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PROPOSED METHOD FOR DETERMINING MARK V TRAJECTORIES BY AID OF GRID STORAGE

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ABSTRACT

It has been proposed that the trajectories of particles in an FFAG accelerator of the spirally-ridged (Mark V) type may conveniently be obtained with an electronic digital computer by storage of values from which the fields may be derived. Based on the scaling properties of the assumed structure, the present report suggests suitable coordinates to be used in representing the field, develops the differential equation describing the magnetostatic problem in these coordinates, indicates the means of obtaining the field-components from the solution of this differential equation, and discusses the use of these results in the dynamical equations for the trajectories. Proposed limits for the parameters are indicated, certain cases in which $1/w$ vanishes being admissible.

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1 References are given in Section IX.

I. INTRODUCTION

The desirability of high-speed orbit computation, for both radial and axial motion in the Mark V FFAG accelerator, and the recent increase of interest[1] in the use of fields not conveniently representable in terms of a simple Fourier expansion in the median plane, has given increased importance to Dr. R. Christain's suggestion[2] that the magnetostatic potential or fields be stored on a grid for immediate use in trajectory computations. As Kerst has pointed out[1], the scaling property of the Mark V accelerator permits these data to be stored as a two-dimensional array. The solution of the potential problem may then, moreover, be reduced to the solution of a second-order differential equation in two independent variables only, thus reducing tremendously the computational work required. We present here some details pertaining to computations of this nature.

II. THE POTENTIAL PROBLEM

A. The Scaling Properties:

The character of the potential representing the magnetic field of a Mark V type accelerator is conveniently seen from a series expansion similar to that given by Powell[3]:

$$\Psi(x,y,\theta)=(1+x)^{K+1}\sum_{m}\sum_{j=0}^{\infty}\left(\frac{y}{1+x}\right)^{2j+1}\left(M_{j,m}\cos m\Phi+N_{j,m}\sin m\Phi\right),$$

where[3,4]

$$\begin{aligned}
\Phi &\equiv \tfrac{1}{w}\ln(1+x)-N\theta\\
x &\equiv (r-r_1)/r_1\\
y &\equiv z/r_1\\
N\theta &= N\phi-\ln(r_1/r_0),
\end{aligned}$$

and the quantities $M_{j,m}$, $N_{j,m}$ are constants obtainable from recursion relations given by Powell[3]. From this development it is evident that $(1+x)^{-(k+1)}\Psi$ has the same value at all points for which both $y/(1+x)$ and Φ have the same values. Also, with $y/(1+x)$ constant, $(1+x)^{-(k+1)}\Psi$ is periodic in Φ with period 2π.

The potential, with its scaling factor, is conveniently expressed in terms of a function of the two variables

$$\xi \equiv \frac{1}{2\pi}\left[\frac{\ln(1+x)}{W} - N\theta\right]$$
$$\eta \equiv \frac{\sqrt{1+(WN)^2}}{2\pi W}\,\frac{y}{1+x} \;:$$

$$\begin{aligned}\Psi(x,y,\theta) &= \frac{1}{A}(1+x)^{k+1}\,\Omega(\xi,\eta)\\ &= \frac{1}{A}(1+x)^{k+1}\,\Omega\left(\frac{\ln(1+x)}{2\pi W}-\frac{N\theta}{2\pi},\ \frac{\sqrt{1+(WN)^2}}{2\pi W}\,\frac{y}{1+x}\right).\end{aligned}$$

From this relationship the differential equation for Ω may be derived and, secondly, expressions found for the field-components in terms of Ω and its derivatives. Ω is periodic in ξ with period unity.

The equation may alternatively be interpreted (i) as referring $\Psi/(1+x)^{k+1}$ to the value obtaining in the $\theta = 0$ plane at $x_0 = (1+x)\exp[-NW\theta] - 1$, $y_0 = y(1+x_0)/(1+x)$, or (ii) relating it to the value at $x_0 = 0$, $y_0 = y/(1+x)$, $\theta_0 = \theta - \left(\frac{1}{WN}\right)\ln(1+x)$.

