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LONGITUDINAL RESISTIVITY WALL INSTABILITIES IN AN INTENSE COASTING BEAM FOR ELLIPTIC GEOMETRY

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1. Introduction

Circular and rectangular geometries were used to evaluate the longitudinal resistive instability for an intense coasting beam in Ref. [1].

In the following analysis an approach similar to Ref. [1] will be applied to an elliptic geometry. The results for the different geometries will be compared.

We assume an elliptic vacuum chamber of finite conductivity σ of the walls.

We call the ratio of the axis ε_0 and ε_1, the former applies for the chamber, the latter for the beam placed in the center of the first. The vacuum chamber is considered as straight.

2. Solutions of Maxwell's Equations

We apply elliptic cylinder coordinates η, ψ, z (Ref. [2]). The coordinate surfaces are given in the cartesian frame by

$$\left(\frac{x}{a \cosh \eta}\right)^2 + \left(\frac{y}{a \operatorname{senh} \eta}\right)^2 = 1 \qquad (1)$$

(elliptic cylinders, η = const.)

$$\left(\frac{x}{a \cos \psi}\right)^2 - \left(\frac{y}{a \operatorname{sen} \psi}\right)^2 = 1$$

(hyperbolic cylinders, ψ = const.)

$$z = z .$$

The η values corresponding to the surface of the beam and of the pipe are given by

$$\operatorname{tgh} \eta_0 = \varepsilon_0$$

$$\mathrm{tgh}\ \eta_1 = \varepsilon_1$$

$$a = b/\cosh \eta_o,$$

a is a scaling constant and b is the major half axis of the pipe.

It should be noted that equ. (1) represents a certain manifold of ellipses which cannot in all cases represent the surface of the beam as well as the surface of the vacuum chamber.

The perturbed charge per unit volume in the beam may be written

$$\rho = \rho_1\, e^{\,i(kz - \omega t)} \tag{2}$$

where k is the wave number equal to n/R. n is the harmonic number and $2\pi R$ is the length of the closed orbit.

From (2) we get the current density

$$j_\psi = j_\eta = 0,\ j_z = \frac{\omega\rho}{k}\ . \tag{3}$$

Maxwell's equations yield

$$\Delta \vec{E} - \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} = 4\pi\ (\mathrm{grad}\,\rho + \frac{1}{c^2}\frac{\partial \vec{J}}{\partial t}) \tag{4}$$

of which we need only the third component. In Ref. [2] it is shown that

$$\Delta_z \vec{E} = \Delta E_z ,$$

then (4) becomes

$$\Delta E_z - \frac{1}{c^2}\ \frac{\partial^2 E_z}{\partial t^2} = 4\ \pi i\ \frac{k\rho}{\gamma_w^{\ 2}}\ \mathcal{H}(\eta_1 - \eta) \tag{5}$$

with

$$\gamma_w^{\,2} = (1 - \beta_w^{\,2})^{-1} ,$$

$$\beta_w = \frac{\omega}{ck} ,$$

$\mathcal{H}(\eta - \eta_1)$ is the Heaviside function

$$\mathcal{H}(\eta_1 - \eta) = 1 \quad \text{for} \quad \eta < \eta_1$$
$$= 0 \quad \text{for} \quad \eta > \eta_1 .$$

We write the general solution of the homogeneous equation of (5) in the following way:

$$E_z = f(\eta, \psi)\, e^{i(kz - \omega t)}$$

this yields

$$\frac{1}{a^2 (\cosh^2 \eta - \cos^2 \psi)} \left\{ \frac{\partial^2 f}{\partial \eta^2} + \frac{\partial^2 f}{\partial \psi^2} \right\} + \left(\frac{\omega^2}{c^2} - k^2\right) f = 0 \qquad (5a)$$

or

$$\frac{\partial^2 f}{\partial \eta^2} + \frac{\partial^2 f}{\partial \psi^2} - \frac{a^2 k^2}{\gamma_w^{\,2}} (\cosh^2 \eta - \cos^2 \psi)\, f = 0 . \qquad (6)$$

In order to separate variables we put

$$f = H(\eta)\, \Psi(\psi)$$

and we get from (6) the following Mathieu equations

$$\frac{d^2 H}{d\eta^2} - (c_1 + 2q \cosh 2\eta)\, H = 0 \qquad (6a)$$

$$\frac{d^2 \Psi}{d\psi^2} + (c_1 + 2q \cos 2\psi)\, \Psi = 0 \qquad (6b)$$

with

$$q = \frac{a^2 k^2}{4 \gamma_w^2} , \tag{7}$$

c_1 is a constant. The general solutions of (6a) and (6b) are

$$H = A_1 \, Ce_m (\eta, -q) + A_2 \, Fe_m (\eta, -q) \tag{8a}$$

$$\Psi = B_1 \, ce_m (\psi, -q) + B_2 \, fe_m (\psi, -q). \tag{8b}$$

Ce_m and Fe_m are the modified Mathieu functions of the first and second kind and the order m ; ce_m and fe_m are the ordinary Mathieu functions.

