MURA Notes

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MURA # 94

Interpolation Formulas For a 2-Dimensional Net

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1. Motivation:

For digital computation of trajectories in a Mark V FFAG accelerator, it may prove convenient to work with fields or potentials stored on a net. It is believed that use of a net is particularly appropriate if complicated fields are to be employed or if speed of computation is to be emphasized.

With limited storage it may prove expedient to store, on a two-dimensional net, the magnetostatic potential and to obtain the three field-components therefrom. It is expected that the potential will approximately satisfy the two dimensional Cartesian Form of Laplace's equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0.$$

It is the purpose of the present notes to indicate an interpolation formula of possible utility for this purpose, it being noted that reasonable accuracy should be particularly sought for the interpolation formula if differentiation as well as interpolation is contemplated.

2. The Interpolation Formula:

Based on the net shown, but not employing the four extreme corner points, a central interpolation formula for two variables may be developed of the Bessel type:

-12	02	12	22
-11	01	11	21
-10	00	10	20
-1-1	0-1	1-1	2-1

$$\begin{aligned}
V(x_0 + uh,\ y_0 + vh) &= V_{00} + u\,(V_{10} - V_{00}) + v\,(V_{01} - V_{00}) \\
&+ \frac{u(u-1)}{4}\left(V_{20} - V_{10} - V_{00} + V_{-10}\right) + \frac{v(v-1)}{4}\left(V_{02} - V_{01} - V_{00} + V_{0-1}\right) \\
&+ uv\left(V_{11} - V_{01} - V_{10} + V_{00}\right)
\end{aligned}$$

$$+\frac{u(u-1)v}{4}\begin{bmatrix} V_{21}-V_{11}-V_{01}+V_{-11} \\ -V_{20}+V_{10}+V_{00}-V_{-10}\end{bmatrix}$$

$$+\frac{uv(v-1)}{4}\begin{bmatrix} V_{12}-V_{02}-V_{11}+V_{01} \\ -V_{10}+V_{00}+V_{1-1}-V_{0-1}\end{bmatrix}$$

$$+\frac{u(u-\frac{1}{2})(u-1)}{6}\left[V_{20}-3V_{10}+3V_{00}-V_{-10}\right]$$

$$+\frac{v(v-\frac{1}{2})(v-1)}{6}\left[V_{02}-3V_{01}+3V_{00}-V_{0-1}\right]$$

$$= V_{00}+\frac{u}{6}\left[-V_{20}+6V_{10}-3V_{00}-2V_{-10}\right]+\frac{v}{6}\left[-V_{02}+6V_{01}-3V_{00}-2V_{0-1}\right]$$

$$+\frac{uv}{4}\begin{bmatrix} -V_{21}+V_{20}-V_{12}+6V_{11}-4V_{10}-V_{1-1} \\ +V_{02}-4V_{01}+2V_{00}+V_{0-1}-V_{-11}+V_{-10}\end{bmatrix}$$

$$+\frac{u^2}{2}\left[V_{10}-2V_{00}+V_{-10}\right]+\frac{v^2}{2}\left[V_{01}-2V_{00}+V_{0-1}\right]$$

$$+\frac{u^3}{6}\left[V_{20}-3V_{10}+3V_{00}-V_{-10}\right]$$

$$+\frac{uv^2}{4}\begin{bmatrix} V_{12}-V_{02}-V_{11}+V_{01} \\ -V_{10}+V_{00}+V_{1-1}-V_{0-1}\end{bmatrix}+\frac{u^2v}{4}\begin{bmatrix} V_{21}-V_{11}-V_{01}+V_{-11} \\ -V_{20}+V_{10}+V_{00}-V_{-10}\end{bmatrix}$$

$$+\frac{v^3}{6}\left[V_{02}-3V_{01}+3V_{00}-V_{0-1}\right].$$

This development has the following properties:

(1) For any functional Form,

$$u = 0, \quad v = 0 \quad \rightarrow V_{00}$$
$$u = 1, \quad v = 0 \quad \rightarrow V_{10}$$
$$u = 0, \quad v = 1 \quad \rightarrow V_{01}$$
$$u = 1, \quad v = 1 \quad \rightarrow V_{11}$$

