quantization" will be given. This corresponds to the polarizations suggested by (4.4). Z^α and $Z^\dagger_\alpha$ are regarded as canonical variables and one adopts the quantization rule

$$Z^\dagger_\alpha \to -i\frac{\partial}{\partial Z^\alpha}$$

with constraints given by the quantized and normal-ordered form of (4.3), that is

$$(E+2)=0 \quad , \quad E=Z^{\alpha}\frac{\partial}{\partial Z^{\alpha}}$$

and

$$\pi^{A}\frac{\partial}{\partial\omega^{A}}=0 \quad .$$

The quantum states are holomorphic functions

$$F(Z^{\alpha})=F(\omega^{A},\pi_{A})$$

homogeneous of degree -2 and subject to

$$\pi^{A}\frac{\partial F}{\partial\omega^{A}}=0 \quad . \tag{4.5}$$

The states are subject to two holomorphic differential constraints and may be regarded as arbitrary holomorphic functions of just two independent complex variables. This may be accomplished directly. Using the homogeneity condition and (4.5) one can show that F must be of the form

$$F=\pi_{1}^{2}H(\eta,\zeta) \quad , \tag{4.6}$$

where ζ and η are defined as in section two, that is,

$$\zeta=\frac{\pi_{0}}{\pi_{1}} \quad , \quad \eta=\frac{\omega^{1}}{\pi^{1}}+\zeta\frac{\omega^{0}}{\pi^{1}} \quad . \tag{4.7}$$

The corresponding contour integral defines a massless field $\phi(x)$ as

$$\phi(x)=\oint d\zeta H(\eta,\zeta) \quad , \tag{4.8}$$

where η is restricted to x, i.e.

$$\eta=t+z+2x\zeta+\zeta^{2}(t-z) \quad . \tag{4.9}$$

Such fields satisfy the $\mathcal{M}_{1,2}$ wave equation:

$$\frac{\partial^{2}\phi}{\partial t^{2}}-\frac{\partial^{2}\phi}{\partial x^{2}}-\frac{\partial^{2}\phi}{\partial z^{2}}=0 \quad .$$

In section six the polarizations suggested by (4.1) will be considered.

5. The Twistor Transform of a Classical Open String.

An open string in $\mathcal{M}_{1,2}$ corresponds to a quasi-periodic null curve in $\mathcal{M}_{1,2}$. If the world sheet of the string is given as $X^a(\tau,\sigma)$ and one works in the conformal gauge, where the induced metric of the world-sheet is conformal to the flat metric

$$ds^2 = d\tau^2 - d\sigma^2 \quad ,$$

then one can write

$$X^a(\tau,\sigma) = \phi^a(\tau-\sigma) + \phi^a(\tau+\sigma) \tag{5.1}$$

where $\dot{\phi}^a(s)$ is both null and periodic:

$$\dot{\phi}^a(s)\dot{\phi}_a(s) = 0 \quad , \quad \dot{\phi}^a(s+2\pi) = \dot{\phi}^a(s) \quad . \tag{5.2}$$

Here and throughout the operation $\dot{}$ denotes $\frac{d}{ds}$. The periodicity condition is equivalent to

$$\phi^a(s+2\pi) = \phi^a(s) + P^a \quad , \tag{5.3}$$

where P^a is indeed the total momentum of the string.

Consider now the twistorial picture of such an object. Note that the original characterization of the string as a timelike minimal surface is not conformally invariant (where the conformal invariance refers to rescalings of $\mathcal{M}_{1,2}$). In flat space-time the open string corresponds to a null curve, a notion which is conformally invariant. (In a general background the twistor approach would involve more than this simple observation. One would go further and regard the null curves, rather than the strings, as the basic objects of interest. This is an appealing approach since one can construct the conformal metric given a knowledge of the null curves.) This change of viewpoint makes it possible to introduce twistor methods. Firstly, since $\dot{\phi}^a(s)$ is null there is a spinor field $\pi_A(s)$ such that

$$\dot{\phi}^{AB}(s)\pi_B(s) = 0 \quad . \tag{5.4}$$

Since the curve is real one imposes

$$\pi^A(s)\pi^\dagger_A(s) = 0 \quad . \tag{5.5}$$

As before, this condition *could* be solved by choosing $\pi^A(s)$ to be real. One normalizes π^A so that $\dot{\phi}^{AB} = \pi^{\dagger A}\pi^B$. It follows from (5.3) that

$$P^{AB} = \int_{-\pi}^{\pi} ds \; \pi^{\dagger A}\pi^B \quad . \tag{5.6}$$

Now the periodicity of $\dot{\phi}^a$ implies that one can expand $\pi^A(s)$ in a Fourier series as

$$\pi^A(s) = \sum_{n=-\infty}^{\infty} \pi^A_n \exp(-ins) \quad . \tag{5.7}$$

The reality condition (5.5) implies that for all q the following constraints are satisfied:

$$\gamma(q) = \sum_{n=-\infty}^{\infty} \pi_n^A \pi^\dagger_{A,n+q} = 0 \quad . \tag{5.8}$$

These constraints may be solved by demanding that

$$\pi_n^{\dagger A} = \pi_{-n}^A \quad . \tag{5.9}$$

A curve $Z^\alpha(s)$ in twistor space may be defined by imposing the incidence relations at every point on the curve:

$$\omega^A(s) + \phi^{AB}(s)\pi_B(s) = 0 \quad . \tag{5.10}$$

The relation (5.4) implies that the space-time curve is also incident with $\dot{Z}^\alpha$:

$$\dot{\omega}^A(s) + \phi^{AB}(s)\dot{\pi}_B = 0 \quad . \tag{5.11}$$

From (2.3) it follows that $Z^\alpha(s)$ must satisfy:

$$Z^\alpha(s)\dot{Z}_\alpha(s) = 0 \quad . \tag{5.12}$$

Similarly the reality of ϕ^a implies that the following constraints must hold:

$$Z^\alpha(s)Z^\dagger_\alpha(s) = 0 \quad , \tag{5.13}$$

$$Z^\alpha(s)\dot{Z}^\dagger_\alpha(s) = 0 \quad , \tag{5.14}$$

$$\dot{Z}^\alpha(s)\dot{Z}^\dagger_\alpha(s) = 0 \quad . \tag{5.15}$$

These last three constraints and (5.8) can all be solved by requiring that $Z^\alpha(s)$ is real and satisfies (5.12).