Because Ψ should be a periodic function of ψ with the period π the constant m must be an even positive integer or zero. For symmetry reasons we have to put $m = 0$.

The general theory of the Mathieu functions is treated in Ref [3] and [4]. Some properties of the modified functions Ce_o and Fe_o are given for convenience in appendix A.

As shown in Ref [3] we have

$$ce_o (\psi, -q) = ce_o (\tfrac{\pi}{2} - \psi, q)$$

and

$$ce_o (\psi, q) = 1 - \frac{1}{2} q \cos 2\psi + \frac{q^2}{32} \cos 4\psi + O(q^3),$$

but because normally q is a very small quantity and ψ is real, we can write with a good approximation,

$$ce_o (\psi, -q) = 1 .$$

For example : the values for the elliptical sections of the pipe in CESAR are

$$\varepsilon_o = 0{,}5$$
$$b = 4{,}7 \text{ cm}$$
$$\gamma = 4{,}5$$
$$R = 380 \text{ cm}$$
$$a = 4{,}1 \text{ cm}$$

and

$$q = 1{,}4 \cdot 10^{-6} n^2 .$$

As the second solution of the Mathieu equation (6b) we take

$$fe_o (\psi, -q) = ce_o (\psi, -q) \int_o^{\psi} \frac{d\bar{\psi}}{ce_o^2(\bar{\psi}, -q)}$$

$$= \psi ,$$

thus (8b) can be replaced by

$$\Psi = B_1 + B_2 \psi ,$$

but as $\Psi(\psi)$ should be a periodic function in ψ it follows necessarily

$$B_2 = 0.$$

Then for the longitudinal field we have, finally,

$$E_z = \left[A_1 \, Ce_o (\eta, -q) + A_2 \, Fe_o (\eta, -q) \right] e^{i(kz - \omega t)} \qquad (9)$$

where A_1 and A_2 will be used to satisfy the boundary conditions. Eq. (9) gives the field for $\eta < \eta_1$. In the region $\eta > \eta_1$ we must solve the inhomogeneous equation of (5) and find a particular integral f_o of f .

From (5) and (5a) we get

$$\frac{\partial^2 f_o}{\partial \eta^2} + \frac{\partial^2 f_o}{\partial \psi^2} - a^2 (\cosh^2 \eta - \cos^2 \psi) \frac{k^2}{\gamma_w^2} f_o =$$

$$= 4 \pi i \frac{ka^2 \rho_1}{\gamma_w^2} \mathcal{H} (\eta_1 - \eta) (\cosh^2 \eta - \cos^2 \psi) .$$

It is easy to see that the function

$$\bar{f} = f_o + 4 \pi i \frac{\rho_1}{k}$$

fulfils also the homogeneous equation (6). Then applying the same procedure for f we get

$$f_o = C_1 Ce_o (\eta, -q) + C_2 Fe_o (\eta, -q) - 4\pi i \frac{\rho_1}{k}$$

where C_1 and C_2 are to be chosen to satisfy the boundary conditions.

Then the field for $\eta < \eta_1$ becomes

$$E_z = \left[C_1 Ce_o (\eta, -q) + C_2 Fe_o (\eta, -q) \right] e^{i (kz - \omega t)} - \frac{4\pi i \rho}{k}. \tag{10}$$

3. <u>Boundary conditions and expressions for the longitudinal field.</u>

From Maxwell's equations we get

$$\mathrm{rot}_\psi \vec{E} = i \frac{\omega}{c} H_\psi$$

$$\mathrm{rot}_\eta \vec{H} = -i \frac{\omega}{c} E\eta ,$$

It is clear that H_z is zero everywhere. Then we find a relation between E_η, H_ψ and E_z

$$E_\eta = -i \frac{ka}{4q} (\cosh^2 \eta - \cos^2 \psi)^{-1/2} \frac{\partial E_z}{\partial \eta} , \tag{10a}$$

$$H_\psi = \beta_w \, \varepsilon_\eta .$$

E_η and H_ψ should be zero at the point $\eta = 0$, $\psi = \pi/2$ for reasons of symmetry. This yields $C_2 = 0$ because $Ce_o{}'(0, -q) = 0$ and $Fe_o{}'(0, -q) = 1$, where the prime denotes derivation with respect to η. The only constants left to define are C_1, A_1 and A_2.

