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THE EFFECTIVE COUPLING IMPEDANCE FOR BUNCHED BEAM INSTABILITIES

by

B. Zotter

Abstract

For both longitudinal and transverse oscillations of coasting charged-particle beams, the interaction with the surroundings can be described by a coupling impedance which is a continuous function of frequency. For bunched beams, this impedance is sampled at an infinite number of discrete frequencies given by the mode spectrum. An "effective coupling impedance" can then be defined as the sum over the product of the coupling impedance and the normalized spectral density.

For resonant impedances and sinusoidal bunch oscillation modes, these infinite sums can be evaluated analytically, while the direct numerical summation may lead to difficulties even with high speed computers as the imaginary part of the sum usually converges very poorly. This method has been incorporated into a computer program for bunched beam instabilities which is presently used for the design of LEP-70 and its injectors.

Geneva - July 1978

1. INTRODUCTION

For the calculation of both longitudinal and transverse stability of bunched beams an "effective coupling impedance" is required, which can be defined as [1]

$$\left(\frac{Z}{\omega}\right)_{eff} = \frac{\sum\limits_{p=-\infty}^{+\infty} \frac{Z(\omega_p)}{\omega_p} \; h_m(\omega_p)}{\sum\limits_{p=-\infty}^{\infty} h_m(\omega_p)} \tag{1}$$

where ω_0 is the (angular) revolution frequency. For the longitudinal case, the mode frequencies are given by

$$\omega_p = \omega_0 (kp + n + m\,\nu_s) \tag{2}$$

where k is the number of bunches

n is the coupled-bunch mode number ($0 \le n \le k$) which determines the phase-shift between bunches

$m \ge 1$ determines the type of oscillation ($m = 1$ dipole, $m = 2$ quadrupole, etc.)

$\nu_s = \omega_s/\omega_0$ is the ratio of synchrotron to revolution frequency.

For transverse oscillations, $Z_\perp$ is proportional to Z/ω [2], and we can still use Eq. (1) if we replace $h_m(\omega_p)$ by $h_m(\omega_p - \omega_\xi)$ where

$$\omega_p = \omega_0 (kp + n + Q) \tag{3}$$

and ω_ξ is the chromatic frequency shift. We also have to include the case $m = 0$, which describes the single-bunch "head-tail" instability. For the longitudinal case, there is no (or negligibly little) chromaticity. Single bunches can only become unstable if two or more modes with different m oscillate at the same frequency. Usually this occurs at quite high frequencies and the single-bunch instability is therefore often called "microwave instability".

For sinusoidal modes with a perturbed charge density proportional to

$$\lambda_m(\phi) = \left\{ \begin{array}{ll} \cos \frac{\pi}{2} (m+1) \frac{\phi}{\phi_0} & m \text{ even} \\ \\ \cos \frac{\pi}{2} (m+1) \frac{\phi}{\phi_0} & m \text{ odd} \end{array} \right| \tag{4}$$

the spectral density is given by [2]

$$h_m(\omega) = (m + 1)^2 \frac{1 + (-)^m \cos \omega\tau}{\left[\left(\frac{\omega\tau}{\pi}\right)^2 - (m + 1)^2\right]^2} \tag{5}$$

where $2\phi_0 = \omega_0\tau$ describes the bunch length, and the oscillation frequencies are given by Eqs. (2) or (3) for longitudinal and transverse oscillations, respectively.

Sinusoidal modes actually describe only oscillation of bunches with a parabolic density distribution, which is a reasonable approximation for protons. For electrons, the distribution is usually Gaussian, and we should use oscillation modes described by Hermite polynomials. However, measurements on existing machines indicate that reality lies somewhere in-between [4], and we thus use the simpler sinusoidal modes for both cases.

2. RESONANT CAVITY IMPEDANCE

The high-frequency coupling impedance of the vacuum chamber and other surrounding equipment can be described by a superposition of resonant modes in many planned or accidental cavities. The impedance of a single resonator is described by

$$Z(\omega) = R \frac{1 - j\,Q\left(\frac{\omega}{\omega_r} - \frac{\omega_r}{\omega}\right)}{1 + Q^2\left(\frac{\omega}{\omega_r} - \frac{\omega_r}{\omega}\right)^2} \tag{6}$$

where R is the shunt impedance

Q is the quality factor

ω_r is the resonant (angular) frequency of a particular cavity mode.

The total effective coupling impedance can be found by adding the contributions of all significant modes. In practice, it is simpler just to select the strongest modes of the RF cavities and to lump all accidental cavities into one low-Q resonator.

Attempts to evaluate Eqs. (1) to (4) by computer have not been very successful. For high Q-values, the real part of the sum over the impedances converges quite fast, but the imaginary part converges badly and needed over hundred thousand terms for reasonable accuracy in our test cases (see Figs. 1 to 3). For low-Q resonators, the situation becomes even worse. However, the direct summation is not only time-consuming, but there is also no easy convergence test as can be best seen from Fig. 3: there the sum as function of the number of terms flattens off at an intermediate value before it descends to the correct result.

3. ANALYTIC SUMMATION

Fortunately, it is possible to sum the series in Eq. (1) analytically for a resonator impedance and sinusoidal modes [5]. The derivation of the two sums is shown in the Appendices. The sum over the spectral modes becomes quite simple

$$\sum_{p=-\infty}^{\infty} h_m(\omega_p) = \frac{\pi^2 d}{2} \tag{7}$$

independent of mode number m which appears explicitely in $h_m(\omega)$. Incidently, the same result is obtained when the summation is replaced by an integral, although this is generally only an approximation.

