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CALCULATION OF THE PARAMETERS OF A QUADRUPOLE TRIPLET FROM THE TRANSFER MATRICES

by

B. Zotter

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1. INTRODUCTION

In the design of beam transfer channels - and insertions for synchrotrons or storage rings - one often encounters the problem of finding a combination of quadrupole lenses that will produce particular transfer matrices in both the horizontal and vertical planes. Traditionally, that problem has been solved by computer "matching" routines that operate by minimizing the deviations from the desired values as function of the lens strengths, lengths, and distances. Although powerful routines are available nowadays, the minimization of a function of many variables becomes quite time consuming with increasing dimensions, and the results depend strongly on the initial values which have to be guessed.

In this report we derive analytic solutions to the problem using three quadrupole lenses and four interquadrupolar distances. The first method is valid for infinitely short lenses, and leads to a second order equation for one of the parameters, from which all others can be calculated. The second method is for lenses of finite lengths, and leads to a higher order non-linear algebraic equation for one of the variables. While this still has to be solved by computer, the number of variables is reduced drastically and considerable saving in computer time can be achieved. In addition, the analytic solutions give insight into the influence of the various parameters, and into the limits of the regions where solutions are possible.

CALCULATION OF THE TRANSFER MATRIXES

Usually, the known quantities are the values (and derivatives) of the beta-functions and the momentum compaction at both ends of the channel. The transfer matrices in both planes can be found from these quantities by the expressions [1)]

$$M = \begin{pmatrix} \sqrt{\frac{\beta_2}{\beta_1}}(\cos\mu + \alpha_1 \sin\mu) & \sqrt{\beta_1\beta_2}\sin\mu \\ \frac{(1+\alpha_1\alpha_2)\sin\mu + (\alpha_2 - \alpha_1)\cos\mu}{\sqrt{\beta_1\beta_2}} & \sqrt{\frac{\beta_1}{\beta_2}}(\cos\mu - \alpha_2\sin\mu) \end{pmatrix} \qquad (1)$$

where $\alpha = -\frac{1}{2}\frac{d\beta}{ds}$, and μ is the phase shift. The horizontal phase shift can be found from the momentum compaction function (unless it is identically zero as for the vertical plane in the ISR). The calculation becomes quite simple when the channel begins - and/or ends - with a cross-over or a parallel beam. E.g. for $\alpha_{p1} = \alpha_{p2} = 0$ (cross-over on both ends) we find $M_{12} = 0$, and hence $\mu = m\pi$.

In the general case we can use the matrix transformation

$$\begin{pmatrix}\alpha_{p2} \\ \alpha_{\hat{p}2}\end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22}\end{pmatrix}\begin{pmatrix}\alpha_{p1} \\ \alpha_{\hat{p}1}\end{pmatrix} \qquad (2)$$

Upon substitution of the matrix elements, we can solve the two equations for cos μ and sin μ (this method is preferable to those using only one line of the matrix equations, which determine the phaseshift only up to a multiple of π). We write the two equations (2) as

$$\begin{aligned} u_2 &= u_1 \cos\mu + v_1 \sin\mu \\ v_2 - \alpha_2 u_2 &= (v_1 - \alpha_2 u_1)\cos\mu - (u_1 + \alpha_2 v_1)\sin\mu \end{aligned} \qquad (3)$$

where

$$u \equiv \frac{\alpha_p}{\sqrt{\beta}}, \qquad v \equiv \frac{\alpha\alpha_p + \beta\alpha_{\hat{p}}}{\sqrt{\beta}} \qquad (4)$$

Adding α_2 times the first to the second equation (3) yields

$$v_2 = v_1 \cos\mu - u_1 \sin\mu \qquad (5)$$

and the solutions become

$$\cos\mu = \frac{u_1u_2 + v_1v_2}{W}$$

$$\sin\mu = \frac{u_2v_1 - u_1v_2}{W} \qquad (6)$$

where

(7)

$$W = u_1{}^2 + v_1{}^2 = u_2{}^2 + v_2{}^2 \qquad (7)$$

is the invariant of the transformation (which has to be the same at each end of the channel when no bending magnets are present).

