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MURA-105
(Notes)

# APPLICATION OF WALKINSHAW'S EQUATION TO THE $2\sigma_y = \sigma_x$ RESONANCE

L. Jackson Laslett*

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A method of analysis which appears to account for the behavior of the axial motion, in the presence of appreciable radial oscillation, has been developed by Walkinshaw, [W. Walkinshaw, "A Spiral Ridged Bevatron", A.E.R.E., Harwell (1956)]. The differential equation characterizing the axial motion is treated as linear, but contains a coefficient which involves the radial motion. As is well-known, the forced radial motion enhances the A-G focussing which appears in the axial equation -- now, however, the additional effect of the free radial betatron oscillations is also included in the axial equation. The super-position of the comparitively-long-wavelength radial oscillations on the forced motion in effect modulates the smooth-approximation coefficient in the axial equation, to yield a Mathieu equation with a coefficient having the period of the radial motion. Under "resonant" conditions, which will be seen to include the case of interest here, this equation may have unstable solutions, and in such cases, the characteristic exponent of the solution appears to compare reasonably in magnitude with the lapse-rate characterizing the exponential growth of the ILLIAC solutions of the "Feckless Five" equations.

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* At the University of Illinois, on leave from Iowa State College.

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Walkinshaw's analysis pertains to differential equations which, in the MURA notation [f. ex., LJL (MURA)-5], are taken to be of the form

$$x'' + (k+1)\,x = -f \sin(x/w - N\theta),$$

$$y'' + \left[-k - (f/w)\cos(x/w - N\theta)\right] y = 0$$

[cf. LJL MURA Notes 6-22 Oct. 1956, Sect. 6, for $y/w \ll 1$] .

A solution for the axial motion, representing a free oscillation of amplitude A superposed on the forced motion, is taken of the form

$$x = A\cos(\nu_x\theta + \epsilon) - (f/\Omega^2)\sin\int \Omega\, d\theta\,,$$

where $\Omega \cong N + A\,(\nu_x/w)\sin(\nu_x\theta + \epsilon)$ and $\nu_x \doteq (k+1)^{1/2}$.

This solutions is substituted into the axial equation to yield, after some approximation ( and a shift of the origin of $\theta$ which we introduce for convenience),

$$y'' + \left[-k + \frac{f^2}{w^2 N^2}\left(1 + \frac{2A\nu_x}{wN}\cos\nu_x\theta\right)\right] y = 0.$$

It is noted that, when $A = 0$, this equation reduces to that given by the smooth-approximation -- we accordingly write

$$y'' + \left[\nu_y^2 + \frac{2A\, f^2 \nu_x}{w^3 N^3}\cos\nu_x\theta\right] y = 0,$$

to obtain an equation of the Mathieu type with a coefficient of period $2\pi/\nu_x$ in $\theta$. By the transformation $\nu_x\theta = 2t$, we have the standard form

$$d^2y/dt^2 + \left[(2\nu_y/\nu_x)^2 + \frac{8f^2}{w^3N^3}\frac{A}{\nu_x}\cos 2t\right] y = 0$$

with a coefficient of period $\pi$ in the independent variable t.

A solution of the Mathieu equation

$$d^2y/dt^2 + \left[a + b\cos 2t\right] y = 0,$$

for b small but not zero, will exhibit instability when the coefficient a is equal or close to the square of an integer. In the present application stop-bands may thus be expected at operating points such that $2\nu_y/\nu_x = m$, the broad band of instability at $2\nu_y/\nu_x = 1$ (or $2\sigma_y/\sigma_x = 1$) being of chief interest in connection with the work presented here. It appears, moreover, possible to employ the Mathieu equation to account semi-quantitatively for (i) the range of b, and hence of the amplitude of free radial oscillation, which may be permitted when the oscillation frequencies depart by a specified amount from the resonant condition, and (ii) the lapse-rate found to characterize the growth of the axial motion when the radial oscillations exceed this limit.

The numerical application of the Mathieu equation to specific problems of stability or instability may be accomplished by reference to ILLIAC solutions for the stability boundaries/or for the characteristic exponent characterizing the solution.