The sum over the product of the impedance (divided by frequency) and the spectral modes is much more complicated, and the effective impedance is given by

$$\left(\frac{Z}{n}\right)_{eff} = \frac{j\ b\ R}{K\ Q}\left[\frac{(m+1)^2\ d^3}{\pi\ u} Z_2 - Z_1\right] \tag{8}$$

where

$$Z_1 = \frac{1}{2}\sum_{i=3}^{4}\frac{1}{(p_1 - p_i)(p_2 - p_i)} = \frac{(m+1)^2 d^2 - b^2}{\left[(m+1)^2 d^2 - b^2\right]^2 + (m+1)^2\ \frac{b^2 d^2}{Q^2}} \tag{9}$$

and

$$Z_2 = \sum_{i=1}^{2} (-)^{i+1}\ \frac{\cot \pi p_i \left[1 + (-)^m \cos\frac{\pi}{d}(p_i + c)\right] + (-)^m \sin\frac{\pi}{d}(p_i + c)}{(p_i - p_3)^2\ (p_i - p_4)^2} \tag{10}$$

with

$$\left.\begin{aligned} p_{1,2} &= -\ a \pm u + j\ v \\ p_{3,4} &= -\ c \pm (m+1) d \end{aligned}\right| \tag{11}$$

and

$$\left.\begin{aligned} u &= b\sqrt{1 - 1/4Q^2} \\ v &= b/2Q \end{aligned}\right| \tag{12}$$

$$\left.\begin{aligned} b &= f_r/k\ f_0 \\ d &= 1/2\ k\ f_0\ \tau = 1/2\ B \end{aligned}\right\} \tag{13}$$

where B is the bunching factor.

For the longitudinal case

$$a = c = \frac{n + m\ \nu_s}{k} \tag{14a}$$

while for transverse oscillations

$$a = \frac{n + Q}{k}\ , \quad c = a - \frac{\omega_\xi}{k\ \omega_0} \tag{13b}$$

where Q is the betatron tune, and ω_ξ the chromatic frequency shift.

While the expression for Z_1 is always real, the more complicated term Z_2 is in general complex. However, this expression can be evaluated quite easily on a computer, and some results for a test-case are shown in Figs. 4 and 5. It appears that the real part of the effective impedance oscillates around zero as a function of the resonator frequency, while the imaginary part oscillates with approximately the same maximum amplitude (for different mode number n), but has also a large constant part (see Figs. 4 and 5) which is the average over all possible mode numbers $0 \leq n \leq k$. This average is shown as function of the bunch length in Fig. 6. For very short bunches it is usually capacitive, while it becomes inductive when the bunch length is longer than the inverse resonator frequency.

4. POTENTIAL WELL BUNCH LENGTHENING

The incoherent synchrotron frequency of particles in a bunch depends on the total number of charged particles and their interaction with the coupling impedance, described by the effective coupling impedance [6]

$$\omega_s^2 = \omega_{s0}^2 \left[1 + A \left(\frac{\ell_0}{\ell} \right)^3 \mathrm{Im} \left(\frac{Z}{n} \right)_{\mathrm{eff}} \right] \tag{14}$$

where

$$A = \frac{(2\pi\ R)^3\ I_0}{3h\ k\ \ell_0^3\ V_{RF}\ \cos\ \phi_s} \tag{15}$$

is proportional to the current per bunch I_0/k. Above transition energy, $\cos\ \phi_s < 0$ and A is negative. For the stationary distribution m = 0 the impedance of a resonator is always inductive $\mathrm{Im}\ \frac{Z}{n} > 0$, and we get a decrease of the synchrotron frequency with increasing current. Furthermore, in a potential well the bunch length is related to the synchrotron frequency. For protons we have constant phase space area or

$$\frac{\ell}{\ell_0} = \frac{\omega_s^2}{\omega_{s0}^2} \tag{16a}$$

while for electrons radiation damping leads to

$$\frac{\ell}{\ell_0} = \frac{\omega_{s0}}{\omega_s} \tag{16b}$$

where ℓ_0 and ω_{s0} are the bunch length and synchrotron frequency for zero current. Combining Eqs. (14) and (16) we obtain the equations for the bunch length

$$\left(\frac{\ell}{\ell_0}\right)^4 - \left(\frac{\ell}{\ell_0}\right)^3 = A \cdot I_m \left(\frac{Z}{n}\right)_{eff} \quad (17a)$$

$$\left(\frac{\ell}{\ell_0}\right)^3 - \left(\frac{\ell}{\ell_0}\right) = - A \cdot I_m \left(\frac{Z}{n}\right)_{eff} \quad (17b)$$

for protons and electrons, respectively. Since the coupling impedance is in general a direct function of the bunch length, and also of the synchrotron frequency which is related to the bunch length by Eqs. (16), the Eqs. (17) are in general transcendental and have to be solved by approximate methods. Since $\left(\frac{Z}{n}\right)_{eff}$ then has to be evaluated many times, it is important that it is given by a relatively simple expression, and not by slowly converging infinite series which would make the solution of Eqs. (17) a major undertaking.