3. SHORT LENS TRIPLET

If we assume quadrupole lenses of vanishing lengths - but with a finite value of $g = \lim_{s \to 0} K.s$ - the transfer matrices simplify considerably. In the horizontal plane we have

$$M_Q = \begin{pmatrix} 1 & 0 \\ g & 1 \end{pmatrix}$$

while we only have to replace g by - g in the vertical plane. However, straightforward multiplication of three quadrupole matrices and four inter-quadrupolar driftspaces

$$M_s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$$

in the right order still yields very complicated expressions. We can simplify this by multiplying from the centre outwards as shown in Appendix A. We then find simple expressions for the quadrupole strengths g_2, g_4, g_6, and the two inner distances s_3, s_5 in terms of s_1 and s_7. These two are determined by the solutions of two nonlinear equations of second and third degree, if we assume a given length for the complete channel. However, we can prescribe one of the lengths s_1 or s_7, and determine the other by the solution of the second order equation. The total length is then the second unknown and is given by the other equation which is linear in this variable.

This procedure is much simpler, and permits determination of the regions where real solutions exist. A small computer program has been written that first calculates the transfer matrices (for several values of the vertical phase shift), then evaluates the limits of s_1 for which solutions are possible, and finally calculates the triplet parameters for several values inside this region. Naturally, solutions with negative distances are mathematically possible and have to be discarded.

While the short lens triplet is actually not realizable, it is a good starting point for finding a long lens triplet either by one of the existing matching routines, or with the program described in the next section.

4. LONG LENS TRIPLET

For a horizontally focussing quadrupole of finite length s, the transfer matrices in the two planes are given by

$$M_F = \begin{pmatrix} \cos\phi & s\,\dfrac{\sin\phi}{\phi} \\ -\,g\,\dfrac{\sin\phi}{\phi} & \cos\phi \end{pmatrix}$$

$$M_F^x = \begin{pmatrix} \cosh\phi & s\,\dfrac{\sinh\phi}{\phi} \\ g\,\dfrac{\sinh\phi}{\phi} & \cosh\phi \end{pmatrix}$$

where $\phi = s\sqrt{|K|}$, $g = s.K$. In this form we see the transition to the short lens form best $(s \to 0,\ \phi \to 0,\ \cos\phi \sim \cosh\phi \sim \frac{\sin\phi}{\phi} \sim \frac{\sinh\phi}{\phi} \to 1,$

g finite). For a defocussing quadrupole the two matrices are simply interchanged.

The method used for short lenses becomes rather complicated because there are no more zeros in the quadrupole matrices *). However, by using a trick described by Regenstreif [2] we can find manageable expressions for the product of a string of quadrupole and straight section matrices. This is used in Appendix B to reduce the problem to a single, nonlinear equation in the two straight-section lengths s_1 and s_7, when we prescribe the three values ϕ_i ($i = 2,4,6$). We can assume either s_1 or s_7, and calculate the other by a zero-finding routine on the computer. All other triplet parameters can then be calculated simply from these two quantities.

In the limit of $\phi_i \to 0$, the results agree with those for the short triplet. In practice we then have to vary the values of ϕ_i until we get acceptable quadrupole lengths and strenghts. Tracking of trajectories in the resultant triplet with any one of the existing routines (BEATCH, modified AGS) has been used to verify the correctness of the solutions.

5. CONCLUSIONS

The calculation of the parameters of a quadrupole triplet for given horizontal and vertical transfer matrices has been solved analytically. For (infinitely) short lenses, the problem can be reduced to a single quadratic equation, and both the solutions and their region of existence can be found directly. For lenses of finite length, the problem leads to a single equation of higher order, which can be solved rapidly by computer. There remain three free parameters in that case

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*) Still, the computer program NEWT uses just this method.

for which we have chosen the three values of the quadrupole length times the square roots of their strengths. These parameters can be varied to find acceptable values of the quadrupole strengths which can be realized in practice.

The main advantages of the analytic approach are saving in computation time, and independence from arbitrary initial guesses which often lead to difficulties in existing matching routines. Furthermore, the discriminant of the second order equation for the short lens case gives direct information about the existence of real solutions.

APPENDIX A

Short Lens Triplet

The horizontal transfer matrix is given by

$$M_H = (SS7)(QF6)(SS5)(QD4)(SS3)(QF2)(SS1) \qquad (A\ 1)$$

where we assume arbitrarily an FDF structure. This is not restrictive, however, as the sign of the focussing strength can alter an F lens into a D lens and vice-versa. The vertical transfer matrix is

