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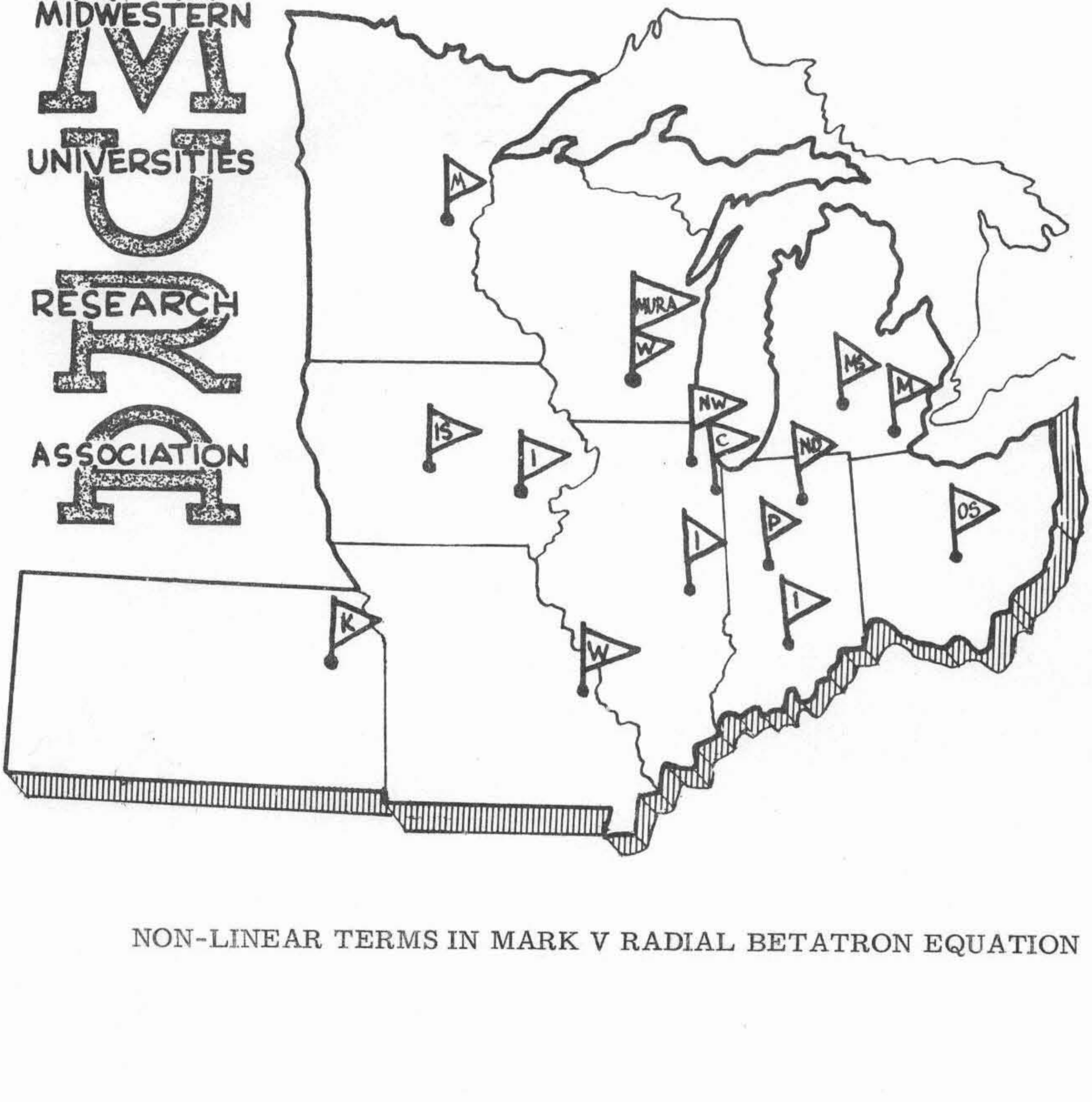