(2) The expression fits any third order polynominal in u, v :

$$a + b_1 u + b_2 v + c_1 u^2 + c_2 uv + c_3 v^2 + d_1 u^3 + d_2 u^2 v + d_3 uv^2 + d_4 v^3.$$

(3a) The sum of the coefficients of u^2 and v^2 will equal zero, corresponding to the (2-dimensional Cartesian) harmonic combination $u^2 - v^2$, <u>provided</u> $V_{00} = \frac{1}{4}[V_{10} + V_{01} + V_{0-1} + V_{-10}]$ as it will be through 3rd order for a harmonic function or if V_{00} is determined by a relaxation process employing the usual algorithm for Laplace's two-dimensional equation.

(3b) The cubic terms, moreover, are harmonic if the foregoing condition is satisfied for each of the four interior points $(V_{00}, V_{01}, V_{10}, \& V_{11})$ <u>and</u> if $V_{21} + V_{1-1} + V_{-10} + V_{02} = V_{20} + V_{0-1} + V_{-11} + V_{12}$. This latter condition is also automatically satisfied through third order for a harmonic function.

In the special case that $u = \frac{1}{2}$ $v = \frac{1}{2}$, this interpolation formula gives

$$V\left(\tfrac{1}{2}, \tfrac{1}{2}\right) = \frac{1}{32}\Big[-V_{21} - V_{20} - V_{12} + 10V_{11} + 10V_{10} - V_{1-1} - V_{02} + 10V_{01} + 10V_{00} - V_{0-1} - V_{-11} - V_{-10}\Big].$$

3. <u>The First Derivatives:</u>

From the interpolation formula of Section 2 one would infer that

$$h\frac{\partial V}{\partial x} \equiv \frac{\partial V}{\partial u} = \frac{1}{6}\left[-V_{20} + 6V_{10} - 3V_{00} - 2V_{-10}\right]$$
$$+ \left[V_{10} - 2V_{00} + V_{-10}\right]u$$
$$+ \frac{1}{4}\left[\begin{matrix} -V_{21} + V_{20} - V_{12} + 6V_{11} - 4V_{10} - V_{1-1} \\ + V_{02} - 4V_{01} + 2V_{00} + V_{0-1} - V_{-11} + V_{-10} \end{matrix}\right]v$$
$$+ \frac{1}{2}\left[V_{20} - 3V_{10} + 3V_{00} - V_{-10}\right]u^2$$

$$+\tfrac{1}{2}\begin{bmatrix} V_{21} - V_{11} - V_{01} + V_{-11} \\ -V_{20} + V_{10} + V_{00} - V_{-10} \end{bmatrix} uv$$

$$+\tfrac{1}{4}\begin{bmatrix} V_{12} - V_{02} - V_{11} + V_{01} \\ -V_{10} + V_{00} + V_{1-1} - V_{0-1} \end{bmatrix} v^2;$$

$$h\,\frac{\partial V}{\partial y} \equiv \frac{\partial V}{\partial v} = \frac{1}{6}\left[-V_{02} + 6V_{01} - 3V_{00} - 2V_{0-1}\right]$$

$$+\frac{1}{4}\begin{bmatrix} -V_{21} + V_{20} - V_{12} + 6V_{11} - 4V_{10} - V_{1-1} \\ +V_{02} - 4V_{01} + 2V_{00} + V_{0-1} - V_{-11} + V_{-10} \end{bmatrix} u$$