$$M_V = (SS7)(QD6)(SS5)(QF4)(SS3)(QD2)(SS1) \qquad (A\ 2)$$

and the transfer matrices of the single elements are

$$SS_i = \begin{pmatrix} 1 & s_i \\ 0 & 1 \end{pmatrix} \qquad (A\ 3)$$

$$QF_i = \begin{pmatrix} 1 & 0 \\ -g_i & 1 \end{pmatrix} \qquad (A\ 4)$$

$$QD_i = \begin{pmatrix} 1 & 0 \\ g_i & 1 \end{pmatrix}$$

Rather than multiplying all seven elements directly, it appears expedient to do it in three steps for each:

$$M_H = (SS7)(\ P\ )(SS1)$$

$$P_H = (QF6)(\ Q\ )(QF2) \qquad (A\ 5)$$

$$Q_H = (SS5)(QD4)(SS3)$$

resp. QF's interchanged with QD's for the vertical case. In this manner we obtain the relations

$$
\begin{aligned}
m_{11} &= p_{11} + s_7 p_{21} \\
m_{12} &= p_{12} + s_1 p_{11} + s_7 p_{22} + s_1 s_7 p_{22} \\
m_{21} &= p_{21} \\
m_{22} &= p_{22} + s_1 p_{21}
\end{aligned}
\qquad (A\ 6)
$$

$$
\begin{aligned}
p_{11} &= q_{11} - g_2 q_{12} \\
p_{12} &= q_{12} \\
p_{21} &= q_{21} - g_6 q_{11} - q_2 q_{22} + g_2 g_6 q_{12} \\
p_{22} &= q_{22} - g_6 q_{12}
\end{aligned}
\qquad (A\ 7)
$$

$$
\begin{aligned}
q_{11} &= 1 + s_5 g_4 \\
q_{12} &= s_3 + s_5 - s_3 s_5 g_4 \\
q_{21} &= g_4 \\
q_{22} &= 1 + s_3 g_4
\end{aligned}
\qquad (A\ 8)
$$

and similar ones for the vertical matrix elements - which we characterize by asterisks - with all signs of g_i inverted. Of each group of four equations one is recurrent due to the condition of unity determinant for all transfer matrices.

Inversion of the equation is straightforward and yields

$$
\begin{aligned}
p_{11} &= m_{11} - s_7 m_{21} \\
p_{12} &= m_{12} - s_1 m_{11} - s_7 m_{21} + s_1 s_7 m_{21} \\
p_{21} &= m_{21} \\
p_{22} &= m_{22} - s_1 m_{21}
\end{aligned}
\qquad (A\ 9)
$$

$$q_{11} = p_{11} + g_2 p_{12}$$
$$q_{12} = p_{12} \qquad \text{(A 10)}$$
$$q_{21} = p_{21} + g_6 p_{11} + g_2 p_{22} + g_2 g_6 p_{12}$$
$$q_{22} = p_{22} + g_6 p_{12}$$

The last set of equations (A 8) permits calculation of the three parameters

$$s_3 = \frac{q_{22} - 1}{q_{21}} = \frac{q_{22}^x - 1}{q_{21}^x}$$
$$\text{(A 11)}$$
$$s_5 = \frac{q_{11} - 1}{q_{21}} = \frac{q_{11}^x - 1}{q_{21}^x}$$

$$g_4 = \quad q_{21} \quad = - q_{21}^x \qquad \text{(A 12)}$$

where the last column comes from the vertical transfer matrix calculation. Substitution of eq. (A 12) into both eqs. (A 11) yields

$$q_{11} + q_{11}^x = 2$$
$$\text{(A 13)}$$
$$q_{22} + q_{22}^x = 2$$

or, with eqs. (A 10)

$$g_2 = \frac{p_{11} + p_{11}^x - 2}{p_{12}^x - p_{12}}$$
$$\text{(A 14)}$$
$$g_6 = \frac{p_{22} + p_{22}^x - 2}{p_{12}^x - p_{12}}$$

Introducing these expressions into (A 12) then yields one equation involving only s_1 and s_7 through the matrix elements p_{ik}

$$(p_{21} + p_{21}^x) + \frac{(p_{11} + p_{11}^x - 2)(p_{22} ++ p_{22}^x - 2)}{(p_{12} - p_{12}^x)^2} = \qquad \text{(A 15)}$$

$$= \frac{p_{22} + p_{22}^x - 2}{p_{12} - p_{12}^x}(p_{11} - p_{11}^x) + \frac{p_{11} + p_{11}^x - 2}{p_{12} - p_{12}^x}(p_{22} - p_{22}^x)$$

We can rewrite this equation as

$$(p_{21} + p_{21}^x)(p_{12} - p_{12}^x)^2 - (p_{11} - p_{11}^x)(p_{22} - p_{22}^x)(p_{12} + p_{12}^x) + \qquad \text{(A 16)}$$

$$+ 4p_{12}(p_{11}^x - 1)(p_{22}^x - 1) + 4p_{12}^x(p_{11} - 1)(p_{22} - 1) = 0$$

in which form we see that the equation is quadratic both in s_1 and s_7, since $p_{11}^{(x)}$ and $p_{22}^{(x)}$ are linear in s_7 or s_1, respectively, while $p_{12}^{(x)}$ is linear in the product $s_1 s_7$ (and $p_{21}^{(x)}$ is constant).