NON-LINEAR TERMS IN MARK V RADIAL BETATRON EQUATION

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NON-LINEAR TERMS IN MARK V RADIAL BETATRON EQUATION

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Starting from the rigorous equations of motion

$$x' = (1+x)\,p \Big/ \sqrt{1-p^2}$$

$$p' = \sqrt{1-p^2} - (1+x)^{k+1}\left[1 + f \sin\left\{\frac{1}{\omega}\ln(1+x) - N\phi^*\right\}\right];$$

$$x' = \frac{dx}{d\phi}, \text{ etc.}$$

for the motion of a particle of mechanical momentum $p = eB_0(r_1/r_0)^k r_1$ in the median plane of the field $B = B_0(r/r_0)^k\left[1 + f\sin\left\{\frac{1}{\omega}\ln(r/r_0) - N\phi\right\}\right]$, where $r \equiv r_1(1+x)$ and $N\phi^* \equiv N\phi + \frac{1}{\omega}\ln(r_1/r_0)$, one may obtain an expanded equation of motion (see DLJ/1)

$$x'' + (k+1)x = f\sin\xi - \left[\frac{f}{\omega}\cos\xi - (k+2)f\sin\xi\right]x$$

$$+\left[(k+1)(k+2) + f\left\{\left[(k+1)(k+2) - \frac{1}{\omega^2}\right]\sin\xi - \frac{2k+3}{\omega}\cos\xi\right\}\right]\frac{x^2}{2} +$$

$$+\left[1 + 2f\sin\xi\right]\frac{x'^2}{2} + \left[k(k+1)(k+2) + f\left\{\left[k(k+1)(k+2) - \right.\right.\right.$$

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$$-\frac{2(k+1)}{w^2}\Big]\sin\xi - \Big[\frac{3k^2+6k+2}{w} - \frac{1}{w^3}\Big]\cos\xi\Big\}\Big]\frac{x^3}{6} + \Big[-(3k+4)$$

$$+3f\Big\{(k+3)\sin\xi - \frac{1}{w}\cos\xi\Big\}\Big]\frac{x x'^2}{2} + O\Big(\frac{x^4}{4!\,w^2},\ k^4x^4,\ x'^4\Big)$$

$$= f\sin\xi - A(\xi)x + B(\xi)\frac{x^2}{2} + C(\xi)\frac{x'^2}{2} + D(\xi)\frac{x^3}{6} + E(\xi)\frac{x x'^2}{2}$$

+ small terms

here $\xi \equiv N\phi$*. The closed orbit is given by a solution x_s having the property $x_s(\phi + \frac{2\pi}{N}) = x_s(\phi)$. This solution has been obtained and evaluated to an accuracy of better than 1% of its dominant terms in DLJ/1. To study radial betatron equations one sets $x = x_s + \rho$, substitutes into the differential equation above, and collects terms by powers of ρ. Terms independent of ρ cancel since x_s is a solution. The remaining terms are:

$$\rho'' - [C(\xi) + E(\xi)x_s(\xi) + \cdots]\,x_s'(\xi)\rho' + [(k+1) - A(\xi) - B(\xi)x_s(\xi)$$

$$-\frac{1}{2}D(\xi)x_s^2(\xi) - \frac{1}{2}E(\xi)x_s'^2(\xi) + \cdots]\rho = \frac{1}{2}[B(\xi) + D(\xi)x_s(\xi) + \cdots]\rho^2$$

$$+[E(\xi)x_s'(\xi) + \cdots]\rho\rho' + \frac{1}{2}[C(\xi) + E(\xi)x_s(\xi) + \cdots]\rho'^2$$

$$+\frac{1}{6}[D(\xi) + \cdots]\rho^3 + \frac{1}{2}[E(\xi) + \cdots]\rho\rho'^2 + \cdots.$$

Terms not written correspond to terms not explicitly written in the original differential equation. The term linear in ρ' may be removed by the change of dependent variable

$$\rho = v \exp \frac{1}{2} \int^{\phi^*} \left[C + E x_s + \cdots \right] x_s' \, d\phi^* \equiv v F.$$

