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LONGITUDINAL RESISTIVE WALL INSTABILITIES OF AN INTENSE COASTING BEAM FOR AXIALLY ASSYMMETRIC GEOMETRY

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1. Introduction

The longitudinal resistive wall instability for azimuthally uniform beam moving in a straight vacuum chamber has been investigated for different arrangements of beam and pipe but the beam was always assumed to be centered in the pipe.

The case of a round and rectangular geometry is treated in Ref. [1] whereas the elliptic geometry is used in Ref. [2].

Here we will study the case of a beam placed asymmetrically in a round pipe.

2. Solution of Maxwell Equations

The approach is the same as in Ref. [2]. We assume a perturbation in the charge distribution of the form

$$\rho = \rho_1 e^{i(kz - \omega t)},$$

$$k = R/n$$

$2\pi R$ - length of the closed orbit

n - harmonic number.

We use the following parameters:

r - radius of the round beam

b - radius of the pipe

d - distance between the axis of the two parallel cylinders.

Further we use Bi-cylinder coordinates η, ϑ, z (Ref. [3]). The coordinate surfaces in the cartesian frame may be written

a) $\eta = \text{const.}, -\infty \leqslant \eta \leqslant +\infty$,

$$x^2 + y^2 + a^2 = 2ax \coth \eta, \qquad (1)$$

this is a cylinder with the axis at $x = a \operatorname{cotgh} \eta$, $y = 0$.

b) θ = const. (this is not the azimuthal variable along the closed orbit), $-\pi \leq \theta \leq \pi$

$$x^2 + (y - a \operatorname{cotg}\theta)^2 = \frac{a^2}{\operatorname{sen}^2\theta}$$

c) z = const.

The pipe is represented by the cylinder $\eta = \eta_0 > 0$, the beam by $\eta = \eta_1 > 0$. $\eta = +\infty$ is identical with $x = a$ which is chosen to be near the axis of the beam $x = a \operatorname{cotgh} \eta_1$, $y = 0$. Using (1) we get for the radii

$$r = a/\operatorname{senh} \eta_1$$

$$b = a/\operatorname{senh} \eta_0$$

and

$$\operatorname{cotgh} \eta_0 - \operatorname{cotgh} \eta_1 = d/a .$$

As r, b and a are known for a particular problem we figure out a, η_0, η_1. For CESAR with $b = 5$ cm and $r = 0{,}15$ cm we get the following table.

TABLE I

d	a	η_1	η_0	q/n
1 cm	12,00 cm	5,1	1,6	$7{,}0 \cdot 10^{-3}$
2 cm	5,25 cm	4,2	0,92	3,1
3 cm	2,67 cm	3,6	0,52	1,5
4 cm	1,12 cm	2,7	0,22	0,7
4,8 cm	0,14 cm	0,84	0,028	0,09

For the source of the electromagnetic field we take the following: the charge density

$$\rho = \rho_1 e^{i(kz - \omega t)} \quad \text{for } \eta > \eta_1$$

$$= 0 \quad \text{for } \eta < \eta_1$$

and the current density

$$j_\eta = j_\theta = 0, \; j_z = \omega\rho/k \, .$$

From Maxwell equations we get the Helmotz equation for the longitudinal field E_z. If we use the following expression for E_z

$$E_z = f\,(\eta, \theta)\, e^{i(kz - \omega t)}$$

we can write

$$\frac{\partial^2 f}{\partial \eta^2} + \frac{\partial^2 f}{\partial \theta^2} - \frac{k^2}{\gamma_w^2} g_{11}\, f = 4\pi i \, \frac{k\rho_1}{\gamma_w^2}\, g_{11} \qquad (2)$$

where g_{11} is the length element of the bi-polar coordinates

$$g_{11} = \frac{a^2}{(\cosh\eta - \cos\theta)^2}$$

and

$$\gamma_w^{\;2} = 1/(1-\beta_w^{\;2})$$

$$\beta_w = \omega/kc \, ,$$

In the region outside the beam, $\eta < \eta_1$, we have to solve the homogeneous equation of (2)

$$\frac{\partial^2 f}{\partial \eta^2} + \frac{\partial^2 f}{\partial \theta^2} - \frac{k^2}{\gamma_w^2} \; \frac{a^2 f}{(\cosh\eta - \cos\theta)^2} = 0 \, , \qquad (3)$$

