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The Bethe - Salpeter Equation

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Starting point: Consider a field ψ operator for protons and a field φ for neutrons coupled to each other through a real scalar meson field A. This is of course only a simple model, but the argument can be generalized to any kind of interacting fields. We use the following Lagrangian

$$\mathcal{L} = \mathcal{L}_\psi + \mathcal{L}_\varphi + \mathcal{L}_A + \mathcal{L}_{int}$$

$$\mathcal{L}_\psi = -\tfrac{1}{4}\left[\bar\psi(x), \left(\gamma\frac{\partial}{\partial x} + m\right)\psi(x)\right] - \tfrac{1}{4}\left[-\frac{\partial\bar\psi(x)}{\partial x_\nu}\gamma_\nu + m\bar\psi(x), \psi(x)\right]$$

$$\mathcal{L}_\varphi = -\tfrac{1}{4}\left[\bar\varphi(x), \left(\gamma\frac{\partial}{\partial x} + m\right)\varphi(x)\right] - \tfrac{1}{4}\left[-\frac{\partial\bar\varphi(x)}{\partial x_\nu}\gamma_\nu + m\bar\varphi(x), \varphi(x)\right]$$

$$\mathcal{L}_A = -\tfrac{1}{2}\left[\frac{\partial A}{\partial x_\nu}\frac{\partial A}{\partial x_\nu} + \mu^2 A^2\right]$$

$$\mathcal{L}_{int} = \frac{ig}{2} A\left(\left[\bar\psi(x), \psi(x)\right] + \left[\bar\varphi(x), \varphi(x)\right]\right).$$

The equations of motion can now be written down

$$\left(\gamma\frac{\partial}{\partial x} + m\right)\psi = igA\psi \equiv f^P(x)$$

$$\left(\gamma\frac{\partial}{\partial x} + m\right)\varphi = igA\varphi \equiv f^N(x)$$

$$(\Box - \mu^2)A = -\frac{ig}{2}\left([\bar\psi, \psi] + [\bar\varphi, \varphi]\right) \equiv -j$$

Canonical commutators:

For $x_0 = x_0'$ we have

$$[A(x), \psi(x')] = [A(x), \varphi(x')] = [A(x), A(x')] = 0$$

$$[\dot A(x), \psi(x')] = [\dot A(x), \varphi(x')] = 0$$

$$[A(x), \dot A(x')] = i\delta(\bar x - \bar x')$$

$$\{\bar\psi(x), \varphi(x)\} = \{\bar\psi(x), \bar\varphi(x)\} = \cdots = 0$$

$$\{\bar{\psi}_\alpha(x), \psi_\beta(x')\} = (\gamma_4)_{\beta\alpha}\, \delta(\bar{x} - \bar{x}')$$

$$\{\bar{\varphi}_\alpha(x), \varphi_\beta(x')\} = (\gamma_4)_{\beta\alpha}\, \delta(\bar{x} - \bar{x}')$$

Define displacement operators P_μ from the Lagrangian and introduce such a representation that every vector in the Hilbert space is an eigenvector of all P's.

Definition: The fundamental quantity in the B.S.equation is the "wave-function" X defined by

$$\chi_n(1,2) = \langle 0 | [\psi(1), \varphi(2)] | n \rangle$$

where $\langle 0|$ is the vacuum and $|n\rangle$ is an arbitrary state.

Differential equation for X:

We obviously have

$$\left(\gamma^P \frac{\partial}{\partial x_1} + m\right)\left(\gamma^N \frac{\partial}{\partial x_2} + m\right)\chi_n(1,2) = \langle 0 | [f^P(1), f^N(2)] | n \rangle$$

$$= -g^2 \langle 0 | [A(1)\psi(1), A(2)\varphi(2)] | n \rangle .$$

We now have identically

$$[A(1)\psi(1), A(2)\varphi(2)] = \tfrac{1}{2}\Big(\{A(1)A(2)\}[\psi(1)\varphi(2)] +$$
$$+ \{\psi(1)\varphi(2)\}[A(1)A(2)] + A(1)[\psi(1)A(2)]\varphi(2) -$$
$$- A(2)[\varphi(2)A(1)]\psi(1)\Big) .$$