One then notes that $F'/F = \frac{1}{2} (C + E x_s + \cdots) x_s'$. The equation for v then becomes

$$v'' + \left[k + 1 + A - B x_s - \tfrac{1}{2} D x_s^2 - \tfrac{1}{2} E x_s'^2 - \tfrac{1}{4} \left\{ (C + E x_s) x_s' \right\}^2 \right.$$

$$\left. - \tfrac{1}{2} \frac{d}{d\phi^*} \left\{ (C + E x_s) x_s' \right\} \right] v = F \left[\left\{ \frac{B + D x_s}{2} + E \left(\frac{C + E x_s}{2} \right) x_s'^2 \right. \right.$$

$$\left. + \left(\frac{C + E x_s}{2} \right)^3 x_s'^2 \right\} v^2 + \left\{ E x_s' + 2 \left(\frac{C + E x_s}{2} \right)^2 x_s' \right\} v v' + \left. \left\{ \frac{C + E x_s}{2} \right\} v'^2 \right]$$

$$+ F^2 \left[\left\{ \frac{D}{6} + \frac{E}{2} \left(\frac{C + E x_s}{2} \right)^2 x_s'^2 \right\} v^3 + \left\{ E \left(\frac{C + E x_s}{2} \right) x_s' \right\} v^2 v' + \left\{ \frac{E}{2} \right\} v v'^2 \right] + \cdots$$

It is now possible to use the results from the smooth approximation, for reasonable σ_x and σ_y to supply inter-relationships among the constants $\frac{1}{w}$, k, f, and N so as to examine the relative orders of magnitude of the various non-linear terms. The values used for this purpose are $k + 1 \sim N^2/16$; $f \sim 1/4$; $1/w \sim N^2$, and we have set $3/4 \simeq 1$ in several places since only order of magnitudes are being evaluated.

$$B \sim N^4\left[-\tfrac{1}{4}\sin\xi - \tfrac{1}{32}\cos\xi + \cdots\right]$$

$$C \sim 1 + \sin\xi$$

$$D \sim N^6\left[\tfrac{1}{4}\cos\xi - \tfrac{1}{16}\sin\xi + \cdots\right]$$

$$E \sim N^2\left[-\cos\xi - \tfrac{1}{4} + \tfrac{1}{16}\sin\xi + \cdots\right]$$

$$x_s \sim \tfrac{1}{N^2}\left[-\tfrac{1}{4}\sin\xi - \tfrac{1}{32} + \cdots\right]$$

$$x_s' \sim \tfrac{1}{N}\left[-\tfrac{1}{4}\cos\xi + \cdots\right]$$

It will be shown in an appendix that $F \sim 1 + \frac{1}{8N^2}\sin\xi - \frac{1}{32N^2}\cos 2\xi$. The coefficient of v^2 is then $-\frac{N^4}{8}\left[\sin\xi + \frac{1}{8}\cos\xi + \frac{1}{8}\sin 3\xi\right] + O(1) + O(\frac{1}{N^2})$, whence only the term $\frac{1}{2}(B + Dx_s)$ need be kept. The coefficient of $v\,v'$ is $\sim \frac{N}{8}(1 + \cos 2\xi + \frac{1}{2}\cos\xi) + O(\frac{1}{N})$, whence only (Ex_s') is important. The coefficient of v^3 is $\frac{N^6}{24}\left[\frac{1}{4}\cos\xi - \frac{1}{16}\sin\xi\right] + \mathcal{O}(1)$, so one keeps only $D/6$. The coefficient of v^2v' is $\mathcal{O}(N/8)$, which is the same as the coefficient of vv'; we will only be interested in $v << 1$, this entire term may be dropped. Finally, the coefficient of vv'^2 is $\mathcal{O}(N^2/2)$. With the various neglects,

$$v'' + [\ ]v \simeq \frac{B + Dx_s}{2}v^2 + Ex_s' vv' + \frac{C + Ex_s}{2}v'^2 + \frac{D}{6}v^3 + \frac{E}{2}vv'^2.$$

