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SHELL

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ONE-LOOP FINITENESS OF QUANTUM GRAVITY OFF MASS SHELL.

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ABSTRACT

The dependence of the counterterms  $\sqrt{g} R^2$  and  $\sqrt{g} R_{\mu\nu}^2$  on gauge parameters is calculated in the one-loop approximation in pure quantum gravity. Gauges are found in which these counterterms are absent, i.e. the theory is finite even off shell. The role of classical equations of motion and possible applications of our method to other theories and, in particular, to supergravity are discussed.

In recent years investigations in quantum gravity and supergravity [1] have been carried out mainly in the framework of the background method of De Witt [2-6]. However some theoretical questions connected with this method, in particular, a complete renormalization procedure and the role of classical equations of motion, have not been thoroughly investigated. These questions become very important now in connection with the investigation of renormalizability (or non-renormalizability), of gravity and supergravity. For example, 't Hooft and Veltman [4] have shown that in the background method the one-loop divergences of the S-matrix of pure gravity can be removed since the divergences vanish on the classical equations of motion (this means, that pure gravity in the one-loop approximation is finite on mass shell). These one-loop divergences can be removed also by a somewhat unusual renormalization of metric  $g_{\mu\nu} \rightarrow g_{\mu\nu} + \epsilon (a R_{\mu\nu} + b g_{\mu\nu} R)$ . The corresponding procedure of eliminating divergences, which vanish on the equations of motion, in higher approximations has not been investigated in detail. Nevertheless usually it is supposed that the divergences, which vanish on the equations of motion, are irrelevant, and for this reason in the investigation of supergravity in the one-, two- and three-loop approximation these divergences have not been analyzed at all. <sup>[4,5]</sup> In this connection we would like to emphasize a special aspect of counterterms which vanish on the equations of motion - their dependence on gauge conditions, which makes it possible to choose gauges where the corresponding divergences are absent even off shell. The simplest example is

quantum electrodynamics where the counterterm  $Z_1 \bar{\psi} \gamma^M D_M \psi$  vanishes on the equation of motion and  $Z_1$  is gauge-dependent. As is known in the lowest approximation in the transverse gauge  $Z_1 = Z_2 = 1$ . One can show also that in each order of perturbation theory a gauge can be found where  $Z_1 = 1$ . At the same time the charge renormalization  $1/Z_3 e^2 F_{\mu\nu}^2$  is gauge-invariant since the corresponding counterterm does not vanish on the equations of motion.

The dependence of the background functional  $W$  on gauge has been investigated in refs. [5,6]. In the Yang-Mills vector field theory the counterterm  $Z F_{\mu\nu}^2$ , that does not vanish on the equations of motion, turned out to be gauge-independent, while the counterterm  $D_\mu F_{\mu\nu} A_\nu = \frac{\delta S}{\delta A_\mu} A_\mu$ , which vanish on the equations of motion in background gauges does not appear due to gauge invariance. In quantum gravity in the one-loop approximation two counterterms  $\int g R^2$  and  $\int \sqrt{-g} R_{\mu\nu}^2$  [4] are possible from general covariance and on dimensional grounds, and they both vanish on the equations of motion  $S^{iM\nu} \equiv \sqrt{-g} (R^{M\nu} - \frac{1}{2} g^{M\nu} R) = 0$ , since they may be represented in the form  $\frac{1}{2} S^{iM\nu} g_{\mu\nu} R$  and  $S^{iM\nu} (R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R)$ , respectively. From the results of ref. [5] (eq. (3.18)) it follows that every counterterm which vanishes on the equations of motion is not, in general, gauge-invariant. Therefore, when beginning this work, we expected to find gauge dependence in the abovementioned counterterms. It was not clear, however, whether this dependence is nontrivial enough to remove both these divergences in some gauges.

The present paper is thus aimed at investigating the

dependence of W-functional on gauge parameters in pure gravity <sup>x)</sup>. This requires cumbersome calculations. In all previous calculations with internal graviton <sup>lines</sup> propagators were taken in the simplest form, in the Feynman gauge. In this case one can use the 't Hooft-Veltman algorithm [4]. When the gauge parameters, that define the graviton propagator, change, the problem becomes technically more complicated. Therefore we do not perform direct calculations of the diagrams for W in an arbitrary gauge. In one of the diagrams e.g. there are approximately  $2 \cdot 10^6$  terms. A roundabout way is to calculate the variation  $\delta W$  at infinitely small change of gauge parameters. The corresponding diagrams are much simpler and can be calculated in arbitrary gauge. The variations obtained can be integrated, and counterterms in an arbitrary gauge can be determined with the aid of a boundary condition-calculation in the Feynman gauge [4].

The paper is organized as follows. Section 2 gives the result of calculation of counterterms in an arbitrary gauge. This result is analysed, and the gauges are found where the theory is finite. For the practical reader it is sufficient to read this sec. and Conclusion. In sec.3 the method of calculations is presented and the general equations for the

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x) It has been shown in ref. [6] that a nontrivial dependence on the gauge parameter connected with the scalar field exists in counterterms of the background functional in the theory of gravitons interacting with the scalar field.

variation of  $W$  under the change of gauge parameters are derived. In Sec.4 the results of calculations of separate diagrams are obtained. In Conclusion the consequences of the result obtained in pure gravity are considered<sup>ed</sup> and the question of applicability of the approach suggested to other theories, and, in particular, to supergravity is discussed. Appendix A presents the Slavnov-Taylor identity in the background formalism. In Appendix B a variation of counterterms in pure gravity in a  $2 + \epsilon$  -dimensional space is calculated.

§ 2. The divergences-free gauges.

In this section we present the results of calculation of the one-loop counterterms and obtain the gauges where the divergences are absent also off mass shell. The generating functional of the Green functions in the background method in gravity is equal to [2-6] (in the expression presented below the tree approximation is absent):

$$\Omega(g) = \exp\{iW(g)\} = \int d\alpha_{\mu\nu} d\psi^M d\bar{\psi}^N \exp\{i[S(g) - S^{iM\nu}(g)\alpha_{\mu\nu} + \frac{\alpha}{2\sqrt{g}} g^{M\nu} C_M C_\nu + \bar{\psi}^M T_{M\nu}(g)\psi^\nu]\} \quad (2.1)$$

where  $g_{\mu\nu} = \delta_{\mu\nu} + k_{\mu\nu}$  is an external gravitational field,  $\alpha_{\mu\nu}$  is a quantum field and

$$S(g) = \int dx \sqrt{-g} R(g) \quad (2.2)$$

$$S^{iM\nu}(g) \equiv S^{iM\nu}(g, x) = \frac{\delta S(g)}{\delta g_{\mu\nu}(x)} = \sqrt{-g(x)} \left( R^{M\nu}(x) - \frac{1}{2} g^{M\nu}(x) R(x) \right) \quad (2.3)$$

$$C_M = -\sqrt{g} (D^\nu(g)\alpha_{\mu\nu} + \beta D_M^\nu g^{\lambda\sigma} \alpha_{\lambda\sigma}) \quad (2.4)$$

$$T_{\mu\nu}\psi^\nu = \sqrt{-g}(\delta_\mu^\lambda \vec{D}^\sigma(g) + p \vec{D}_\mu(g) g^{\lambda\sigma}) (G_{\lambda\nu} D_\sigma(g) + G_{\sigma\nu} D_\lambda(g)) \psi^\nu \quad (2.5)$$

$$G_{\mu\nu} = g_{\mu\nu} + \alpha_{\mu\nu} \quad (2.6)$$

$\psi^\nu$  are ghost particles, the arrow above the derivatives means that they act on all the functions to the right of them.

In order to obtain S-matrix elements with the aid of (2.1) one should take the external field  $g_{\mu\nu}$  satisfying the classical equations of motion.

