On the Concept of Similiarity in the Theory of Isotropic Turbulence

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MUCH recent work has been done in the study of isotropic turbulence, particularly from the point of view of its spectrum. But the underlying concept is still the assumption of the similarity of the spectrum during the process of decay, which is equivalent to the idea of self-preservation of the correlation functions introduced by the senior author. It is however generally recognized that the correlation function does change its shape during the process of decay, and hence the concept of self-preservation or similarity must be interpreted with suitable restrictions. Under the limitation to low Reynolds numbers of turbulence, the original idea of Kármán-Howarth has been confirmed. Then the decay consists essentially of viscous dissipation of energy separately in each individual frequency interval. However, when turbulent diffusion of energy, i.e., transfer of energy between frequency intervals, occurs at a significant rate, the interpretation of the decay process and the spectral distribution is quite varied. This can be seen by a comparison of the recent publications of Heisenberg,[1] Batchelor,[2] Frenkiel,[3] and the present authors.[4,5] The purpose of the present paper is an attempt to clarify this situation.

Since some of these discussions are presented in terms of the correlation functions and others in terms of the spectrum, we shall begin by giving a systematic demonstration of the relation between these two theories. This can be easily done by a three-dimensional Fourier transform of the equation for the change of the double correlation tensor:

$$\partial/\partial t(u'^2R_{ik})-u'^3(\partial/\partial\xi_j)(T_{ijk}+T_{kji})=2\nu u'^2R_{ik},\quad (1)$$

where u'^2 is the mean square of the turbulent of velocity, t is the time, ν is the kinematic viscosity coefficient, and $R_{ik}(\xi_l, t)$ and $T_{ijk}(\xi_l, t)$ are the double and the triple correlation tensor defined by Kármán and Howarth for two points P and P' separated by a space vector ξ_l. By contracting the resultant equation and multiplying it with $4\pi\kappa^2/3$, where κ is the wave number, we obtain the following equation for the change of spectrum:

$$\partial F/\partial t+W=-2\nu\kappa^2F,\quad (2)$$

where

$$\begin{aligned}
&F=(4\pi\kappa^2/3)F_{nn},\\
&F_{ik}(\kappa_l)=u'^2/(2\pi)^3\int\int\int R_{ik}(\xi_l, t)e^{i(\kappa_n\xi_n)}d\tau(\xi),\\
&W=(4\pi\kappa^2/3)\cdot 2i\kappa_jW_{njn},\\
&W_{ijk}(\kappa_l)=u'^3/(2\pi)^3\int\int\int T_{ijk}(\xi_l, t)e^{i(\kappa_n\xi_n)}d\tau(\xi).
\end{aligned}\quad (3)$$

Evaluating these integrals in terms of spherical coordinates in the ξ-space we obtain

$$\begin{aligned}
&F=\tfrac{1}{3}\{\kappa^2F_1''(\kappa)-\kappa F_1'(\kappa)\},\\
&F_1(\kappa)=2u'^2/\pi\int_0^\infty f(r, t)\cos\kappa r dr;\\
&W=\tfrac{2}{3}\{\kappa^2H_1''(\kappa)-\kappa H_1'(\kappa)\},\\
&\kappa H_1(\kappa)=2u'^3/\pi\int_0^\infty h(r, t)\sin\kappa r dr.
\end{aligned}\quad (4)$$

where $f(r, t)$ and $h(r, t)$ are the double and triple correlation functions satisfying the Kármán-Howarth equation:

$$\partial/\partial t(u'^2f)+2u'^3(\partial h/\partial r+4h/r)=2\nu u'^2(\partial^2f/\partial r^2+(4/r)(\partial f/\partial r)).\quad (5)$$

The relations (4) which connect (2) and (5) have been obtained previously (1947) by the junior author.[6] The

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[1] W. Heisenberg, Proc. Roy. Soc. London, A **195**, 402 (1948).

[2] G. K. Batchelor, "Recent developments on turbulence research," general lecture at the Seventh International Congress for Applied Mechanics, September, 1948.

[3] F. N. Frenkiel, "Comparison between theoretical and experimental laws of decay of turbulence," presented at the Seventh International Congress for Applied Mechanics, September, 1948.

[4] Th. v. Kármán, Comptes Rendus **226**, 2108 (1948); Proc. Nat. Acad. Sci. **34**, 531 (1948); Sverdrup anniversary volume (1949).