B. <u>The Equation for the Quantity Ω Characterizing the Potential:</u>

By straight-forward differentiation of the relation given at the end of the preceding sub-section, one finds

$$Ar_1^2\nabla^2\Psi = \frac{(1+x)^{k+1}}{4\pi^2}\left[\left(\frac{1}{W}\right)^2+N^2\right]\left\{\begin{aligned}&\Omega_{\xi\xi}+\left(1+\frac{4\pi^2\eta^2}{(\frac{1}{W})^2+N^2}\right)\Omega_{\eta\eta}-\frac{4\pi(\frac{1}{W})}{(\frac{1}{W})^2+N^2}\eta\Omega_{\xi\eta}\\ &+\frac{4\pi k'(\frac{1}{W})}{(\frac{1}{W})^2+N^2}\Omega_{\xi}-\frac{4\pi^2(2k'-1)}{(\frac{1}{W})^2+N^2}\eta\Omega_{\eta}+\frac{4\pi^2k'^2}{(\frac{1}{W})^2+N^2}\Omega\end{aligned}\right\}$$

where $k' \equiv k + 1$.

(This result will also be found to follow by transformation of Sessler's equation,[5] which is expressed in terms of the variables $u \equiv \exp(2\pi W \xi)$, $v \equiv \frac{2\pi}{\sqrt{(\frac{1}{W})^2+N^2}} \eta \exp(2\pi W\xi)$, $\Psi(u,v) \propto \Omega \cdot \exp(2\pi W \xi)$.)

It is visualized that the differential equation for Ω , obtained by setting $\nabla^2\Psi = 0$, be solved on a ξ, η net by a relaxation method, <u>employing the periodicity property cited</u> and noting that Ω will be an <u>odd</u> function of η , vanishing at $\eta = 0$ (the median plane). For the variable ξ we have selected a range of unit width, as, for example, $-1 \leq \xi \leq 0$. It may prove expedient first to solve the relaxation problem on a coarse net, covering the complete magnetic gap above the median plane, and then pass to a finer net covering a region above the median plane not much greater than that of dynamical interest.

The <u>boundary</u> may be characterized in the reference plane by the curve

$$\eta = \frac{1}{2\pi}\sqrt{(\tfrac{1}{W})^2+N^2}\, G(\xi)$$

or

$$y = (1+x)\, G\left(\frac{\ln(1+x)}{2\pi W} - \frac{N\theta}{2\pi}\right)$$

$$z = r\, G\left(\frac{\ln(r/r_1)}{2\pi W} - \frac{N\theta}{2\pi}\right)$$

$$= r\, G\left(\frac{\Phi}{2\pi}\right),$$

where

$$G(\xi+1) = G(\xi) \quad .$$

To define the potential problem completely it is necessary to specify the boundary conditions on the aforementioned curve. It is perhaps most natural to specify the quantity Ω itself, although other forms of boundary conditions involving the normal derivative of Ω might be visualized. Specification of Ω must be in conformance with the periodicity condition (periodic in ξ , with period unity). A particularly simple specification would take Ω to be constant on the boundary, corresponding to the potential Ψ varying along the pole as r^{k+1} .

It may be mentioned at this point that presumably many, if not most, cases of practical interest may involve a mirror symmetry, Ω and G being in such cases even functions of ξ about planes spaced one-half unit apart. By making provision to take advantage of such symmetry whenever present, the computational effort can be reduced or the accuracy of the computations increased.

C. The Pole-Face Currents:

The potential function which results from solution of the problem defined in sub-section B will require the presence of pole-face currents for its realization. Considering these currents as current sheets,

$$4\pi \bar{J}' = \overline{\text{grad}\, \Psi} \times \hat{n} \quad \text{(unrat, emu)},$$

where $\hat{n}$ is the unit normal to the pole-surface (directed into the gap). These currents will be source-free. The current density can then be written explicitly in terms of Ω and its derivatives, evaluated on the pole surface, and in terms of the function G and its derivative. To accomplish this computation, one may note that, in cylindrical coordinates,

$$\hat{n} = \frac{-\hat{e}_z + [G + G'/(2\pi w)]\,\hat{e}_r - [NG'/(2\pi)]\,\hat{e}_\theta}{\{1 + [G + G'/(2\pi w)]^2 + [NG'/(2\pi)]^2\}^{\frac{1}{2}}} .$$

The results do not appear to be particulary simple, but one observes that the current density (current per unit width of conductor) scales as r^k with the dominant term being

$$4\pi J_\theta \doteq (k+1)(\Omega/A)\, r^k$$
$$= (k+1)(\Psi/r).$$

This last result is understandable in light of the fact that the potential increases by a factor of the order $1 + (k+1)\lambda/r$ in one wavelength and so gives rise to an average tangental field $-(k+1)\Psi/r$ along the pole. The nature of this term moreover suggests the suitability from a practical standpoint of taking Ω as constant, or substantially so, along the pole boundary.