ACKNOWLEDGEMENT

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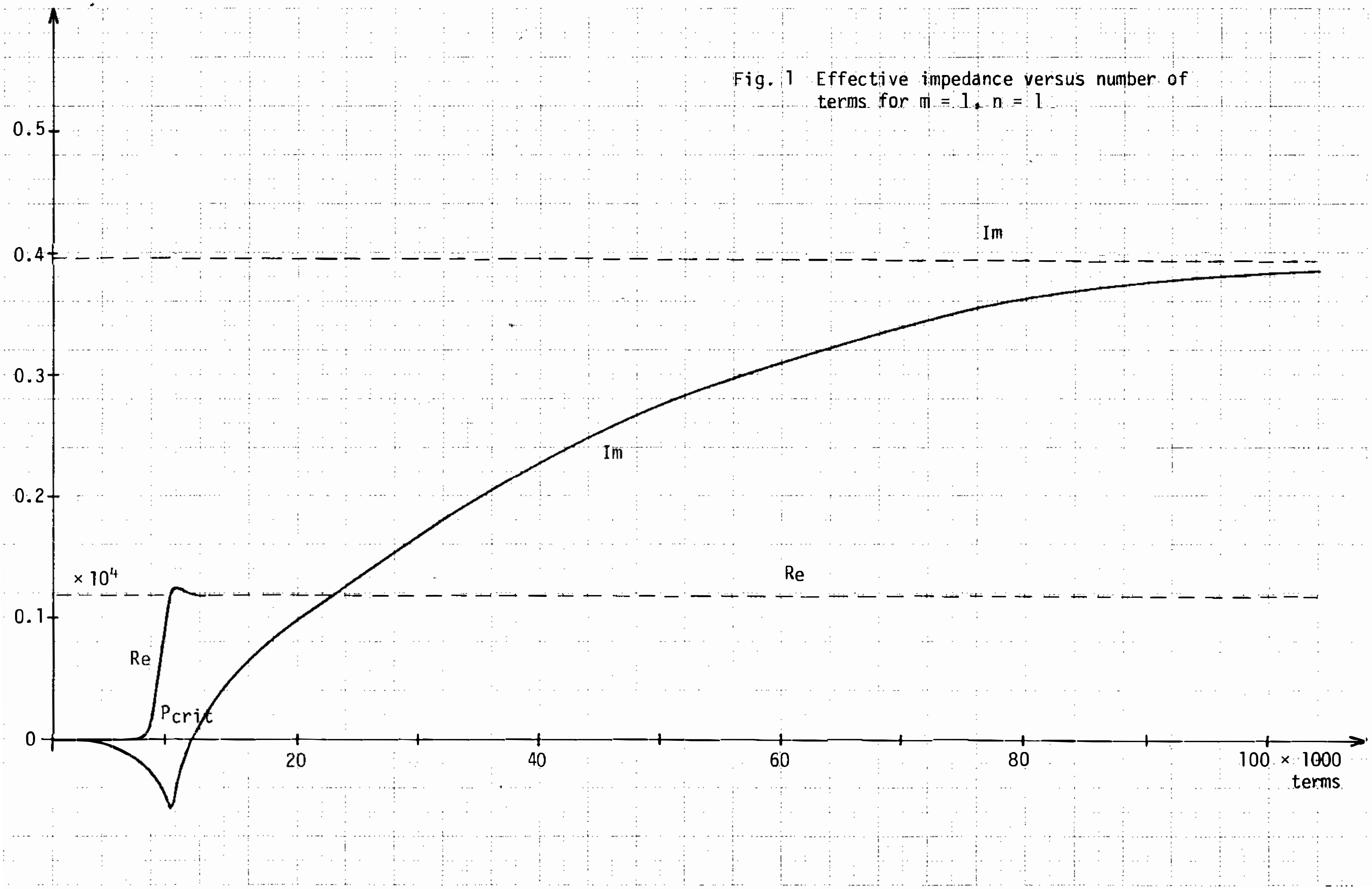