In order to make equation (3) separable we introduce an approximation. The smallest value of cosh η occurs at the surface of the pipe and is always bigger than one. If we put in equation (3)

$$(\cosh\eta - \cos\theta) \sim \cosh\eta \qquad (5)$$

which is strictly valid for all η at $\theta = \pm \frac{\pi}{2}$ and which is a good approximation for $\theta \neq \pm \frac{\pi}{2}$ and η large, we obtain

$$\frac{\partial^2 f}{\partial \eta^2} + \frac{\partial^2 f}{\partial \theta^2} + \frac{k^2}{\gamma_w^2} \frac{a^2 f}{\cosh^2 \eta} = 0. \qquad (6)$$

Equ. (6) now separates. If we put

$$f = H(\eta)\Theta(\theta)$$

we get the following two equations

$$\frac{d^2 H}{d\eta^2} - \left(\frac{k^2 a^2}{\gamma_w^2 \cosh^2 \eta} + c_1\right) H = 0 \qquad (7a)$$

$$\frac{d^2\Theta}{d\theta^2} + c_1\Theta = 0 \; . \qquad (7b)$$

Because E_z is a periodic function of θ c_1 cannot be negative. Further as E_z cannot change its sign for symmetry reasons we get $c_1 = 0$ and Θ is a constant. (7a) yields

$$\frac{d^2 H}{d\eta^2} - \frac{q^2 H}{\cosh^2 \eta} = 0 \qquad (7c)$$

with

$$q = ka/\gamma_w \; . \qquad (8)$$

In CESAR it is $\gamma_w = 4.5$ and $R = 380$; the quantity q/n is tabulated in table I.

Equ. (7c) can be solved by the series method as shown in Appendix A. For large value of η a first integral is given approximatively by

$$H_1 = 1 + \frac{q^2}{4 \cosh^2 \eta} + \dots \qquad (9)$$

Because q is very small H_1 is a function always near one. The second integral of (7a) H_2, dan be found by the Euler method.

3. Boundary conditions and expressions for the longitudinal field.

The longitudinal field E_z outside the beam, $\eta < \eta_1$ is represented by

$$E_z = \left[A_1 H_1 (\eta) + A_2 H_2 (\eta) \right] e^{i(kz - \omega t)} . \tag{13}$$

As in the beam, $\eta > \eta_1$, E_z must remain finite we have to drop $H_2(\eta)$ in the homogeneous solution and we obtain the following general solution for $\eta > \eta_1$.

$$E_z = B_1 H_1 (\eta) e^{i(kz - \omega t)} - 4\pi i \frac{\rho}{k} . \tag{14}$$

The last term in the right of the equ. (14) is the particular integral of the inhomogeneous equation (3), valid for $\eta > \eta_1$.

From the following Maxwell equations

$$\mathrm{rot}_\theta \vec{E} = i \frac{\omega}{c} H_\theta$$

$$\mathrm{rot}_\eta \vec{H} = - i \frac{\omega}{c} E_\eta$$

we get a relation between H_θ and E_z taking into account that for reasons of symmetry H_z is zero:

$$H_\theta = - \frac{i \beta_w \gamma_w^2}{ak} (\cosh \eta - \cos \theta) \frac{\partial E_z}{\partial \eta} . \tag{15}$$

The boundary conditions which have to be satisfied are

a) continuity of E_z at $\eta = \eta_1$,
b) continuity of H_θ at $\eta = \eta_1$, and
c) normalization between E_z and H_θ at $\eta = \eta_o$, which means

$$E_z = (1 - i)\,\mathcal{R}\,H_\vartheta \qquad (16)$$

where

$$\mathcal{R} = \sqrt{\frac{\omega}{8\pi\sigma}}$$

σ - conductivity of the walls of vacuum chamber. Condition (16) cannot be satisfied completely because there is the dependence on $\cos\vartheta$ in the expression (15) for H_ϑ. But in the following we shall apply the approximation (5) again. This means that the boundary condition at $\eta = \eta_o$ is satisfied only in the regions of the surface for which $\cos\vartheta$ is near zero.