III. THE FIELD COMPONENTS

In terms of the coordinates ξ, η introduced previously, the field components may be obtained by differentiation of the identity of Section II A which related the potential Ψ to the quantity Ω :

$$B_z = -(1/r_1)\frac{\partial \Psi}{\partial y} = \frac{-1}{r_1 A}\,\frac{\sqrt{\left(\frac{1}{w}\right)^2 + N^2}}{2\pi}\,(1+x)^k\left[\frac{\Omega}{\eta} + \eta\frac{\partial}{\partial\eta}\left(\frac{\Omega}{\eta}\right)\right]$$

$$B_r = -(1/r_1)\frac{\partial \Psi}{\partial x} = -\frac{1}{r_1 A}(1+x)^k\left[k\left(\frac{\Omega}{\eta}\right) + \frac{1}{2\pi w}\frac{\partial}{\partial\xi}\left(\frac{\Omega}{\eta}\right) - \eta\frac{\partial}{\partial\eta}\left(\frac{\Omega}{\eta}\right)\right]\eta$$

$$B_\theta = -(1/r_1)\frac{\partial\Psi/\partial\theta}{1+x} = \frac{1}{r_1 A}\left(\frac{N}{2\pi}\right)(1+x)^k\;\eta\frac{\partial}{\partial\xi}\left(\frac{\Omega}{\eta}\right) .$$

The field-components have been expressed in terms of $\frac{\Omega}{\eta}$ in the thought that this quantity, being more constant than Ω itself, would be the quantity to be stored (as 1/3 -words) in the computer memory, and differentiation as well as interpolation could then best be performed on the quantity Ω/η . This quantity will be an even function of η and periodic in ξ with period unity. B_z will also be an even function of η , while B_r and B_θ will be odd.

IV. THE BASIC GRID CELL IN THE DYNAMICAL REGION

The quantity corresponding to the potential, as Ω/η , is to be stored on a grid which in effect covers unit range for the variable ξ . Since Ω/η is an even function of η, Ω itself being an odd function, the range of the variable η may extend from zero to some maximum value. This maximum value preferably should be left somewhat flexible, but will be substantially $Z\ max/\lambda$, where within the dynamical region this quantity may be expected to have a value (denoted $\frac{1}{\tau}$) lying in the neighborhood of 1/6 or 1/7 for practicable structures. The useful range of η should be broken up into M intervals, each of size $1/M\tau$, with M preferably at least as large as 18. If the intervals of ξ are taken, for convenience, to be of the same size as those for η , the number of such intervals would be $M\tau$. If P represents the maximum number of net points for which storage (in the form of 1/3 -words) is possible within the fast memory of the computer, the parameters determining the mesh size must then be selected so that

$$\tau M(M+1) \leq P.$$

If τ were 6.5, a possible value for M might be 18 in that 6.5 x 18 x 19 = 2223 = 3(741); similarly, for τ = 6.0 and M = 19, we find 6.0 x 19 x 20 = 2280 = 3(760).

V. THE INTERPOLATION FORMULAE

As noted previously, the fields are to be obtained from quantities stored on a net by processes of interpolation and interpolation-differentiation. The suggestion has been made that, in addition to giving reasonable accuracy, an interpolation-differentiation formula suitable for the present purpose should exhibit a continuous derivative upon crossing from the region covered by one cell to that covered by an adjacent cell. Sessler has proposed for this purpose an interpolation formula which, in one-dimensional form, is

$$\begin{aligned}\Lambda(t_0+uh) &= u\Lambda_1+(1-u)\Lambda_0+\tfrac{1}{4}u(1-u)(-\Lambda_2+\Lambda_1+\Lambda_0-\Lambda_{-1}) \\ &\quad +\tfrac{1}{2}u(\tfrac{1}{2}-u)(1-u)(\Lambda_2-3\Lambda_1+3\Lambda_0-\Lambda_{-1}) \\ &= \Lambda_0+\tfrac{1}{2}u(\Lambda_1-\Lambda_{-1})+\tfrac{1}{2}u^2(-\Lambda_2+4\Lambda_1-5\Lambda_0+2\Lambda_{-1}) \\ &\quad +\tfrac{1}{2}u^3(\Lambda_2-3\Lambda_1+3\Lambda_0-\Lambda_{-1})\end{aligned}$$

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Dr. R. Christain (L.A.S.L.) has kindly pointed out that the existence of mirror symmetry in the boundary conditions of the potential problem considered in MURA-LJL-8 does not necessarily imply the existence of mirror symmetry in the solution Ω. This situation is evidently the result of the appearance of both even and odd derivatives of Ω with respect to ξ in the governing differential equation. The suggestion made in the paragraph at the top of page 5 may therefore be inapplicable in cases of practical interest.