Another equation for ℓ_1 and ℓ_7 can be obtained from the condition

$$s_1 + s_3 + s_5 + s_7 = \ell \qquad \text{(A 17)}$$

where ℓ is the (known) total length of the channel. Substituting into this expressions s_3 and s_5 from (A 11), (A 10) and (A 14) yields a third order equation in s_1 and s_7 to be solved together with the second order eq. (A 16). It is simpler to avoid that problem by considering ℓ as an unknown, and specifying either s_1 or s_7. Then the other quantity can be calculated from a single second order equation.

Assuming we know s_1, we introduce the unknown variable

$$z = m_{21}^x \, p_{11} \qquad \text{(A 18)}$$

to get the equation

$$\alpha z^2 + \beta z + \gamma = 0 \qquad \text{(A 19)}$$

with the coefficients

$$\alpha = 2(p_{22} + p_{22}^{x} - 2)\varepsilon$$

$$\beta = 4\beta_1 - (m_{21} - m_{21}^{x})(\beta_2 + 3\beta_3) + \delta\varepsilon(p_{22} + 3p_{22}^{x} - 4) \qquad \text{(A 20)}$$

$$\gamma = 4m_{21}m_{21}^{x}\,\gamma_1 + (m_{21} + m_{21}^{x})\gamma_2 + \delta[4\gamma_3 - p_{22}m_{21}^{x}(m_{21} + m_{21}^{x}) - p_{22}^{x}\,\gamma_4)] + \delta^2\varepsilon\, p_{22}^{x}$$

and

$$\delta = m_{21}m_{11}^{x} - m_{21}^{x}m_{11}$$

$$\varepsilon = m_{21}p_{22}^{x} + m_{21}^{x}p_{22} \qquad \text{(A 21)}$$

$$\beta_1 = m_{21}^{2} - 2m_{21}m_{21}^{x}p_{22}p_{22}^{x} + m_{21}^{x2}$$

$$\beta_2 = m_{21}p_{22} - m_{21}^{x}p_{22}^{x} \qquad \text{(A 22)}$$

$$\beta_3 = m_{21}p_{22}^{x} - m_{21}^{x}p_{22}$$

$$\gamma_1 = m_{21}p_{22} + m_{21}^{x}p_{22}^{x}$$

$$\gamma_2 = m_{21}^{2} - 6m_{21}m_{21}^{x} + m_{21}^{x2}$$

$$\gamma_3 = m_{21}^{x2} - m_{21}m_{21}^{x}p_{22}p_{22}^{x} \qquad \text{(A 23)}$$

$$\gamma_4 = 2m_{21}^{2} - 5m_{21}m_{21}^{x} + m_{21}^{x2}$$

Although these coefficients are rather complicated, it is quite simple to evaluate them on a computer and to find the solutions

$$z = \frac{-\beta \pm \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha} \qquad \text{(A 24)}$$

after testing that $\beta^2 > 4\alpha\gamma$, condition for the existence of real solutions. Finally, we find

$$s_7 = \frac{m_{11}m_{21}^{x} - z}{m_{21}m_{21}^{x}} \qquad \text{(A 25)}$$

from which we can then calculate the other distances s_3 and s_5, the total length ℓ and, the three quadrupole strengths g_2, g_4 and g_6. However, we have to discard solutions for which one or more distances are negative. This often leaves us with a unique solution for a given set of transfer matrices, and we can adjust the total length of the channel by altering s_1. The computer program TRIP performs all these operations, after calculating the transfer matrixes from given input and output values of the trajectory functions. One also has to assume a value for vertical phaseshift, which can be varied to find acceptable solutions. In order not to be asked to perform impossible transformations the program calculates the correct value of α_p' at the end of the channel from the invariant, which is not altered by quadrupoles or straight sections. The program requires as input six values at the beginning (β_H, α_H, β_V, α_V, α_p, α_p'), five values of the end (same except α_p'), initial values for s_1 and ϕ_V, steps Δs and $\Delta\phi_V$, and the number of steps K_s and K_ϕ It will then print the transfer matrices, phaseshift and possibly solutions for the triplet parameters $s_{1,3,5,7}$, $g_{2,4,6}$.

