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PROPOSAL TO REFINE THE DETERMINATION OF THE CHARACTERISTIC EXPONENT IN A HILL'S EQUATION PERTAINING TO THE MARK V FFAG

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1. Purpose:

The proposal has been made by the MURA group [1-3] to prepare tables giving the characteristic exponent in the solution of the Hill's equation

$$d^2Y/dt^2 + (A + B\cos 2t + C\cos 4t + D\cos 6t)Y = 0$$

for values of the coefficients more representative of those characterizing particle motion in typical Mark V FFAG structures than are listed in the coarse-mesh tables prepared previously.[4] Although this proposal may have become less pressing as a result of recent re-emphasis within the MURA program, the present notes are intended to outline a method for preparing the tables in the event that such tables should be felt to be of value. As a by-product, the derivative of the phase-function appearing in the phase-amplitude form[5] of the Floquet solutions would be available.

2. General Method:

Although variational methods have been devised[6,7,1,8] to relate the characteristic exponent to the coefficients of the differential equation, and one of these methods employed in constructing the original tables,[4] it is felt that direct integration of the differential equation may prove equally if not more satisfactory.

By virtue of the symmetry of the equation the numerical integration need be performed, for two sets of initial conditions, only over one half period[1] -- i.e., for the range from $t = 0$ up to and including $t = \pi/2$. The two solutions which are to be found in the interval $0, \pi/2$ may conveniently be those solutions (Y_1, Y_2) for which the following initial conditions are prescribed:

$$Y_1(0) = m \qquad Y_2(0) = 0$$
$$Y_1'(0) = 0 \qquad Y_2'(0) = n ,$$

the values m and n being introduced merely for possible convenience in scaling. The characteristic exponent σ is then given (for details see Appendix) most directly in terms of the solutions at $t = \pi/2$ by evaluating

$$\cos \sigma = \frac{1}{mn} \left[Y_1(\tfrac{\pi}{2})Y_2'(\tfrac{\pi}{2}) + Y_1'(\tfrac{\pi}{2})Y_2(\tfrac{\pi}{2}) \right] ,$$

and
$$\sigma/\pi = \frac{1}{\pi} \cos^{-1}\left\{ \left[Y_1(\tfrac{\pi}{2})Y_2'(\tfrac{\pi}{2}) + Y_1'(\tfrac{\pi}{2})Y_2(\tfrac{\pi}{2}) \right] /(mn) \right\}$$

for the stable case $[-1 \leqslant \cos\sigma \leqslant 1]$.

In order to retain information concerning the nature of the solutions to the differential equation, without printing out an excessive number of quantities it seems most appropriate to list quantities representing ϕ', the derivative of the phase function. This quantity, from which the amplitude function w and the phase function itself may be obtained, is perhaps also of particular physical interest because it enters directly in the expression for the displacement resulting from an angular scattering. If the values of the two solutions (Y_1, Y_2) found at the successive steps of the integration are retained in the computer memory until the integration has been carried through the complete range 0 to $\pi/2$, one may then evaluate

$$\frac{\sin\sigma}{\phi'(t_i)} = \frac{2}{m^2n^2}\left\{ Y_2(\tfrac{\pi}{2})Y_2'(\tfrac{\pi}{2})\left[Y_1(t_i)\right]^2 - Y_1(\tfrac{\pi}{2})Y_1'(\tfrac{\pi}{2})\left[Y_2(t_i)\right]^2\right\}.$$

This quantity may be directly interpreted as describing the square of the amplitude function, since $w^2\phi'$ is invariant.

It is accordingly suggested that the following quantities be printed out for each problem:

(1) As table-headings, the values of the coefficients A, B, C, D;

(2) $\frac{\sin\sigma}{\phi'(t_i)}$, $i = 0, 1, \ldots N$,
where N denotes the number of steps between $t = 0$ and $t = \pi/2$ (e.g., 7 values for N = 6);

(3) $\cos\sigma$

(4) σ/π , provided $-1 \leq \cos\sigma \leq 1$, the values then being selected such that $0 \leq \sigma/\pi \leq 1$. [If $|\cos\sigma| > 1$, it might be of interest to print out $-(1/\pi)\cosh^{-1}(\cos\sigma)$, the negative sign being a marker to denote instability and the quantity itself representing the average rate of increase for the logaritmm of the ascending eigenfunction.]

3. The Expected Range-of-Values of the Solutions:

For S.H.M. solutions, with $\omega \leq 1$, we would expect the magnitudes of Y_1, Y_2, Y_1', Y_2' not to exceed m, $(\pi/2)n$, m, and n, respectively. In typical A-G problems the magnitudes may be expected to be possibly some three times greater than for the S.H.M. case and limits of 2^5m, 2^5n, 2^5m, 2^5n would appear to be generally very liberal.