In Appendix B it is shown how the three constants A_1, A_2 and B_1 have been evaluated from the boundary conditions a), b) and c). The results for the longitudinal field E_z evaluated on the polar axis $\eta = \infty$, which is very near to the beam axis are

$$E_z(\infty) = 4\pi i \frac{\rho}{k}\left[H_2'(\eta_1) - 1 + \right.$$
$$\left. - H_1'(\eta_1)\,\frac{H_2(\eta_o) - T\,H_2'(\eta_o)}{H_1(\eta_o) - T\,H_1'(\eta_o)}\right] \qquad (17)$$

with

$$T = -(1 + i)\,\frac{\mathcal{R}\,\beta_w\,\gamma_w^2}{ak}\cosh\eta_o = -(1 + i)\,T_1\,. \qquad (18)$$

4. <u>Discussion of the dispersion relation and results for CESAR.</u>

Following Ref. [1] we write $\omega = n\omega_o$. For CESAR we have

$$\omega_o = 7{,}5 \cdot 10^7\ cs^{-1}$$

$$\sigma = 1{,}2 \cdot 10^{16}\ s^{-1}$$

this results in

$$T_1 = \frac{2{,}4 \cdot 10^{-2}}{\sqrt{n}}\ \mathrm{cotgh}\ \eta_o\,.$$

As one can see from table I $T_1 \lesssim 1$ and $\eta_o \gg \frac{q^2}{4}$ is fulfilled. Introducing in (17) the results of Appendix A and (18) we obtain

$$E_z(\infty) = - i \frac{k}{\gamma_w^2} \lambda \, \mathrm{senh}^2 \eta_1 \left[\epsilon(\eta_1) - (\eta_1 - \eta_o) \epsilon'(\eta_1) + i \epsilon'(\eta_1) T_1 \right] \qquad (19)$$

$\epsilon(\eta)$ is a function described in Appendix A. λ is the charge per unit length.

$$\lambda = \lambda_1 e^{i(kz - \omega t)}.$$

In Ref. [1] the Vlasov equation for the distribution function $\Psi(w, \theta, t)$ in the beam is solved and a dispersion relation is obtained.

With the same procedure as applied in Ref. [1] and Ref. [2] we get for the complex quantity (U - iV)

$$U = Ne^2 \frac{k}{\gamma_w^2} \mathrm{tgh}^2 \eta_1 \left[1 + 2 (\eta_1 - \eta_o) \mathrm{tgh}\, \eta_1 \right] \qquad (20a)$$

$$V = 2 Ne^2 \mathcal{R} \beta_w \mathrm{tgh}^3 \eta_1 \mathrm{cotgh}\, \eta_o / b \qquad (20b)$$

The results in Ref. [1] for a round beam placed in the centre of a round pipe are

$$U_c = Ne^2 \frac{k}{\gamma_w^2} \left[1 + 2 \ln (b/r) \right] \qquad (21a)$$

$$V_c = 2 Ne^2 \mathcal{R} \beta_w / b \qquad (21b)$$

To check the result (20a), (20b) we take for the excentricity d zero. As

$$\eta_1 - \eta_o \rightarrow \ln \frac{b}{r} \text{ for } \eta_1, \eta_o \rightarrow \infty$$

is obvious, we find that (21a), (21b) is just a special case of (20a), (20b).

When the beam is moved away from the centre, the quantity U decreases monotonically and V increases as shown in Fig. 1.

We can write also

$$U = U_c \frac{tgh^2 \eta_1 \left[1 + 2 (\eta_1 - \eta_o) \, tgh \, \eta_1\right]}{1 + 2 \ln (^b/r)} \qquad (22a)$$

and

$$V = V_c \, tgh^3 \eta_1 \, cotgh \, \eta_o . \qquad (22b)$$

For CESAR, b = 5 cm, r = 0,15 cm, (22a) and (22b) give the following values:

TABLE II

d	V/V_c	U/U_c	$V/U \cdot \sqrt{n} \cdot 10^3$
0 cm	1.0	1.00	5.6
1 cm	1.1	1.00	6.2
2 cm	1.4	0.95	8,3
3 cm	2.1	0.90	13
4 cm	4.5	0.75	34
4,8 cm	19	0.19	560