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and for which the derivative is then given by

$$h\Lambda'(t_0+uh) = \tfrac{1}{2}(\Lambda_1-\Lambda_{-1}) + u(-\Lambda_2+4\Lambda_1-5\Lambda_0+2\Lambda_{-1})$$
$$+\tfrac{3}{2}u^2(\Lambda_2-3\Lambda_1+3\Lambda_0-\Lambda_{-1})$$
$$= \tfrac{1}{2}u(\Lambda_2-\Lambda_0)+\tfrac{1}{2}(1-u)(\Lambda_1-\Lambda_{-1})$$
$$+\tfrac{3}{2}u(1-u)(-\Lambda_2+3\Lambda_1-3\Lambda_0+\Lambda_{-1}). \qquad \text{(i)}$$

This formula has the following properties:

$$\Lambda(0)=\Lambda_0 \qquad h\Lambda'(0)=\tfrac{1}{2}(\Lambda_1-\Lambda_{-1})$$
$$\Lambda(\tfrac{1}{2})=\tfrac{1}{16}(-\Lambda_2+9\Lambda_1+9\Lambda_0-\Lambda_{-1}) \qquad h\Lambda'(\tfrac{1}{2})=\tfrac{1}{8}(-\Lambda_2+11\Lambda_1-11\Lambda_0+\Lambda_{-1})$$
$$\Lambda(1)=\Lambda_1 \qquad h\Lambda'(1)=\tfrac{1}{2}(\Lambda_2-\Lambda_0),$$

it being observed that the expressions for $\Lambda'(0)$ and $\Lambda'(1)$ are consistent with the requirement of continuity for the derivative in addition to continuity of the function itself.

For direct interpolation of a function, rather than the formation of its interpolated derivative, a more conventional Bessel type of central-difference would seem preferable[6.]:

$$\Lambda(t_0+uh) = u\Lambda_1+(1-u)\Lambda_0+\tfrac{1}{4}u(1-u)(-\Lambda_2+\Lambda_1+\Lambda_0-\Lambda_{-1})$$
$$+\tfrac{1}{6}u(\tfrac{1}{2}-u)(1-u)(\Lambda_2-3\Lambda_1+3\Lambda_0-\Lambda_{-1})$$
$$=\Lambda_0+\tfrac{1}{6}u(-\Lambda_2+6\Lambda_1-3\Lambda_0-2\Lambda_{-1})$$
$$+\tfrac{1}{2}u^2(\Lambda_1-2\Lambda_0+\Lambda_{-1})+\tfrac{1}{6}u^3(\Lambda_2-3\Lambda_1+3\Lambda_0-\Lambda_{-1}). \qquad \text{(ii)}$$

In double interpolation, we would propose the use of (ii) twice to obtain an interpolated value of the function itself, and the use of (i) in conjunction with (ii) to obtain an estimate of a first derivative. The formulae may be applied in the ξ and η directions in either order.

VI. THE DYNAMICAL EQUATIONS

The dynamical equations conventionally might be written [7,3] in cylindrical coordinates as

$$\frac{d}{d\theta}\left(\frac{x'}{R}\right) = \frac{1+x}{R} + \frac{er_1}{p}\left[(1+x)\,B_z - y'B_\theta\right]$$

$$\frac{d}{d\theta}\left(\frac{y'}{R}\right) = \frac{er_1}{p}\left[x'\,B_\theta - (1+x)\,B_r\right],$$

where primes denote differentiation with respect to the independent variable θ and $R \equiv \sqrt{(1+x)^2 + x'^2 + y'^2}$, or, expressed as a set of first order differential equations,

$$x' = \frac{(1+x)p_x}{\sqrt{1-p_x^2-p_y^2}}$$

$$y' = \frac{(1+x)\,p_y}{\sqrt{1-p_x^2-p_y^2}}$$

$$p_x' = \sqrt{1-p_x^2-p_y^2} + \frac{er_1}{p}(1+x)\left[B_z - \frac{p_y}{\sqrt{1-p_x^2-p_y^2}}\cdot B_\theta\right]$$

$$p_y' = \frac{er_1}{p}(1+x)\left[\frac{p_x}{\sqrt{1-p_x^2-p_y^2}}\cdot B_\theta - B_r\right].$$

If these equations were employed, the quantities to be printed out would be x, y, x', y' or, perhaps better,[8] x, y, $x'/R = p_x$, $y'/R = p_y$.