Fig. 1 Effective impedance versus number of terms for m = 1, n = 1

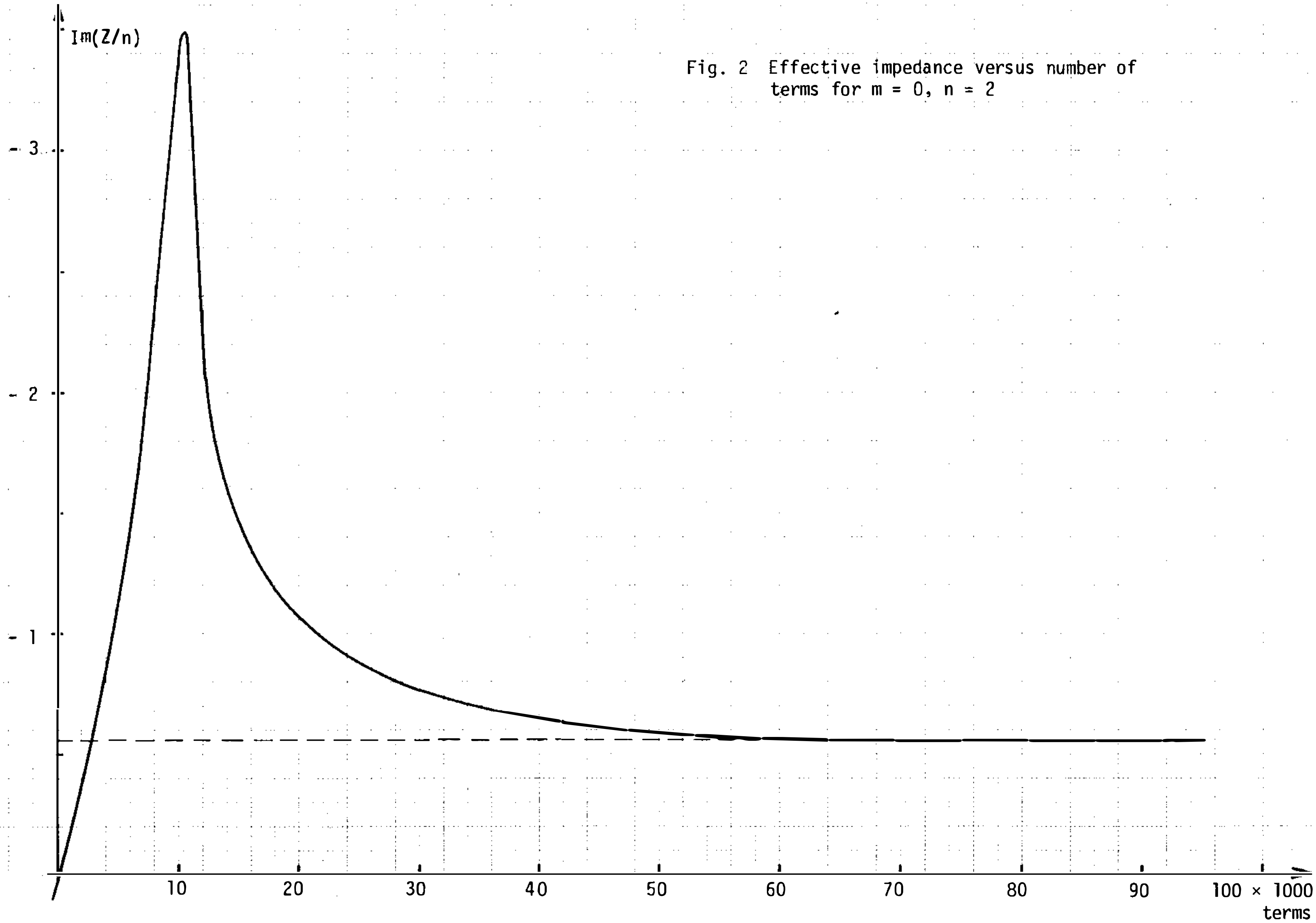

Fig. 2 Effective impedance versus number of terms for m = 0, n = 2

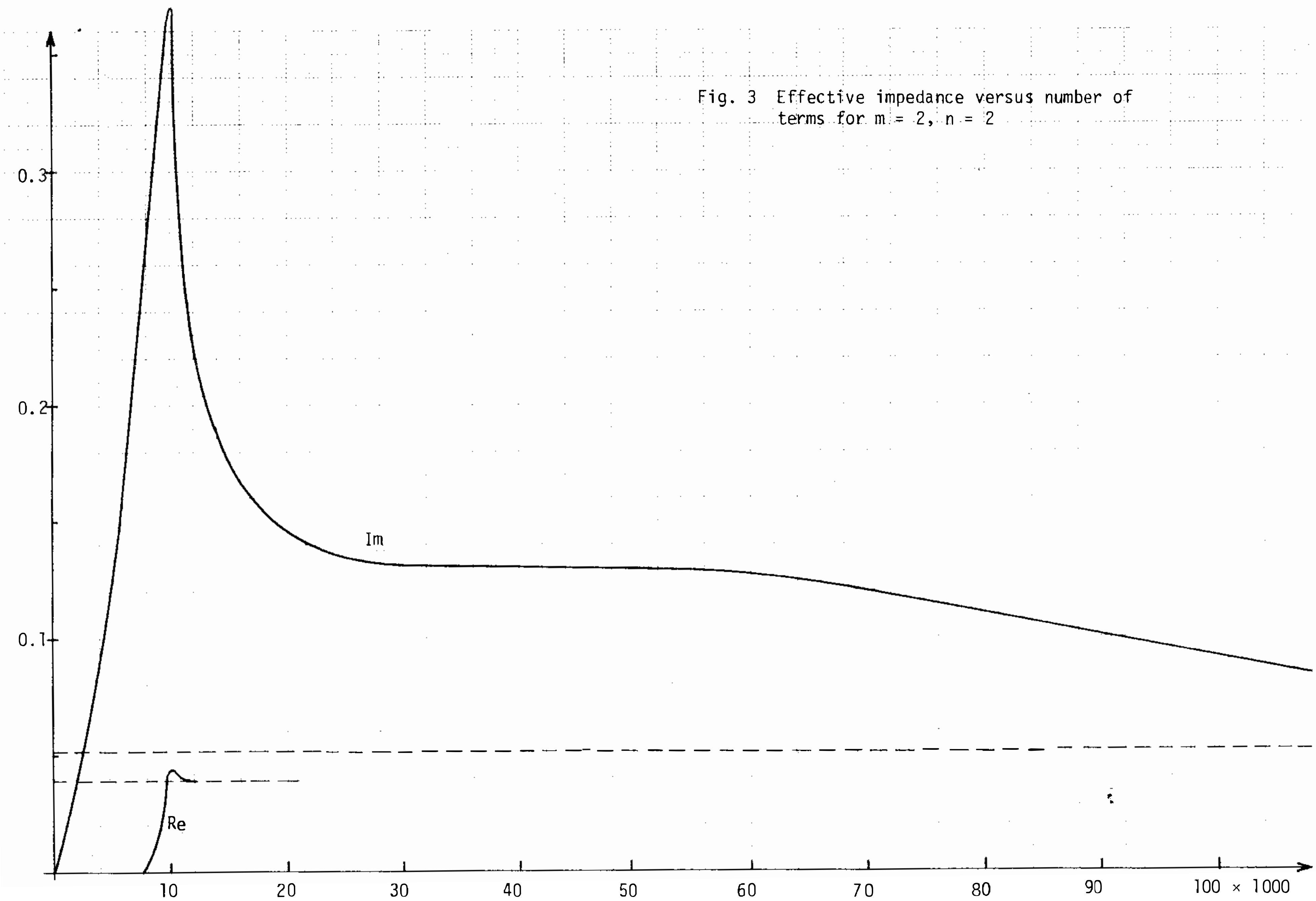

Fig. 3 Effective impedance versus number of terms for m = 2, n = 2

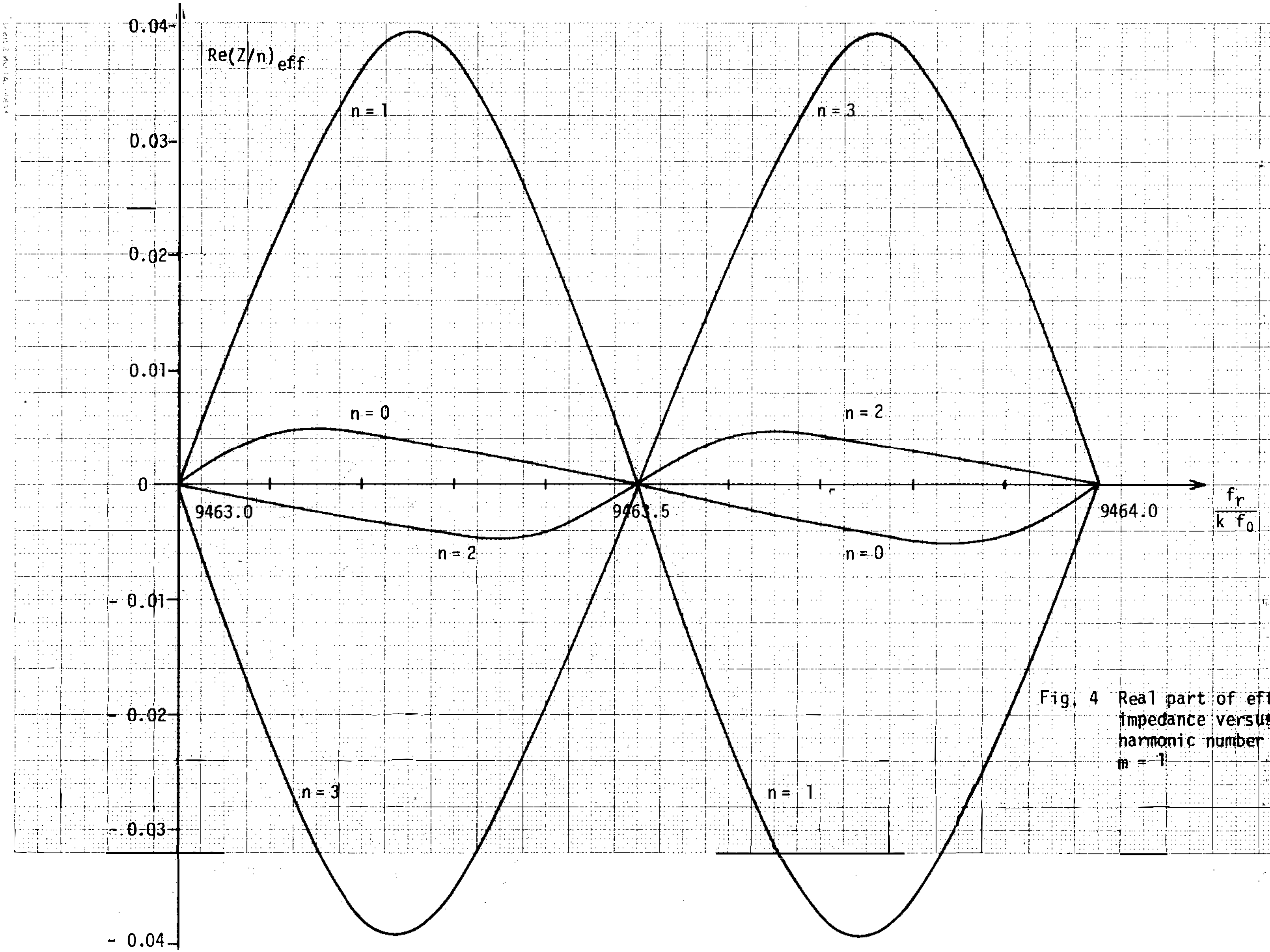

Fig. 4 Real part of effective impedance versus harmonic number
m = 1

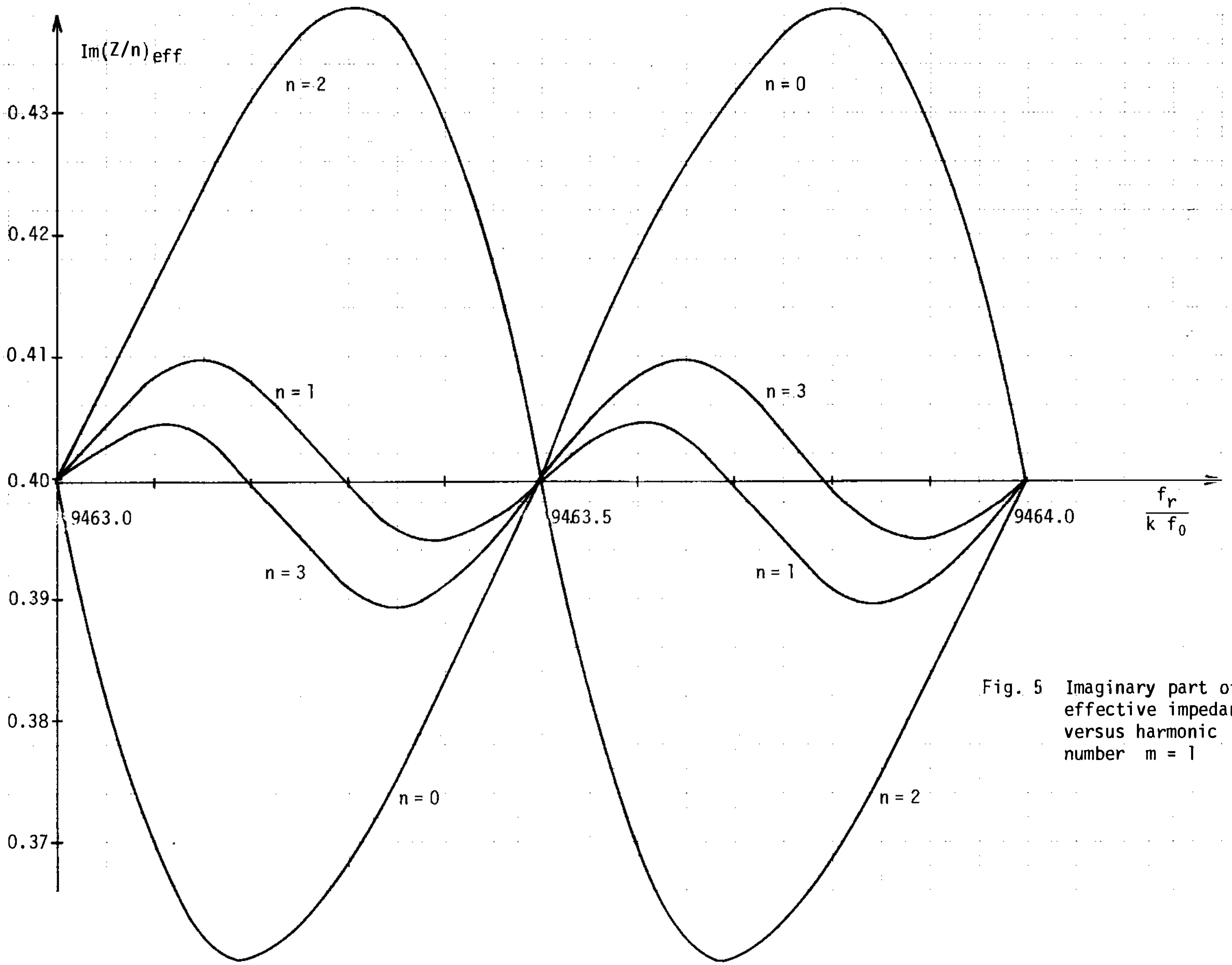

Fig. 5 Imaginary part of effective impedance versus harmonic number m = 1

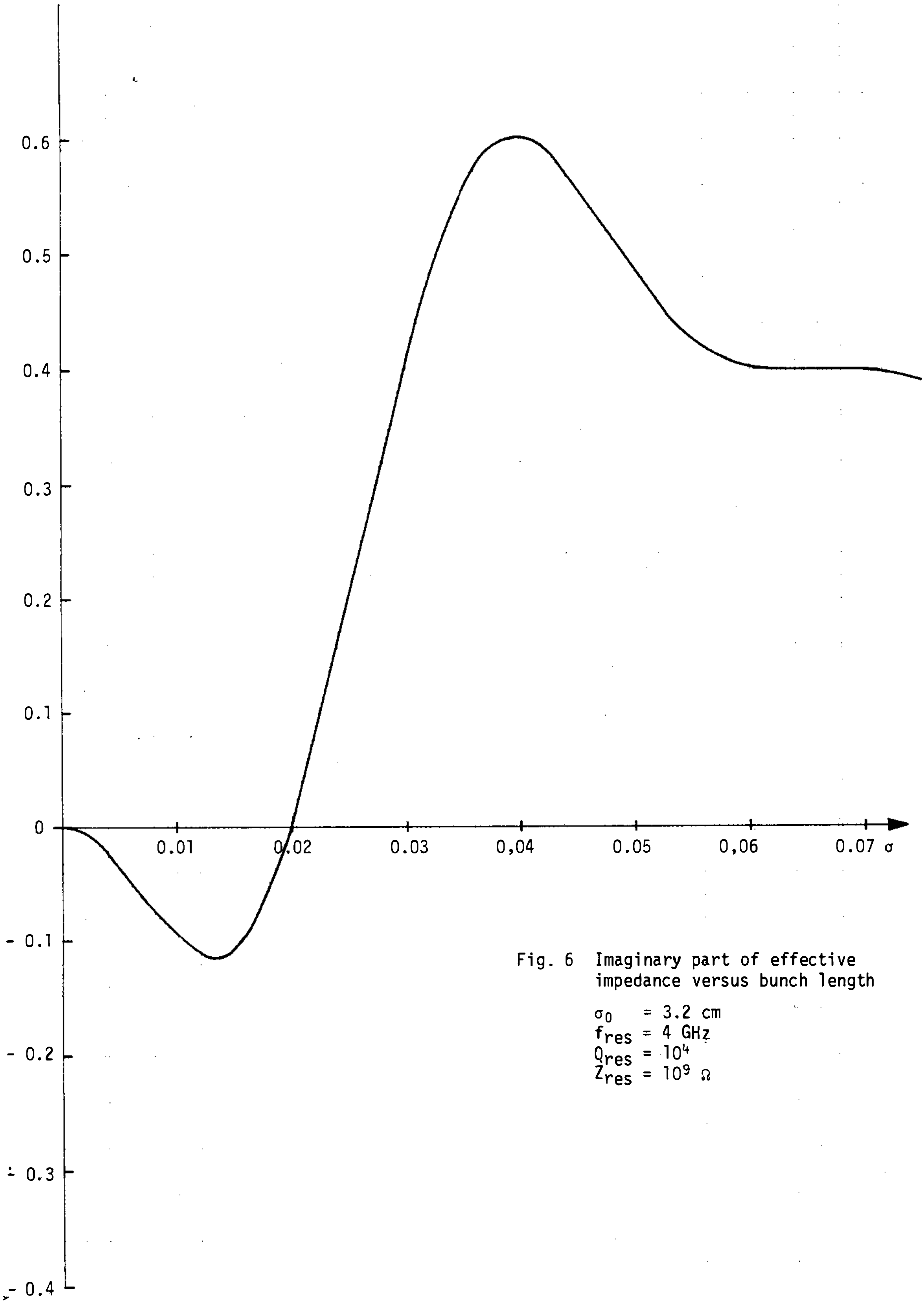

Fig. 6 Imaginary part of effective impedance versus bunch length

σ_0 = 3.2 cm
f_{res} = 4 GHz
Q_{res} = 10^4
Z_{res} = 10^9 Ω

APPENDIX A

Summation of the bunch spectrum

(1) We want to evaluate

$$H_m = \sum_{p=-\infty}^{\infty} h_m(\omega_p) \tag{A1}$$

where