In contrast to the case of real null geodesics, the complex twistor picture of a real string is significantly more complicated than the real twistor picture. In the real picture the only constraint is (5.12). In the complex picture one must also impose (5.8), (5.13), (5.14) and (5.15). Recall that in the case of null geodesics the complex twistor picture required two extra constraints. In the case of the string there are four, which are straightforward to solve. At this point it is useful to recall the corresponding situation in four dimensions (Shaw, 1986a). In four dimensions (Minkowskian signature) the real null curves are represented by complex twistor curves $Z^\alpha(s)$ subject to (cf. section four)

$$Z^\alpha(s)\bar{Z}_\alpha(s) = 0 \quad , \tag{5.16}$$

$$Z^\alpha(s)\dot{\bar{Z}}_\alpha(s) = 0 \quad , \tag{5.17}$$

$$\dot{Z}^\alpha(s)\dot{\bar{Z}}_\alpha(s) = 0 \ . \tag{5.18}$$

At present it is not known how to solve these equations in a covariant manner corresponding to what one can do in three dimensions, where one chooses the twistor to be real. This represents a significant difference between the two dimensions and causes difficulties when one considers the quantization.

Now consider again the theory in three dimensions. Equation (5.7) defines a Fourier expansion for $\pi^A(s)$ and one wants to do the same for $\omega^A(s)$. However, this is not possible directly since the curve $\omega^A(s)$ is not periodic. Indeed, from (5.3) it follows that

$$\omega^A(s+2\pi) = \omega^A(s) - P^{AB}\pi_B(s) \ . \tag{5.19}$$

Another difficulty is that the translations do not act properly on the twistor fields as given. Under a space-time translation of the string: $X^a \to X^a + V^a$ the null curve ϕ^a transforms according to

$$\phi^a \to \phi^a + \frac{1}{2}V^a \ ,$$

and so

$$\omega^A(s) \to \omega^A(s) - \frac{1}{2}V^{AB}\pi_B(s) \ . \tag{5.20}$$

This is *half* the displacement in twistor space that one expects. Both these problems can be solved as follows. First define

$$\Pi^{AB} = \frac{1}{2\pi}P^{AB} = \sum_{n=-\infty}^{\infty} \pi_n^{\dagger A}\pi_n^B \tag{5.21}$$

and

$$\gamma^a = \frac{1}{2\pi}\int_{-\pi}^{\pi} ds \ \phi^a(s) \ , \tag{5.22}$$

the latter quantity being the average $< \phi^a >_s$ with respect to the given parameterization. Now define $\Omega^A(s)$ by

$$\omega^A(s) = \{\gamma^{AB} - s\Pi^{AB}\}\pi_B(s) + \Omega^A(s) \ . \tag{5.23}$$

The new curve $P^\alpha(s) = \{\Omega^A(s), \ \pi_A(s)\}$ has several useful properties. Note first that $\Omega^A(s)$ is periodic, so that $P^\alpha(s)$ defines a *loop* on twistor space. Also, under translations Ω^A transforms correctly:

$$\Omega^A(s) \to \Omega^A(s) - V^{AB}\pi_B(s) \ .$$

Some other useful properties of this new curve will be noted shortly. The constraints are modified when expressed in terms of $P^\alpha(s)$. For example, (5.12) and (5.14) are replaced, respectively, by

$$P^\alpha\dot{P}_\alpha + \Pi^{AB}\pi_A\pi_B = 0 \ , \ \ P^\alpha\dot{P}^\dagger_\alpha + \Pi^{AB}\pi_A\pi^\dagger_B = 0 \ , \tag{5.24}$$

and similar modifications are made to equations (5.13) and (5.14).

The principal use of the loops $P^\alpha(s)$ is that they can be used to cast the symplectic structure into a particularly straightforward form. From any Lagrangian field theory one can construct an associated (not necessarily non-degenerate) symplectic 2-form (see e.g. Woodhouse, 1980). Applying these ideas to string theory, suppose that one has a string $X^a(\tau,\sigma)$ and two neighbouring strings $X^a + V_1^a$, $X^a + V_2^a$. Working in the conformal gauge the symplectic form is described by an integral:

$$2\omega(V_1, V_2) = \int_0^\pi d\sigma \{V_2^a V_{1a,\tau} - V_1^a V_{2a,\tau}\} \quad . \tag{5.25}$$

The integral can be taken over the $\tau = 0$ cross-section, for the equations of motion and boundary condtions ensure that ω is independent of τ. The immediate goal is to translate (5.25) into an integral on twistor space. First one writes it in terms of variations of the associated null curves. With (5.1) holding let

$$V_i^a = \delta_i \phi^a(\tau - \sigma) + \delta_i \phi^a(\tau + \sigma) \quad .$$

Substitution and some reorganisation gives

$$2\omega(V_1, V_2) = \int_{-\pi}^{\pi} ds\{\delta_2\phi^a(s)\delta_1\dot{\phi}_a(s) - \delta_1\phi^a(s)\delta_2\dot{\phi}_a(s)\} + \delta_2\phi^a(-\pi)\delta_1\phi_a(\pi) - \delta_1\phi^a(-\pi)\delta_2\phi_a(\pi) \quad . \tag{5.26}$$

Note that the boundary terms do not vanish since one can vary the total momentum of the curve.