For A we can use the equation

$$A(x) = A^{(0)}(x) - \int_{-\infty}^{x} dx'\, \Delta(x-x')\, j(x') =$$
$$= -\int_{x'}^{x} \Delta(x-x'')\, j(x'')\, dx'' - \int_{x_0''=x_0'} d^{(3)}x'' \left(\dot{A}(x'')\Delta(x-x'') + A(x'')\dot{\Delta}(x-x'') \right)$$

with $[A^{(0)}(x), A^{(0)}(x')] = -i\,\Delta(x'-x)$

$$\langle 0 | \{A^{(0)}(x), A^{(0)}(x')\} | 0 \rangle = \Delta^{(1)}(x'-x) .$$

So far everything is exact. We now approximate the right-hand side of our equation for X by expanding it in powers of $A^{(0)}$ (or in powers of Δ and $\Delta^{(1)}$!) and dropping everything except the first non-vanishing term. The physical meaning of this expansion is somewhat obscure, but it nevertheless gives us a definite mathematical scheme to work with. We now have

$$[\psi(1)A(2)] = -\int_{x'}^{x''}\Delta(23)[\psi(1)j(3)]\,dx'''$$

$$A(1)[\psi(1)A(2)] \approx -\int_{x'}^{x''}\Delta(23)A^{(0)}(1)[\psi(1)j(3)]\,dx''' .$$

If we further take for $|n\rangle$ a state with no free mesons in it (for $t=-\infty$!) we get

$$\langle 0|A(1)[\psi(1)A(2)]\varphi(2)|n\rangle \sim \Delta\cdot\langle 0|A^{(0)}(1)[\cdots]\cdots|n\rangle \sim \Delta^2 = 0$$

The last factor appears, because the meson created by $A^{(0)}$ must be annihilated again and thus must yield another Δ-function.

In a similar way we obtain

$$\langle 0|A(2)[\varphi(2)A(1)]\psi(1)|n\rangle \sim \Delta^2 \sim 0$$

$$\langle 0|\{A(1),A(2)\}\cdot[\psi(1)\varphi(2)]|n\rangle \sim \langle 0|\{A^{(0)}(1)A^{(0)}(2)\}[\psi(1)\varphi(2)]|n\rangle \sim$$

$$\sim \Delta^{(1)}(12)\langle 0|[\psi(1)\varphi(2)]|n\rangle = \Delta^{(1)}(12)\chi_n(1,2)$$

$$[A(1),A(2)] \sim -i\Delta(21)$$

$$\{\psi(1)\varphi(2)\} = \int_{x'}^{x''}\{\psi(1),S^N(23)f^N(3)\}\,dx'''$$

$$= \int_{x'}^{x''}dx'''\int_{x'''}^{x'}dx^{IV}\{S^P(14)f^P(4),S^N(23)f^N(3)\} -$$

$$- i\int_{x'}^{x''}dx'''\int_{x_0^{IV}=x_0'''}d^{(3)}x^{IV}\{S^P(14)\gamma_4^P f^P(4),S^N(23)f^N(3)\} \sim \Delta$$

$$\therefore\ \langle 0|[A(1)A(2)]\cdot\{\psi(1)\varphi(2)\}|n\rangle \sim \Delta^2 \sim 0 .$$

Hence

$$(\gamma^P \frac{\partial}{\partial x_1} + m)(\gamma^N \frac{\partial}{\partial x_2} + m)\chi_m(1,2) = -\frac{g^2}{2}\Delta^{(1)}(12)\chi_m(1,2).$$

Note: The definition of $\chi_m(1,2)$ which we have used is not the only one possible. It is e.g. also reasonable to take

$$\tilde{\chi}_m(1,2) = \langle 0|P(\psi(1)\varphi(2))|m\rangle \varepsilon(12)$$

where P is the usual time-ordering product. In that case we get

$$(\gamma^P \frac{\partial}{\partial x_1} + m)(\gamma^N \frac{\partial}{\partial x_2} + m)\tilde{\chi}_m(1,2) = -\frac{g^2}{2}\Delta_F(1,2)\tilde{\chi}_m(1,2).$$

We observe that χ and $\tilde{\chi}$ are the same function of x' and x'' for space-like distances (apart from a factor $\frac{1}{2}$ which is not very interesting) but that they differ for time-like distances. The fact that the "wave-function" is not unique for time-like distances must be remembered when one discusses the physical interpretation of the formalism.