$$\omega_p = \left| \begin{array}{ll} (kp + n + m\,\nu_s)\ \omega_0 & \text{(longitudinal)} \\ (kp + n + Q)\ \omega_0 - \omega_\xi & \text{(transverse)} \end{array} \right| \tag{A2}$$

and

$$h_m(\omega) = (m+1)^2 \frac{1 + (-)^m \cos \omega\tau}{\left[\left(\frac{\omega\tau}{\pi}\right)^2 - (m+1)^2\right]^2} \tag{A3}$$

(2) We substitute Eqs. (A2) and (A3) into Eq. (A1) and obtain

$$H_m = (m+1)^2\, d^4 \sum_{p=-\infty}^{\infty} \frac{1 + (-)^m \cos \frac{\pi}{d}(p + c)}{(p - p_1)^2\,(p - p_2)^2} \tag{A4}$$

where

$$p_{1,2} = -\,c \pm (m+1)\, d \tag{A5}$$

$$c = \left\{ \begin{array}{ll} \dfrac{n + m\,\nu_s}{k} & \text{(longitudinal)} \\ \dfrac{n + Q}{k} - \dfrac{\omega_\xi}{k\omega_0} & \text{(transverse)} \end{array} \right| \tag{A6}$$

and

$$d = \frac{1}{2k\, f_0\, \tau} = \frac{1}{2B} \tag{A7}$$

when B is the bunching factor.