(i) A useful estimate of the expected restrictions on the radial motion may be obtained, however, by appeal to the fact that near a = 1, b = 0 the stability boundaries can be represented rather well by the condition

$$|b| \doteq 2\,|a - 1| .$$

We find in this way the following estimate for the limiting amplitude:

$$A_1 = \frac{\omega^3 N^3}{4f^2}\,\nu_x\,\left|(2\nu_y/\nu_x)^2 - 1\right|$$

$$\cong \frac{\omega^3 N^3}{2f^2}\,\left|2\nu_y - \nu_x\right| \quad \left(\text{for } \frac{2\nu_y}{\nu_x} - 1 \ll 1\right).$$

It may be noted that this result, although expressed in terms of $\nu_x$ and $\nu_y$, concerns an inherent <u>sector</u> <u>resonance</u> which arises when $2\sigma_y/\sigma_x = 1$.

(ii) An estimate of the lapse-rate characterizing unstable solutions near a = 1, b = 0 may, moreover, be made by taking

$$\mu \doteq \frac{\pi}{4}\sqrt{b^2-4(a-1)^2} \text{ nepers for } \Delta t = \pi \quad \text{when } (|b|>2|a-1|)$$

$$= \frac{\pi}{4}\frac{\nu_x}{N}\sqrt{b^2-4(a-1)^2} \quad \text{nepers per sector}$$

$$= \frac{\pi/4}{N}\sqrt{\left(\frac{8f^2A}{\omega^3N^3}\right)^2-4\left[(2\nu_y)^2-\nu_x^{\,2}\right]^2/\nu_x^{\,2}} \quad \text{nepers per sector}$$

$$= \frac{1.57}{N}\sqrt{\left(\frac{4f^2A}{\omega^3N^3}\right)^2-\left[(2\nu_y)^2-\nu_x^{\,2}\right]^2/\nu_x^{\,2}} \quad \text{nepers per sector}$$

$$= \frac{0.68}{N}\sqrt{\left(\frac{4f^2A}{\omega^3N^3}\right)^2-\left[(2\nu_y)^2-\nu_x^{\,2}\right]^2/\nu_x^{\,2}} \quad \text{decades per sector.}$$

A convenient alternative form for this last result is

$$\mu = \frac{2\pi f^2}{\omega^3N^4}\sqrt{A^2-A_1^{\,2}} \quad \text{nepers/sector}$$

$$= \frac{2.73 f^2}{\omega^3N^4}\sqrt{A^2-A_1^{\,2}} \quad \text{decades/sector.}$$

Results obtained with the ILLIAC, for 5-sector machines with model-like parameters such that $0.5\pi < \sigma_{xo} < 0.6\pi$ and $0.2\pi < \sigma_{yo} < 0.4\pi$, appear fairly close to these estimates. In all the ILLIAC runs the radial amplitudes were measured, however, near the center of a focusing region, at $N\theta = 0$ (Mod. $2\pi$), where the amplitudes of the non-sinusoidal A-G oscillations can exceed those corresponding to the smooth approximation representation of the motion. By way of example we present here the results for an accelerator for which

$$k = 0.6436 \qquad 1/w = 20.82 \qquad f = 1/4 \qquad N = 5 :$$

In this case the oscillation frequencies are such that

$$\left.\begin{aligned} \sigma_{xo} &= 0.5388\,\pi \\ \sigma_{yo} &= 0.2855\,\pi \end{aligned}\right\} \quad \text{or} \quad \left\{\begin{aligned} \nu_{xo} &= 1.347 \\ \nu_{yo} &= 0.714 \end{aligned}\right.$$

and the limiting amplitude for $x$ appeared to be some 0.0075 units to the left of the stable fixed point ($N\theta = 0$, mod. $2\pi$). For these machine parameters the equation for $A_1$ yields

$$A_1 = \frac{500}{(20.82)^3} \; 1.347 \; \left[(1.06)^2 - 1\right]$$

$$= \frac{500 \times 1.347 \times 0.1236}{9025}$$

$= 0.0092$, the observed limiting amplitude at $N\theta = 0$ (Mod. $2\pi$) thus being within 20% of this estimate.

With respect to the lapse-rate, we continue this example by consideration of the case $A = 0.0225$. Then $\sqrt{A^2 - A_1^2} = 0.02035$, and one expects

$$\mu = \frac{0.171\ (20.82)^3}{625}\ (0.02035)$$

$$= 2.73 \times 10^{-4} \times 9025 \times 0.02035$$

$$= 0.050 \text{ decades/sector,}$$

in close agreement with the value 0.055 decades/sector found from the ILLIAC work.

[For this case the coefficients in the Mathieu equation are $a = 1.12$, $b = 0.604$, for which an independent extrapolation of coarse tables extending to $a \ngtr 1$ suggests $\mu = 0.107$ nepers/sector = 0.046 decades/sector.]

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