The precise translation of (5.26) into twistor space depends on whether the real or complex viewpoint is adopted. In the real case one sets $\dot{\phi}^{AB} = \pi^A \pi^B$. Now let V_i^a correspond to variations $\delta_i P^\alpha$ in P^α. Then some calculation leads to:

$$\omega(V_1, V_2) = \int_{-\pi}^{\pi} ds \; \delta_1 P^\alpha \delta_2 P_\alpha \quad . \tag{5.27}$$

Thus if the open strings in space-time are represented by loops in twistor space subject to (5.24) then the symplectic structure on the space of open strings is just the integral around the loop of the standard twistor symplectic structure. This formula is significantly more straightforward than its space-time counterpart, although it does not have the *manifest* reparameterization invariance of the latter.

In the complex picture a similar result is obtained [cf. (4.1) vs. (4.4)]. When all the reality constraints are satisfied, writing $\dot{\phi}^{AB} = \pi^A \pi^{\dagger B}$ leads to

$$2\omega(V_1, V_2) = \int_{-\pi}^{\pi} ds \; \{\delta_1 P^\alpha \delta_2 P_\alpha^\dagger - \delta_2 P^\alpha \delta_1 P_\alpha^\dagger\} \quad . \tag{5.28}$$

Clearly (5.28) reduces to (5.27) when the reality conditions are solved. It is important to realize that the second expression (5.28) is actually quite general, and a corresponding expression can be written down in all dimensions for which the twistor description of an open string is known. For example, in four dimensions, when the constraints (5.16-18) are satisfied, one has

$$2\omega(V_1, V_2) = i\int_{-\pi}^{\pi} ds\ \{\delta_1 P^\alpha \delta_2 \bar{P}_\alpha - \delta_2 P^\alpha \delta_1 \bar{P}_\alpha\}\ . \tag{5.29}$$

A similar expression exists in six dimensions also (Hughston and Shaw, 1986).

The form of the symplectic structure suggests natural canonical twistor variables, namely the Fourier coefficients in an expansion of the loop. Although the form of ω shows that these coefficients always define a natural symplectic frame, it is not necessarily the case that these coefficients are the appropriate variables to be used in canonical quantization. Their viability depends to a great extent on the behaviour of the constraints, and we anticipate that the complex picture may be more problematic due to the profusion of constraints in that picture. The essential difficulty is the phase invariance in the complex twistor picture. The symplectic form given by (5.28) or (5.29) is obviously highly degenerate. In four dimensions, for example, if one sets $\delta_j P^\alpha = i\phi_j P^\alpha$ then restricted to such tangent directions $\omega \equiv 0$. Before quantization it is necessary then to examine in detail the proposed "canonical variables" and the Poisson bracket algebra of the constraints, with respect to the real and complex twistor variables.

One expands $P^\alpha(s)$ as

$$P^\alpha(s) = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} P^\alpha_n \exp(-ins)\ . \tag{5.30}$$

In the real case the reality conditions are

$$P^\alpha_n = P^{\dagger\alpha}_{-n}\ .$$

These coefficients define a symplectic frame, by inspection of (5.27) and (5.28). In the real case one obtains

$$\omega = \sum_{n=-\infty}^{\infty} dP^\alpha_n \wedge dP_{-n,\alpha} = dP^\alpha_0 \wedge dP_{0,\alpha} + 2\sum_{n=1}^{\infty} dP^\alpha_n \wedge dP^\dagger_{n,\alpha}\ , \tag{5.31}$$

and in the complex case,

$$\omega = \sum_{n=-\infty}^{\infty} dP^\alpha_n \wedge dP^\dagger_{n,\alpha}\ . \tag{5.32}$$

This *suggests* what the natural canonical variables should be. Note that if one restricts all P^α_n to be zero except for $n = 0$ one recovers the symplectic form appropriate to null geodesics. The constraints are easily expanded in terms of the Fourier coefficients. The condition $P^\alpha \dot{P}_\alpha + \Pi^{AB}\pi_A\pi_B = 0$ becomes

$$\mathcal{L}_r = \sum_{n=-\infty}^{\infty} [-inP^\alpha_{r-n}P_{n,\alpha} + P^{AB}\pi_{r-n,A}\pi_{n,B}] = 0\ . \tag{5.33}$$

In the complex case one also has (5.8), and the Fourier versions of $P^\alpha P^\dagger_\alpha = 0$, i.e.

$$\mathcal{A}_r = \sum_{n=-\infty}^{\infty} P^\alpha_n P^\dagger_{n+r,\alpha} = 0\ ; \tag{5.34}$$

$P^\alpha \dot{P}^\dagger_\alpha + \Pi^{AB}\pi_A\pi^\dagger_B = 0$, i.e.

$$\mathcal{B}_r = \sum_{n=-\infty}^{\infty} [i(n+r)P^\alpha_n P^\dagger_{n+r,\alpha} + P^{AB}\pi_{n,A}\pi^\dagger_{n+r,B}] = 0 \quad ; \tag{5.35}$$

finally $\dot{P}^\alpha \dot{P}^\dagger_\alpha + \Pi^{AC}(\pi^\dagger_C\dot{\pi}_A - \pi_C\dot{\pi}^\dagger_A) = 0$, i.e.

$$\mathcal{C}_r = \sum_{n=-\infty}^{\infty} in(n+r)P^\alpha_n P^\dagger_{n+r,\alpha} + 2nP^{AB}\pi_{n,A}\pi^\dagger_{n+r,B} = 0 \quad . \tag{5.36}$$

Consider now the Poisson brackets of these various quantities. In the *complex* case one tries to consider the canonical variables as the set $\{P^\alpha_n, P^\dagger_{n,\alpha}, \ -\infty < n < \infty\}$. The Poisson bracket in the complex case is therefore

$$\{f,g\} = \sum_{r=-\infty}^{\infty} \left\{ \frac{\partial f}{\partial P^\alpha_r}\frac{\partial g}{\partial P^\dagger_{r,\alpha}} - \frac{\partial g}{\partial P^\alpha_r}\frac{\partial f}{\partial P^\dagger_{r,\alpha}} \right\} \quad .$$