[5] C. C. Lin, Proc. Nat. Acad. Sci. **34**, 540 (1948); "On the law of decay and the spectrum of isotropic turbulence," presented at the Seventh International Congress for Applied Mechanics, September, 1948.

[6] C. C. Lin, "Remarks on the spectrum of turbulence," presented at the First Symposium of Applied Mathematics, American Mathematical Society, August, 1947; to appear in the Proceedings of the symposium.

function F is essentially identical with Heisenberg's spectral function, whereas F_1 is the spectrum function introduced by Taylor about a decade ago on the basis of a one-dimensional Fourier analysis of wind-tunnel turbulence. The tensor F_{ik} in (3) was introduced and studied by Batchelor and Kampé de Fériet in 1948.

In this note, we shall restrict ourselves to the spectral theory. We propose to analyse the spectrum and its change during the process of decay. We distinguish two extreme cases: (a) the Reynolds number of turbulence is initially very large, and (b) very small initial Reynolds numbers. The latter case is very much simpler, and can be explained in a few words after the first case is investigated. We shall therefore now consider the case of very large Reynolds numbers.

There is no general principle known which would determine the most probable energy distribution over the spectrum. The problem we deal with is not a question of statistical equilibrium in the proper sense. We shall base our investigations on the concept that during the process of decay the spectrum shows a tendency to become similar. Similarity in this case means that the spectrum can be expressed in the form

$$F=U^2 l\varphi(\kappa l), \tag{6}$$

where U is a typical velocity and l a typical length. The problem is to connect these typical quantities with measurable ones, such as the kinematic viscosity ν, Loitsiansky's invariant J_0, and the rate of energy dissipation ϵ. Full similarity would mean that it is possible to express U and l by unique relations for all values of κ and for the whole process of decay. Dealing with the experimental evidences and by dimensional considerations, one readily recognizes that this is not possible. Hence, one looks for a solution which best satisfies the similarity requirement.

All the authors agree in the following picture: the low frequency ranges contain the bulk of energy, while the viscous dissipation is negligible. They furnish energy by the action of inertial forces to the high frequency ranges, where it is converted into heat. Physically, this was seen by Taylor in the early stages of the development of the theory, but Kolmogoroff made these ideas more precise.

Kolmogoroff recognized that based on this physical concept the parameters which determine U and l for the high frequency range are the coefficient of kinomatic viscosity ν and the rate of energy dissipation ϵ. The rate of dissipation in any case is equal to $10\nu u'^2/\lambda^2$, where λ is the microscale in the dissipation mechanism. It is essential in Kolmogoroff's concept that u' and λ do *not* appear explicitly in the similarity analysis. Thus,

$$U=(\nu\epsilon)^{\frac{1}{4}}, \quad l=(\nu^3/\epsilon)^{\frac{1}{4}}. \tag{7}$$

It is reasonable to assume that at the lower end of this κ-range, the spectrum becomes independent of ν. Then necessarily

$$F\sim\epsilon^{\frac{2}{3}}\kappa^{-5/3}, \tag{8}$$

as it was concluded by several authors.

On the other hand, concerning the low frequencies, the junior author has first shown[6] that the lowest frequencies involve a fixed parameter, i.e., Loitsansky's invariant J_0, and indeed that the spectrum is of the form $J_0\kappa$.[4]

This situation clearly shows the infeasibility of one similarity range extending from $\kappa=0$ to $\kappa=\infty$, and furthermore makes it necessary to consider at least three ranges: that of the lowest frequencies essentially determined by Loitsiansky's invariant, an intermediary range which not necessarily complies with the similarity characteristics of the high range but is significantly influenced by the rate of dissipation ϵ (though not directly by ν), and the high frequency range in the sense of Kolmogoroff.

We therefore introduce three sets of characteristic quantities, namely, V^*, L^*; V, L; and v, η for the lowest, the medium, and the high frequency ranges respectively. The problem is how to connect them with each other and with other physical quantities.

Equation (7) gives

$$v=(\nu\epsilon)^{\frac{1}{4}}, \quad \eta=(\nu^3/\epsilon)^{\frac{1}{4}}. \tag{7a}$$

For the lowest and the medium ranges, we postulate the existence of a parameter (in general variable with time) which is common to these ranges, and governs the complicated mechanism of energy transfer in these low frequency ranges as the viscosity governs transfer of energy into heat in the high range. The formal statement of this hypothesis is

$$V^*L^*=VL=D. \tag{9}$$

We may call this parameter D the transfer coefficient or eddy diffusion coefficient of the turbulence mechanism. Obviously, it has a significance only when turbulent diffusion is active.

