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ON THE NUMERICAL SOLUTION OF HELMHOLTZ'S EQUATION AND THE EIGEN-VALUE PROBLEM OF THE LAPLACE OPERATOR BY THE CAPACITANCE MATRIX METHOD

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The thesis consists of this summary and the following papers:

1. Proskurowski W. and Widlund O., *On the Numerical Solution of Helmholtz's Equation by the Capacitance Matrix Method,* Part I, TRITA-NA-7602, Rep. RIT Stockholm, 1976. To appear in a slightly different form in Math. Comp., July 1976.

2. Ibid, Part II, TRITA-NA-7603, Rep. RIT Stockholm, 1976.

3. Pereyra V., Proskurowski W. and Widlund O., *On a family of Elliptic Difference Schemes Suggested by Heinz-Otto Kreiss,* TRITA-NA-7608, Rep. RIT Stockholm, 1976.

4. Proskurowski W., *On the Numerical Solution of the Eigenvalue Problem of the Laplace Operator by the Capacitance Matrix Method,* TRITA-NA-7609, Rep. RIT Stockholm, 1976.

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1. INTRODUCTION

The problem of solving Helmholtz's equation

$$-\Delta u + cu = f \ ,$$

where c is a real constant, and the corresponding eigenvalue problem

$$\Delta u + \lambda u = 0 \ ,$$

are of considerable interest and arise in a variety of applications, see Courant and Hilbert [9] and Garabedian [11].

Since the first implementation of the Fast Fourier Transform in 1965, see Cooley and Tukey [8], several fast methods have been developed for solving finite difference analogues of the Helmtoltz's equation, see Hockney [12], Buneman [5] and also Buzbee, Golub and Nielson [7]. All these methods can be regarded as efficient computer implementations of the separation of variables method. That method can only be used for regions which, after a possible change of independent variables, are rectangular and for differential operators of special form, see Widlund [16]. For a square region with n^2 mesh points only $n^2(1 + \mathcal{O}(1/n))$ storage locations are needed, while the operation count is $c_1 n^2 \log_2 n(1 + o(1))$, where c_1 is a constant less than 10. Similarily, fast solvers exist for infinite parallel strips, see [1] and Fischer, Golub, Hald, Leiva and Widlund [10], and the whole plane, see Hockney [12].

In this thesis we consider a problem on an arbitrary bounded plane region Ω. The region Ω is imbedded in a rectangle, an infinite parallel strip or the whole plane. A uniform mesh is imposed on the enlarged region. An expanded linear system of algebraic equations is derived which has a reducible matrix. This new matrix contains the matrix of our original problem as an irreducible component. The resulting matrix is a rank p modification of a separable problem which allows for the use of a fast solver. Here p is the number of irregular mesh points, i.e. those mesh points which do not have all its next neighbors in the open set of Ω. For two dimensional problems we thus have a value of p of the order n. The problem can be solved with the aid of the Woodbury formula or one of its variants, see Buzbee, Dorr, George and Golub [6] and Hockney [13]. The operation count for their algorithms is $c_2 pn^2 \log_2 n(1 + o(1))$.

2. SUMMARY OF THE PAPERS

In [1] we consider the solution of the interior Direchlet and Neumann problems for Helmholtz's equation on an arbitrary bounded plane region. This work grew out of an observation by Widlund [16] of a formal analogy between the Woodbury formula and a clasical solution formula for the Neumann problem for Laplace's equation, see Courant and Hilbert [9] or Garabedian [11]. In this potential theoretical approach an Ansatz is made in terms of a single layer potential. The charge density is then found by solving a Fredholm integral equation of the second kind. This suggests that iterative methods might compete successfully with Gaussian elimination for solving the capacitance matrix equation which corresponds to this integral equation. We have found the conjugate gradient method quite effective; this method results in considerable savings compared to previous implementations of the capacitance matrix method in cases where the number of variables is large and only one or a few problems are solved for a given region Ω. The proper Ansatz for the continuous Direchlet problem is a layer of dipoles. Changing our Ansatz to a finite difference analog of a dipole layer, the capacitance matrices are also well-conditioned and the conjugate gradient method performs very satisfactorily. We note that our Ansatz falls outside the algebraic framework of Buzbee, Dorr, George and Golub [6]. Our treatment also differs from theirs in that we allow the capacitance matrix to become singular.

Another main improvement is the fast generation of the capacitance matrix. The basic idea is the use of translation invariance which can be achieved by imposing a periodicity condition as a boundary condition for the problem for which the fast Helmholtz's solver is applied. The matrix representing the discrete fundamental solution is then a circulant and therefore by knowing one of its columns we know the entire matrix.

Taken together these improvements result in an operation count of $c_3 n^2 \log_2 n(1 + o(1))$, i.e. a solution to our problem can be obtained at an expense which grows no faster than that for a fast solver on a rectangle when the mesh size is refined.

This paper also contains a discussion of previous work on capacitance matrix methods.

A FORTRAN program which implements the above method is given in [2].

The method for the Direchlet problem which is presented in [1] is of second order. Methods of solving the interior Direchlet problem for Laplace's equation with higher accuracy are developed in [3]. These methods were suggested by H.-O. Kreiss. Here, the approximation at the irregular mesh points is chosen so that several terms of an asymptotic error expansion exist. This expansion justifies the use of deferred correction methods, see Pereyra [14]. A convergence result for these methods is established and confirmed by numerical experiments. The deferred correction method does not require mesh refinement and therefore the capacitance matrix, once generated and factored, can be used inexpensively to generate the corrections. For problems with sufficiently smooth solutions the gain in accuracy of several orders of magnitude is observed while the increase of computational expense is moderate.

The problem of finding several eigenfunctions and eigenvalues of the interior Direchlet problem for the reduced wave equation is considered in [4]. We combine two fast algorithms, the iterative Block Lanczos method to compute eigenvalues and eigenvectors of a given matrix, see Underwood [15], and our capacitance matrix method. The capacitance matrix is generated and factored only once for a given problem. In each iteration of the Block Lanczos method the discrete Helmholtz's equation is solved at a cost of $c_3 n^2 \log_2 n$ operations. The cost of this step is only about twice that of the fast solver on the rectangle in which our region is imbedded.

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