For a small value of d we get $V \ll U$. Above transition growth time and stability criterion against longitudinal instability can be written in the form

$$\tau = \left[n \, |k_o| \, U\right]^{-1/2} \qquad (23a)$$

$$\delta > \left[U/n \, |k_o|\right]^{1/2} \qquad (23b)$$

as shown in Ref. [1] :

2 - total frequency spread in the beam,

k_o - constant related to the momentum compaction factor, For a large d , V becomes larger than U and (23a), (23b) are no longer valid. For $V \gg U$ the equations for growth time and stability become (see Ref. [4])

$$\tau = \left[2/n \, |k_o| \, V\right]^{1/2} \tag{24a}$$

$$\delta > \left[V/n \, |k_o|\right]^{1/2} \tag{24b}$$

5. Conclusions

Taking into account (23) and (24) we can infer from Fig. 1 or table II that above a certain value of d the beam is less stable than for the case d = 0. The growth time is plotted against d in Fig. 2.

This could explain qualitatively why the beam in CESAR is more unstable at injection orbit which is very near to the plate of the pulsed inflector than at central orbit.

Appendix A

The functions $H_1(\eta)$, $H_2(\eta)$

We consider the equation

$$\frac{d^2H}{d\eta^2} - \frac{q^2 H}{\cosh^2\eta} = 0 . \qquad (A1)$$

The quantity q^2 is very small. For CESAR and for $n = 10$ we get a q^2 quantity of the order of 10^{-4} for $d = 3$cm. If η increases, the second term goes to zero. Thus we can omit the second term for $\eta \to \infty$ and write

$$\frac{d^2H}{d\eta^2} = 0 , \qquad (A2)$$

Two particular solutions are

$$H_1 = 1 \text{ and } H_2 = \eta . \qquad (A3)$$

We are interested in the particular integrals of (A1) of which (A3) are the assymptotic expressions.

The differential equation (A1) can be integrated by the method of series. We look for an integral of the type

$$H_1(\eta) = \sum_0^{\infty}{}_i \frac{\alpha_i}{\cosh^i \eta} \qquad (A4)$$

that, if we take $\alpha_o = 1$, for $\eta \to$ tends to unit.

Substitution of (A4) in (A1) yields $\alpha_1 = 0$ and the following correlation between α_i and α_{i+1}

$$i\,(i + 1)\alpha_i + q^2\alpha_i - (i + 2)^2\alpha_{i+2} = 0 , \qquad (A5)$$

If $q \ll 1$ and $i > 0$ is fulfilled we obtain from (A5) with good approximation

$$\alpha_{i+2} = \frac{i\,(i+1)}{(i+2)^2}\,\alpha_i\,. \tag{A6}$$

For odd i it is $\alpha_i = 0$. $\alpha_2 = -\frac{q^2}{4}$ results from (A5) for $i = 0$.

It is easily verified that (A6) and even i give

$$\alpha_i = \frac{(i-1)!}{\left[i\,(i-2)\,(i-4)\,(i-6)\ldots 4\right]^2}\,\alpha_2 \tag{A7}$$

then if we put

$$\delta_i = \frac{(i-1)!}{\left[i\,(i-2)\,(i-4)\ldots 4\right]^2}\,. \tag{A8}$$

we get for $H_1(\eta)$

$$H_1(\eta) = 1 + \frac{q^2}{4}\sum_{1}^{\infty} i\,\frac{\delta_{2i}}{\cosh^{2i}\eta} \tag{A9}$$

with

$$\delta_2 = 1.$$

The serie

$$S = \sum_{1}^{\infty} i\;\delta_{2i}$$

is convergent which was checked numerically and S is equal to 2,8. This implies that the series (A9) or (A4) are always convergent. Further the convergence becomes very fast for $\eta \to \infty$.

For all values of η, the second term on the right-hand side in (A9) is always small in comparison to one. Therefore we can write

$$H_1(\eta) = 1 + \frac{q^2}{4}\,\varepsilon(\eta)\,. \tag{A10}$$

The maximum value of $\varepsilon(\eta)$ is 2,8 and the function decreases rapidly to zero by increasing η.