APPENDIX B

Thick Lens Triplet

The transfer matrices for the quadrupoles can be written

$$QF_i = \begin{pmatrix} a_i & b_i \\ c_i & a_i \end{pmatrix}$$

$$QD_i = \begin{pmatrix} a_i^x & b_i^x \\ c_i^x & a_i^x \end{pmatrix} \qquad \text{(B 1)}$$

where

$$a_i = \cos \phi_i \qquad a_i^x = \cosh \phi_i$$

$$b_i = \frac{\sin \phi_i}{\sqrt{|K_i|}} \qquad b_i^x = \frac{\sinh \phi_i}{\sqrt{|K_i|}} \qquad \text{(B 2)}$$

and

$$\phi_i = s_i \sqrt{|K_i|} \qquad \text{(B 3)}$$

In extension of an approach described by Regenstreif [2], we define the quantities

$$X = s_1 + \frac{a_2}{c_2}$$

$$Y = s_3 + \frac{a_2}{c_2} + \frac{a_4^x}{c_4^x}$$

$$Z = s_5 + \frac{a_6}{c_6} + \frac{a_4^x}{c_4^x} \qquad \text{(B 4)}$$

$$T = s_7 + \frac{a_6}{c_6}$$

and the corresponding starred equivalents ($s_i^x = s_i$, $a_i^{xx} = a_i$, $c_i^{xx} = c_i$).

The elements of the overall transfer matrices

$$M_H = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$
$$M_V = \begin{pmatrix} A^x & B^x \\ C^x & D^x \end{pmatrix} \qquad \text{(B 5)}$$

are then given by the rather simple expressions

$$A = c_2\ c_4^x\ c_6\ [Y(ZT - \frac{1}{c_2^2}) - \frac{T}{c_4^{x2}}]$$

$$B = c_2\ c_4^x\ c_6\ [(XY - \frac{1}{c_6^2})(ZT - \frac{1}{c_6^2}) - \frac{XT}{c_4^{x2}}]$$

$$C = c_2\ c_4^x\ c_6\ [YZ - \frac{1}{c_4^{x2}}] \qquad \text{(B 6)}$$

$$D = c_2\ c_4^x\ c_6\ [(XY - \frac{1}{c_2^2})Z - \frac{X}{c_4^{x2}}]$$

and the corresponding expressions with all terms starred (double star = no star). We form the simpler combinations

$$A - CT = - \frac{c_2\ c_4^x}{c_6}\ Y$$
$$D - CX = - \frac{c_4^x\ c_6}{c_2}\ Z \qquad \text{(B 7)}$$

As third equation we use the expressions for C. The fourth relation is recurrent because of the requirement of unity determinant.

If we assume the values of ϕ_i, we can calculate a_i and a_i^x, and - with $\delta_i \equiv \frac{\sin \phi_i}{\sinh \phi_i}$ -

$$c_i = - \delta_i\ c_i^x \qquad \text{(B 8)}$$

We shall rewrite the equations with the definition

$$x \equiv c_2\, X, \quad y \equiv c_2\, c_4^x\, Y, \quad z \equiv c_6\, c_4^x\, Z, \quad t = c_6\, T \tag{B 9}$$

(and starred equivalents). The six main equations then become

$$\begin{aligned} y &= Ct - Ac_6 & y^x &= C^x t^x - A^x c_6^x \\ z &= Cx - Dc_2 & z^x &= C^x x^x - D^x c_2^x \\ yz &= Cc_4^x + c_2\, c_6 & y^x z^x &= C^x c_4 + c_2^x\, c_6^x \end{aligned} \tag{B 10}$$

in which we can immediately replace all c_i^x by c_i with equation (B 8). The relations between the starred and unstarred auxiliary quantities follow from their definition (B 4) and equations (B 10). With

$$\alpha_i \equiv a_i + \delta_i\, a_i^x \tag{B 11}$$

$$\begin{aligned} x + \delta_2\, x^x &= \alpha_2 \\ \delta_4\, y - \delta_2\, y^x &= \alpha_4\, c_2 - \alpha_2\, c_4 \\ \delta_4\, z - \delta_6\, z^x &= \alpha_4\, c_6 - \alpha_6\, c_4 \\ t + \delta_6\, t^x &= \alpha_6 \end{aligned} \tag{B 12}$$