$$+\left[V_{01} - 2V_{00} + V_{0-1}\right] v$$

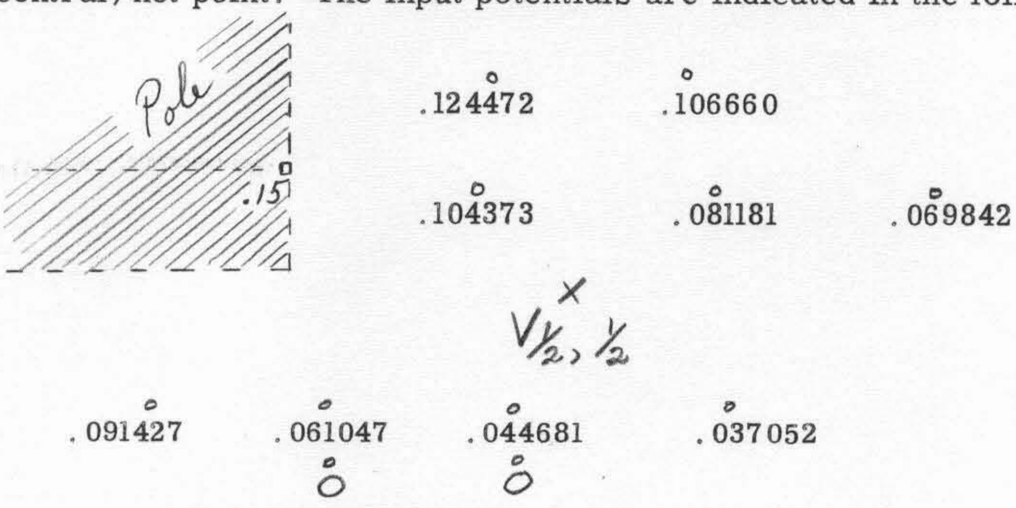
$$+\tfrac{1}{4}\begin{bmatrix} V_{21} - V_{11} - V_{01} + V_{-11} \\ -V_{20} + V_{10} + V_{00} - V_{-10} \end{bmatrix} u^2$$

$$+\tfrac{1}{2}\begin{bmatrix} V_{12} - V_{02} - V_{11} + V_{01} \\ -V_{10} + V_{00} + V_{1-1} - V_{0-1} \end{bmatrix} uv$$

$$+\tfrac{1}{2}\left[V_{02} - 3V_{01} + 3V_{00} - V_{0-1}\right] v^2.$$

4. Examples:

(a) An example for the application of the interpolation formula to the potential is afforded by the results of Illiac run FL0004, in which a 2-dimensional Laplace problem is solved for a rectangular pole. We take as given the potentials at every other net point and compare the interpolated and Illiac values for an interior (central) net point. The input potentials are indicated in the following diagram:



For the central point 1/2, 1/2, indicated by X, we find

Calc. by Interpolation,	$V_{\frac{1}{2}\,\frac{1}{2}}$ =	.072918*
" " Illiac		.073347
Error (difference)		.000429
Fractional Error		.00585

*If one just averaged the 4 central points, one would obtain $.07282\,0_5$, with an error $.00052 6_5$ and a fractional error .00718.

(b) A second example considers the relatively smooth 2-dimensional harmonic function V = sin x Sink y on a net including the values 0.882, 1.008, 1.134, and 1.260 for x and the same set of values for y.

One finds (including in the coefficients 2 figures which are not significant):

$$V = 1.00441817 + .07981752\,u + .16542506\,v$$
$$-.00796157\,u^2 + .01321427\,uv + .00798319\,v^2$$
$$-.00019001\,u^3 -.00142752\,u^2v$$
$$+ .00061894\,uv^2 + .0046094\,v^3 \quad ;$$

then, with h = 0.126,

$$\frac{\partial V}{\partial x} = \begin{bmatrix} .07981752 & -.01592313\,u + & .01321427\,v \\ -.00057003\,u^2 & -.00285504\,uv + & .00061894\,v^2 \end{bmatrix} \cdot \frac{1}{0.126},$$

$$\frac{\partial V}{\partial y} = \begin{bmatrix} .16542506 + & .01321427\,u + & .01596637\,v \\ -.00142752\,u^2 + & .00123788\,uv + & .00138282\,v^2 \end{bmatrix} \cdot \frac{1}{0.126}.$$

For the potentials one finds:

$V_{1/2,\,1/2}$	1.130281	by interp.	$V_{3/4,\,3/4}$	1.195568_5	by interp.
	1.130289	by exact. fcn.		1.195575_6	by exact fcn
	.000008	error		$.000007_1$	error
	.000007	frac. error		.000006	frac. error

Similarly, for the fields

u	v	V_x interp	V_x exact	Error	Frac. Error	V_y interp.	V_y exact	Error	Frac. Error
0	0	.633472	.633639	.00167	.000264	1.312897	1.313079	.000182	.000139
1/2	1/2	.617155	.617181	.000026	.000042	1.431061	1.431041	.000020	.000014
3/4	3/4	.604821	.604751	.000070	.000116	1.491918	1.491868	.000050	.000034