Derivation of $H_1(\eta)$ with respect to η is

$$H_1'(\eta) = \frac{q^2}{4} \varepsilon'(\eta) . \tag{A11}$$

For $\eta \rightarrow \infty$ $H_1'(\eta)$ tends to 0. Note again that $\varepsilon'(\eta)$ is a negative quantity. The second integral $H_2(\eta)$ of (A1) can be derived by the Euler method:

$$H_2(\eta) = H_1(\eta) \int_0^\eta \frac{d\eta}{H_1^2(\eta)} . \tag{A12}$$

From (A10) and with $q \ll 1$ it results in

$$H_1^2(\eta) = 1 + \frac{q^2}{2} \varepsilon(\eta)$$

and

$$\frac{1}{H_1^2(\eta)} = 1 - \frac{q^2}{2} \varepsilon(\eta)$$

then we get

$$\int_0^\eta \frac{d\eta}{H_1^2} = \eta - \frac{q^2}{2} g(\eta) \tag{A13}$$

wherein

$$g(\eta) = \int_0^\eta \varepsilon(\eta)\, d\eta .$$

Inserting (A13) in (A12) we get

$$H_2(\eta) = \left[1 + \frac{q^2}{4} \varepsilon(\eta)\right] \left[\eta - \frac{q^2}{2} g(\eta)\right] .$$

Omitting term in q^4 we get

$$H_2(\eta) = \eta - \frac{q^2}{2} \left[g(\eta) - \frac{1}{2} \eta \varepsilon(\eta)\right] . \tag{A14}$$

If η goes to infinity the function $H_2(\eta)$ tends to ∞ linearly, whereas for $\eta = 0$ the same function is zero. Thus we can write

$$\lim_{\eta \to \infty} H_2 H_1' = 0 .$$

Finally we get from (A14)

$$H_2' (\eta) = 1 - \frac{q^2}{4} \left[\varepsilon (\eta) - \eta \varepsilon' (\eta) \right] ; \qquad (A15)$$

this quantity is always near to unity and the quantity in brackets is always positive.

From the relation above we can infer that the Wronskian W is a constant and can be evaluated for $\eta = \infty$. It turns out that W is equal to one.

Appendix B

Evaluation of the constant A_1, A_2, B_1.

From the continuity of the longitudinal field E_z (13) (14) at $\eta = \eta_1$, the beam boundary, we get

$$A_1 H_1 (\eta_1) + A_2 H_2 (\eta_1) = B_1 H_1 (\eta_1) - 4\pi i \frac{\rho_1}{k} . \qquad (B1)$$

The continuity of H_θ (15) at $\eta = \eta_1$ demands

$$A_1 H_1' (\eta_1) + A_2 H_2' (\eta_1) = B_1 H_1' (\eta_1) \qquad (B2)$$

where the prime denotes derivation with respect to η.

From (B1) and B2) we obtain the constants A_1 and A_2 as functions of the B_1. Now we bear in mind that the wronskian of the two functions H_1 and H_2 is one as shown in App. A. This yields

$$A_1 = B_1 - 4\pi i \frac{\rho_1}{k} H_2'(\eta_1) \qquad (B3)$$

$$A_2 = 4\pi i \frac{\rho_1}{k} H_1'(\eta_1) \qquad (B4)$$

B_1 can now be determined applying the boundary condition (16) at $\eta = \eta_o$. From (13) and (15) it results

$$A_1 H_1(\eta_o) + A_2 H_2(\eta_o) =$$

$$= -(1+i)\mathcal{R} \frac{\beta_w \gamma_w^2 \cosh\eta_o}{a\,k} \left[A_1 H_1'(\eta_o) + A_2 H_2'(\eta_o) \right]. \qquad (B5)$$

Deriving (B5) we applied the approximation (5) for the reasons mentioned above. Putting (B3) and (B4) in (B5) we get finally

$$A_1 \left[H_1(\eta_o) - TH_1'(\eta_o) \right] + A_2 \left[H_2(\eta_o) - TH_2'(\eta_o) \right] = 0$$

or

$$B_1 = 4\pi i \frac{\rho_1}{k} \left[H_2'(\eta_1) - H_1'(\eta_1) \frac{H_2(\eta_o) - TH_2'(\eta_o)}{H_1(\eta_o) - TH_1'(\eta_o)} \right]. \qquad (B6)$$

As $H_1(\eta)$ at $\eta = \infty$ is one, using (B6) we get the expression (17) for the longitudinal field E_z near the beam centre.