4. The Recommended Ranges of the Coefficients:

Based on the potential application of the proposed tables to problems concerned with Mark V FFAG synchrotron designs, we propose the following values for the coefficients:

D = 0

C: -0.5(0.1)0.5

B: 0(0.2)0.8(0.1)1.6(0.2)2.0

A: -0.5(0.1)-0.2(0.05)-0.15(0.03)0.15(0.05)0.2(0.1)0.5 ;

D: -0.05(0.05)0.10

C = 0

B: as before

A: as before .

This proposed schedule involves 4275 different combinations of values for the coefficients.

5. References:

(1) Letter from L. J. Laslett to J. N. Snyder, dtd. 13 August 1955.
(2) Letter from L. W. Jones and D. W. Kerst, August 1955.
(3) Letter from L. J. Laslett to J. N. Snyder, dtd. 15 August 1955.
(4) Laslett, Snyder, & Hutchinson, MURA Notes, 20 April 1955.
(5) E. Courant & H. Snyder, BNL Report EDC-15 (2 September 1954).
(6) L. J. Laslett, MURA Notes, dtd. 15 March 1955.
(7) Letter from L. J. Laslett to J. M. Snyder, dtd. 11 July 1955.
(8) Summarized in Appendix III of LJL(MURA)-5 (30 July 1955).
(9) L. J. Laslett, MURA Notes, dtd. 28 October 1954, esp. expression for $P(\tau)$ at top of p.15.

APPENDIX

Derivation of Relations Cited in Section 2

If we denote the matrix which carries the motion from 0 to t by $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and that which carries the motion from 0 to the mid-point $\pi/2$ by $\begin{pmatrix} a_s & b_s \\ c_s & d_s \end{pmatrix}$, then appeal to symmetry permits matrices covering other ranges to be written. We list the matrices of interest below.

0 to $\pi/2$: $M_1 = \begin{pmatrix} a_s & b_s \\ c_s & d_s \end{pmatrix}$

$\pi/2$ to π: $M_2 = \begin{pmatrix} d_s & b_s \\ c_s & a_s \end{pmatrix}$

0 to π: $M_2M_1 = \begin{pmatrix} a_sd_s + b_sc_s & 2b_sd_s \\ 2a_sc_s & a_sd_s + b_sc_s \end{pmatrix}$

0 to t (or π to $\pi+t$): $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

t to $\pi/2$: $M_3 = \begin{pmatrix} a_s d - b_s c & b_s a - a_s b \\ c_s d - d_s c & d_s a - c_s b \end{pmatrix}$

t to $\pi+t$:
$$M_4 = M\, M_2\, M_3$$
$$= \begin{pmatrix} a_s d_s + b_s c_s + 2(a_s c_s bd - b_s d_s ac) & 2(b_s d_s a^2 - a_s c_s b^2) \\ 2(a_s c_s d^2 - b_s d_s c^2) & a_s d_s + b_s c_s + 2(b_s d_s ac - a_s c_s bd) \end{pmatrix}.$$

The elements are given in terms of the fundamental solutions, for which

$$Y_1(0) = m \qquad Y_2(0) = 0$$
$$Y_1'(0) = 0 \qquad Y_2'(0) = n,$$

by

$$a = (1/m)Y_1(t) \qquad b = (1/n)Y_2(t)$$
$$c = (1/m)Y_1'(t) \qquad d = (1/n)Y_2'(t)$$

and

$$a_s = (1/m)Y_1(\tfrac{\pi}{2}) \qquad b_s = (1/n)Y_2(\tfrac{\pi}{2})$$
$$c_s = (1/m)Y_1'(\tfrac{\pi}{2}) \qquad d_s = (1/n)Y_2'(\tfrac{\pi}{2}).$$

We then have, firstly,

$$\cos\sigma = \frac{1}{2}\,\mathrm{Tr}(M_2 M_1)$$
$$= a_s d_s + b_s c_s$$
$$= \frac{1}{mn}\left[Y_1(\tfrac{\pi}{2})Y_2'(\tfrac{\pi}{2}) + Y_1'(\tfrac{\pi}{2})Y_2(\tfrac{\pi}{2})\right].$$

Secondly, the upper right hand element of M_4 may be indentified[9] with $\frac{\sin\sigma}{\phi'(t)}$, which we may term $[w(t)]^2$:

$$w^2 \equiv \frac{\sin\sigma}{\phi'} = 2(b_s d_s a^2 - a_s c_s b^2)$$
$$= \frac{2}{m^2 n^2}\left\{Y_2(\tfrac{\pi}{2})Y_2'(\tfrac{\pi}{2})\left[Y_1(t)\right]^2 - Y_1(\tfrac{\pi}{2})Y_1'(\tfrac{\pi}{2})\left[Y_2(t)\right]^2\right\}.$$