Equations (B 10) and (B 12) form 10 equations for 11 unknowns. As last equation we can add the requirement $\sum s_i = \ell$, if the total length ℓ is known. Again it is expedient not to prescribe ℓ, but either s_1 or s_7. We can then add one of the two equations following from (B 4)

$$\begin{aligned} x &= s_1\, c_2 + a_2 \\ t &= s_7\, c_6 + a_6 \end{aligned} \tag{B 13}$$

From the equations for x and t (B 12) we get

$$\begin{aligned} x^x &= a_2^x + s_1\, c_2^x = a_2^x - s_1\, c_2/\delta_2 \\ t^x &= a_6^x + s_7\, c_6^x = a_6^x - s_7\, c_6/\delta_6 \end{aligned} \tag{B 14}$$

and hence from (B 10)

$$y = y_o + y_1 c_6 \qquad y^x = y_o^x - y_1^x c_6/\delta_6$$
$$z = z_o + z_1 c_2 \qquad z^x = z_o^x - z_1^x c_2/\delta_2 \qquad \text{(B 15)}$$

where

$$y_o = Ca_6 \qquad y_o^x = C^x a_6^x$$
$$z_o = Ca_2 \qquad z_o^x = C^x a_2^x$$
$$y_1 = \ell_7 C - A \qquad y_1^x = \ell_7 C^x - A^x \qquad \text{(B 16)}$$
$$z_1 = \ell_1 C - D \qquad z_1^x = \ell_1 C^x - D^x$$

Substitution of the expressions (B 15) for y and z into the 2nd and 3rd equation (B 12) yields

$$\alpha_4 c_2 - \alpha_2 c_4 = q_1 + p_1 c_6$$
$$\alpha_4 c_6 - \alpha_6 c_4 = q_2 + p_2 c_2 \qquad \text{(B 17)}$$

where

$$q_1 = \delta_4 y_o - \delta_2 y_o^x$$
$$q_2 = \delta_4 z_o - \delta_6 z_o^x$$
$$p_1 = \delta_4 y_1 + \frac{\delta_2}{\delta_6} y_1^x \qquad \text{(B 18)}$$
$$p_2 = \delta_4 z_1 + \frac{\delta_6}{\delta_2} z_1^x$$

We can solve equations (B 17) for c_2 and c_6

$$c_2 = c_{2o} + c_{21} c_4$$
$$c_6 = c_{6o} + c_{61} c_4 \qquad \text{(B 19)}$$

where

$$c_{2o} = (\alpha_4\ q_1 + q_2\ p_1)/c_{oo}$$

$$c_{21} = (\alpha_4\ \alpha_2 + \alpha_6\ p_1)/c_{oo}$$

$$c_{6o} = (\alpha_4\ q_2 + q_1\ p_2)/c_{oo} \qquad \text{(B 20)}$$

$$c_{61} = (\alpha_4\ \alpha_6 + \alpha_2\ p_2)/c_{oo}$$

and

$$c_{oo} = \alpha_4{}^2 + p_1\ p_2$$

The expressions (B 19) we can substitute into the two quadratic equations (B 12) to get

$$Pc_4{}^2 + Qc_4 + R = 0$$

$$P^x c_4{}^2 + Q^x c_4 + R^x = 0 \qquad \text{(B 21)}$$

where

$$P = uv - \varepsilon_1 \qquad P^x = u^x v^x - \varepsilon_1$$

$$Q = uq + vp - \varepsilon_2 \qquad Q^x = u^x q^x + v^x p^x - \varepsilon_2^x \qquad \text{(B 22)}$$

$$R = pq - \varepsilon_3 \qquad R^x = p^x q^x - \varepsilon_3^x$$

and

$$u = y_1\ c_{61} \qquad u^x = y_1{}^x\ c_{61}$$

$$v = z_1\ c_{21} \qquad v^x = z_1{}^x\ c_{21} \qquad \text{(B 23)}$$

$$p = y_o + y_1\ c_{61} \qquad p^x = y_o{}^x + y_1{}^x\ c_{61}$$

$$q = z_o + z_1\ c_{61} \qquad q^x = z_o{}^x + z_1{}^x\ c_{61}$$

$$\varepsilon_1 = c_{21}\ c_{61} \qquad \varepsilon_2 = (c_{2o}\ c_{61} + c_{6o}\ c_{21}) - C/\delta_4 \qquad \text{(B 24)}$$

$$\varepsilon_3 = c_{2o}\ c_{6o} \qquad \varepsilon_2{}^x = (c_{2o}\ c_{61} + c_{6o}\ c_{21}) + \delta_2\delta_6 c^x$$