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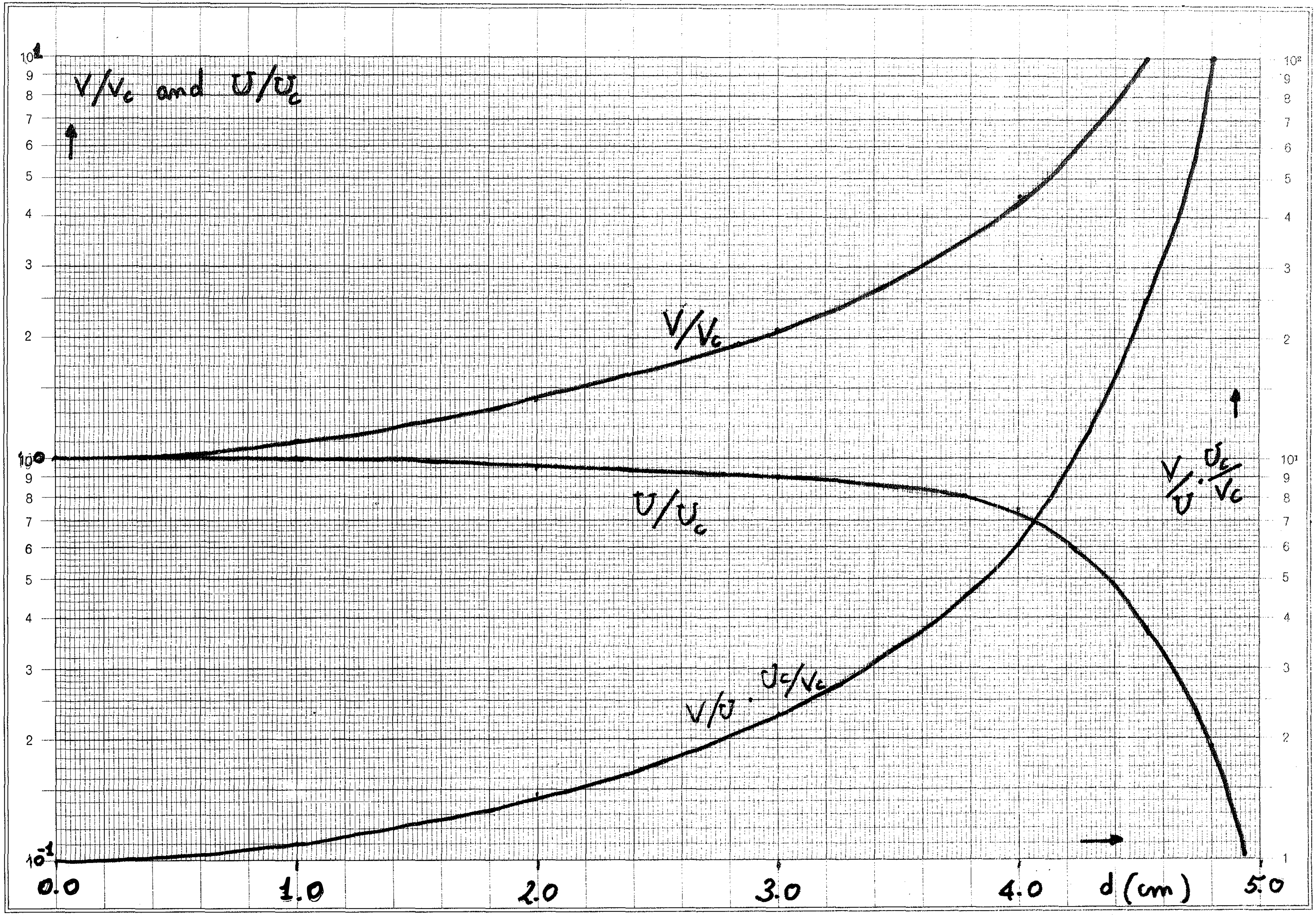