Now define $\mathcal{U}_{n,q} = P^\alpha_n P^\dagger_{n+q,\alpha}$ and $\mathcal{V}_{n,q} = P^{AB}\pi_{n,A}\pi^\dagger_{n+r,B}$ so that $\mathcal{A}_r = \sum \mathcal{U}_{n,r}$ etc. The Poisson brackets of the $\mathcal{U}$ and $\mathcal{V}$ functions are

$$\{\mathcal{U}_{n,q}, \mathcal{U}_{m,s}\} = \delta_{n,m+s}\mathcal{U}_{m,s+q} - \delta_{m,n+q}\mathcal{U}_{n,s+q} \quad ,$$

$$\{\mathcal{U}_{n,q}, \mathcal{V}_{m,s}\} = \delta_{n,m+s}\mathcal{V}_{m,s+q} - \delta_{m,n+q}\mathcal{V}_{n,s+q} \quad , \tag{5.37}$$

$$\{\mathcal{V}_{n,q}, \mathcal{V}_{m,s}\} = 0 \quad ,$$

from which the constraint algebra can be calculated. It is at this point that one runs into difficulties with the complex picture, for the constraint algebra fails to close. That is, with respect to the hypothesized canonical variables, the constraints are *second class.* The problematic commutator is that of two $\mathcal{C}$ constraints. Explicitly, one finds that

$$\{\mathcal{C}_q, \mathcal{C}_s\} = i(s^2 - q^2)\mathcal{C}_{s+q} + 2i(s-q)\mathcal{D}_{s+q} \quad , \tag{5.38}$$

where $\mathcal{D}_p$ is a new constraint, given by

$$\mathcal{D}_p = \sum_{-\infty}^{\infty} in^2(n+p)\mathcal{U}_{n,p} + (3n^2 + np)\mathcal{V}_{n,p} \quad . \tag{5.39}$$

Repeated application of the Poisson brackets gives another constraint $\mathcal{E}_p$, and so on. It is not clear whether one can quantize the theory in this form, since the quantum constraints would generate, by commutation, an infinite tower of constraints above those present at the classical level. This sort of behaviour also occurs in four dimensions. The constraint (5.18) turns out to be second class in exactly the same way, when one Fourier expands the fields and tries the canonical variables suggested by (5.28). An interesting escape route is suggested by enlarging the classical theory to allow solutions not satisfying the $\mathcal{D}$ constraints. This is a mild complexification of the system which appears to be quantizable-the remaining constraints (5.16) and (5.17) generate a Poisson bracket algebra which is the semi-direct sum of the Lie algebra of $Diff(S^1)$ with a commutative algebra. The consequences of this are under consideration, as is the situation when one considers the quantization of finite-dimensional subsets of the classical phase space. (Hughston and Shaw, 1986b). Alternatively, one might consider adding the additional constraints to the quantum theory. In this case it is not clear whether one is left with a non-trivial theory containing an appropriate spectrum of states.

Returning to the dimension three case, and in contrast to the dimension four case, one is in the fortunate position of being able to solve the reality conditions at the classical level. Thus the appearance of equations like (5.38) does not matter in this dimension. One has to investigate one set of constraints, given by (5.33). (A major concern therefore is how one can achieve a corresponding reduction in four dimensions.)

Now consider the situation where the reality conditions have been solved, so that $P_n^{\dagger\alpha} = P_{-n}^{\alpha}$ and the canonical variables are those suggested by (5.31): one considers only the positive values of n. The only constraints are those given by (5.33), and one wants to express these conditions in terms of the positive-n coefficients. Note first that

$$\mathcal{L}_{-r} = \mathcal{L}_r^{\dagger} \quad , \tag{5.40}$$

so that one need only consider non-negative r initially. In this case it follows that

$$\begin{aligned}\mathcal{L}_r = &\sum_{m=1}^{\infty}[i(2m+r)P_{m+r}^{\alpha}P_{m,\alpha}^{\dagger} + 2P^{AB}\pi_{m+r,A}\pi_{m,B}^{\dagger}] \\ &+ \sum_{m=0}^{r}[imP_m^{\alpha}P_{r-m,\alpha} + P^{AB}\pi_{r-m,A}\pi_{m,B}] \quad .\end{aligned} \tag{5.41}$$

Thus

$$\begin{aligned}\mathcal{L}_{-r} = &\sum_{m=1}^{\infty}[-i(2m+r)P_{m+r}^{\dagger\alpha}P_{m,\alpha} + 2P^{AB}\pi_{m+r,A}^{\dagger}\pi_{m,B}] \\ &+ \sum_{m=0}^{r}[-imP_m^{\dagger\alpha}P_{r-m,\alpha}^{\dagger} + P^{AB}\pi_{r-m,A}^{\dagger}\pi_{m,B}^{\dagger}] \quad .\end{aligned} \tag{5.42}$$

In the real case one takes as the definition of Poisson brackets that which is suggested by (5.31):

$$2\{f,g\} = \left\{\frac{\partial f}{\partial\omega_0^A}\frac{\partial g}{\partial\pi_{0,A}} - \frac{\partial g}{\partial\omega_0^A}\frac{\partial f}{\partial\pi_{0,A}}\right\} + \sum_{r=1}^{+\infty}\left\{\frac{\partial f}{\partial P_r^{\alpha}}\frac{\partial g}{\partial P_{r,\alpha}^{\dagger}} - \frac{\partial g}{\partial P_r^{\alpha}}\frac{\partial f}{\partial P_{r,\alpha}^{\dagger}}\right\} \quad . \tag{5.43}$$

It is then a straightforward but somewhat lengthy calculation (which is therefore omitted) to verify that with this definition of the Poisson bracket, the algebra of the $\mathcal{L}_n$ closes, with, for all integers m and n,

$$\{\mathcal{L}_m, \mathcal{L}_n\} = i(m-n)\mathcal{L}_{m+n} \quad . \tag{5.44}$$

One recovers the Lie algebra of $Diff(S^1)$ from the twistor Poisson bracket algebra. The two sets of constraints with $n \geq 0$ and $n \leq 0$ each generate a closed sub-algebra under Poisson brackets. Also, the constraint $\mathcal{L}_0$ defines the mass m of the system, via

$$\mathcal{L}_0 = \frac{1}{2\pi}m^2 + \sum_{n=1}^{\infty} 2inP_n^\alpha P_{n,\alpha}^\dagger = 0 \ . \tag{5.45}$$

Thus $m = 0$ if only P_0^α is non-vanishing, giving the correct correspondence with the case of null geodesics discussed before. This defines sensible canonical variables with which to consider the quantization: the only constraints are first class and the theory reduces appropriately to the null geodesic case.

