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Reduction of Multiple Scattering Displacement by a Magnetic Field Parallel to the Beam*

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Abstract

A charged particle of mass m moving with velocity V_z through a continuous medium approximately parallel to a uniform magnetic field H suffers many small-angle Coulomb scatterings. The net square displacement in one traversal can be expressed as a sum of the square displacements from single scatterings occurring in many independent traversals. These single displacements have been evaluated by beam transport methods, and the ratio of the net rms displacement projected on a plane perpendicular to the magnetic field to the zero field displacement is found to be $F = \left[\frac{6}{\phi^2}\left(1 - \frac{\sin\phi}{\phi}\right)\right]^{\frac{1}{2}}$, where $\phi = eHz/(mV_z c)$ is the angle of the helical trajectory, e.g. for a single-turn helical trajectory, $F(2\pi) = 0.4$, and the effect of the magnetic field is to reduce the multiple scattering to 40%. This confirms the solution of the diffusion equation for the distribution function due to Farley, Fiorentini and Stocks.

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A charged particle moving through a continuous medium will be deflected by many independent small-angle Coulomb scatterings with the nuclei of the medium[1]). An estimate of the net deflection (in the absence of large-angle scatters and assuming no energy loss) can be made by the conventional expressions

$$\sqrt{\langle\theta^2_{proj}\rangle} = \frac{15\ \mathrm{MeV/c}}{p\beta}\sqrt{\frac{L}{X}}\,,$$

$$\sqrt{\langle\theta^2\rangle} = \sqrt{2\langle\theta^2_{proj}\rangle}\,, \qquad (1)$$

where $\sqrt{\langle\theta^2\rangle}$ is the root mean square deflection angle, $\sqrt{\langle\theta^2_{proj}\rangle}$ is the root mean square deflection angle projected onto any plane containing the initial trajectory, p is the particle's momentum, βc is its speed, L is the length of the trajectory in the material, and X is the radiation length of the material.

In the absence of a magnetic field, the resulting rms displacement is

$$\sqrt{\langle u^2(H=0)\rangle} = \sqrt{\langle\theta^2\rangle}\, L/\sqrt{3}\,. \qquad (2)$$

It is the purpose of this note to investigate how expression (2) for $\sqrt{\langle u^2\rangle}$ is changed by the presence of a uniform magnetic field parallel to the initial direction of the particle. Previous work on this problem has indicated that $\sqrt{\langle u^2\rangle}$ is reduced by a magnetic field: Čerenkov[2]) and Farley et al.[3]) derived a diffusion equation for the distribution function and solved it for the particular case of particles directed parallel to the magnetic field. Daniel[4]) developed a more intuitive model in which energy losses were also included. The present work shows (in the Appendix) that the net square displacement in one traversal of the scattering medium can be expressed as the sum of the square displacements from single scatterings in many independent traversals. These displacements are evaluated by beam transport methods assuming no energy loss. The result agrees with previous work[3]. The results are applicable to gas-filled detectors such as cloud chambers and time projection chambers, and also to higher-density detectors with restrictions on the path length being imposed to approximate constant energy conditions.

The helical trajectory of a positive particle has been conveniently parameterized by Edwards and Rose[5]: taking the z-axis along the magnetic field, the x- and y-displacements and $x' \equiv dx/dz$ and $y' \equiv dy/dz$ at a distance L measured along the trajectory can be written in terms of their values at an earlier point ℓ by (see fig. 1)

$$\begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix}_L = M \begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix}_\ell , \qquad (3)$$

where $M \equiv \begin{pmatrix} 1 & (z \sin\phi)\phi & 0 & z(1-\cos\phi)/\phi \\ 0 & \cos\phi & 0 & \sin\phi \\ 0 & -z(1-\cos\phi)/\phi & 1 & (z\sin\phi)/\phi \\ 0 & -\sin\phi & 0 & \cos\phi \end{pmatrix}$

ϕ is the angle between ℓ and L on the trajectory projected on the xy-plane, and z is the axial distance between ℓ and L. Eq. (3) is exact even for large displacements. There is only one independent variable in (3) since ϕ and z are connected by the relation

$$\phi = V_T z/(V_z R) , \qquad (4)$$

where $V_T = \sqrt{V_x^2 + V_y^2}$ and V_z are the tangential and axial velocities, and R is the radius of the trajectory projected on the xy-plane. Eq. (4) can be obtained by removing time from the equations of radial and

transverse motion. Alternatively

$$\phi = eHz/(\gamma m_o V_z c) , \qquad (5)$$

where the symbols have their usual meaning.

We will use this formalism to study single independent scatters and subsequently combine their effects, as described in the Appendix. But first to obtain some insight into the focusing effect of the magnetic field consider a single scatter at ℓ in the particular case $\phi = 2n\pi$, where n is an integer: M is seen to become the unit matrix, indicating that there is point-to-point focusing from ℓ to L. This means that a single scatter at ℓ will not contribute any displacement in the xy-plane at L in the particular case of $2n\pi$ separation.