Integral equation for χ_m.

The differential equation above can be formally integrated to give e.g.

$$\chi_m(1,2) = F(1,2) - \frac{g^2}{2}\iint dx''' dx^{iv} S_R^P(13) S_R^N(24)\chi_m(34)\Delta^{(1)}(34)$$

where $(\gamma^P \frac{\partial}{\partial x_1} + m)(\gamma^N \frac{\partial}{\partial x_2} + m)F(1,2) = 0$.

The physical meaning of the function $F(1,2)$ is, however, not easy to find in this way. It is therefore more convenient to start with the integrated equations of motion

$$\psi(x) = \psi^{(0)}(x) - \int S_R^P(x-x')f^P(x')dx'$$
$$\varphi(x) = \varphi^{(0)}(x) - \int S_R^N(x-x')f^N(x')dx'$$

$$[\psi(1)\varphi(2)] = [\psi^{(0)}(1)\varphi^{(0)}(2)] - \int S_R^P(13)[\rho^P(3)\varphi^{(0)}(2)]\,dx''' - \int [\psi^{(0)}(1), S_R^N(24)\rho^N(4)]\,dx^{iv} + \iint S_R^P(13)\,S_R^N(24)[\rho^P(2), \rho^N(4)]\,dx'''dx^{iv}$$

$$\chi_m(1,2) \cong \chi_m^{(0)}(1,2) - \int S_R^P(13)\langle 0|[\rho^P(3)\varphi^{(0)}(2)]|m\rangle dx''' - \int \langle 0|[\psi^{(0)}(1), S_R^N(24)\rho^N(4)]|m\rangle dx^{iv} - \frac{g^2}{2}\iint S_R^P(13)\,S_R^N(24)\Delta^{(1)}(34)\chi_m(34)\,dx'''dx^{iv}$$

$$\chi_m^{(0)}(1,2) = \langle 0|[\psi^{(0)}(1), \varphi^{(0)}(2)]|m\rangle .$$

If no free nucleons are present in the state $|m\rangle$ we get $\chi_m^{(0)} = 0$. The second and the third terms, however, are not zero. If we treat them in the same way as above we obtain

$$\chi_m(1,2) = -\frac{g^2}{2}\iint dx'''dx^{iv}\Big(S_R^P(13)\,S_R^N(24)\,\Delta^{(1)}(34) + S_R^P(13)\,S^{N(1)}(24)\,\Delta_R(34) + S^{P(1)}(13)\,S_R^N(24)\,\Delta_A(34)\Big)\chi_m(34).$$

<u>The eigenvalue problem:</u>

We now introduce the "center-of-gravity" coordinate $X = \frac{x_1 + x_2}{2}$, the relative coordinate $x = x_1 - x_2$ and write

$$\chi_m(1,2) = e^{iKX} f_m(x).$$

If we further take only the large components into account we get

$$e^{iKX_{12}} f(x_{12}) \cong -\frac{g^2}{2}\frac{4m^2}{(2\pi)^{11}} \iiint dp\,dp'\,d\kappa \iint dX_{34}\,dx_{34} \cdot$$

$$\cdot\, e^{iKX_{34}} f(x_{34})\, e^{ip(13)+ip'(24)+i\kappa(34)} \times$$

$$\times \left\{ \left(\frac{\delta(p^2+m^2)}{p'^2+m^2} + \frac{\delta(p'^2+m^2)}{p^2+m^2} \right) \frac{1}{\kappa^2+\mu^2} + \frac{\delta(\kappa^2+\mu^2)}{(p^2+m^2)(p'^2+m^2)} \right\}.$$