Before discussing quantization, it is useful to consider the sense in which the framework described above is a prototype for other types of theory. If one restricts attention to space-times of Minkowskian signature, then one cannot adapt directly the covariant analysis given above to four dimensions, because there is no covariant way of eliminating the phase invariance. The same remarks apply to six dimensions. To use the same kind of approach one needs to consider dimensions in which the action of complex conjugation on spinors is similar to that in dimension three. The next even dimension in which this occurs is actually ten! It turns out that with a slight shift in our interpretation of the equations (such as allowing the spinor indices A to run from 1 to 16 instead of 1 to 2) and the insertion of some invariant 10-dimensional spinors, the formalism can be carried over in detail to the dimension ten case. Because of the periodicity associated with the reality conditions one anticipates similar constructions in Minkowski space in dimensions 18, 26 and so on.

In ten dimensions there are two kinds of twistor: real twistors and pure twistors. Whilst the latter type is probably the most appropriate geometrical notion of a twistor, here the first kind of quantity will be considered (see Shaw 1986b for further discussion of these two spaces). The two reduced spin spaces are 16-dimensional and complex conjugation is involutory and maps each reduced space to itself. The two spaces are naturally dual to one another, so a Dirac spinor can be thought of as a pair (α^A, β_A). The Γ-matrices are block anti-diagonal, with entries γ_{AB} and γ^{AB}, and each γ matrix is symmetric and real. The Clifford algebra is

$$\gamma^a_{AB}\gamma^{bBC} + \gamma^b_{AB}\gamma^{aBC} = 2g^{ab}\delta^C_A \ ,$$

where a runs from 1 to 10 and g_{ab} is the metric. The quantities

$$G^{ABCD} = \frac{1}{16}\gamma^{aAB}\gamma_a^{CD}$$

and corresponding quantities with indices raised and lowered with G and its inverse define the spinor translation of the metric. It satisfies the conditions

$$G^{A(BCD)} = 0 \ , \quad G^{ABCD} = G^{(AB)(CD)} \ .$$

A consequence of these relations is that a real vector V^a is null if and only if its spinor equivalent V^{AB} can be written as

$$V^{AB} = G^{ABCD}\pi_A\pi_B \quad ,$$

where π_A is real. (This is *not* equivalent to $V_{AB} = \pi_A\pi_B$.) One may define a real twistor as a pair $Z^\alpha = \{\omega^A, \pi_A\}$, and incidence with a space-time point x^{AB} is expressed, as before, by the condition

$$\omega^A + x^{AB}\pi_B = 0 \quad .$$

As before, a space-time point may be simultaneously incident with two twistors Z_1^α and Z_2^α if and only if

$$Z_1^\alpha Z_{2,\alpha} \equiv \omega_1^A\pi_{2,A} - \omega_2^A\pi_{1,A} = 0 \quad ,$$

and the symplectic from translates to twistor space though the identity

$$2d\omega^A \wedge d\pi_A = dp_{AB} \wedge dx^{AB} \quad , \quad p_{AB} = G_{AB}^{CD}\pi_C\pi_D \quad .$$

To deal with an open string one procedes in the same way, defining quasi-periodic curves $Z^\alpha(s)$ subject to

$$Z^\alpha(s)\dot{Z}_\alpha(s) = 0 \quad ,$$

with incidence relations corresponding to (5.10) and (5.11), and a symplectic structure formally identical to (5.27). The 3-dimensional theory does indeed carry over to the ten-dimensional case, provided one makes appropriate insertions of the G_{ABCD} spinor at various points. The details of the calculation may be easily reconstructed from the 3-dimensional results and therefore need not be given here.

6. Twistor Quantization II: Open Strings.

At this point it is helpful to collect together some relevant facts about the classical theory. In doing so it is convenient to rescale the twistor variables. Let $Z_n^\alpha = \sqrt{2}P_n^\alpha$, where $Z_0^{\dagger\alpha} = Z_0^\alpha$ and $Z_n^\alpha = \{\omega_n^A, p_{n,A}\}$. Then the symplectic structure is given by

$$\omega = d\omega_0^A \wedge dp_{0,A} - \sum_{n=1}^{\infty} dZ_{n,\alpha}^\dagger \wedge dZ_n^\alpha \tag{6.1}$$

and the constraints are given by

$$\begin{aligned}\mathcal{L}_r = &\sum_{m=1}^{\infty} i(m+\frac{r}{2})Z_{m+r}^\alpha Z_{m,\alpha}^\dagger + \frac{1}{2\pi}P^{AB}p_{m+r,A}p_{m,B}^\dagger \\ &+\sum_{m=0}^{r}\frac{im}{2}Z_m^\alpha Z_{r-m,\alpha} + \frac{1}{4\pi}P^{AB}p_{r-m,A}p_{m,B}\end{aligned} \tag{6.2}$$

and their complex conjugates. In (6.2) the total momentum P^{AB} is given by

$$P^{AB} = \frac{1}{2}p_0^A p_0^B + \sum_{n=1}^{\infty} p_n^{(A}p_n^{\dagger B)} \ . \tag{6.3}$$