For an alternative formulation, avoiding the inclusion of a logarithm routine in the computational program, Dr. Symon has suggested that the following coordinates might be employed:

$$S = \ln(r/r_1) = \ln(1+x),$$

$$T = z/r = y/(1+x), \text{ and}$$

θ (as before).

Introducing $P_s \equiv \dfrac{S'}{\sqrt{1+S'^2+(T'+TS')^2}}$ and $P_T \equiv \dfrac{T'+TS'}{\sqrt{1+S'^2+(T'+TS')^2}}$,

the differential equations which were expressed in cylindrical coordinates transform to

$$S' = \frac{P_s}{\sqrt{1-P_s^2-P_T^2}}$$

$$T' = \frac{P_T - TP_s}{\sqrt{1-P_s^2-P_T^2}}$$

$$P_s' = \sqrt{1-P_s^2-P_T^2} - \frac{e}{AP}\exp(K'S)\left\{\frac{\sqrt{(\frac{1}{W})^2+N^2}}{2\pi}\left[\left(\frac{\Omega}{\eta}\right)+\eta\frac{\partial}{\partial\eta}\left(\frac{\Omega}{\eta}\right)\right]\right.$$

$$\left. + \frac{P_T}{\sqrt{1-P_s^2-P_T^2}}\,\frac{N}{2\pi}\,\eta\frac{\partial}{\partial\xi}\left(\frac{\Omega}{\eta}\right)\right\}$$

$$P_T' = \frac{e}{AP}\exp(K'S)\left\{(K'-1)\left(\frac{\Omega}{\eta}\right)+\frac{1}{2\pi W}\frac{\partial}{\partial\xi}\left(\frac{\Omega}{\eta}\right)-\eta\frac{\partial}{\partial\eta}\left(\frac{\Omega}{\eta}\right)\right.$$

$$\left. + \frac{P_s}{\sqrt{1-P_s^2-P_T^2}}\,\frac{N}{2\pi}\,\frac{\partial}{\partial\xi}\left(\frac{\Omega}{\eta}\right)\right\}\eta \, ,$$

where, as before, $k' \equiv k+1$, the constant $\frac{e}{AP}$ is normally chosen to be

$$\frac{2\pi}{\sqrt{\left(\frac{1}{W}\right)^2+N^2}\,\left\langle\frac{\Omega}{\eta}\right\rangle_{\eta\to 0}}\, ,$$ and

$$\xi = \frac{(1/W)S - N\theta}{2\pi}\, , \qquad \eta = \frac{\sqrt{\left(\frac{1}{W}\right)^2+N^2}}{2\pi}\,T\, .$$

The quantities which would be most directly interpretable in light of past experience would be $\exp(S)-1$, P_s, $T\exp(S)$, P_T. These quantities, therefore, might well be the quantities to be printed out during a run. At the end of a run it would be useful to print the quantities S and T themselves, as well as P_S and P_T, to good accuracy in order to provide adequate starting values in case continuation of the run should prove desirable.

It should perhaps be mentioned at this time that room should be left in the code to permit the inclusion of "bumps".

VII. ALGORISMS AND RANGES OF PARAMETERS

A. Algorisms:

(i) Dr. Snyder has brought to our attention an algorism for $\Omega_{\xi\xi} + \Omega_{\eta\eta}$ which is particularly accurate if, as in the present case, the dependent variable satisfies or very-nearly satisfies Laplace's two-dimensional equation in Cartesian coordinates:

$$\Omega_{\xi\xi} + \Omega_{\eta\eta} \doteq \frac{4(\Omega_{10}+\Omega_{-10}+\Omega_{01}+\Omega_{0-1})+(\Omega_{11}+\Omega_{1-1}+\Omega_{-11}+\Omega_{-1-1})-20\Omega_{00}}{6h^2}$$

A check of this algorism for a harmonic function simulating the potential in the neighborhood of a pole-corner suggests that the improvement in accuracy would be desirable and would not be compromised at such points by the inaccuracies resulting from the use of simple algorisms for the remaining comparitively-small terms which enter in the differential equation for Ω.