We can solve equations (B 21) for c_4 in two ways by multiplying the top equation with P^x, resp. R^x, the bottom with P, resp. R, to get

$$c_4 = -\frac{PR^x - RP^x}{PQ^x - QP^x} = -\frac{QR^x - RQ^x}{PR^x - RP^x} \qquad \text{(B 25)}$$

from which we finally get a single (nonlinear) equation for ℓ_1 or ℓ_7

$$(PR^x - RP^x) = (PQ^x - QP^x)(QR^x - RQ^x) \qquad \text{(B 26)}$$

which has been programmed to be solved by computer (TRIL). The analytic solutions for thin lenses can be used as initial values for a zero finding routine.

Once s_1 and s_7 are known, all other variables can be found directly: c_4 from equation (B 25), c_2 and c_6 from equations (B 19), s_3 and s_5 by combining equations (B 4), (B 10), and (B 16)

$$s_3 = \frac{\delta_4\ a_4^x}{c_4} - \frac{a_2}{c_2} - \frac{\delta_4}{c_2\ c_4}(y_o + y_1\ c_6)$$

$$s_5 = \frac{\delta_4\ a_4^x}{c_4} - \frac{a_6}{c_6} - \frac{\delta_4}{c_4\ c_6}(z_o + z_1\ c_2) \qquad \text{(B 27)}$$

Furthermore, for i = 2,4,6,

$$g_i = -\ c_i\ \phi_i/\sin\phi_i$$

$$|K_i| = c_i^2/\sin^2\ \phi_i \qquad \text{(B 28)}$$

$$s_i = \phi_i^2/g_i = -\frac{\phi_i\ \sin\ \phi_i}{c_i}$$

and the total length is

$$\ell = \sum_{i=1}^{7} s_i \qquad \text{(B 29)}$$

REFERENCES

1) Courant - Snyder; Theory of the AG Synchrotron, Annals of Physics 3, 1-48 (1958)

2) Regenstreif; Phase Space Transformations by Means of Quadrupole Multiplets, CERN Report 67-6 (1967)

FIGURES and TABLES

Table 1: Results for a typical testcase. The desired trajectory functions at the end of a channel are obtained with triplets found by the computer program TRIP. The first column under point 3) is the short-lens solution found from a second order equation, the next columns are solutions with increasing lens lengths - and hence decreasing strengths - derived from the short-lens solutions. The lens-lengths are increased either until one of the intermediate length becomes negative, or until the strengths are low enough.

Figure 1: Graph of the trajectory functions β_H, β_V, α_p through the channel obtained by analytic matching in the example of the table, corresponding to the case $\phi_2 = 0.6$.

Table 1: Results for a typical testcase

1) Desired trajectory functions on both ends of channel

	β_H	α_H	β_V	α_V	α_p	α_p'
in	11.45	- 4.68	11.45	- 4.68	0.	0.097
out	12.80	0.	21.30	0.	0.86	- 0.0625 *)

desired total length of channel 21.8 m.

2) Matrix elements for vertical phaseshift $\phi_V = 0.5$

$$M_H = \begin{pmatrix} -4.344 & 8.866 \\ 0.203 & -0.644 \end{pmatrix} \qquad \phi_H = 2.319 \text{ *)}$$