One may write down a corresponding expression for the angular momentum tensor M^{ab} as a sum over modes. This is given in the space-time by

$$M^{ab} = \int_0^\pi d\sigma\ \{X^a X_{,\tau}^b - X^b X_{,\tau}^a\} \ , \tag{6.4}$$

the integral being taken over any constant-τ cross-section. This tensor may be expressed in spinor form as

$$M^{ab} = M^{(AK)(BL)} = \mu^{AB}\varepsilon^{KL} + \mu^{KL}\varepsilon^{AB} \ , \tag{6.5}$$

where μ^{AB} is both symmetric and real. In terms of the twistor loop coordinates defined by (5.23) one finds that

$$\mu^{AB} = \int_{-\pi}^{\pi} ds\ \Omega^{(A}\pi^{B)} \ , \tag{6.6}$$

and hence that

$$\mu^{AB} = \frac{1}{2}\omega_0^{(A}p_0^{B)} + \sum_{n=1}^{\infty}\frac{1}{2}\{\omega_n^{(A}p_n^{\dagger B)} + \omega_n^{\dagger(A}p_n^{B)}\} \ . \tag{6.7}$$

This result is another useful consequence of the change of variables given by (5.23): both the total momentum and the total angular momentum are a diagonal sum over the momenta associated with each mode.

One may adopt as the quantization rule the holomorphic one suggested by (6.1). The quantum states are "functions" $f(p_0^A, Z_n^\alpha)$ of a spinor and a sequence of twistors which are holomorphic in their twistor arguments. One makes the replacements

$$\omega_0^A \to i\frac{\partial}{\partial p_{0,A}} \ , \quad Z_{n,\alpha}^\dagger \to -i\frac{\partial}{\partial Z_n^\alpha} \ , \tag{6.8}$$

and require that the states $f(p_0, Z_n)$ satisfy the quantized versions of the constraints.

At this point one must face various issues concerning the quantum-mechanical forms of the constraint and momentum functions. Clearly both the momentum and angular momentum operators are free of factor-ordering ambiguities. Of the constraints only $\mathcal{L}_0$ contains a factor-ordering problem, resulting at this stage in an ambiguity in the mass (squared) operator. A related problem concerns the details of how one elevates the remaining constraints to the quantum level. Recall that at the classical level, since we have a real theory, then if $\mathcal{L}_r = 0$ for $r \geq 0$ then $\mathcal{L}_r = 0$ for $r \leq 0$, and vice versa. The constraints for r non-negative form a closed algebra under the action of Poisson brackets, as do those for r non-positive. Thus one considers quantization with the imposition of just "half" the constraints at the quantum level, since just half the constraints and reality characterize the classical theory.

To deal with both of these issues it is helpful to take a step back and consider again the quantization of the real null geodesics, but this time within the real picture. In this case the symplectic form is

$$\omega = d\omega_0^A \wedge dp_{0,A}$$

with no constraints and a momentum operator $P^{AB} = \frac{1}{2} p_0^A p_0^B$. There is absolutely no difficulty in quantizing this system. With the first part of (6.8) one realizes the states of the theory as functions $f(p_0^A)$. These represent the spinorial Fourier transform of the corresponding space-time fields, which one may recovers as

$$\phi(x) = \iint dp_{0,A} \wedge dp_0^A f(p_{0,C}) \exp\{-\frac{i}{2} p_{0,A} p_{0,B} x^{AB}\} \quad . \tag{6.9}$$

The integral is over real $p_{0,A}$ and $\phi(x)$ is manifestly positive-frequency. It is also massless. Now at the classical level the real null geodesics are contained within the set of quasi-periodic null curves (= open strings). It is therefore appropriate to make the same requirement at the quantum level, and so the following hypothesis is made:

The functions $f(p_0^A)$ are allowed quantum string states, and these states are massless.

This is a stringent requirement, and has a number of interesting consequences. First, it requires that the quantum-mechanical form of $\mathcal{L}_0$ is ordered precisely as in (5.44). Denoting the operator forms by carets, then

$$\hat{\mathcal{L}}_0 = \frac{1}{2\pi} \hat{m}^2 + \sum_{n=1}^{\infty} n \hat{E}_n \quad , \tag{6.10}$$

where $\hat{E}_n = Z_n^\alpha \frac{\partial}{\partial Z_n^\alpha}$ is the nth homogeneity operator. The operator $\hat{\mathcal{L}}_0$ differs from its normal-ordered form by an *infinite* constant:

$$: \hat{\mathcal{L}}_0 : = \hat{\mathcal{L}}_0 + 2 \sum_{n=1}^{\infty} n \quad , \tag{6.11}$$

and that this particular constraint has the consequence that any state which is homogeneous in each of its twistor arguments is necessarily a mass eigenstate.

Second, if $r < 0$ then

$$\hat{\mathcal{L}}_r f(p_{0,A}) \equiv \hat{\mathcal{L}}^\dagger_{-r} f(p_{0,A}) \equiv 0 \quad , \tag{6.12}$$

but if $r > 0$ then

$$\hat{\mathcal{L}}_r f(p_{0,A}) = \sum_{m=0}^{r} [\frac{im}{2} Z^\alpha_m Z_{r-m,\alpha} + \frac{1}{4\pi} P^{AB} p_{r-m,A} p_{m,B}] f(p_{0,A}) \quad . \tag{6.13}$$

Thus the hypothesis demands that one does not admit the *positive* r constraints when one chooses the holomorphic polarization. If instead one had chosen the anti-holomorphic polarization then one would have excluded the negative r constraints. Thus the constraints to impose are therefore the $\hat{\mathcal{L}}^\dagger_r$ for $r \geq 0$, and these are given by the quantized conjugates of (6.2); that is,

$$\hat{\mathcal{L}}^\dagger_r = \sum_{m=1}^{\infty} (m + \frac{r}{2}) Z^\alpha_m i \hat{Z}^\dagger_{m+r,\alpha} + \frac{1}{2\pi} P^{AB} p_{m,B} \hat{p}^\dagger_{m+r,A}$$

$$+ \sum_{m=0}^{r} \frac{-im}{2} \varepsilon^{\alpha\beta} \hat{Z}^\dagger_{m,\beta} \hat{Z}^\dagger_{r-m,\alpha} + \frac{1}{4\pi} P^{AB} \hat{p}^\dagger_{r-m,A} \hat{p}^\dagger_{m,B} \quad , \tag{6.14}$$

where $\hat{Z}^\dagger_{m,\beta} = -i\frac{\partial}{\partial Z^\beta_m}$ if $m > 0$ and $\{p_{0,B}, -i\frac{\partial}{\partial p_{0,B}}\}$ if $m = 0$. Thus $\hat{p}^\dagger_{m,B}$ is given explicitly as $-i\frac{\partial}{\partial \omega^B_m}$ if $m > 0$ and is multiplication by $p_{0,B}$ if $m = 0$.