The only boundary condition that must be satisfied at $\eta = \eta_o$ is

$$E_z = -(1 - i) \quad H \mathcal{R} \tag{11}$$

with

$$\mathcal{R} = \left(\frac{\omega}{8\pi\sigma}\right)^{1/2}.$$

To eliminate the dependence of H_ψ on ψ (E_z is independent of ψ) we use for the quantity

$$b_o = a\,(\cosh^2 \eta_o - \cos^2 \psi)^{1/2} \tag{12}$$

an average value on the ellipse $\eta = \eta_o$.

Applying (11) at $\eta = \eta_o$ and the condition of continuity for E_z and H_ψ at $\eta = \eta_1$, as it is shown in appendix B, we determine the constants A_1, A_2 and C_1 and we get for the field E_z in the median plane of the beam, $\eta = 0$, the following expression

$$E_z\,(\eta = 0) = 4\pi i\,\frac{\rho}{k}\quad Fe_o'\,(\eta_1, -q) - 1 \quad + \\ - Ce_o'\,(\eta_1, -q\,)\;\frac{Fe_o\,(\eta_o, -q) - T\,Fe_o'\,(\eta_o, -q)}{Ce_o(\eta_o, -q) - T\,Ce_o'\,(\eta_o, -q)} \tag{13}$$

where T is the complex quantity

$$T = (1 + i)\mathcal{R}\;\beta_w\,\frac{\gamma_w^2}{k\,b_o} = (1 + i)\;T_1 .$$

In order to simplify the expression for E_z (13) we introduce some assumptions.

As mentioned before the coordinate system applied can represent rigorously either the surface of the beam or of the vacuum chamber.

We will consider two cases:

a) The parameters are chosen such that the surface of the elliptical tank is one of the family of the coordinate ellipses and the minor axis of the ellipse describing the beam surface is equal to the minor axis of the real beam.

b) Fitting the family of ellipses to the beam surface and preserving the vertical dimensions of the tank yields a large value of η_o provided that the ratio of beam size and height of the vacuum chamber is small. This means that the ellipse $\eta = \eta_o$ is nearly a circle and may describe in this way the case of an elliptic beam in a circular tank.

4. Elliptic Beam in a Tank with Elliptic Cross Section

For CESAR we get

$$\eta_o = 0{,}55, \ \cosh \eta_o = 1{,}15$$

and

$$a = 4{,}1 \text{ cm} .$$

The vertical half size of the beam is 0,1 cm. This yields

$$\eta_1 = 0{,}025, \ \cosh \eta_1 = 1 .$$

For $\eta \lesssim 1$ and for $q \lesssim 10^{-2}$, which for CESAR corresponds to $n \lesssim 100$, we can write in good approximation

$$Ce_o (\eta, - q) = 1$$

$$Fe_o (\eta, - q) = \eta$$

$$Ce_o' (\eta, - q) = q \text{ senh } 2\eta$$

$$Fe_o' (\eta, - q) = 1 + q\eta \text{ senh } 2\eta - \frac{q}{2} \cosh 2\eta .$$

With the parameters

$$\omega_o = 2\,\pi \text{ x revolution frequency} = 7{,}5 \cdot 10^7 \ s^{-1}$$

$$\sigma = 1{,}2 \ . \ 10^{16} \ S^{-1}$$

$$b_o = 3{,}5 \ \text{cm}$$

we get

$$T_1 = \frac{3{,}4 \ . \ 10^{-2}}{\sqrt{n}} \ .$$

Writing for CESAR

$$T_1 \ll 1 \text{ and } q \ll \frac{2}{\cosh^2 \eta_o}$$

and taking into account all the considerations above, we get the following expression for E_z in the median plane of the beam

$$E_z \ (\eta = 0) = - 2i \ \frac{\lambda k}{\gamma_w^{\ 2}} \left[\frac{1}{2} \ \text{cotgh} \ 2\,\eta_1 + \eta_o - \eta_1 - i \ T_1 \right] \quad (15)$$

with λ the perturbed charge per unit length

$$\lambda = \lambda_1 \ e^{i\,(kz \,-\, \omega t)} \ .$$

Following Ref. [1] in the treatment of the Vlasof equation for the distribution function $\Psi\,(w, \theta, t)$ of the particles in phase space, we have the dispersion relation