$$M_V = \begin{pmatrix} -1.863 & 7.487 \\ -0.294 & 0.643 \end{pmatrix}$$

3) Triplet solutions for $s_1 = 2.5$ m

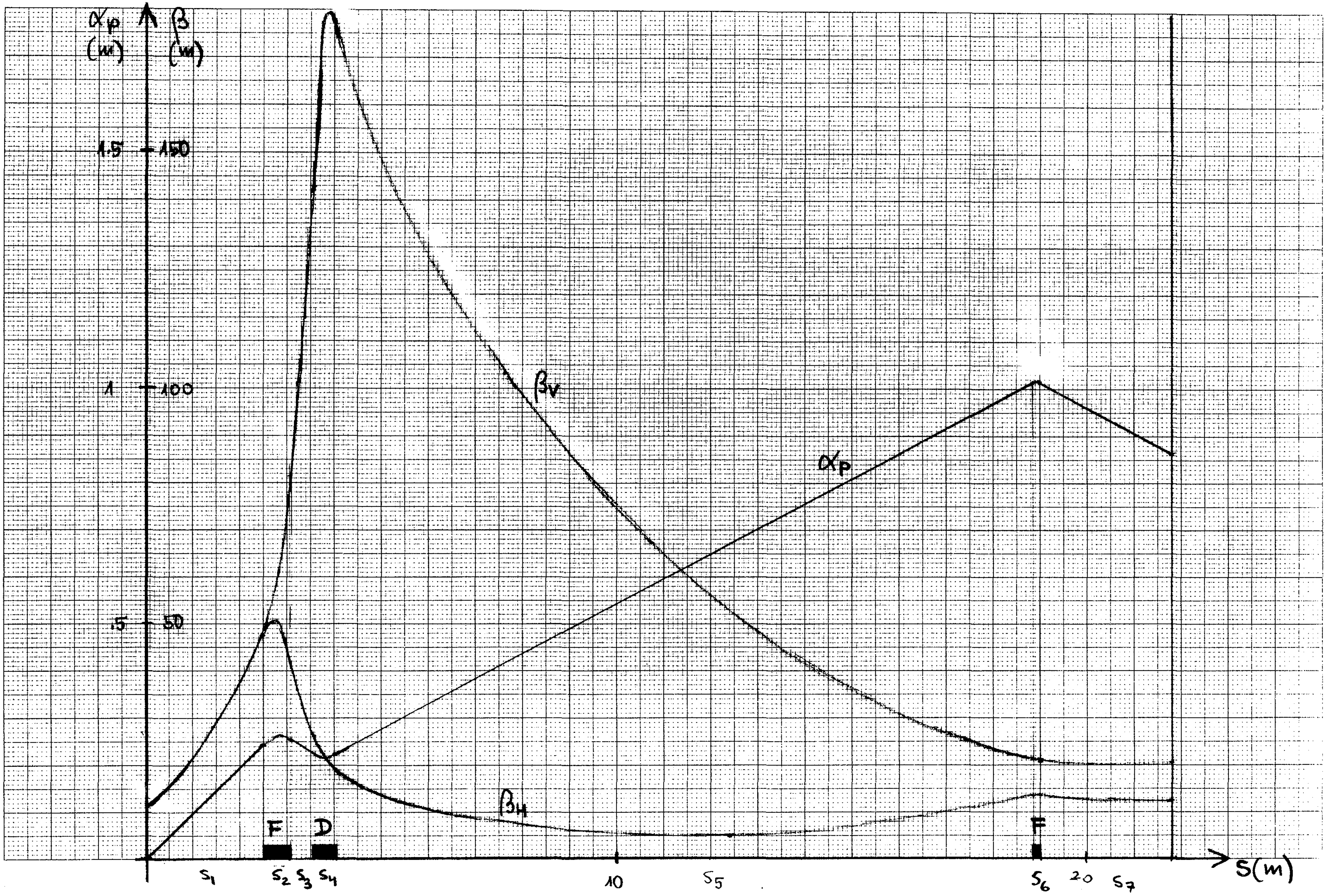
$\phi_2 = s_2\sqrt{K_2}$ **)	0	0.2	0.4	0.6	units
s_2	0.	0.076	0.289	0.586	m
s_3	1.270	1.170	0.888	0.474	"
s_4	0.	0.062	0.237	0.487	"
s_5	14.317	14.390	14.585	14.822	"
s_6	0.	0.017	0.064	0.123	"
s_7	3.074	3.032	2.922	2.788	"
ℓ	21.160	21.248	21.484	21.779	"
K_1 (g_1)	(- 0.516)	- 6.890	- 1.917	- 1.049	m^{-2} (m^{-1})
K_2 (g_2)	(0.422)	6.936	1.946	1.074	" "
K_3 (g_3)	(- 0.116)	- 6.691	- 1.782	- 0.922	" "

4) Verification of solution for $\phi_2 = 0.6$ with tracking routine (BEATCH)

	β_H	α_H	β_V	α_V	α_p	α_p'
out	12.86	0.003	20.65	0.058	0.861	- 0.0627

*) Calculated by the program.

**) ϕ_4 and ϕ_6 chosen such that $K_4 \approx K_6 \approx K_2$.



α_p (m)
β (m)
1.5
150
1
100
.5
50
β_V
α_p
β_H
F
D
F
s_1
s_2 s_3 s_4
10
s_5
s_6
20
s_7
s(m)
FABRICATION SUISSE

FIG. 1