Separations other than $\phi = 2n\pi$ will not be so favourable, and we now study the general case where a single scatter of amount $\delta x'_\ell$ and $\delta y'_\ell$ at ℓ is responsible for displacements δx_L and δy_L at L, viz.:

$$\begin{pmatrix} x + \delta x \\ x' + \delta x' \\ y + \delta y \\ y' + \delta y' \end{pmatrix}_L = M \begin{pmatrix} x \\ x' + \delta x' \\ y \\ y' + \delta y' \end{pmatrix}_\ell .$$

Subtracting eq. (3), we obtain

$$\delta x_L = z \frac{\sin\phi}{\phi} \delta x'_\ell + z \frac{(1 - \cos\phi)}{\phi} \delta y'_\ell, \text{ and}$$

$$\delta y_L = -z \frac{(1 - \cos\phi)}{\phi} \delta x'_\ell + z \frac{\sin\phi}{\phi} \delta y'_\ell .$$

As a measure of the displacement at L due to the scattering at ℓ we define

$$u \equiv [(\delta x_L)^2 + (\delta y_L)^2]^{\frac{1}{2}}$$

substituting δx_L, δy_L and using eq. (4)

$$u = \left(2(1 - \cos\phi)[(\delta x'_\ell)^2 + (\delta y'_\ell)^2]\right)^{\frac{1}{2}} V_z R/V_T . \qquad (6)$$

In eq. (6) we see that u vanishes at $\phi = 2n\pi$, as expected.

A single elastic scatter changes the velocity components V_x, V_y, V_z but not the magnitude of the velocity V, so the angle from the i^{th} scatter can be written as $\theta_i = [(\delta V_x)^2 + (\delta V_y)^2 + (\delta V_z)^2]^{1/2}/V$. Noting that $x' \equiv dx/dz = \frac{dx}{dt}/\frac{dz}{dt} = V_x/V_z$, etc. and restricting our attention to trajectories which are approximately parallel to the magnetic field so that V_z can be considered constant,

$$\theta_i = [(\delta x'_\ell)^2 + (\delta y'_\ell)^2]^{1/2} V_z/V . \tag{7}$$

Then from (6) the displacement at L due to a single scatter of angle θ_i at ℓ is

$$u_i = D_{helix}(\ell)\ \theta_i \tag{8}$$

where $D_{helix} = \sqrt{2(1 - \cos\phi)}\ VR/V_T$. The net mean square displacement at L from scatters at all previous points ℓ on the trajectory can then be obtained from eqs. (8) and (A1) as discussed in the Appendix:

$$\langle u^2\rangle = \frac{\langle\theta^2\rangle}{L}\int_0^L D^2_{helix}(\ell)\ d\ell . \tag{9}$$

As before, $\langle\theta^2\rangle$ is the accumulated mean square angle. Using eq. (4) and the relations $\ell^2 = z^2 + (R\phi)^2 = z^2\left(1 + (V_T/V_z)^2\right) = (zV/V_z)^2$, we obtain the final result: the rms displacement in the xy-plane due to multiple scattering along the complete track L is

$$\sqrt{\langle u^2\rangle} = \sqrt{\langle\theta^2\rangle}\ LF(\Phi)/\sqrt{3} , \tag{10}$$

where

$$F(\Phi) \equiv \left[\frac{6}{\Phi^2}\left(1 - \frac{\sin\Phi}{\Phi}\right)\right]^{1/2} \tag{11}$$

is plotted in fig. 2, and Φ is the value of ϕ for the complete trajectory. A similar result holds for the variable u_{proj} and θ_{proj} projected onto any plane through the trajectory. Comparing this result with eq. (2), we see that F is the ratio of the rms displacement in the xy-plane with magnetic field to the zero field displacement for the same track length. We see in fig. 2 that F is always less than unity, which means that the magnetic field acts to reduce the transverse displacement due to multiple scattering. F is seen to drop quickly as Φ approaches 2π and the focusing effects become appreciable.

Alternatively the result (10) can be written using (5) in a form convenient for large Φ,

$$\sqrt{\langle u^2\rangle} = \sqrt{\frac{\langle\theta^2\rangle}{3}}\ \frac{\gamma m_o c V_z}{eH}\ \Phi F .$$

$\Phi F(\Phi)$ is plotted in fig. 2, and we see that for $\Phi > \pi$, ΦF oscillates about 0.77. Thus the effective length of the trajectory to use in eq. (2) attains the approximately constant value $0.77\ \gamma m_o c V_z/(eH)$ at large values of $\Phi = eH/(\gamma m_o V_z c)z$. Thus for trajectories with $\Phi > \pi$, $\sqrt{\langle u^2\rangle}$ increases as $L^{1/2}$, as opposed to $L^{3/2}$ in the field-free case.

Eq. (10) is the identical result obtained by Farley et al.[3]) using quite different methods: they solved the diffusion equation for the distribution function in this case assuming the scattered trajectories make only small angles with the trajectory which was parallel to the magnetic field.