$$\lambda = -2\,\pi i \ e^2 \ \langle RE_z \rangle \int \frac{d\Psi_o}{dw} \ \frac{dw}{(\omega - n\dot{\theta})} \ , \quad (16)$$

this can be written as

$$- 1 = (U - iV) \ I \ , \quad (17)$$

$\langle R E_z \rangle$ in (16) should be the average value over the cross section of the beam; but we will take for it the value at $\eta = 0$ using (15). Thus it follows:

$$U = Ne^2 \frac{k}{\gamma_W^2} \left[\cotgh 2\eta_1 + 2 (\eta_o - \eta_1) \right] \qquad (18)$$

and

$$V = 2 Ne^2 \Re \beta_W / b_o \, . \qquad (19)$$

5. Elliptic Beam in a Tank with Round Cross Section

For a straight section in CESAR, which is a round pipe, we use the following parameters

$\epsilon_1 = 0,4$

$\eta_1 = 0,42$, $\cosh \eta_1 = 1,1$

$a = 0,23$ cm

$b_o = 5,0$ cm (radius of the pipe)

$\eta_o = 3,8$, $\cosh \eta_o = 22$,

As we have $\eta_1 \lesssim 1$, $\eta_o > 1$, for $n < 10$, we see that the results of the foregoing chapter are applicable.

6. Comparison of Different Geometries for CESAR

The analysis in Ref [1] yields for a round beam in a round pipe the following values

$$U = Ne^2 \frac{n}{R} \frac{1 + 2 \ln (bo/b_1)}{\gamma_W^2} \qquad (20)$$

$$V = 2 Ne^2 \Re \beta_W / b_o \qquad (21)$$

where b_o and b_1 are the radii of the pipe and of the beam.

Comparison between (18), (19) and (20), (21) shows that V is the same for all geometries considered. Whereas for U we can write

$$U_{round} / U_{elliptic} = \frac{1 + 2 \ln (b_o / b_1)}{\operatorname{cotgh} 2\eta_1 + 2(\eta_o - \eta_1)}. \qquad (22)$$

For CESAR:

a) Elliptical cross sections,

$\eta_1 = 0{,}025, \quad \eta_o = 0{,}55$

$b_o = 3{,}5$ cm

$b_1 = 0{,}1$ cm

$U_{round} / U_{elliptic} = 0{,}4.$

The radii to evaluate U_{round} were taken equal to the mean values between the major and minor half axis.

b) Elliptical beam in the round straight section,

$\eta_1 = 0{,}42, \quad \eta_o = 3{,}8$

$b_o = 5{,}0$ cm

$b_1 = 0{,}2$ cm

$U_{round} / U_{elliptic} = 0{,}8$.

To evaluate U_{round} the elliptic beam was supposed to be round with radius b_1.

When the focal distance 2a becomes zero the term in brackets in (18) becomes $(1 + 2 \ln (b_o / b_1))$.

Formula (18) $U_{ell.}$ is then identical with (20) U_{round}. From (22) it is obvious that $U_{round} \leqslant U_{elliptic}$ is always valid.

The growth time in the case $U \gg V$ is given by

$$\tau = \frac{1}{\left[n\,|k_o|\,U\right]^{1/2}}$$

for an accelerator above transition (see Ref. [1]).

Thus we can infer

$$\tau_{round} \gg \tau_{elliptic} .$$

If we refer to the stability criteria in Ref. [1] we can conclude that a round beam in a round tank is more stable than an elliptic beam in an elliptic vacuum chamber.

Appendix A

The Modified Mathieu functions

The modified Mathieu function $Ce_m(\eta, -q)$ $(m = 0, 1, 2...)$ is defined by the series

$$Ce_m(\eta, -q) = \sum_{r=0}^{\infty} A_{2r}^{(m)} \cosh 2r\eta . \qquad (A1)$$

For $m = 0$ and q small it is

$$A_{2r}^{(0)} = \frac{2}{(r!)^2}\left(\frac{q}{4}\right)^2 , \quad A_o^{(0)} = 1 .$$

Derivation with respect to η of (A1) gives

$$Ce_m'(\eta, -q) = \sum_{r=0}^{\infty} 2r\, A_{2r}^{(m)} \operatorname{senh} 2r\eta . \qquad (A2)$$

For $\eta = 0$ and q small we get

$$Ce_o(0, -q) = 1 + \frac{q}{2} + \frac{q^2}{32} + 0\,(q^3) \simeq 1 \qquad (A3)$$

$$Ce_o{}' (0, -q) = 0 . \qquad (A4)$$

The second modified Mathieu function is defined by the integral (Euler method)

$$Fe_m (\eta, -q) = Ce_m (\eta, -q) \int_o^\eta \frac{d\eta}{\left[Ce_m (\eta, -q)\right]^2} . \qquad (A5)$$