(c) A horrible example can be contrived, as might be expected for a 3rd order formula, by taking the potential to be a pure 4th order harmonic--e.g.:

$$V = x^4 + 4x^3y - 6x^2y^2 - 4xy^3 + y^4.$$

For this function our interpolation formula would suggest

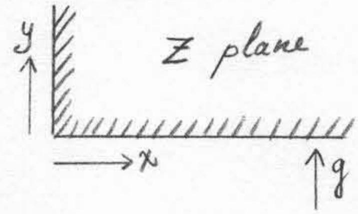
$$V = -2u - 2v + u^2 + 6uv + v^2$$
$$+ 2u^3 - 12uv^2 + 2v^3,$$

consistent with $V_{00} = 0,\ V_{10} = 1,\ V_{01} = 1,\ V_{11} = -4.$

For $u = \frac{1}{2},\ v = \frac{1}{2}$ we find however -1 while the true value is -.25;

for $u = \frac{3}{4},\ v = \frac{3}{4},$ we similarly find -1.875 while the true value is -1.265625.

(d) A somewhat realistic analytic example, similar to example (a) can be contrived by considering the two-dimensional potential in the neighborhood of a corner. (Another similar example might be based on the slotted plane, illustrated by Fig. 4.23 of Smythe's text.)



For a corner, a conformal transformation leads to the result

$$Z = -\frac{g}{\pi}\left[2\sqrt{1-Z_1} + \ln \frac{\sqrt{1-Z_1}-1}{\sqrt{1-Z_1}+1}\right], \text{ with}$$

$$W = -\frac{V_0}{\pi}\left(\ln Z_1 - i\pi\right).$$

By expansion of this result one is led, following a suggestion of Dr. Sessler, to consider that there would be interest in considering the following simpler, but similar potential;

$$V = \frac{V_0}{\pi}\left[\frac{3\pi}{2}\left(\frac{r}{g}\right)^{2/3} \sin\frac{2\phi}{3}\right],$$

from which

$$E_x \equiv -\frac{\partial V}{\partial x} = \frac{V_0}{g}\,\frac{\sin \phi/3}{\left[\frac{3\pi}{2}\left(\frac{r}{g}\right)\right]^{1/3}} \quad \text{and}$$

$$E_y \equiv -\frac{\partial V}{\partial y} = \frac{V_0}{g}\,\frac{\cos \phi/3}{\left[\frac{3\pi}{2}\left(\frac{r}{g}\right)\right]^{1/3}}.$$

We consider a grid of basis cell size $\frac{g}{5\pi}$, for which the potentials are believed to be as illustrated:

$\frac{g}{5\pi}, 180^\circ$
.388101

0

$\sqrt{5}\frac{g}{5\pi}, 2.677945^r$ — .748748

$\sqrt{2}\frac{g}{5\pi}, \frac{3\pi}{4}$ — .564622

$\frac{g}{5\pi}, 90^\circ$ — .388101

$\sqrt{2}\frac{g}{5\pi}, 45^\circ$ — .282311

$2\sqrt{2}\frac{g}{5\pi}, \frac{3\pi}{4}$ — .896281

$\sqrt{5}\frac{g}{5\pi}, 2.034444^r$ — .748748

$\frac{2g}{5\pi}, 90^\circ$ — .616072

$\sqrt{5}\frac{g}{5\pi}, 1.107149^r$ — .515637

$\sqrt{10}\frac{g}{5\pi}, 1.892547^r$ — .919733

$\frac{3g}{5\pi}, 90^\circ$ — .807283

Interior points of interest may include

1/2, 1/2	:	$r = \sqrt{2.5}\ \dfrac{g}{5\pi}$	$\phi = 1.89254\dot{7}^{r}$
1, 1/2	:	$1.5\ \dfrac{g}{5\pi}$	90°
1/2, 1	:	$\sqrt{1.25}\ \dfrac{g}{5\pi}$	$2.03444\dot{4}^{r}$
3/4, 3/4	:	$\sqrt{1.625}\ \dfrac{g}{5\pi}$	$1.76819\dot{2}^{r}$