One may check directly that the algebra of these operators is given by

$$[\hat{\mathcal{L}}^\dagger_r, \hat{\mathcal{L}}^\dagger_s] = (r - s)\hat{\mathcal{L}}^\dagger_{r+s} \quad , \tag{6.15}$$

for $r \geq 0$ and $s \geq 0$. Thus to define the states $f(p_0, Z_p)$ it is sufficient to impose just *three* constraints:

$$\hat{\mathcal{L}}^\dagger_0 f = 0 \quad , \tag{6.16}$$

$$\hat{\mathcal{L}}^\dagger_1 f = 0 \quad , \tag{6.17}$$

$$\hat{\mathcal{L}}^\dagger_2 f = 0 \quad , \tag{6.18}$$

the remaining constraints being generated by commutation of the operators and the use of (6.15).

It is straightforward to construct a host of exact solutions of the quantum constraint equations. The functions $f(p_{0,A})$ are the basic allowed states. The next set of states to consider are functions $f(p_{0,A}, Z^\alpha_1)$, independent of Z^α_n for $n \geq 2$. It is easy to see that in this case

$$\hat{\mathcal{L}}^\dagger_k f \equiv 0 \ , \ k \geq 3 \ . \tag{6.19}$$

Writing $Z_1^\alpha = \{\omega_1^A,\ \pi_{1,A}\}$ the momentum operator acting on such functions is

$$P_{AB} = \frac{1}{2} p_{0,A} p_{0,B} - i p_{1,(A} \frac{\partial}{\partial \omega_1^{B)}} \tag{6.20}$$

so that the mass-squared operator is

$$\hat{m}^2 = -i p_0^A p_0^B p_{1,A} \frac{\partial}{\partial \omega_1^B} + \frac{1}{2} p_1^B p_1^C \frac{\partial}{\partial \omega_1^B} \frac{\partial}{\partial \omega_1^C} \ . \tag{6.21}$$

The constraints are then

$$\hat{E}_1 f + \frac{1}{2\pi} \hat{m}^2 f = 0 \ , \tag{6.22}$$

$$p_{0,A} \frac{\partial f}{\partial p_{1,A}} + i \frac{\partial^2 f}{\partial p_{0,A} \partial \omega_1^A} + \frac{1}{2\pi} p_{1,A} \frac{\partial}{\partial \omega_1^A} p_{0,B} \frac{\partial f}{\partial \omega_1^B} = 0 \ , \tag{6.23}$$

$$\left(p_{0,A} \frac{\partial}{\partial \omega_1^A} \right)^2 f = 0 \ . \tag{6.24}$$

These equations can be simplified as follows. Define η by

$$\eta = \frac{p_{0,A} \omega_1^A}{p_{0,B} p_1^B} \ . \tag{6.25}$$

This variable is obviously homogeneous of degree zero in each of $p_{0,A}$ and Z_1^α. The condition (6.24) may be solved by writing

$$f = G(\eta, p_{0,A}, p_{1,A}) \ . \tag{6.26}$$

Furthermore, the Z_1 homogeneity operator acting on f is now

$$p_{1,A} \left(\frac{\partial G}{\partial p_{1,A}} \right)_{\eta, p_0} \tag{6.27}$$

and the mass-squared operator is now given by

$$\hat{m}^2 f = \frac{1}{2} \left(\frac{\partial^2 G}{\partial \eta^2} \right)_{p_0, p_1} \ . \tag{6.28}$$

The remaining constraints are then (dropping the obvious subscripts)

$$p_{1,A} \frac{\partial G}{\partial p_{1,A}} + \frac{1}{4\pi} \frac{\partial^2 G}{\partial \eta^2} = 0 \ , \tag{6.29}$$

$$\left(p_{0,A} \frac{\partial}{\partial p_{0,A}} + 1 \right) \frac{\partial G}{\partial \eta} - i (p_{0,B} p_1^B)^{-1} p_{0,A} \frac{\partial G}{\partial p_{1,A}} = 0 \ . \tag{6.30}$$

The solution space to this system contains both massless and massive states. For a massless state G must be linear in η:

$$G = A(p_{0,A}, p_{1,A}) + \eta B(p_{0,A}, p_{1,A}) \ , \tag{6.31}$$

where A and B are each homogeneous of degree zero in $p_{1,A}$. Some manipulations with (6.30) shows that A and B are in fact independent of $p_{1,A}$ and that B must be homogeneous of degree -1 in $p_{0,A}$:

$$G = A(p_{0,A}) + \eta B(p_{0,A}) \ , \quad p_{0,A}\frac{\partial B}{\partial p_{0,A}} = -B \ . \tag{6.32}$$

To obtain massive states one proceeds similarly. Let G be homogeneous of degree $-k$ (k non-negative) in its $p_{1,A}$ dependence. The constraints give

$$\frac{\partial^2 G}{\partial \eta^2} = 4\pi k G \tag{6.33}$$

and the mass $m = \sqrt{2\pi k}$. Now set $G = \exp(\pm\sqrt{2}m\eta)H(p_{0,A}, p_{1,A})$, where

$$\left(p_{1,A}\frac{\partial}{\partial p_{1,A}} + k\right) H = 0 \ , \tag{6.34}$$

$$\left(p_{0,A}\frac{\partial}{\partial p_{0,A}} + 1\right) H \pm \frac{i}{\sqrt{2}m}\frac{1}{p_{0,B}p_1^B} p_{0,A}\frac{\partial H}{\partial p_{1,A}} = 0 \ . \tag{6.35}$$

These last two conditions can be solved easily by writing

$$H = (p_{0,A}p_1^A)^{-k}\phi(p_{0,A}) \ , \tag{6.36}$$

where ϕ is homogeneous of degree $k-1$. Thus the massive states of this type correspond to functions homogeneous in each of their spinor arguments. This is a particular case of a quite general feature. Although the P^α are coordinates for R^4, the quantization is actually taking place on RP^3. The symplectic form given by (5.27) is degenerate in directions given by vectors $\delta_i P^\alpha(s) = \phi_i(s)P^\alpha(s)$, so that it is only non-degenerate on the tangent space of the loop space of the projective space RP^3. It is, however, more convenient to quantize on the non-projective space and observe the homogeneous character of the resulting states.