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Appendix

Consider a beam in a scattering medium with magnetic focusing elements and let the vector displacement at a distance L along the trajectory due to a single scatter of angle $\overline{\theta}_i$ at a distance ℓ_i along the trajectory be $\overline{u}_i = D(\ell_i)\overline{\theta}_i$. $D(\ell_i)$ is determined by the magnetic focusing elements between ℓ_i and L. We make use of the fact that small angles are vectors[6] so that the total displacement from N scatters is

$$\overline{u} = \sum_{i=1}^{N} D(\ell_i)\overline{\theta}_i \ .$$

Then

$$u^2 = \sum_{i=1}^{N} D^2(\ell_i)\theta_i^2 + \sum_{i \neq j} D(\ell_i)D(\ell_j)\overline{\theta}_i \cdot \overline{\theta}_j \ .$$

Averaging over many traversals and noting $\overline{\theta}_i$ is positive as often as it is negative,

$$\langle u^2 \rangle = \sum_{i=1}^{N} D^2(\ell_i)\langle \theta_i^2 \rangle \ .$$

Noting from eq. (1) that the mean square accumulated angle $\langle \theta^2 \rangle$ is proportional to L, we obtain the general result suggested by Thiessen[7],

$$\langle u^2 \rangle = \frac{\langle \theta^2 \rangle}{L} \int_0^L D^2(\ell)\, d\ell \ . \qquad \text{(A1)}$$

This is the basis of eq. (9).

Eq. (2), which gives the field-free displacement from multiple scattering, is conventionally derived[1] by solving the diffusion equation for the Fermi distribution function. However, to demonstrate the use of eq. (A1), we take the one-dimensional form

$$\langle \delta x^2 \rangle = \frac{\langle \theta^2 \rangle}{L} \int_0^L D^2(z)\, dz$$

and insert the fact from beam transport that for a drift length of magnitude L-z

$$D(z) = L-z \ .$$

This immediately gives eq. (2),

$$\sqrt{\langle (\delta x)^2 \rangle} = \sqrt{\langle \theta^2_{proj} \rangle}\, L/\sqrt{3} \ .$$

Eq. (A1) is of general use in predicting multiple scattering by gas in magnetic spectrometers, beam lines, etc.

References

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5) D.N. Edwards and B. Rose, Nucl. Instr. & Meth. 7 (1960) 135.

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7) A. Thiessen, Cal.Tech. Thesis (1967).

Figure Captions

1. The projection of the helical trajectory of a positive particle onto the xy-plane where the magnetic field is parallel to the z-axis. The i^{th} single scatter (θ_i does not lie in the xy-plane) takes place where the track length is ℓ, and causes a displacement u_i in the xy-plane at the end of the track where the length is L. ℓ and L are separated axially by z and in angle by ϕ; Φ is the angle for the complete trajectory.
2. $F = \left[\frac{6}{\Phi^2}\left(1 - \frac{\sin \Phi}{\Phi}\right)\right]^{1/2}$ and ΦF versus Φ. F is the ratio of the rms displacements due to multiple scattering in the presence of a magnetic field to the zero-field displacement for the same trajectory length.

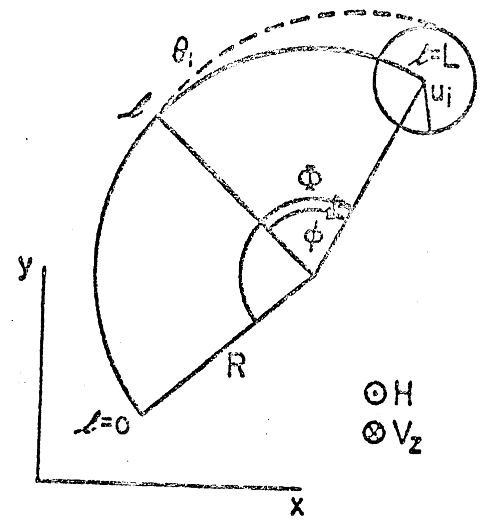

Fig. 1

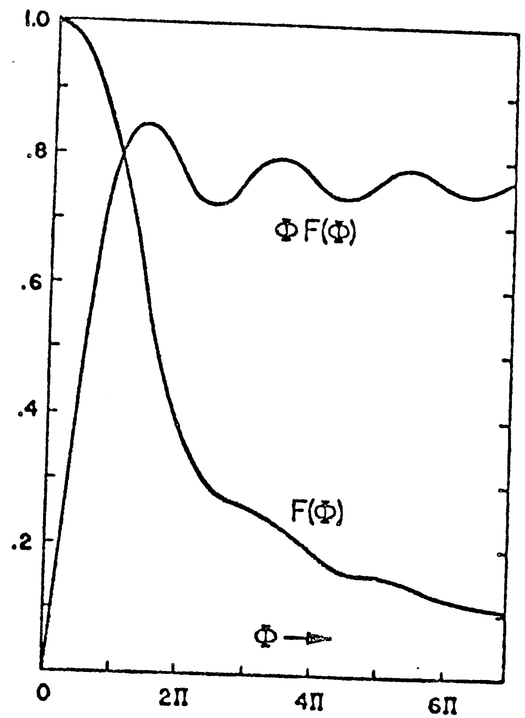

Fig. 2