For $m = 0$, q small and $\eta = 0$ we get

$$Fe_o (0, -q) = 0 . \qquad (A6)$$

Derivation of (A5) with respect to η results in

$$Fe_m{}' (\eta, -q) = Ce_m{}' (\eta, -q) \int_o^\eta \frac{d\eta}{\left[Ce_m (\eta, -q)\right]^2} + \frac{1}{Ce_m (\eta, -q)}$$

and

$$Fe_o{}' (0, -q) = 1 . \qquad (A7)$$

For $\eta \to \infty$ the integral on the right hand of (A5) tends to be a constant quantity α_∞. Thus for large value of η, $Fe_m (\eta, -q)$ is equal to $Ce_m (\eta, -q)$ multiplied by α_∞.

The assymptotic behavior of the two modified Mathieu functions is such that for $\eta \to \infty$ both diverge. The wronskian

$$W (\eta, -q) = Ce_o (\eta, -q) Fe_o{}' (\eta, -q) - Fe_o (\eta, -q) Ce_o{}' (\eta, -q)$$

turns out to be a constant and for q small we get

$$W (0, -q) = 1. \qquad (A8)$$

When the condition

$$q \ll \frac{2}{\cosh 2\eta}$$

is satisfied we have with a good approximation

$$\begin{aligned} Ce_o(\eta, -q) &= 1 + \frac{q}{2}\cosh 2\eta \\ Fe_o(\eta, -q) &= \eta\left(1 + \frac{q}{2}\cosh 2\eta\right) - \frac{q}{2}\,\mathrm{senh}\, 2\eta \\ Fe_o'(\eta, -q) &= 1 + q\,\eta\,\mathrm{senh}\, 2\eta - \frac{q}{2}\cosh 2\eta \\ Ce_o'(\eta, -q) &= q\,\mathrm{senh}\, 2\eta \ . \end{aligned} \qquad (A9)$$

Finally when η is large we can apply the followingassymptotic expansion for $Ce_o(\eta, -q)$:

$$Ce_o(\eta, -q) \sim I_o(\sqrt{q}\ e^{\eta}) \qquad (A10)$$

where I_o is the modified Bessel function of the first kind some values of which are tabulated in Ref. [5] .

Appendix B

Evaluation of the constants A_1, A_2, C_1

From the continuity of E_z and H_ψ at $\eta = \eta_1$ and by (9), (10), (10a) and (10b) we get the following system of linear equations

$$\begin{aligned} C_1\, Ce_o(\eta_1, -q) - 4\pi i \frac{\rho_1}{k} &= A_1\, Ce_o(\eta_1, -q) + A_2\, Fe_o(\eta_1, -q) \\ C_1\, Ce_o'(\eta_1, -q) &= A_1\, Ce_o'(\eta_1, -q) + A_2\, Fe_o'(\eta_1, -q) \ . \end{aligned} \qquad (B1)$$

Resolving (B1) with respect to C_1 by the Kramer method, and by (A8) we get

$$A_1 = C_1 - 4\pi i \frac{\rho_1}{k}\, Fe_o'(\eta_1, -q) \qquad (B2)$$

$$A_2 = 4\pi i \frac{\rho_1}{k}\, Ce_o'(\eta_1, -q) \ . \qquad (B3)$$

From the boundary condition (11) at $\eta = \eta_2$ and applying (12) it results

$$A_1 \, Ce_o \, (\eta_o, - q) + A_2 \, Fe_o \, (\eta_o, - q) = \tag{B4}$$

$$= (1 + i) \mathcal{R} \beta_w \frac{\gamma_w^2}{kb_o} \left[A_1 \, Ce_o' \, (\eta_o, - q) + A_2 \, Fe_o' \, (\eta_o, - q) \right].$$

Putting (B2) and (B3) in (B4) we obtain finally

$$C_1 = 4\pi i \frac{\rho_1}{k} \left[Fe_o'(\eta_1, - q) - Ce_o' \, (\eta_1, - q) \, \frac{Fe_o \, (\eta_o, - q) - TFe_o'(\eta_o, - q)}{Ce_o \, (\eta_o, - q) - TCe_o' \, (\eta_o, -q)} \right].$$

Substituting (B5) in (10), taking into account $C_2 = 0$ and using (A9), we get for $\eta = 0$ the expression (13) for the longitudinal field E_z.

References

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