One can build up more complicated states by considering functions of a spinor and several twistors. In this way twistor states of various mass and spin can be constructed. A more thorough examination of the resulting spectrum of states will be given elsewhere. Note however that by making appropriate restrictions on the homogeneities of the functions involved it appears that we one exclude tachyonic states from the theory. It is not known at present whether there is some natural and minimal universal restriction on the twistor states which excludes tachyonic states. Of course, by making the functions have zero or negative homogeneity in each of the twistors Z_n^α the mass-squared becomes non-negative, but some weaker condition may suffice.

In the present approach the classical ground state (null geodesic) is quantized correctly as a massless system, which is in sharp contrast with other conventional approaches. In three dimensions this can also be achieved within the light-cone gauge approach. The fact that the Lorentz algebra closes identically when there is only one transverse state means that it is *not* necessary that the theory contain a tachyon. This is certainly consistent with the preliminary results from the present approach.

The other issue of considerable importance in this context is the notion of an appropriate inner product on the states. At present it is not known what form this might take. The inner products constructed in twistor theory, when written down in the twistor space, are of a quite different character from those constructed in conventional quantum theory, and in the case of simple massless fields arise as contour integrals over regions of twistor space, rather than as inner products with respect to some L^2 norm. The conventional notion of a norm fails for example in four dimensions because the inner product on twistor space is not positive definite, and one is dealing with homogeneous functions for which the usual idea of a norm is inadequate. One anticipates some generalization of the contour integral inner product to the string states, and then one may be able to discuss in detail the status of negative-norm states within the twistor model.

It is also of considerable interest to generalize the model described here to other dimensions. The generalization to dimension ten appears to be particularly straightforward. The difficulties with dimensions four and six parallel closely those with the complex picture in dimension three. The essential difficulty is that the infinite set of complex twistors represent far too many variables to describe the true degrees of freedom of the string. This is mainly the result of the phase invariance of the complex twistor description. As well as the reparameterizations which lead to the Virasoro algebra one has an additional invariance group of maps of the circle into $U(1)$, since the twistor representation of the string is invariant under multiplication by $\exp\{i\phi(s)\}$ for $\phi(s)$ an arbitrary periodic function of s. In the case of three dimensions we may eliminate this invariance before quantization by making the twistors real. This could be called the "real gauge". An analogue of this choice of gauge is needed for dimension four before the twistor quantization can be discussed in that dimension. The possible gauge choices and the nature of the resulting theory will be discussed elsewhere, as will the possibility of defining the quantum theory by imposing all the constraints generated by commutation. The phase invariance is also reflected in the huge degeneracy of the symplectic structure (5.29), which is degenerate with respect to tangent vectors defining phase rotations or real rescalings of $P^{\alpha}(s)$. In three dimensions one had only to deal with the real rescalings. The quantization in four dimensions is really of loops in the projective space CP^3, and the prevailing difficulties stem mainly from attempting to use the complex homogeneous coordinates as canonical variables.

It is appropriate to conclude by describing in more detail the quantum constraint algebra. This is one of the more curious features of the twistor model. The algebra (6.15) has been given for $r \geq 0$ and $s \geq 0$ only, and involves the operators $\hat{\mathcal{L}}_r^\dagger = \hat{\mathcal{L}}_{-r}$. The full algebra is a central extension of the Poisson bracket algebra (5.44), and takes the form

$$[\hat{\mathcal{L}}_r, \hat{\mathcal{L}}_s] = (s-r)\hat{\mathcal{L}}_{r+s} + c(r)\delta_{r,-s} \quad . \tag{6.37}$$

One may compute $c(r)$ by evaluating the commutator against the simple states $f(p_{0,A})$, using expressions (6.13) and (6.14). One obtains a non-zero $c(r)$ because the second derivatives in the finite sum in (6.14) act on the quadratic terms in (6.13) to give a scalar contribution to the commutator. One finds that

$$c(r) = -\frac{r}{6}(r^2+2) \quad . \tag{6.38}$$

Now the central charge in the Virasoro algebra is always an odd cubic, and the linear term can be adjusted by adding a constant to $\hat{\mathcal{L}}_0$. However, the coefficient of r^3 is an invariant and is given here as 1/6. In a standard *space-time* covariant quantization in dimension D the coefficient is $D/12$. Thus the (twistor) Virasoro algbra in dimension *three* corresponds to the spacetime algebra obtained in dimension two.

The algebra given by (6.37-8) is not quite in standard form. If one defines

$$V_m = \hat{\mathcal{L}}_{-m} = \hat{\mathcal{L}}^{\dagger}_m, \quad m \neq 0 \ ,$$

$$V_0 = \hat{\mathcal{L}}_0 + \frac{1}{4} \ ,$$

then the algebra becomes

$$[V_m, V_n] = (m-n)V_{m+n} + \frac{d}{12}(m^3 - m)\delta_{m,-n} \tag{6.39}$$

and the states $f(p, Z)$ satisfy

$$[V_0 - h]f(p,Z) = 0 \ , \quad V_m f(p,Z) = 0 \ , \quad m \geq 1 \ , \tag{6.40}$$

where $d = 2$ and $h = \frac{1}{4}$.

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