Our interpolation formulas then read:

$$\frac{V}{V_0/\pi} = .748748 \quad -.143002\,u \quad -.181013\,v$$
$$+.007429\,u^2 \quad -.003816\,uv \quad -.006571\,v^2$$
$$+.002897\,u^3 \qquad +.007810\,u^2 v$$
$$-.047839\,uv^2 + .003458\,v^3 \quad , \qquad \text{with}$$

$$\frac{E_x}{V_0/g} = 5(.143002 - .014858\,u + .003816\,v$$
$$-.008691\,u^2 - .015620\,uv + .047839\,v^2),$$

$$\frac{E_y}{V_0/g} = 5(.181013 + .003816\,u + .013142\,v$$
$$-.007810\,u^2 + .095678\,uv - .010374\,v^2).$$

For the potentials we find:

u	v	$\frac{\pi}{V_0}$ V interp.	$\frac{\pi}{V_0}$ Vexact	Error	Frac. Error
1/2	1/2	.581792	.579396	.002396	.00412
1	1/2	.514392	.508556	.005836	.01135
1/2	1	.471465	.471682	.000217	.00046
3/4	3/4	.489867	.486899	.002968	.00606

Likewise for the fields:

u	v	$\frac{g}{V_0} E_x$ interp.	exact	error	frac. error	$\frac{g}{V_0} E_y$ interp.	exact	error	frac. error
1/2	1/2	.716815	.756304	.039	.052	1.044328	1.035444	.0088	.0086
1	1/2	.627555	.652478	.025	.038	1.144178	1.130125	$.014\dot{1}$	.0124
1/2	1	.886225	.902927	.017	.018	1.157878	1.120809	$.037_1$	$.033_1$
3/4	3/4	.739775	.765801	.026	.034	1.186610	1.145239	$.041_4$	$.036_1$

5. Further Orientation by 1-dimensional Examples:

In view of the possible difficulties suggested by the last example of the preceding section, it was proposed by Dr. Sessler that useful orientation could be obtained by considering analogous one-dimensional problems.

(a) A problem similar to that of § 5d is provided by taking

$$V = (0.3x)^{2/3}, \quad -E_x = \frac{0.2}{(0.3x)^{\cancel{1/3}}} \quad \text{and}$$

employing the interval indicated:

Here

x	0	1	2	3
v	0	.4481405	.7113787	.9321698

3rd order interpolation, based on values at x = 0, 1, 2, and 3 gives

$$V = .4481405 + .3319468u - .0924512u^2 + .0237425u^3 ,$$

$$-E_x = .3319468 - .1849023u + .0712276u^2.$$

We find

x	u	V interp	V true	Error	Frac. Error	-E interp	-E true	Error	Frac Error
1 1/3	1/3	.5493964	.5428835	.0065129	.0120	.2729508	.2714418	.0015090	.0056
1 1/2	1/2	.5939689	.5872302	.0067387	.0115	.2573026	.2609912	.0036886	.0141
1 2/3	2/3	.6353838	.6299605	.0054233	.0086	.2403353	.2519842	.0116489	.0462

The errors are of the order 1.2% in V and 4.6% in E, similar to the 2-dimensional example.

(b) The example of sub-section (a) was repeated, using higher-order interpolation.

$$V = A + Bu + Cu^2 + Du^3 + Eu^4, \text{ where}$$

$$A = V_0$$

$$B = \frac{1}{12}\left[-3V_{-1} - 10V_0 + 18V_1 - 6V_2 + V_3\right]$$

$$C = \frac{1}{24}\left[11V_{-1} - 20V_0 + 6V_1 + 4V_2 - V_3\right]$$

$$D = \frac{1}{12}\left[-3V_{-1} + 10V_0 - 12V_1 + 6V_2 - V_3\right]$$

$$E = \frac{1}{24}\left[V_{-1} - 4V_0 + 6V_1 - 4V_2 + V_3\right].$$

With

x	0	1	2	3	4
V	$V_{-1} = 0$	.4481405	.7113787	.9321698	1.1292432

$$V = .4481405 + .3216363u - .0872959u^2 + .0340530u^3 - .0051552u^4$$

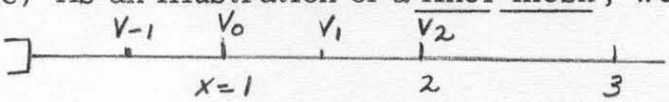
$$-E = .3216363 - .1745918u + .1021590u^2 - .0206210u^3 .$$