It is easily seen [2-6] that  $\Omega(g)$  is invariant under gauge transformations of the external field

$$g_{\mu\nu} \rightarrow g_{\mu\nu} - (g_{\mu\sigma} D_\nu(\xi) + g_{\nu\sigma} D_\mu(\xi)) \xi^\sigma \quad (2.7)$$

where  $\xi^\nu(x)$  are infinitely small co-ordinate-dependent parameters of transformation. A simple calculation of the divergency index shows that all the diagrams of one-loop approximation diverge as  $\Lambda^4$  (where  $\Lambda$  is a cut-off), so that the counter-terms must contain zero, second and fourth power of the momentum. We are interested in divergences containing an external field only. Since they must be invariant under gauge transformations (2.7) they can be represented in the form

$$\Delta S_{div} = \sqrt{-g} (c + dR + a(R^2 + 2R_{\mu\nu}^2) + bR_{\mu\nu}^2) \quad (2.8)$$

where  $c$  diverge as  $\Lambda^4$ ,  $d$  - quadratically,  $a$  and  $b$  - logarithmically. We shall use dimensional regularization. At such a regularization  $c=d=0$ . Therefore the divergences have the following structure

$$\Delta S_{div} = \epsilon \sqrt{-g} \left[ \alpha (R^2 + 2 R_{\mu\nu} R^{\mu\nu}) + \beta R_{\mu\nu} R^{\mu\nu} \right] \quad (2.9)$$

$$\epsilon = \frac{1}{8\pi^2(4-n)}$$

The calculation described in the next two Sections gives the following values for the coefficients  $\alpha$  and  $\beta$  :

$$\alpha = \frac{1}{\alpha^2} \left[ \frac{1}{384(1+\beta)^4} + \frac{1}{8} \right] + \frac{1}{\alpha} \left[ \frac{1}{64(1+\beta)^4} - \frac{1}{24(1+\beta)^3} + \frac{1}{16(1+\beta)^2} + \frac{1}{6} \right] + \frac{3}{128(1+\beta)^4} - \frac{1}{8(1+\beta)^3} + \frac{1}{3(1+\beta)^2} - \frac{1}{2(1+\beta)} + \frac{7}{15} \quad (2.10)$$

$$\beta = \frac{1}{12(1+\beta)^2} \left( 6\beta + 3 - \frac{1}{\alpha} \right) \quad (2.11)$$

The condition of the absence of divergences gives two equations for  $\alpha$  and  $\beta$  :

$$\alpha = 0, \quad \beta = 0 \quad (2.12)$$

These equations have 6 solutions (the first four solutions were found numerically):

$$\begin{aligned} \beta_1 &\approx -0,63, & \frac{1}{\alpha_1} &\approx -0,78 \\ \beta_2 &\approx -1,05, & \frac{1}{\alpha_2} &\approx -3,4 \\ \beta_{3,4} &\approx -0,77 \pm 0,25i, & \frac{1}{\alpha_{3,4}} &\approx -1,6 \pm 1,5i \\ \beta_{5,6} &= \infty, & \frac{1}{\alpha_{5,6}} &= -\frac{2}{3} \pm 2\sqrt{\frac{37}{45}}i \end{aligned} \quad (2.13)$$

The first two solutions are exactly real (have no imaginary part). The next four solutions correspond to complex values of the gauge parameters. The use of complex gauge parameters is possible, in principle, since on one hand even at real values of gauge parameters an effective action (at the exponent in expression (2.1)) is not Hermitian, and on the other



hand physical quantities do not depend on the value of gauge parameters. The use of complex gauge parameters could lead to difficulties if particle masses depended on gauge parameters (as is the case with a spontaneous breaking of gauge symmetry). But this is not so in the considered case, and the use of complex values of the parameters  $\alpha$  and  $\beta$  entails no difficulties. An exigent reader may restrict himself to the first two real solutions.

Thus, in the background method in pure gravity (2.1) in the gauge described by the gauge function (2.4) and by the values of the parameters (2.13) one-loop divergences are absent even off shell.

### § 3. General equations for $\mathcal{R}$

Here we obtain an expression for derivative of the divergences over the gauge parameters. The procedure used here can be applied to arbitrary gauge theories.

As has been mentioned in Sec.1 it turned out to be easier not to calculate directly the diagram divergences but to calculate the derivatives of these diagrams with respect to gauge parameters. This is connected with the fact that under variation of the gauge condition the variation of the generating functional  $\mathcal{R}$  contains a gauge function  $C_M$ . The Green functions containing  $C_M$  are connected by the Slavnov-Taylor identity with the Green functions of ghost particles whose calculation is much easier. The equation for the variation of  $\mathcal{R}$  with a change of gauge conditions was obtained in Ref. [5] and has the form (in application to gravity)

$$\begin{aligned} S(\Omega) &= i \int \delta^{(4)}(q, \lambda) \langle S \mathcal{X}(\lambda) \rangle \equiv \\ &= i \int \delta^{(4)}(q; x) \langle (G_{\mu\lambda} D_\nu(G) + G_{\nu\lambda} D_\mu(G))_x D^{\lambda\sigma}(q, G; x, y) \delta C_\sigma(y) \rangle \quad (3.1) \end{aligned}$$

The brackets in (3.1) imply functional averaging with the weight  $\exp(-L^* \int \mathcal{L} \text{eff})$ , where  $\int_{\mathcal{L} \text{eff}}$  is the expression in the exponent of eq. (2.1), and summation (integration) over the repeating indices (coordinates) is meant. The D-function in (3.1) is connected with the Green function of ghost particles by the relation

$$\langle \psi^\lambda(x) \bar{\psi}^\sigma(y) \rangle = L^* \langle D^{\lambda\sigma}(q, G; x, y) \rangle \quad (3.2)$$

and is equal to  $[T_{\lambda\sigma}]^{-1}$ :

$$T_{\mu\sigma} D^{\sigma\nu} = \delta_\mu^\nu \quad (3.2')$$

Equation (3.1) was obtained in Ref. [5] by means of a change of variables [2] in the functional integral

$$\mathcal{X}_{\mu\nu}(x) \rightarrow \mathcal{X}_{\mu\nu}(x) - (G_{\mu\sigma} D_\nu(G) + G_{\nu\sigma} D_\mu(G))_x \int dy D^{\sigma\lambda}(q, G; x, y) \delta C_\lambda(y) \quad (3.3)$$

and the corresponding change in  $\psi, \bar{\psi}$ . Appendix A contains the derivation of relation (3.1) based on the use of the Slavnov-Taylor identities. The analysis of eq. (3.1) shows that in the one-loop approximation on dimensional grounds and from general covariance the contribution into the diverging part has the form

$$\langle \delta^\nu \mathcal{X}_{\mu\nu} \rangle_{\text{div}} \sim \frac{1}{n-4} (a_1 R_{\mu\nu} + a_2 g_{\mu\nu} R) \quad (3.4)$$

Now we shall take into account that the action  $S(g)$  is gauge-invariant (i.e. the covariant derivative of  $S^{i\mu\nu}$  is zero) and that we calculate in the one-loop approximation. This means that in the function  $D^{\lambda\sigma}$  one can take  $G_{\mu\nu} = g_{\mu\nu}$ . The expression (3.1) for the variation  $\delta\Omega$  is now simplified

$$\delta\Omega = -i S^{i\mu\nu}(g, x) D^{\lambda\sigma}(g, g; x, y) \langle (g \overset{\leftarrow}{\partial}_\nu \overset{\leftarrow}{\partial}_\mu \overset{\leftarrow}{\partial}_\lambda(x) + \partial_\lambda \overset{\leftarrow}{\partial}_\nu \overset{\leftarrow}{\partial}_\mu(x)) \delta C_\sigma(y) \rangle \quad (3.5)$$

The arrow above the derivative in (3.5) implies that it acts on the function  $D^{\lambda\sigma}$ . The variation of the parameter  $\beta$  corresponds to the following variation of the gauge function

$$\delta C_\sigma = -\delta\beta \sqrt{g} \partial_\sigma (g^{\lambda\delta} \overset{\leftarrow}{\partial}_\lambda \delta) \quad (3.6)$$

We obtain

$$\frac{\delta\Omega}{\delta\beta} = i S^{i\mu\nu}(g, x) D^{\lambda\sigma}(g, g; x, y) \sqrt{g(y)} \langle (g \overset{\leftarrow}{\partial}_\nu \overset{\leftarrow}{\partial}_\mu \overset{\leftarrow}{\partial}_\lambda(x) + \partial_\lambda \overset{\leftarrow}{\partial}_\nu \overset{\leftarrow}{\partial}_\mu(x)) \partial_\sigma (g^{\lambda\delta}(y) \overset{\leftarrow}{\partial}_\lambda \delta(y)) \rangle \quad (3.7)$$