(3) Partial fraction decomposition yields

$$\frac{1}{P(p)} = \frac{1}{(p - p_1)^2\,(p - p_2)^2} = \sum_{i=1}^{2} \left[\frac{a_i}{p - p_i} + \frac{b_i}{(p - p_i)^2}\right] \tag{A8}$$

where

$$\left.\begin{aligned} a_{1,2} &= \mp \frac{2}{(p_1 - p_2)^3} \\ b_{1,2} &= \frac{1}{(p_1 - p_2)^2} \end{aligned}\right| \qquad \text{(A9)}$$

(4) Summation formulae

$$\sum_{p=-\infty}^{\infty} \frac{1}{P(p)} = -\pi \sum_{i=1}^{2} a_i \cot \pi p_i + \pi^2 \sum_{i=1}^{2} \frac{b_i}{\sin^2 \pi p_i} \qquad \text{(A10)}$$

From the general summation formula for $\sum_{-\infty}^{\infty} \frac{\cos \theta (p + c)}{P(p)}$, which can be found from Ref. 5), we get for $\theta = \frac{\pi}{d}$ $\theta(p_i + c) = \pm (m + 1)\pi$, and hence $\cos \theta(p_i + c) = (-)^{m+1}$, and $\sin \theta(p_i + c) = 0$. Then