We then find

x	u	Vinterp	Vtrue	Error	Frac. Error	-Einterp	-Etrue	Error	Frac. Error
1 1/3	1/3	.5468506	.5428835	.0039671	.00731	.2740263	.2714418	.0025845	.00952
1 1/2	1/2	.5910691	.5872302	.0038389	.00654	.2573025	.2609912	.0036887	.01413
1 2/3	2/3	.6328380	.6299605	.0028775	.00457	.2445359	.2519842	.0074483	.02866

It is seen that the errors are <u>not</u> markedly reduced, being just somewhat under 1% for V and about 3% for E.

(c) As an illustration of a <u>finer mesh</u>, we take



Now

$V_{-1} = .2823108$ $V_o = .4481405$ $V_1 = .5872302$ $V_2 = .7113787$

$$V = .4481405 + .1504932\,u - .0133700\,u^2 + .0019665\,u^3$$

$$-E_x = 2(.1504932 - .0267400\,u + .0058994\,u^2)$$

At $u = 2/3$ (x = 1 1/3)

Vinterp	Vtrue	Error	Frac. Error	-E interp	-E true	Error	Frac. Error
.5431097	.5428835	.0002262	.000417	.2705770	.2714418	.0008648	.003186

The error is seen to have been reduced considerably in this example, being about 0.04% in V and 0.3% in E.

(d) It may be also of interest to examine the accuracy resulting from storing the <u>field</u> directly, then interpolating without differentiation.

With
$$\left.\begin{aligned} E_{-1} &= .3764144 \\ E_o &= .2987603 \\ E_1 &= .2609912 \\ E_2 &= .2371262 \end{aligned}\right\}$$ using the mesh of sub-section (c),

third order, Newtonian 2nd order, and linear interpolation give, respectively,

$$E \doteq .2987603 - .05338145\,u + .019925\,u^2 - .00433015\,u^3 ,$$

$$E \doteq .2987603 - .0447211_5\,u + .0069520_5\,u^2 ,$$

$$E \doteq .2987603 - .0377691\,u .$$

We then compare the interpoluted fields at $x = 1/3$, $1/2$, and $2/3$ with the true fields

$$E_{1/3} = .2837967, \quad E_{1/2} = .2773445, \quad E_{2/3} = .2714418$$

and find

For the 3rd order and Newtonian 2nd order cases, a fractional error of about 0.3% (similar to that of subsection (c) above) and, for the linear case, about 1%.

In summary, the grid size seems all-important. Extensive interpolation formulas, by contrast, are of little advantage. Direct interpolation on the fields themselves appears to be of little help, perhaps because the fields become singular.

6. Example to Illustrate a Suggestion:

In taking stock of our position with respect to our ultimate application, it seems likely that one would do best to store the potential, or some other single quantity, rather than the field components themselves. This scheme would economize on storage. An adequately fine net is most important. The size of the net, and the number of figures stored per net point, should be adjusted together within the limitations of the computer memory to optimize the overall accuracy. It appears that improvement in the accuracy of interpolation, and possibly adequately accurate results with a feasible net, may be obtained by storing something like the potential divided by the axial distance from the median plane.

To illustrate this suggestion, we consider a case exhibiting the type of singularity considered earlier, feeling that cases with smoother pole-contours will be no worse.

$$V/A = \left[\left(1 + \frac{x}{g}\right)^{2/3} - \left(1 - \frac{x}{g}\right)^{2/3}\right] \text{ where } A = V_0\, 2^{-2/3}$$

We introduce

$$\Lambda/A \equiv \frac{g\, V/A}{x} = \frac{\left(1+\frac{x}{g}\right)^{2/3} - \left(1-\frac{x}{g}\right)^{2/3}}{x/g}$$

Then

$$\frac{g}{A}(-E_x) \equiv g\,\frac{\partial V/A}{\partial x} = \frac{2}{3}\left[\left(1+\frac{x}{g}\right)^{-1/3} + \left(1-\frac{x}{g}\right)^{-1/3}\right] \qquad \text{analytically}$$

and also equals $\dfrac{\Lambda}{A} + x\dfrac{\partial}{\partial x}\left(\dfrac{\Lambda}{A}\right)$.