The variation of the parameter  $\alpha$  corresponds to the variation  $\delta C$ :

$$\delta C_\sigma = \frac{\delta\alpha}{2\alpha} C_\sigma \quad (3.8)$$

In this case in expression (3.5) one can use the Slavnov-Taylor identity once again. Finally we come to the following equations.

$$\begin{aligned} \frac{\delta\Omega}{\delta\alpha} &= \frac{i}{2\alpha^2} S^{i\mu\nu}(g, x) S^{i\mu'\nu'}(g, x') D^{\lambda\sigma}(g, g; x, y) D^{\lambda'\sigma'}(g, g; x', y) \times \quad (3.9) \\ &\times \sqrt{g(y)} g_{\sigma\sigma'}(y) \langle (2 \overset{\leftarrow}{\partial}_\nu \overset{\leftarrow}{\partial}_\mu \overset{\leftarrow}{\partial}_\lambda(x) + \partial_\lambda \overset{\leftarrow}{\partial}_\nu \overset{\leftarrow}{\partial}_\mu(x)) (2 \overset{\leftarrow}{\partial}_{\nu'} \overset{\leftarrow}{\partial}_{\mu'} \overset{\leftarrow}{\partial}_{\lambda'}(x') + \partial_{\lambda'} \overset{\leftarrow}{\partial}_{\nu'} \overset{\leftarrow}{\partial}_{\mu'}(x')) \rangle - \\ &- \frac{1}{2\alpha^2} S^{i\mu\nu}(g, x) \left[ \delta_\mu^\lambda \delta_\sigma^\delta + \delta_\mu^\sigma \delta_\delta^\lambda \right] \partial_\nu D^{\lambda\sigma}(g, g; x, y) \sqrt{g(y)} g_{\delta\beta}(y) (g \overset{\leftarrow}{\partial}_\lambda \delta) \end{aligned}$$

$$+ \partial_\alpha g_{\lambda\epsilon}^{(n)} D^{\alpha\beta} (g, g; x, y) + D^{\delta\delta} (g, g; x, y) \sqrt{g(y)} g_{\delta\beta}(y) \partial_\beta \left( 2g_{\mu\alpha}(x) \partial_\nu + \partial_\alpha g_{\mu\nu}(x) \right) D^{\alpha\beta} (g, g; x, y) ]$$

To perform the one-loop calculations it is sufficient to know, the graviton Green function  $\langle \mathcal{X}_{\mu\nu} \mathcal{X}_{\lambda\epsilon} \rangle$  that appears in eqs. (3.7), (3.9) in the tree approximation.

It is easily seen that the divergences in expressions (3.7) and (3.9) are directly connected with the divergences of the initial expression (2.1). Indeed, let us represent the result of the calculation of  $\Omega$  in the one-loop approximation in the form

$$\Omega_1 = e^{iW_1} = e^{iW_R} + i\Delta S_{div} \quad (3.10)$$

where  $W_R$  is a finite function and all the divergences are included into  $\Delta S_{div}$ . Then in the one-loop approximation

$$\delta\Omega_1 = i\delta W_1 = i[\delta W_R + \delta(\Delta S_{div})] \quad (3.11)$$

Expression (3.11) shows that the one-loop divergences in (3.7) and (3.9) coincide with the derivatives with respect to  $\beta$  and  $\alpha$  of one-loop divergences of the functional  $\Omega$ .

Further simplification of eqs. (3.7) and (3.9) is made with the help of the following considerations. Since an exact structure of the divergences is known (see (2.9)), it is sufficient to calculate (3.7) and (3.9) only to the accuracy of the second-order terms in  $h_{\mu\nu}$  and the fourth-order terms in derivatives. Then the structure of eq. (2.9) is re-established by these terms unambiguously. Since  $S^{i\mu\nu}$  is already of the first order of magnitude in  $\partial\partial h$ , it is only the term of the form  $\partial\partial h$  that is to be calculated in the factor before

$S^{iM\nu}$ . The first term in (3.9) contains  $S^{iM\nu}$  two times. Therefore, in the remaining quantities in this term one can put  $g_{\mu\nu} = \delta_{\mu\nu}$ . Further on, to re-establish both structures in (3.4) it is sufficient to consider the transverse external field ( $\partial_\mu h_{\mu\nu} = 0$ ) alone, and one can use the condition  $\partial_\mu S^{iM\nu} = 0$  which is valid in the given approximation with respect to  $h_{\mu\nu}$ . The exact expression for divergences is re-established with the help of the following correspondence rules:

$$S^{iM\nu} (a \delta_{\mu\nu} \delta_{\lambda\sigma} + b \delta_{\mu\lambda} \delta_{\nu\sigma}) D^{i\lambda\sigma} \rightarrow \sqrt{g} (a R^2 + b R_{\mu\nu} R^{\mu\nu}) \quad (3.12)$$

$$S^{iM\nu} (a \square h_{\mu\nu} + b \delta_{\lambda\sigma} \square h_{\lambda\sigma}) \rightarrow \sqrt{g} (-(a+b) R^2 + 2a R_{\mu\nu} R^{\mu\nu})$$

#### § 4. The calculations.

Now we are in a position to present some details of the calculations of  $\partial\Omega/\partial\beta$  (3.7) and  $\partial S/\partial\alpha$  (3.9) in the lowest order approximation with respect to the external field. We use here only lower indices, repeated ones implying Lorentz summation ( $\delta_{\mu\nu} = (+, -, -, -)$ ). In this approximation we have:

$$\begin{aligned} \frac{\partial W_4}{\partial \alpha} = & -\frac{i}{2\alpha^2} S^{iM\nu}(g, x) S^{iM'\nu'}(g, x') D_{\lambda\sigma}^{(0)}(x-y) D_{\lambda'\sigma'}^{(0)}(x'-y) \times \\ & \times (-2\partial_\nu \delta_{\lambda\sigma} + \partial_\lambda \delta_{\nu\sigma}) (-2\partial_{\nu'} \delta_{\lambda'\sigma'} + \partial_{\lambda'} \delta_{\nu'\sigma'}) H_{\mu\lambda\mu'\lambda'}^{(0)}(x-x') - \\ & -\frac{1}{2\alpha^2} S^{iM\nu}(g, x) \left[ D_{\lambda\sigma}^{(0)}(x-z) D_{\sigma\delta}^{(0)}(x-z) \partial_\gamma \partial_\lambda h_{\mu\nu}(x) - D_{\lambda\sigma}^{(0)}(x-z) (h_{\gamma\delta}(z) \delta_{\sigma\delta}^{\gamma\delta} + \right. \end{aligned} \quad (4.1)$$

$$+ 2h_{\alpha\beta}(z) \partial_\mu \partial_\nu \overset{(0)}{D}_{\lambda\sigma}(x-z) - 4 \overset{(1)}{D}_{\lambda\sigma}(x,z) \partial_\mu \partial_\nu \overset{(0)}{D}_{\lambda\sigma}(x-z) ]$$

$$\begin{aligned} \frac{\partial W_1}{\partial \beta} = & i S^{i\nu} (g, \lambda) \left\{ g \partial_\nu \overset{(0)}{D}_{\alpha\beta}(x-z) \partial_\rho \left[ \left( \frac{1}{2} h_{\nu\tau}(z) \delta_{\alpha\sigma} - h_{\lambda\sigma}(z) \right) H_{\mu\alpha, \lambda\sigma}(x-z) \right] - \right. \\ & - \overset{(0)}{D}_{\alpha\beta}(x-z) \partial_\rho \left[ \left( \frac{1}{2} h_{\delta\gamma}(z) \delta_{\alpha\sigma} - h_{\lambda\sigma}(z) \right) H_{\mu\nu, \lambda\sigma}(x-z) \right] - 2 \partial_\nu \overset{(1)}{D}_{\alpha\beta}(x,z) \partial_\rho \overset{(0)}{H}_{\mu\alpha, \lambda\lambda}(x-z) - \\ & - \overset{(1)}{D}_{\alpha\beta}(x,z) \partial_\alpha \overset{(0)}{H}_{\mu\nu, \lambda\lambda}(x-z) + g \partial_\nu \overset{(0)}{D}_{\alpha\beta}(x-z) \overset{(1)}{H}_{\mu\alpha, \lambda\lambda}(x,z) \partial_\rho + \\ & + \overset{(0)}{D}_{\alpha\beta}(x-z) \partial_\alpha \overset{(1)}{H}_{\mu\nu, \lambda\lambda}(x,z) \partial_\rho - \partial_\nu \overset{(0)}{D}_{\alpha\beta}(x-z) \partial_\rho h_{\delta\gamma}(z) \overset{(0)}{H}_{\mu\alpha, \lambda\lambda}(x-z) - \\ & \left. - \frac{1}{g} \overset{(0)}{D}_{\alpha\beta}(x-z) \partial_\rho h_{\delta\gamma}(z) \partial_\alpha \overset{(0)}{H}_{\mu\nu, \lambda\lambda}(x-z) \right\} \end{aligned} \quad (4.2)$$

In eqs. (4.1) and (4.2) the following notations are also used.