(ii) To develop an algorism for the quantity

$\lim_{\eta \to 0} \left(\frac{\Omega}{\eta}\right) \equiv \Omega_\eta \Big|_0$, which is to be stored at those points for which $\eta = 0$, we suggest proceeding in the following manner. We take note of the fact that Ω is an odd function of η and write expansions in terms of derivatives at the point 0, 0. Then

$$\Omega_{01} = h\Omega_{\eta} \qquad\qquad + \frac{h^3}{6}\Omega_{\eta\eta\eta}$$
$$\Omega_{11} = h\Omega_{\eta} + h^2\Omega_{\xi\eta} + \frac{h^3}{6}(3\Omega_{\xi\xi\eta} + \Omega_{\eta\eta\eta})$$
$$\Omega_{-11} = h\Omega_{\eta} - h^2\Omega_{\xi\eta} + \frac{h^3}{6}(3\Omega_{\xi\xi\eta} + \Omega_{\eta\eta\eta}).$$

Moreover, differentiation of the differential equation for Ω and taking the limit $\eta \longrightarrow 0$ yields

$$0 = \frac{4\pi^2K^2}{(\frac{1}{w})^2 + N^2}\Omega_{\eta} + \frac{4\pi K(\frac{1}{w})}{(\frac{1}{w})^2 + N^2}\Omega_{\xi\eta} + \Omega_{\xi\xi\eta} + \Omega_{\eta\eta\eta} .$$

Solution of these simultaneous equations then gives the result:

$$\Omega_{\eta}\Big|_0 = \frac{\left[1 + \frac{2\pi K(\frac{1}{w})}{(\frac{1}{w})^2 + N^2}h\right]\Omega_{11} + 4\Omega_{01} + \left[1 - \frac{2\pi K(\frac{1}{w})}{(\frac{1}{w})^2 + N^2}h\right]\Omega_{-11}}{h\left[6 - \frac{4\pi^2K^2}{(\frac{1}{w})^2 + N^2}h^2\right]}$$

for use with the ξ, η net.

B. Ranges of Parameters:

(i) It is proposed that the following range of machine parameters be accommodated:

$$\frac{\text{[redacted]}}{4} \leqslant \frac{\text{[redacted]}}{\sqrt{(\frac{1}{w})^2 + N^2}} \leqslant 5000$$

$$|\eta| \leqslant 0.2$$

$$0.005 \leqslant h \leqslant 0.02$$

(ii) It is believed that the following limits can then be established:

$$\frac{\left|\frac{\Omega}{\eta}\right|}{\left\langle\frac{\Omega}{\eta}\right\rangle} < 3, \quad \text{so } \frac{1}{4}\frac{\left|\frac{\Omega}{\eta}\right|}{\left\langle\frac{\Omega}{\eta}\right\rangle} < 1;$$

$$\frac{\left|\eta\frac{\partial}{\partial\xi}\left(\frac{\Omega}{\eta}\right)\right|}{\left\langle\frac{\Omega}{\eta}\right\rangle} < 2 \quad \text{and} \quad \frac{\left|\eta\frac{\partial}{\partial\eta}\left(\frac{\Omega}{\eta}\right)\right|}{\left\langle\frac{\Omega}{\eta}\right\rangle} < 2.^{*}$$

(iii)

$$\left|\frac{P_T}{\sqrt{1-P_S^2-P_T^2}}\right|_{Max.} < \left|\frac{P_S}{\sqrt{1-P_S^2-P_T^2}}\right|_{Max.} < 2,$$

$$\frac{1}{\sqrt{1-P_S^2-P_T^2}} < 4.$$

(iv)

$$\left|\frac{NP_T}{2\pi}\right| < \left|\frac{NP_S}{2\pi}\right| < 6,$$ and very probably less than unity.

(v)

$$|K'S| < 2.$$

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* These estimates, based in part on some results obtained with the "FL" potential code, are slightly greater than estimated previously (18 Feb.) in an informal communication (L.J.L.).

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IX. REFERENCES

1. MURA Technical Meeting, Columbus, Ohio (6 - 7 January, 1956).
2. MURA Technical Meeting, Urbana, Illinois (20 Nov. 1955).
3. J. L. Powell, MURA Report MURA-JLP-3 (16 August 1955).
4. L. J. Laslett, MURA Report LJL(MURA)-5 (30 July 1955).
5. A. M. Sessler, MURA Notes (10 January 1956). (The last term within the brackets of line 3 on page 6 evidently should read $(V/u^2)\Psi_V$.) [#6.]
6. L. J. Laslett, MURA Notes (16 ... January 1956).
7. L. J. Laslett, Report LJL (MAC)-4 (March - April, 1954).
8. L. J. Laslett, letter to R. Christian, dtd. 13 December 1955.