We consider net-points such that

$$\frac{x}{g} = 0, \frac{1}{18}, \frac{2}{18}, \cdots \frac{17}{18}, 1;$$

i.e., the net-size h is related to the semi-gap such that $\frac{h}{g} = \frac{1}{18}$.

For this example the following figures are representative:

x/g	Λ/A
0	1.3333
1/18	1.3336
2/18 = 1/9	1.3346
3/18 = 1/6	1.3361
7/18	1.3492
8/18 = 4/9	1.3545
9/18 = 1/2	1.3608
10/18 = 5/9	1.3683
14/18 = 7/9	1.4151
15/18 = 5/6	1.4341
16/18 = 8/9	1.4590
17/18	1.4953

x/g	$\frac{g}{A}(-E_x)$
1/18	1.33425
1.5/18 = 3/36	1.33540
2/18 = 1/9	1.33702
8/18 = 4/9	1.40072
8.5/18 = 17/36	1.41097
9/18 = 1/2	1.42233
15/18 = 5/6	1.75612
15.5/18 = 31/36	1.82930
16/18 = 8/9	1.92603

A Bessel interpolation formula may be applied of the form

$$y = y_0 + \left(\frac{-4y_{-1} - 6y_0 + 12y_1 - 2y_2}{12}\right)u + \left(\frac{y_{-1} - 2y_0 + y_1}{2}\right)u^2 + \left(\frac{y_{-1} + 3y_0 - 3y_1 + y_2}{6}\right)u^3$$

$$x\frac{dy}{dx} = \left(\frac{x}{g}\right)\Big[\left(-6y_{-1} - 9y_0 + 18y_1 - 3y_2\right) + \left(18y_{-1} - 36y_0 + 18y_1\right)u + \left(-9y_{-1} + 27y_0 - 27y_1 + 9y_2\right)u^2\Big]$$

In the first interval, taking $y \equiv \Lambda/A$,

$$y = 1.3336 + .000683u + .00035u^2 - .000033u^3$$

$$x\frac{dy}{dx} = \left(\frac{x}{g}\right)\left[.01230 + .01260u - .00180u^2\right],$$

leading to E_x = 1.33428, 1.33554, and 1.33717 at the points considered.

Similarly in the center interval,

$$y = 1.3545 + .005792\,u + .0005\,u^2 + .000025\,u^3$$

$$x\frac{dy}{dx} = \left(\frac{x}{g}\right) \left[.10395 + .018\,u \quad .00135\,u^2\right]$$

leading to values E_x = 1.40070, 1.41102, and 1.42245 or 1.42247.

In the last interval, nearest the pole,

$$y = 1.4341 + .021033\,u + .00295\,u^2 + .0009166\,u^3$$

$$x\frac{dy}{dx} = \left(\frac{x}{g}\right) \left[.3786 + .1062\,u + .0495\,u^2\right]$$

For the three points of this interval which were considered,

x/g	u	$-E_x$interp	$-E_x$true	Error	Frac. Error
5/6	0	1.74960	1.75612	.00652	.00371
31/36	1/2	1.82787	1.82930	.00143	.00078
8/9	1	1.93393	1.92603	$.0079_0$	.0041

We may then regard the frac. error as typically 0.4%.

Lower-order interpolation (second or first order) did not appear to give as satisfactory results, errors of 1% or 3 1/2% being respectively obtained.

7. Application:

Application of this suggestion indicated in Section 6 to the computation of orbits in a MK, V spirally-ridged FFAG synchrotron would appear to involve the following storage scheme in some r, z plane (quantities out of this plane would be obtainable by virtue of the scaling properties of the structure -- see Sessler's notes of 10 January):

Store a quantity roughly proportional to V/Z, the exact character to be determined by the scaling aspects of the problem. Scale the magnitude of this quantity carefully and store as 1/3-words on a mesh ($z > 0$) 18 x 18 in area. The no. of memory points required for this storage is, then, $\frac{19 \times 109}{3}$ = 691 full words. Third order interpolation and differentiation is imagined (requiring perhaps a few extra net points at the boundaries for perhaps a total of 703 words) and it is hoped that the field-error would then rarely exceed 0.4 percent.