$\overset{(0)}{D}_{\alpha\beta}$  and  $\overset{(1)}{D}_{\alpha\beta}$  mean two first terms of the expansion of the function  $D^{\alpha\beta}$  in the degrees

of  $h_{\mu\nu}$ :

$$D^{\alpha\beta}(g, g; x, y) \approx \overset{(0)}{D}_{\alpha\beta}(x-y) + \overset{(1)}{D}_{\alpha\beta}(x, y) \quad (4.3)$$

$$\overset{(0)}{D}_{\alpha\beta}(x-y) = -\frac{1}{(2\pi)^4} \int d p e^{-i p(x-y)} \frac{1}{p^2} \left( \delta_{\alpha\beta} - \frac{p_\alpha p_\beta}{p^2} \frac{2p+1}{g(1+\beta)} \right) \quad (4.4)$$

$$\overset{(1)}{D}_{\alpha\beta}(x, y) = -\overset{(0)}{D}_{\alpha\lambda}(x-z_1) Q_{\lambda\sigma}(z_1, z_2) \overset{(0)}{D}_{\sigma\beta}(z_2-y) \quad (4.5)$$

$$\begin{aligned} Q_{\alpha\beta}(z_1, z_2) = & \left[ \delta_{\alpha\beta} \left( \frac{1}{2} h_{\gamma\delta} \square + \frac{1}{2} \partial_\mu h_{\gamma\delta} \partial_\mu - h_{\mu\nu} \partial_\mu \partial_\nu \right) + \right. \\ & + \left( \frac{1}{2} + \beta \right) h_{\gamma\delta} \partial_\alpha \partial_\beta + \partial_\beta h_{\mu\alpha} \partial_\mu - \partial_\alpha h_{\mu\beta} \partial_\mu + \partial_\mu h_{\alpha\beta} \partial_\mu + \\ & \left. + h_{\alpha\beta} \square + \left( \frac{1}{2} + \beta \right) \partial_\beta h_{\gamma\delta} \partial_\alpha + \beta \partial_\alpha \partial_\beta h_{\gamma\delta} \right]_{z_1} \delta(z_1 - z_2) \end{aligned} \quad (4.6)$$

(0)  $D_{\alpha\beta}$  coincides with the free ghost propagator.  
 $H_{\mu\nu,\lambda\sigma}$  and  $H_{\mu\nu,\lambda\sigma}^{(1)}$  are two first terms of the expansion of the graviton Green function  $H_{\mu\nu,\lambda\sigma}(x,y) \equiv \langle \mathcal{X}_{\mu\nu}(x) \mathcal{X}_{\lambda\sigma}(y) \rangle$ , calculated in the tree approximation. In this approximation the function  $H_{\mu\nu,\lambda\sigma}$  is equal to:

$$H_{\mu\nu,\lambda\sigma}(x,y) = \left[ \frac{\delta}{\delta \mathcal{X}_{\mu\nu}(x)} \frac{\delta}{\delta \mathcal{X}_{\lambda\sigma}(y)} \left( S'(G) + \frac{\alpha}{2} g^{\mu\nu} C_{\mu\nu} \right) \right]_{x=0}^{-1} \quad (4.7)$$

We have

$$H_{\mu\nu,\lambda\sigma}(x,y) \approx H_{\mu\nu,\lambda\sigma}^{(0)}(x-y) + H_{\mu\nu,\lambda\sigma}^{(1)}(x,y) \quad (4.8)$$

$$H_{\mu\nu,\lambda\sigma}^{(1)}(x-y) = -\frac{i}{(2\pi)^4} \int d^4 p e^{-i p(x-y)} \frac{1}{p^2} \left[ \delta_{\mu\lambda} \delta_{\nu\sigma} + \delta_{\mu\sigma} \delta_{\nu\lambda} - \frac{1+\beta}{p^2} (\delta_{\mu\lambda} p_\nu p_\sigma + \delta_{\mu\sigma} p_\nu p_\lambda + \delta_{\nu\lambda} p_\mu p_\sigma + \delta_{\nu\sigma} p_\mu p_\lambda) + \frac{2(1+2\beta)}{(n-2)(1+\beta)} \frac{1}{p^2} (\delta_{\mu\nu} p_\lambda p_\sigma + \delta_{\lambda\sigma} p_\mu p_\nu) - \frac{2}{n-2} \delta_{\mu\nu} \delta_{\lambda\sigma} + \frac{2(1+2\beta)}{(1+\beta)^2} \left( 1 + \frac{2\beta+3}{2\alpha} - \frac{1+2\beta}{n-2} \right) \frac{p_\mu p_\nu p_\lambda p_\sigma}{p^4} \right] \quad (4.9)$$

(In (4.9)  $n$  is the space-time dimension).

$$H_{\mu\nu,\lambda\sigma}^{(1)}(x,y) = -H_{\mu\nu,\alpha\beta}^{(0)}(x-z_1) Q_{\alpha\beta,\gamma\delta}(z_1, z_2) H_{\gamma\delta,\lambda\sigma}^{(0)}(z_2 - y) \quad (4.10)$$

$$Q_{\mu\nu,\lambda\sigma}(z_1, z_2) = \left[ \frac{1}{4} (\delta_{\mu\nu} \square h_{\lambda\sigma} + \delta_{\sigma\lambda} \square h_{\mu\nu} + \delta_{\mu\nu} \partial_\lambda \partial_\sigma h_{\gamma\gamma} + \delta_{\lambda\sigma} \partial_\mu \partial_\nu h_{\gamma\gamma}) - \frac{1}{4} \delta_{\mu\nu} \delta_{\lambda\sigma} \square h_{\gamma\gamma} + \frac{1}{2} \delta_{\mu\lambda} \delta_{\nu\sigma} \square h_{\gamma\gamma} - \delta_{\mu\lambda} \square h_{\nu\sigma} - \delta_{\mu\lambda} \partial_\nu \partial_\sigma h_{\gamma\gamma} + \left( (\alpha\beta - \frac{1}{2}) \delta_{\mu\alpha} \delta_{\nu\beta} \delta_{\gamma\delta} + \delta_{\mu\sigma} \delta_{\nu\beta} \delta_{\gamma\delta} + \alpha \delta_{\mu\sigma} \delta_{\nu\alpha} \delta_{\gamma\beta} - \frac{1}{2} \delta_{\mu\sigma} \delta_{\nu\sigma} \delta_{\alpha\beta} + (\alpha\beta - \frac{1}{2}) \delta_{\mu\nu} \delta_{\sigma\alpha} \delta_{\gamma\beta} + (\alpha\beta^2 + \frac{1}{2}) \delta_{\mu\nu} \delta_{\alpha\beta} \delta_{\gamma\sigma} \right) \partial_\alpha \Gamma_{\lambda\gamma\beta\delta} \delta_{\gamma\sigma} + \partial_\alpha \Gamma_{\lambda\beta\sigma\delta} \delta_{\gamma\sigma} + \Gamma_{\lambda\beta\sigma\delta} \delta_{\gamma\sigma} \partial_\alpha + \Gamma_{\lambda\beta\sigma\delta} \delta_{\gamma\sigma} \partial_\alpha + \Gamma_{\lambda\mu\sigma} \delta_{\gamma\sigma} \partial_\beta + \Gamma_{\lambda,\alpha\sigma} \delta_{\gamma\sigma} \partial_\beta + \Gamma_{z_1,\alpha\beta} \delta_{\lambda\sigma} \delta_{\gamma\sigma} \partial_\tau \right] - \sqrt{g} \left( (1+\alpha) g^{\mu\alpha} g^{\nu\lambda} g^{\sigma\beta} - \frac{1}{2} g^{\mu\lambda} g^{\nu\sigma} g^{\alpha\beta} + (\alpha\beta - \frac{1}{2}) g^{\mu\alpha} g^{\nu\beta} g^{\lambda\sigma} \right)$$

$$+(\alpha\beta - \frac{1}{2})g^{\mu\nu}g^{\lambda\sigma}g^{\rho\sigma} + (\alpha\beta^2 + \frac{1}{2})g^{\mu\nu}g^{\lambda\sigma}g^{\alpha\rho} \Big]_{Z_1} \delta(Z_1 - Z_2)$$

Further transformations in expression (4.11) were carried out with the help of a computer. Besides, the terms in  $Q_{\mu\nu, \lambda\sigma}$  which differ in the order of the indices  $\mu$  and  $\nu$  (and also  $\lambda$  and  $\sigma$ ) are considered as being equal since the vertex  $Q_{\mu\nu, \lambda\sigma}$  is multiplied on his right and left by the propagator  $D_{\lambda\sigma}^{(1)}$  which is symmetric under the transposition of these indices.