$$\sum_{p=-\infty}^{\infty} \frac{\cos \frac{\pi}{d} (p + c)}{P(p)} = - (-)^{m+1} \sum_{i=1}^{2} a_i \cot \pi p_i + (-)^{m+1} \pi^2 \sum_{i=1}^{2} b_i \left[\frac{1}{\sin^2 \pi p_i} - \frac{1}{d}\right] \qquad \text{(A11)}$$

(5) Adding Eq. (A10) and $(-)^m$ times Eq. (A11) yields $\frac{\pi^2}{d}(b_1 + b_2)$, since all other terms cancel. From Eqs. (A5) and (A9) we get $b_1 = b_2 = 1/(m + 1)^2 d^2$, and thus Eq. (A4) yields

$$H_m = \frac{\pi^2 d}{2} \qquad \text{(A12)}$$

in exact agreement with the result of the (approximate) integral

$$H_m \approx \frac{2}{k \omega_0} \int_0^{\infty} h_m(\omega) d\omega = \frac{(m + 1)^2}{k f_0 \tau} \int_0^{\infty} \frac{1 + (-)^m \cos \pi x}{[x^2 - (m + 1)^2]^2} dx \qquad \text{(A13)}$$

This integral can be found in tables, and equals $\pi^2/4(m + 1)^2$. With Eq. (A7) we thus obtain again Eq. (A12).