Introducing $f(q)$ with the aid of

$$f(x) = \frac{1}{(2\pi)^4} \int e^{iqx} f(q)\,dq$$

$f(q)$ becomes

$$f(q) = -\frac{2m^2 g^2}{(2\pi)^3} \int d\kappa\, f(q+\kappa) \Bigg\{$$

$$\frac{1}{\kappa^2+\mu^2} \left(\frac{\delta((\frac{K}{2}-q)^2+m^2)}{(\frac{K}{2}+q)^2+m^2} + \frac{\delta((\frac{K}{2}+q)^2+m^2)}{(\frac{K}{2}-q)^2+m^2} \right) +$$

$$+ \frac{\delta(\kappa^2+\mu^2)}{[(\frac{K}{2}-q)^2+m^2][(\frac{K}{2}+q)^2+m^2]} \Bigg\}.$$

In the special coordinate system where $K = (0,0,0,i(2m-E))$ we have $\frac{K^2}{4}+m^2 \approx mE$, $(E \ll m)$.
If we put the two times in $x_\mu(1,2)$ equal we get $x_0 = 0$.
The interesting quantity is then $\varphi(\bar{q}) = \int f(q)\,dq_0$.
If we further make the approximation

$$\int d\kappa\, f(q+\kappa) F(\kappa^2+\mu^2) \approx \int d^{(3)}\bar{\kappa}\, \varphi(\bar{q}+\bar{\kappa}) F(\bar{\kappa}^2+\mu^2),$$
$$(E_{rel} \ll \mu),$$

we get

$$\varphi(\bar{q}) = -\frac{2m^2 g^2}{(2\pi)^3} \int \frac{d^{(3)}\bar{\kappa}\,\varphi(\bar{q}+\bar{\kappa})}{\bar{\kappa}^2+\mu^2} \int dq_0 \left(\frac{\delta(A+2mq_0-q_0^2)}{A-2mq_0-q_0^2} + \frac{\delta(A-2mq_0-q_0^2)}{A+2mq_0-q_0^2} \right)$$

with

$$A = \bar{q}^2 + mE > 0$$

After some simple calculations we obtain

$$\varphi(\bar{q}) = -\frac{g^2}{(2\pi)^3} \frac{m}{A} \int \frac{d^{(3)}\bar{\kappa}\,\varphi(\bar{q}+\bar{\kappa})}{\bar{\kappa}^2+\mu^2}$$

or

$$\left(\frac{\bar{q}^2}{m}+E\right)\varphi(\bar{q}) = -\frac{g^2}{(2\pi)^3} \int \frac{d^{(3)}\bar{\kappa}\,\varphi(\bar{q}+\bar{\kappa})}{\bar{\kappa}^2+\mu^2}$$

This is the usual Schrödinger equation with the Yukawa potential. It is an eigenvalue equation for E if we introduce the extra condition

$$\int |\varphi(\vec{q})|^2 d^{(3)}\vec{q} = \text{constant}$$

Note: The term $\frac{\delta(\varkappa^2+\mu^2)}{(\ldots)(\ldots)}$ gives no contribution in this approximation. Our function is thus really a solution of the equation

$$\left(\gamma^P \frac{\partial}{\partial x_1} + m\right)\left(\gamma^N \frac{\partial}{\partial x_2} + m\right)\chi_n(1,2) = 0 \quad !!!$$

The physical significance of this is rather obscure!

What we know about the Bethe-Salpeter method:

1) The formal mathematics is quite clear.

2) The "ladder-approximation" gives in the non-relativistic limit an eigenvalue problem for the binding energy. The equation obtained is identical with the usual "adiabatic" Schrödinger equation.

What we do not know about the Bethe-Salpeter method.

1) Is the equation really an eigenvalue problem also in the extreme relativistic case?

2) What is the physical meaning of the approximations made? How far can the "corrections" to the "adiabatic" result be trusted?

3) What is the physical meaning of the "wave-function "?

Of these three questions the last one seems to be least important and the answer is possibly "none"! The first question is not unimportant. It is possible that a closer inspection of the connection between the Dancoff method and the Bethe-Salpeter method could answer this question. Let us guess that the answer is "yes". The second question is certainly the most important one. Before that one is answered no reliable calculations can be made based on this method.

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