As it follows from eq's (4.1), (4.4), (4.9) the dependence of  $W_1$  on  $\alpha$  is simple and can be determined without any calculations:

$$\frac{\partial W_1}{\partial \alpha} = A \alpha^{-3} + B \alpha^{-2}$$

Now we shall represent all the terms in the form of the Feynman diagrams. For  $\partial W_1 / \partial \alpha$  we have:

$$I_1^{(\alpha)} = -\frac{i}{2\alpha^2} S^{i\mu\nu}(g, x) S^{i\mu'\nu'}(g, x') \bar{D}_{\lambda\sigma}^{(1)}(x-y) \bar{D}_{\lambda\sigma}^{(1)}(x'-y) (2\partial_\nu \delta_{\lambda\sigma} + \partial_\lambda \delta_{\nu\sigma}) \times (2\partial_{\nu'} \delta_{\lambda\sigma'} + \partial_{\lambda'} \delta_{\nu'\sigma'}) H_{\mu\sigma, \mu'\sigma'}(x-x') \quad (4.12)$$

$$I_2^{(\alpha)} = \frac{i}{2\alpha^2} S^{i\mu\nu}(g, x) \bar{D}_{\lambda\sigma}^{(1)}(x-z) \bar{D}_{\lambda\sigma}^{(1)}(x-z) \partial_\lambda h_{\mu\nu}(z) \quad (4.13)$$

$$I_3^{(\alpha)} = -\frac{i}{2\alpha^2} S^{i\mu\nu}(g, x) \bar{D}_{\lambda\sigma}^{(1)}(x-z) \partial_\mu \partial_\nu \bar{D}_{\lambda\sigma}^{(1)}(x-z) h_{\sigma\delta}(z) \quad (4.14)$$

$$I_4^{(\alpha)} = -\frac{i}{\alpha^2} S^{i\mu\nu}(g, x) \bar{D}_{\lambda\sigma}^{(1)}(x-z) \partial_\mu \partial_\nu \bar{D}_{\lambda\sigma}^{(1)}(x-z) h_{\sigma\delta}(z) \quad (4.15)$$

$$I_5^{(\alpha)} = -\frac{2i}{\alpha^2} S^{i\mu\nu}(g, x) \bar{D}_{\lambda\sigma}^{(1)}(x, z) \partial_\mu \partial_\nu \bar{D}_{\lambda\sigma}^{(1)}(x-z) \quad (4.16)$$

$\partial W_1 / \partial \beta$  looks as follows:



$$I_1^{(\beta)} = 2 S^{ij\mu\nu} (g, x) \partial_\nu D_{\alpha\beta}^{(0)}(x-z) \partial_\rho \left[ \left( \frac{1}{2} h_{\delta\delta}^{(z)} \delta_{\lambda\delta}^\mu - h_{\lambda\delta}^{(z)} \right) H_{\mu\nu, \lambda\delta}^{(0)}(x-z) \right] \quad (4.17)$$

$$I_2^{(\beta)} = - S^{ij\mu\nu} (g, x) D_{\alpha\beta}^{(0)}(x-z) \partial_\rho \left[ \left( \frac{1}{2} h_{\delta\delta}^{(z)} \delta_{\lambda\delta}^\mu - h_{\lambda\delta}^{(z)} \right) \partial_\alpha H_{\mu\nu, \lambda\delta}^{(0)}(x-z) \right] \quad (4.18)$$

$$I_3^{(\beta)} = - 2 S^{ij\mu\nu} (g, x) \partial_\nu D_{\alpha\beta}^{(1)}(x, z) \partial_\rho H_{\mu\alpha, \lambda\lambda}^{(0)}(x-z) \quad (4.19)$$

$$I_4^{(\beta)} = - S^{ij\mu\nu} (g, x) D_{\alpha\beta}^{(1)}(x, z) \partial_\alpha \partial_\rho H_{\mu\nu, \lambda\lambda}^{(0)}(x-z) \quad (4.20)$$

$$I_5^{(\beta)} = 2 S^{ij\mu\nu} (g, x) \partial_\nu D_{\alpha\beta}^{(0)}(x-z) H_{\mu\alpha, \lambda\lambda}^{(1)}(x, z) \bar{\partial}_\rho \quad (4.21)$$

$$I_6^{(\beta)} = S^{ij\mu\nu} (g, x) D_{\alpha\beta}^{(0)}(x-z) \partial_\alpha H_{\mu\nu, \lambda\lambda}^{(1)}(x, z) \bar{\partial}_\rho \quad (4.22)$$

$$I_8^{(\beta)} = - \frac{1}{2} S^{ij\mu\nu} (g, x) D_{\alpha\beta}^{(0)}(x-z) \partial_\alpha H_{\mu\nu, \lambda\lambda}^{(0)}(x-z) \partial_\rho h_{\delta\delta}^{(z)} \quad (4.23)$$

$$I_7^{(\beta)} = - S^{ij\mu\nu} (g, x) \partial_\nu D_{\alpha\beta}^{(0)}(x-z) H_{\mu\alpha, \lambda\lambda}^{(0)}(x-z) \partial_\rho h_{\delta\delta}^{(z)} \quad (4.24)$$

The corresponding diagrams are shown in the figures 1-6 and the results of calculations are given in the tables I-III. For example according to the table I  $I_2^{(\alpha)}$  is equal to

$$I_2^{(\alpha)} = - \varepsilon \left[ \sqrt{g} R^2 \left( - \frac{1}{32\alpha^2(1+\beta)^2} - \frac{3}{8\alpha^2} \right) + \sqrt{g} R_{\mu\nu} R^{\mu\nu} \left( \frac{1}{16\alpha^2(1+\beta)^2} + \frac{3}{4\alpha^2} \right) \right] \quad (4.25)$$



Fig. 1



Fig. 2

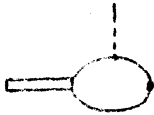


Fig.3



Fig.4







Fig.5



Fig.6

Fig.1 corresponds to the expression  $I_1^{(\alpha)}$ , fig.2 - to  $I_2^{(\alpha)}, I_3^{(\alpha)}, I_4^{(\alpha)}$ , fig.3 - to  $I_5^{(\alpha)}$ , fig.4 - to  $I_1^{(\beta)}, I_2^{(\beta)}, I_7^{(\beta)}, I_8^{(\beta)}$ , fig.5 - to  $I_3^{(\beta)}, I_4^{(\beta)}$ , fig.6 - to  $I_5^{(\beta)}, I_6^{(\beta)}$ . Furthermore, the following designations are used:

-   $g^{\mu\nu}$
  -   $\overset{(0)}{D}$ , the propagator of the fictitious particles
  -   $\overset{(0)}{H}$ , the graviton propagator
  -  the field  $h_{\mu\nu}$ , which either appears explicitly in diagrams or is contained in the vertices
- $Q_{\delta\delta}$  or  $Q_{\mu\nu, \delta\delta}$