By introducing this hypothesis, we come to the following picture. In the lowest range, as it was pointed out, the invariant J_0 has a decisive influence. Hence, the characteristic parameters must be determined by J_0 and D. In the medium range, they must depend not only on D, but also on ϵ, since this range supplies the energy to be dissipated in the high frequency range. Thus, we have

$$V=(D\epsilon)^{\frac{1}{4}}, \quad \text{and} \quad L=(D^3/\epsilon)^{\frac{1}{4}}, \tag{10}$$

for the medium range, analogous to (7a), and

$$V^*=(D^5/J_0)^{\frac{1}{3}} \quad \text{and} \quad L^*=(J_0/D^2)^{\frac{1}{3}}, \tag{11}$$

for the range of lowest frequencies.

The three ranges, with characteristic quantities (7a), (10), and (11) appear clearly separated when their scales are much different from each other. Thus, when

$$\eta/L=(D/\nu)^{-\frac{3}{4}}\ll 1, \tag{12}$$

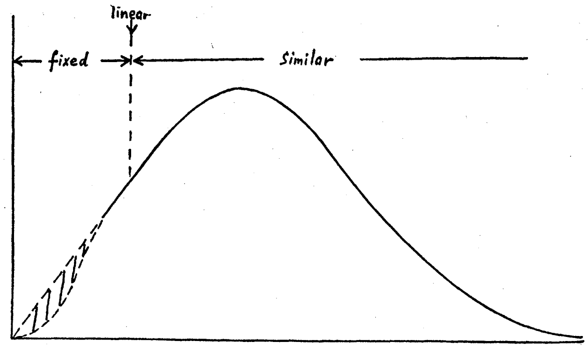

FIG. 1. Spectrum during early period.

the high and intermediate ranges appear clearly defined over significant parts of the whole spectrum. In between, there is a transition range depending only on the parameter ϵ common to both ranges. From dimensional arguments, we have for this transition range

$$F\sim\epsilon^{\frac{2}{3}}\kappa^{-5/3} \tag{13}$$

in accordance with Kolmogoroff's result for high Reynolds numbers. We shall see later that the condition (12) warrants the high value for the Reynolds number. Similarly, when

$$L/L^*=(T/T^*)^{\frac{1}{2}}\ll 1, \quad (T=L/V, \quad T^*=L^*/V^*), \tag{14}$$

we have a transition range between the low and medium frequencies. In this transition range, the spectrum depends only on D. Again, dimensional arguments show that there

$$F\sim D^2\kappa. \tag{15}$$

The physical significance of (14) will be explained below.

The exact behavior in the medium range will be determined by the fact which of the fixed parameters ν or J_0 has the predominating influence. We believe that we can arrive at a satisfactory description of the actual process by assuming that a change over takes place. We may divide the process into three stages: (I) the early stage, in which we shall see that $\eta:L=$constant, (II) the intermediate stage, in which we shall see that $L:L^*$ constant, and (III) the final stage, in which the distinction of several scales is impossible. This last case is the well-understood case of complete similarity at extremely low Reynolds numbers.

(I) *The early stage.* The turbulence field is actually created by some mechanism, natural or artificial, which produce individual eddies. Apparently, these eddies converge toward a kind of statistical balance through exchange of energy. The first period after homogeneity and isotropy are established shall be designated the early stage of the decay process. Experimental evidence on the decay law in these early stages shows that the similarity prevailing at high frequencies extends to the medium range. This statement is identical with the conclusion reached by the junior author,[5] assuming that a perfect similarity of the correlation function exists with the exception of correlations of points between very large distances. Accordingly V and L are constant multiples of v and η. Hence,

$$D=V^*L^*=VL\sim v\eta \tag{16}$$

and because $v\eta=\nu$, D is a constant. On the other hand,

$$V^{*2}L^{*5}=J_0. \tag{17}$$

Thus, in the early stage, the spectral function for the lowest frequency range appears to be independent of time. This fixed range extends as far as the linear part of the spectrum described by (15).