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Fig. 1

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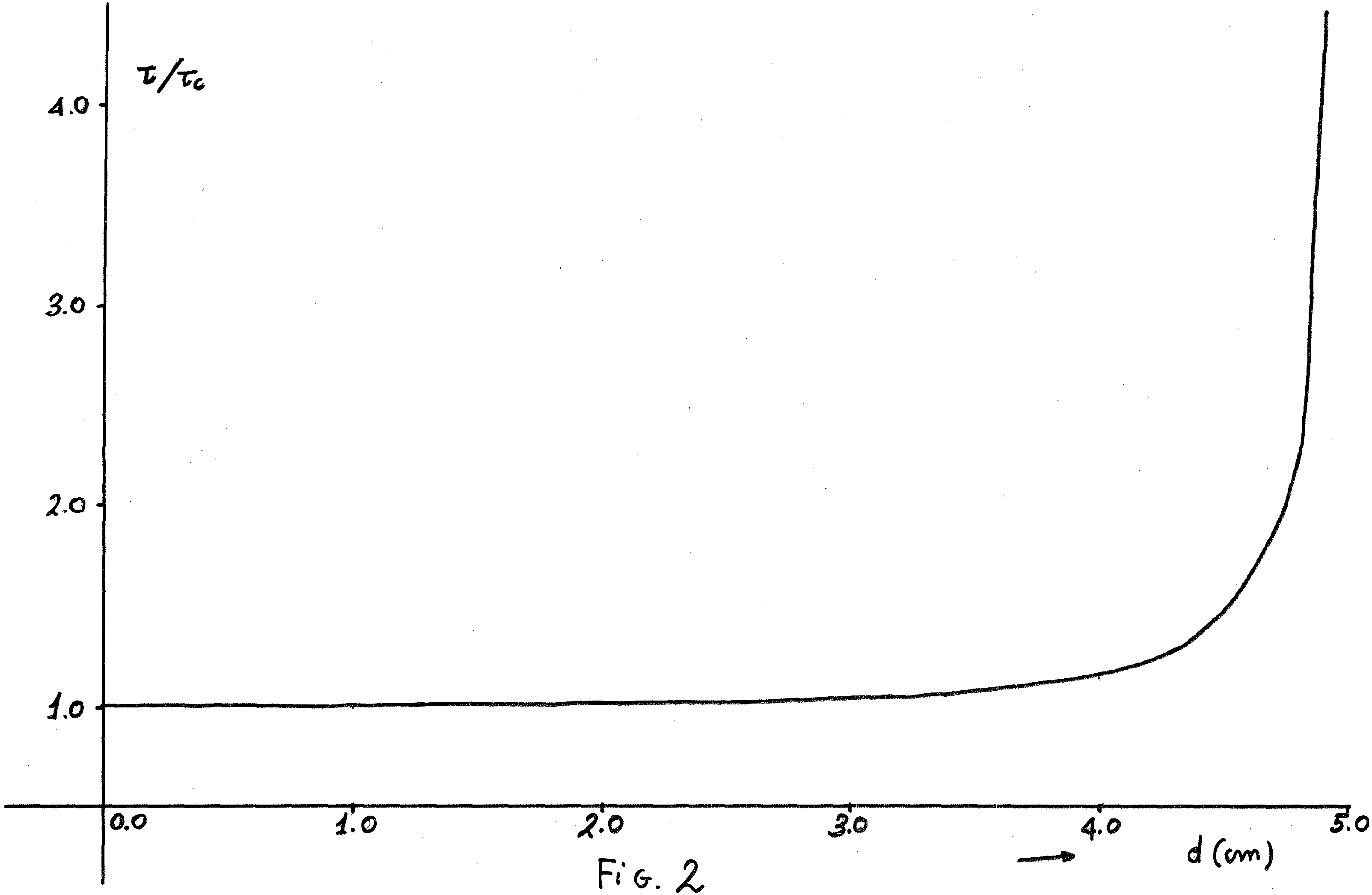

Fig. 2