	$-\epsilon\sqrt{g}R^2$				$-\epsilon\sqrt{g}R_{\mu\nu}R^{\mu\nu}$			
	$1/\alpha^3$	$1/\alpha^2$	$1/\alpha$	$1/\alpha^2$	$1/\alpha^3$	$1/\alpha^2$	$1/\alpha$	$1/\alpha^2$
$I_1^{(a)}$	$\frac{1}{192} \frac{1}{4}$	$\frac{1}{64} \frac{1}{12}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{96} \frac{1}{2}$	$\frac{1}{32}$	$\frac{1}{24}$	$\frac{1}{8}$
$I_2^{(a)}$			$-\frac{1}{32}$	$-\frac{3}{8}$				$\frac{1}{16}$
$I_3^{(a)}$			$-\frac{7}{192}$	$\frac{5}{48} - \frac{11}{48}$				
$I_4^{(a)}$			$-\frac{1}{32}$	$\frac{1}{16} - \frac{1}{12}$				$\frac{1}{24} - \frac{1}{12}$
$I_5^{(a)}$		$\frac{1}{24}$	$\frac{7}{192} - \frac{1}{6}$	$\frac{29}{48}$			$-\frac{1}{24}$	$\frac{1}{16} - \frac{1}{12}$

Table I

$$G = \frac{1}{1+p}$$

		$-\varepsilon\sqrt{g}R^2$									
		$1/d$					$\alpha^0$				
$1/d^2$		$6^5$	$6^4$	$6^3$	$6^2$	$6$	$6^5$	$6^4$	$6^3$	$6^2$	$6$
$I_1^{(p)}$				$\frac{1}{96}$	$\frac{1}{12}$	$\frac{1}{12}$			$\frac{1}{32}$	$\frac{1}{48}$	$\frac{7}{24}$
$I_2^{(p)}$				$-\frac{1}{192}$		$-\frac{7}{48}$			$\frac{1}{64}$	$\frac{1}{6}$	$\frac{1}{16}$
$I_3^{(p)}$			$\frac{1}{12}$	$\frac{7}{96}$	$-\frac{5}{24}$			$\frac{1}{4}$	$-\frac{37}{48}$	$-\frac{3}{4}$	$\frac{3}{4}$
$I_4^{(p)}$			$-\frac{1}{24}$	$-\frac{7}{192}$	$\frac{5}{48}$				$\frac{13}{192}$	$\frac{7}{24}$	$-\frac{1}{24}$
$I_5^{(p)}$	$\frac{1}{48}$	$\frac{1}{8}$	$-\frac{11}{48}$	$\frac{1}{16}$	$-\frac{1}{12}$	$-\frac{1}{12}$	$\frac{3}{16}$	$-\frac{17}{16}$	$\frac{37}{48}$	$-\frac{1}{12}$	$-\frac{7}{24}$
$I_6^{(p)}$	$-\frac{1}{96}$	$-\frac{1}{16}$	$\frac{1}{16}$	$\frac{5}{96}$	$\frac{7}{48}$	$\frac{7}{48}$	$-\frac{3}{32}$	$\frac{3}{16}$	$\frac{7}{96}$	$-\frac{1}{3}$	$\frac{1}{16}$
$I_7^{(p)}$				$-\frac{1}{16}$	$\frac{5}{24}$				$-\frac{3}{16}$	$\frac{5}{6}$	$-\frac{3}{4}$
$I_8^{(p)}$				$\frac{1}{32}$	$-\frac{5}{48}$				$\frac{3}{32}$	$-\frac{3}{8}$	$\frac{1}{24}$

Table II

$-\varepsilon \sqrt{g} R_{\mu\nu} R^{\mu\nu}$										
	$1/\alpha^2$	$1/\alpha$			$\alpha^0$					
	$\epsilon^5$	$\epsilon^4$	$\epsilon^3$	$\epsilon^2$	$\epsilon^1$	$\epsilon^0$	$\epsilon^{-1}$	$\epsilon^{-2}$	$\epsilon^{-3}$	$\epsilon^{-4}$
$I_1^{(p)}$			$-\frac{1}{8}$	$\frac{1}{3}$				$-\frac{3}{8}$	$\frac{11}{12}$	$\frac{5}{6}$
$I_2^{(p)}$			$\frac{1}{16}$	$\frac{5}{12}$				$\frac{3}{16}$	$-\frac{1}{2}$	$-\frac{1}{12}$
$I_3^{(p)}$			$-\frac{1}{12}$	$\frac{1}{8}$				$-\frac{1}{4}$	$-\frac{3}{8}$	$-\frac{3}{4}$
$I_4^{(p)}$			$\frac{1}{24}$	$-\frac{1}{16}$				$\frac{1}{8}$	$-\frac{17}{48}$	$\frac{1}{4}$
$I_5^{(p)}$	$\frac{1}{24}$	$\frac{1}{4}$	$-\frac{11}{24}$	$\frac{1}{3}$	$\frac{1}{3}$			$\frac{3}{8}$	$-\frac{13}{6}$	$-\frac{5}{16}$
$I_6^{(p)}$	$-\frac{1}{48}$	$-\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{4}$	$-\frac{5}{12}$			$-\frac{3}{16}$	$-\frac{17}{12}$	$\frac{5}{4}$

Table III

We remind that we are interested only in the divergent parts of the terms which are proportional to  $\partial\partial h$  and we used the conditions  $\partial_\mu h_{\mu\nu} = \partial_\mu S^{;\mu\nu} = 0$ . The designation  $\Rightarrow$  in the eq's (4.12)-(4.24) indicates that we used the correspondence rules (3.12).

The diagrams  $I_4^{(\alpha)} - I_5^{(\alpha)}$ ,  $I_7^{(\beta)}$ ,  $I_8^{(\beta)}$  were calculated directly. The diagrams  $I_1^{(\alpha)}$ ,  $I_5^{(\alpha)}$ ,  $I_4^{(\beta)} - I_8^{(\beta)}$  were calculated with the help of the computer <sup>(x)</sup> in the framework of dimensional regularization with use of equations:

$$\int \frac{d^L p P_M}{(p^2)^\alpha ((p-k)^2)^\beta} = i(-\pi)^{n/2} \frac{\Gamma(\alpha+\beta-\frac{n}{2})\Gamma(\frac{n}{2}+1-\alpha)\Gamma(\frac{n}{2}-\beta)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(n+1-\alpha-\beta)} \frac{K_M}{(k^2)^{\alpha+\beta-n/2}}$$

$$\int \frac{d^L p P_\mu P_\nu}{(p^2)^\alpha ((p-k)^2)^\beta} = \frac{i(-\pi)^{n/2}}{(k^2)^{\alpha+\beta-n/2} \Gamma(\alpha)\Gamma(\beta)\Gamma(n+2-\alpha-\beta)} \times$$

$$\times \left[ \Gamma(\alpha+\beta-\frac{n}{2})\Gamma(\frac{n}{2}+2-\alpha)\Gamma(\frac{n}{2}-\beta) K_\mu K_\nu + \right.$$

$$\left. + \frac{i}{2} \Gamma(\alpha+\beta-1-\frac{n}{2})\Gamma(\frac{n}{2}+1-\alpha)\Gamma(\frac{n}{2}+1-\beta) K^2 g_{\mu\nu} \right],$$

etc.

Thus we obtain the following equations for the one-loop divergences:

$$\frac{\partial}{\partial \alpha} \Delta S_{div} = \sum_{i=4}^5 I_i^{(\alpha)} \quad (4.27)$$

$$\frac{\partial}{\partial \beta} \Delta S_{div} = \sum_{i=1}^8 I_i^{(\beta)}$$

<sup>x)</sup> The calculations were performed by means of some modification of Veltman's algebraic manipulation program Schoonschip on Dubna computer.

A simple integration of these equations gives:

$$\Delta S_{div}^4 = \epsilon \sqrt{g} \left[ R^2 (a + c_1) + R_{\mu\nu} R^{\mu\nu} (2a + b + c_2) \right] \quad (4.28)$$

where  $c_1$  and  $c_2$  are the integration constants independent on  $\alpha$  and  $\beta$ ,  $a$  and  $b$  are given by the eq's (2.10), (2.11). For the determination of  $c_1$  and  $c_2$  we compare the result obtained above with the expression for  $\Delta S_{div}$  obtained by 't Hooft and Veltman [A] in the gauge  $\alpha = -1, \beta = -1/2$ :

$$\Delta S_{div} \Big|_{\alpha=-1, \beta=-1/2} = \epsilon \sqrt{g} \left( \frac{1}{120} R^2 + \frac{7}{20} R_{\mu\nu} R^{\mu\nu} \right) \quad (4.29)$$

The comparison of the eq's (4.29) and (4.28) gives  $c_1 = c_2 = 0$  and we obtain finally the expression (2.9) for the  $\Delta S_{div}$ .