APPENDIX B

Effective impedance of a resonator

(1) Effective impedance

$$\left(\frac{Z}{n}\right)_{eff} = \frac{\omega_0}{H_m} \sum_{p=-\infty}^{\infty} \frac{Z(\omega_p)}{\omega_p} h_m(\omega_p) \qquad \text{(B1)}$$

where

$$Z(\omega) = \frac{R}{1 + j\,Q\left(\frac{\omega}{\omega_r} - \frac{\omega_r}{\omega}\right)} \qquad \text{(B2)}$$

and ω_p and $h_m(\omega)$ have been defined in Appendix A.

(2) We can combine these expressions to get

$$\left(\frac{Z}{n}\right)_{eff} = -\frac{2\,j\,b\,d^3}{\pi^2 k}(m+1)^2\frac{R}{Q}\sum_{p=-\infty}^{\infty}\frac{1 + (-)^m \cos\frac{\pi}{d}(p+c)}{(p-p_1)(p-p_2)(p-p_3)^2\,(p-p_4)^2} \qquad \text{(B3)}$$

where

$$p_{1,2} = -a \pm u + j\,v$$
$$p_{3,4} = -c \pm (m+1)\,d \qquad \text{(B4)}$$

with

$$u = b\left(1 - \frac{1}{4Q^2}\right)^{\frac{1}{2}}$$
$$v = \frac{b}{2Q} \qquad \text{(B5)}$$

(hence $u^2 + v^2 = b^2$) and

$$a = \frac{n + m\,\nu_s}{k} \quad \left(\text{or } a = \frac{n+Q}{k} \text{ transverse}\right)$$
$$b = \frac{f_r}{k\,f_0}$$
$$c = a - \frac{f_\xi}{k\,f_0} \qquad \text{(B6)}$$
$$d = \frac{1}{2k\,f_0\,\tau}$$

(3) Partial fraction decomposition

$$\frac{1}{P(p)} = \frac{1}{(p - p_1)(p - p_2)(p - p_3)^2 (p - p_4)^2} = \sum_{i=1}^{4} \frac{a_i}{p - p_i} + \sum_{i=3}^{4} \frac{b_i}{(p - p_i)^2} \qquad (B7)$$

where

$$a_{1,2} = \frac{\pm 1}{(p_1 - p_2)(p_{1,2} - p_3)^2 (p_{1,2} - p_4)^2}$$

$$b_{3,4} = \frac{1}{(p_{3,4} - p_1)(p_{3,4} - p_2)(p_3 - p_4)^2} \qquad (B8)$$

($a_{3,4}$ are more complicated, but not required.)

(4) With the summation formulae used in Appendix A we get for $\theta = \frac{\pi}{d}$

$$\sum_{p=-\infty}^{\infty} \frac{\cos \frac{\pi}{d}(p + c)}{P(p)} = - \pi \sum_{i=1}^{2} a_i \frac{\cos\left[\pi p_i - \frac{\pi}{d}(p_i + c)\right]}{\sin \pi p_i} + (-)^{m+1} \times$$

$$\times \left[- \pi \sum_{i=3}^{4} a_i \cot \pi p_i + \pi^2 \sum_{i=3}^{4} b_i \left(\frac{1}{\sin^2 \pi p_i} - \frac{1}{d}\right)\right] \qquad (B9)$$

while

$$\sum_{p=-\infty}^{\infty} \frac{1}{P(p)} = - \pi \sum_{i=1}^{4} a_i \cot \pi p_i + \pi^2 \sum_{i=3}^{4} \frac{b_i}{\sin^2 \pi p_i} \qquad (B10)$$

hence several terms cancel when we calculate

$$\sum_{p=-\infty}^{\infty} \frac{1 + (-)^m \cos \frac{\pi}{d}(p + c)}{P(p)} =$$

$$= - \pi \sum_{i=1}^{2} a_i \frac{(-)^m \cos\left[\pi p_i - \frac{\pi}{d}(p_i + c)\right] + \cos \pi p_i}{\sin \pi p_i} + \frac{\pi^2}{d}(b_3 + b_4) \qquad (B11)$$

which does not contain the coefficients a_3 and a_4.

(5) With $\frac{\cos\left[\pi p_i - \frac{\pi}{d}(p_i + c)\right]}{\sin \pi p_i} = \cot \pi p_i \times \cos \frac{\pi}{d}(p_i + c) + \sin \frac{\pi}{d}(p_i + c)$ we get finally

$$\left(\frac{Z}{n}\right)_{eff} = + j \frac{b R}{k Q}\left[\frac{(m + 1)^2 d^3}{\pi u} Z_2 - Z_1\right] \qquad (B12)$$

where

$$Z_1 = \frac{1}{2} \sum_{i=3}^{4} \frac{1}{(p_1 - p_i)(p_2 - p_i)} \qquad \text{(B13)}$$

and

$$Z_2 = \sum_{i=1}^{2} (-)^{i+1} \frac{\cot \pi p_i \left[1 + (-)^m \cos \frac{\pi}{d} (p_i + c)\right] + (-)^m \sin \frac{\pi}{d} (p_i + c)}{(p_i - p_3)^2 \, (p_i - p_4)^2} \qquad \text{(B14)}$$

(6) Although p_1 and p_2 are complex, it turns out that Z_1 is purely real and can be written

$$Z_1 = \frac{t^2 - b^2}{(t^2 - b^2)^2 + t^2 b^2/Q^2} \qquad \text{(B15)}$$

where

$$t = (m + 1)d \qquad \text{(B16)}$$

However, Z_2 is complex in general, and no simple general expression has been found.