The spectrum is thus as shown in Fig. 1 (after Batchelor). By integration, one arrives at

$$u'^2=\int_0^\infty F(\kappa)d\kappa=\text{const.}V^2-u_D^2, \tag{18}$$

where u_D^2 is represented by the shaded area. By computing $\epsilon=-du'^2/dt$ and using the relation (10), we see that $V^2\sim t^{-1}$, and

$$u'^2=\frac{D_0}{10}t^{-1}-u_D^2, \quad \lambda^2=10\nu t\{1-10u_D^2t/D_0\},$$
$$R_\lambda=R_{\lambda 0}\{1-10u_D^2t/D_0\}, \tag{19}$$

where $R_{\lambda 0}$ is the initial Reynolds number

$$R_{\lambda 0}=\lim_{t\to 0}(u'\lambda/\nu), \tag{20}$$

and D_0 is a quantity proportional to D, defined by

$$D_0=\lim_{t\to 0}(u'^2\lambda^2/\nu). \tag{21}$$

A formula of the type of Eq. (21) for the diffusion coefficient has been suggested previously (1937) by the senior author.[7] The law of decay (19) was given by the junior author.[5] The role of the relatively invariant low frequency components has also been discussed by Heisenberg and Batchelor.

By using the law (19), one can see from (14) that the linear range of the spectrum would exist essentially for small values of t/T^*. This sets a limit to the period of validity of the law of decay (19). One can easily see the same limitation from the law (19) itself.

It can now be seen that the condition (12) is essentially the requirement that the Reynolds number is large (see (21)). It should be noted however that the existence of the $\kappa^{-5/3}$-range is not essential in the above discussion of the decay process. Hence, the Reynolds number need not be large, in order that the law (19) holds. In fact, for small initial Reynolds numbers, the Reynolds number at the end of the early period may be so small that the final period already sets in. This explains the

[7] Th. v. Kármán, J. Ae. Sci. 4, 131 (1937).

agreement obtained by the junior author for almost the whole process of decay in comparing his theory with the experiments of Batchelor and Townsend.[8]

(II) *The intermediate stage.* For large Reynolds numbers, after the disappearance of the linear part of the spectrum, the scales L and L^* become of the same order of magnitude, and it may be expected that the bulk of turbulent energy of scale L shares the behavior of the large eddies of scale L^*. The ratios L/L^* and V/V^* are expected to be constants. These conditions lead at once to the law of decay discussed by the senior author.[4] It is characterized by $d\lambda^2/d(\nu t)=7$.

During this period, the spectrum at low and medium frequencies depends only on the parameters J_0 and ν, and must therefore be of the form

$$F=J_0\kappa^4\Phi(\kappa L), \tag{22}$$

where $\Phi(\kappa L)$ behaves as $(\kappa L)^{-17/3}$ for $\kappa L\gg 1$, and approaches unity when $\kappa L\rightarrow 0$. An interpolation formula for Φ has been suggested and checked by correlation measurements by the senior author.

The diffusion coefficient in this range is easily seen to be proportional to $u'^2\lambda^2/\nu$. Hence, the condition (12) is again that the Reynolds number should be large. When the Reynolds number becomes very small, the scales η and L are of the same order, so that there is only one scale for all frequencies. We then approach a complete similarity, and are at the beginning of the final period. With reference to (12), we see that it should happen when R_λ is of the order of unity. According to the experiments of Batchelor and Townsend, the final period sets in at $R_\lambda \sim 5$.

The intermediate stage is very long, if the initial Reynolds number is very large. It begins with some value of R_λ close to $R_{\lambda 0}$. During this period, R_λ changes according to the power law $t^{-3/14}$. Although the supposed origin of time in this formula is unknown, it must be before the beginning of the early period, since the slope of the λ^2 *versus* νt curve decreases. Thus, R_λ become of the order of unity only when t is of the order of $T^*R_{\lambda 0}{}^{14/3}$. One expects therefore to find an intermediate stage many times the early period for high initial Reynolds numbers.

The above predictions are based on some simple hypotheses and physical picture, and should be confirmed experimentally. Unfortunately, there does not seem to be any experimental data available for sufficiently high Reynolds numbers and over a sufficiently long period. Most of the decay measurements at high Reynolds numbers hardly extend beyond the early period, when the law of decay (19) is quite adequate. Also, it must be kept in mind that the above discussions hold only for an infinite field of turbulence. In an actual experiment, the scale of the apparatus might become comparable with the scale of turbulence. In such cases, the significance of Loitsiansky's invariant becomes uncertain.

[8] G. K. Batchelor and A. A. Townsend, Proc. Roy. Soc. London, A **194**, 527 (1948).