CONCLUSION

Thus, our paper presents an efficient way of removing divergences of the type  $S^{ik} \gamma_k$  in arbitrary gauge theories. This method is based on the use of the fact that under a change of the gauge conditions a variation of the background-functional  $W$  takes the form [5]:  $S^{ik} \langle \delta \phi_k \rangle$ . This makes it possible in the cases where a sufficiently large number of gauge parameters do exist to choose gauges where the abovementioned divergences are absent.

The following useful methodical conclusion can be drawn from this work. In gauge theories it is sometimes necessary to calculate in arbitrary gauges. In the background functional it is more convenient to calculate the necessary quantities in the simplest gauge only, (e.g. in the Feynman gauge) instead of performing calculations in an arbitrary one. Then one must calculate the diagrams corresponding to the variation  $\delta W$  at the change of the gauge parameters. These variations are easily integrated over  $\alpha$  and  $\beta$  etc., and the calculation of  $W$  in the simplest gauge is used as a boundary condition. In this case (i) the structure of the result is known in advance, (3.1) ii) the diagrams for variations  $\delta W$  are much simpler than the diagrams for  $W$ . and, iii) there exists an additional control over the correctness of the calculations: from different diagrams for  $\partial W / \partial \alpha$  and  $\partial W / \partial \beta$  one must obtain equivalent answers for those terms in  $W(\alpha, \beta)$  that depend both on  $\alpha$  and on  $\beta$ .



The main result of the present paper is that in the one-loop approximation in pure quantum gravity such gauges have been obtained, in which the theory is finite even off shell. This result gives an additional explanation of the known result of 't Hooft and Veltman [4], that the sum of all the one-loop radiative corrections to the S-matrix is finite. The proof of this statement is based on the fact that  $R^2$  and  $R_{\mu\nu}^2$  vanish on the classical equations of motion. Although this proof is quite correct, it is somewhat unusual. The present paper shows that the finiteness of the S-matrix follows from the fact of the existence of gauges in which counterterms are absent at all, and both the Green functions and the S-matrix are finite.

One of the most interesting analogs of this phenomenon is known to exist in the theory of the Yang-Mills fields with spontaneous symmetry breaking. In the unitary gauge this theory is formally nonrenormalizable. However, it is well known, that only renormalizable divergences remain on shell. Besides, there exist some gauges (renormalizable gauges) in which the theory is renormalizable off shell too. The present paper suggests something analogous. In pure gravity there exist some gauges in which there are no divergences not only on shell (which is valid for any gauge) but also off shell.

We should emphasize that up to now no theory has been known in which all the counterterms would vanish in some gauge (even in the one-loop approximation).

Usually in all gauges there remain gauge-invariant physical charge renormalizations. The corresponding divergences are not of the type  $S^{ik} \gamma_K$  i.e. they do not vanish on the equations

of motion and must be removed by standard methods. And only in pure gravity in the one-loop approximation there are no physical renormalizations.

It is of interest now to discuss the conclusions of the present paper in connection with the arguments of Ref. [8]

concerning positiveness of the contribution into

$\sqrt{g} \left( R_{\mu\nu}^2 - \frac{1}{3} R^2 \right)$  and  $\sqrt{g} R^2$  from various particles

including gravitons. It seems to us that it is a misunderstanding analogous to the one which has existed before the discovery of the asymptotical freedom in the Yang-Mills theory. Before this discovery it was believed that spectrality leads to the zero-charge problem and to the ghost poles in the propagators.

It turned out, however, that in covariant gauges in the Yang-Mills theory the positiveness is lost due to the presence of ghost particles. It was important also to understand that the renormalization of the vector field wave function is gauge-dependent and, therefore, can acquire arbitrary values. Something analogous has happened in our case in which both counter-terms have been made equal to zero by the choice of the gauge parameters, though in the Feynman gauge the divergences are indeed positive.

In connection with the supergravity theories, investigated intensively at the present time [1,7] we should note the following:

If it turns out indeed, as it is expected now, that the only counterterms which are present in supergravity are of the

type  $S^{\text{ik}} \gamma_{\text{k}} [11, 7]$ , it is quite possible that at a special choice of gauge conditions the theory will be finite. Thus we shall have a quantum gravitational theory which cannot be renormalizable because of the dimensional coupling constant but can be finite. The model of such a theory is described here on the example of the one-loop approximation in the pure gravity.

Appendix A

In this Appendix Ward-Slavnov-Taylor identities are derived in an arbitrary gauge theory in the background gauge. We shall use the abstract notations of De Witt [2,5]

Let us consider the generating functional

$$\Omega(h, \eta) = \int d\alpha d\psi d\bar{\psi} \exp\{i[S(G) - S(h) - \delta^{ij} \alpha_i + \frac{1}{2} \delta^{\mu\nu} C_\mu C_\nu + \bar{\psi}^M T_{\mu\nu}(h, G) \psi^\nu + \eta^i \alpha_i]\} \quad (A.1)$$

Here  $h_i$  is an external gauge field,  $\alpha_i$  - a quantum gauge field,  $\psi^\nu$  and  $\bar{\psi}^M$  are the ghost fields,  $G_i = h_i + \alpha_i$ ,  $S(h)$  is an action invariant under the gauge field transformations

$$h_i \rightarrow h_i + R_{i\mu}(h) \xi^M, \quad R_{i\mu}(h) \delta^{ij}(h) = 0 \quad (A.2)$$

$$\delta^{ij}(h) \equiv \frac{\delta}{\delta h_i} S(h), \quad (A.3)$$

where  $\xi^M$  are infinitesimal parameters of gauge transformations. The operators  $R_{i\mu}(h)$  are linear in  $h_k$ . Then, the gauge function  $C_\mu$  and the operator  $T_{\mu\nu}$  are equal to

$$C_\mu = R_{i\mu}(h) \delta^{ik} \alpha_k \quad (A.4)$$

$$T_{\mu\nu}(h, G) = R_{i\mu}(h) \delta^{ik} R_{k\nu}(G) \quad (A.5)$$

Metric tensors  $\tilde{\gamma}^{\mu\nu}$  and  $\gamma^{ik}$  are arbitrary and may be  $h$ -dependent. They have the group transformation law suggested by the position of its indices. And, finally,  $\eta^i$  is an external source for the quantum field  $\mathcal{X}_i$ . At  $\eta^i = 0$  the functional (A.I) coincides with the background functional [2,5]. After integrating over the ghost fields we obtain

$$\Omega(h, \eta) = \int d\mathcal{X} \exp \left\{ i \left[ S(G) - S(h) - \delta^{ik} \mathcal{X}_k + \frac{1}{2} \tilde{\gamma}^{\mu\nu} C_\mu C_\nu + \eta^k \mathcal{X}_k \right] + \text{Sp} \ln T_{\mu\nu}(h, G) \right\} \quad (\text{A.6})$$

In (A.6) we make the substitution of variables

$$\mathcal{X}_k \rightarrow \mathcal{X}_k + R_{k\mu}(G) D^{\mu\nu}(h, G) \bar{\xi}_\nu, \quad (\text{A.7})$$

$$T_{\mu\nu} D^{\nu\lambda} = \delta_\mu^\lambda, \quad (\text{A.8})$$

where  $\bar{\xi}_\nu$  is an infinitesimal function independent of the field  $\mathcal{X}$ . As usual, it is proved that the variation of the term  $\text{Sp} \ln T$  is compensated by the transformation Jacobian [2] (it is necessary here that the conditions  $R_{k\mu}^{\delta k} = f^{\mu\nu} = 0$  be satisfied, where  $f^{\mu\nu\lambda}$  are the structure constants of the group; in the framework of dimensional regularization it is sufficient to require that  $R_{k\mu}$  and  $f^{\mu\nu\lambda}$  be local functions, i.e. depend on different coordinates only through the  $\delta$ -functions and their derivatives

of a finite-order). As a result we have

$$\langle [\tilde{R}^{\mu\nu} C_{\mu} + (\eta^k S^{ik}(h)) R_{k\mu}(G) D^{\mu\nu}(h, G)] \rangle = 0 \quad (A.9)$$

or taking into account (A.2)

$$\langle [\tilde{R}^{\mu\nu} C_{\mu} + \eta^k R_{k\mu}(G) D^{\mu\nu}(h, G) - S^{ik}(h) \tilde{R}_{k\mu}(\infty) D^{\mu\nu}(h, G)] \rangle = 0 \quad (A.10)$$

where

$$R_{k\mu}(G) = R_{k\mu}(h) + \tilde{R}_{k\mu}(\infty) \quad (A.11)$$

Differentiating (A.9) with respect to  $\eta^i$  and setting then  $\eta^k = 0$  we have

$$\begin{aligned} \langle \tilde{R}^{\mu\nu} C_{\mu} \mathcal{X}_i \rangle \Big|_{\eta=0} &= \\ &= i \langle R_{i\mu}(G) D^{\mu\nu}(h, G) \rangle \Big|_{\eta=0} + S^{ik}(h) \langle \tilde{R}_{k\mu}(\infty) D^{\mu\nu}(h, G) \mathcal{X}_i \rangle \end{aligned} \quad (A.12)$$

The identity (A.12) was used in the derivation of (3.5) and (3.9). The following substitutions should be made here:

$$\begin{aligned} i &\rightarrow (\mu\nu, \kappa) \\ \tilde{R}^{\mu\nu} &\rightarrow \frac{\alpha}{\sqrt{-g(x_{(\mu\nu)})}} g^{\mu\nu}(x_{(\mu\nu)}) \delta(x_{(\mu\nu)} - x_{(\nu\mu)}) \\ \mathcal{X}^{i\kappa} &\rightarrow \frac{1}{4} \sqrt{-g} (g^{\mu\lambda} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\lambda} + 2\beta g^{\mu\nu} g^{\lambda\sigma})_{x_{(\mu\nu)}} \delta(x_{(\mu\nu)} - x_{(\lambda\sigma)}) \\ R_{i\lambda} \xi^{\lambda} &\rightarrow -(g_{\mu\lambda} \partial_{\nu} + g_{\nu\lambda} \partial_{\mu} + \partial_{\lambda} g_{\mu\nu}) \xi^{\lambda} \equiv -(g_{\mu\lambda} D_{\nu}(g) + g_{\nu\lambda} D_{\mu}(g)) \xi^{\lambda} \end{aligned} \quad (A.13)$$

where  $C_M$  and  $T_{\mu\nu}$  are given by expressions (2.4) and (2.5), respectively. Let us show e.g. how (3.5) is derived. The variation of (A.6) with respect to the metric (at  $\mathcal{G}=0$ ) equals to

$$\delta\Omega = i \langle \tilde{\gamma}^{\mu\nu} C_M \delta C_\nu \rangle + \langle \delta T_{\mu\nu} D^{\nu M} \rangle \quad (\text{A.14})$$

where

$$\delta C_\nu = \frac{1}{2} (\tilde{\gamma}^{\nu\lambda})^{-1} \delta \tilde{\gamma}^{\lambda\sigma} C_\sigma + R_{i\nu}(h) \delta \tilde{\gamma}^{ik} \mathcal{X}_k \quad (\text{A.15})$$

Since  $\delta C_\nu$  is linear in  $\mathcal{X}$  one can use identity (A.12).

We have

$$\begin{aligned} \delta\Omega &= i \delta^{ik}(h) \langle \tilde{R}_{k\mu}(\mathcal{X}) D^{\mu\nu}(h, G) \delta C_\nu \rangle - \\ &- \frac{1}{2} (\tilde{\gamma}^{\nu\lambda})^{-1} \delta \tilde{\gamma}^{\lambda\nu} - \\ &- \langle R_{i\nu}(h) \delta \tilde{\gamma}^{ik} R_{kM}(G) D^{\mu\nu}(h, G) \rangle + \langle \delta T_{\mu\nu} D^{\nu M}(h, G) \rangle \end{aligned} \quad (\text{A.16})$$

The last two terms in the right-hand side of (A.16) cancel exactly, while the second term in the r.-h. side is not essential for us, e.g. in a dimensional regularization it is zero.

As a result we come to

$$\delta\Omega|_{\eta=0} = i S^{jk}(h) \langle \tilde{R}_{\kappa\mu}(\alpha) D^{\mu\nu}(h, G) \delta C_\nu \rangle \Big|_{\eta=0} \quad (\text{A.17})$$

Note that (A.17) can be obtained also after a change of variables (2.5) in (A.6) (at  $\eta=0$ ), <sup>[2,5]</sup>:

$$\mathcal{L}_\kappa \rightarrow \mathcal{L}_\kappa + R_{\kappa\mu}(G) D^{\mu\nu}(h, G) \delta C_\nu \quad (\text{A.18})$$

### Appendix B

In this Appendix gauge-dependence of the counterterm  $\sqrt{g} R$  is calculated in the one-loop approximation in pure quantum gravity in the space-time dimension  $n=2+\epsilon$  (in this connection see Ref. [9]).

Equation (3.1) is valid in a space of any dimension. Near two dimensions instead of (3.2) we have

$$\langle \delta\alpha_{\mu\nu} \rangle_{\text{div}}^4 \sim a \frac{g_{\mu\nu}}{n-2} \quad (\text{B.1})$$

i.e.

$$\delta W_1 = -i \frac{a}{n-2} S^{j\mu\nu} g_{\mu\nu} = i \frac{a}{2} \sqrt{g} R \quad (\text{B.2})$$

To determine the variation of counterterms proportional to  $1/(n-2)$  one must in (B.1) calculate the terms, containing  $(n-2)^{-2}$ . In this case we must find, as follows from (B.2), the part of  $\mathcal{L}$ , which is proportional to  $(n-2)^{-1}$ . From eq.(3.9) it follows



that in  $\partial W/\partial \alpha$  there are no such terms and in  $\partial W/\partial \beta$  such terms appear due to the pole  $(n-2)^{-1}$  in the graviton propagator. For calculations we shall use the fact that

$g_{\mu\nu} = \delta_{\mu\nu} + h_{\mu\nu}$  and that by calculating the coefficient in front of  $\delta_{\mu\nu}$  we shall obtain the whole answer from general covariance. It is sufficient to calculate a simple diagram ,

Fig.7.

$$I_{(2+\epsilon)}(\beta) = \frac{\partial W_1}{\partial \beta} = -i \int S^{\mu\nu}(g, x) \left[ -2 \partial_\nu D_{\delta\delta}^{(0)}(x-z) \partial_\delta H_{\mu\alpha, \lambda\lambda}^{(0)}(x-z) - D_{\delta\delta}^{(0)}(x-z) \partial_\delta \partial_\nu H_{\mu\nu, \lambda\lambda}^{(0)}(x-z) \right] \quad (B.3)$$



Fig.7.

The result is

$$\frac{\partial W_{1,div}}{\partial \beta} = -i \frac{1}{n-2} \left( \frac{1}{(1+\beta)^3} - \frac{2}{(1+\beta)^2} \right) \sqrt{-g} R \quad (B.4)$$

This expression is easily integrated over  $\beta$  , and one can determine the values of  $\beta$  at which the counterterm  $\sqrt{-g} R$  is absent. It is important that the calculations showed a nontrivial  $\beta$ -dependence.

REFERENCES

1. M.T.Grisaru and P. van Nieuwenhuizen and J.A.M.Vermaseren, Phys. Rev. Lett. 37, 1662 (1976) and references therein.
2. B. De Witt, Phys. Rev. 160, 1113(1967) 162, 1195, 1239 (1967).
3. J.Honerkaup, Nucl. Phys. B48, 269 (1972); I.Arefieva, L.Paddeev, A.Slavnov, Teor.Mat. Fiz. 21,311 (1974); H.Kluberg-Stern and J.B.Zuber, Phys. Rev. D12, 492 (1975); I.Tyutin, in press.
4. G.'t Hooft and M.Veltman, Ann. Inst. H.Poincare 20, 69(1974).
5. R.Kallosh, Nucl. Phys. B78, 293 (1974).
6. M.T.Grisaru, P. van Nieuwenhuizen, and C.C.Wu, Phys. Rev. D12, 3203 (1975).
7. M.T.Grisaru, Phys. Lett. 66B, 75 (1977); S.Deser, I.H.Kay, K.S.Stelle, Phys. Rev. Lett. 38, 527 (1977).
8. S.Deser and P. van Nieuwenhuizen, Phys. Rev. D10, 401(1974).
9. R.Gastmans, R.Kallosh, C.Truffin, P.N.Lebedev preprint N 34, January 1977 and Nucl. Phys., to be published.

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