Now if $|v|$ is of order p, $|v'|$ is not larger in order of magnitude than $N\,p$. The five right hand terms are then of relative orders

$$\frac{N^4}{8}\cdot p^2 + \frac{N^2}{8}p^2 + \frac{N^2}{2}p^2 + \frac{N^6}{24}p^3 + \frac{N^4}{2}p^3.$$

The second, third, and fifth terms may thus be neglected, so that finally $v'' + [\ ]v \simeq (B + Dx_s)\frac{v^2}{2} + Dv^3$. Inserting the actual quantities, we have

$$v'' + [\ ]v \simeq -\left[\frac{f}{\omega^2}\sin\xi + \frac{(2k+2)f}{\omega}\cos\xi - \frac{f^2}{2\omega^3 D_1}\sin 2\xi\right]\frac{v^2}{2}$$
$$+\left[\frac{f}{\omega^3}\cos\xi - \frac{3(k+1)f}{\omega^2}\sin\xi\right]\frac{v^3}{6}; \quad D_1 \equiv N^2 - (k+1)$$

Neglected terms are less than 1% of those kept in each coefficient, for $N > 25$. Some re-examination would be required for small N models.

It is conventional to replace ξ by a new independent variable $\xi = 2t - \delta_1$, where $\tan\delta_1 = Bs/Bc$ from DLJ/1.

If this is done, and if the small out- of - phase component of the C term in the coefficient of v is neglected, one may write the equation as

$$\frac{d^2 v}{dt^2} + \left[A + B\cos 2t + C\cos 4t\right] v \simeq$$

$$-\frac{4}{N^2}\left[\frac{f}{\omega^2}\sin(2t-\delta_1) + \frac{(2k+3)f}{\omega}\cos(2t-\delta_1)\right.$$

$$\left.-\frac{f^2}{2\omega^3 D_1}\sin(4t-2\delta_1)\right] v^{3/2} + \frac{4}{N^2}\left[\frac{f}{\omega^3}\cos(2t-\delta_1)\right.$$

$$\left.-\frac{3(k+1)f}{\omega^2}\sin(2t-\delta_1)\right]\frac{v^3}{L}$$

Here A, B, and C are to be taken from DLJ/1 or from Laslett's corrected results.

Appendix

An attempt to evaluate approximately the quantity F = $\exp\frac{1}{2}\int^{\phi^*}(c+Ex_s+\cdots)\,X_s'\,d\phi^*$ will at first sight lead the reader to conclude that this quantity will grow exponentially with θ because of terms in the integrand, at any stage of approximation, having average value different from zero.

However, it can be shown, by several different lines of reasoning, that, if the full rigorous expression for the integrand could be evaluated, such terms would cancel precisely. In evaluating an approximate analytical expression for F it is therefore appropriate to ignore terms leading to a non-periodic part in F. In this appendix we will illustrate that the largest terms kept in the approximate analysis of DLJ/1 and the present paper do indeed cancel, and we will show how to determine the largest oscillating parts.

$$(C + Ex_s) \cong (1 + 3f \sin \xi) - \left[-(3k+4) - \frac{3f}{\omega}\cos\xi\right]$$

$$\left[\alpha + \beta \sin \xi + \gamma \cos \xi + \cdots\right]$$

$$\cong \text{const.} + \left[3f + (3k+4)\beta - 3f^2(k+3)\right]\sin\xi$$

$$+ \left[(3k+4)\gamma + \frac{3f}{\omega}\alpha\right]\cos\xi + \cdots$$

$$x_s' \cong -N\gamma \sin\xi + N\beta\cos\xi$$

$$\overline{(C + Ex_s)x_s'} \cong \frac{N}{2}\Big[-3f\gamma - (3k+4)\gamma\beta + 3f\alpha(k+3)\gamma + (3k+4)\gamma\beta$$

$$+ \frac{3f}{\omega}\alpha\beta + \cdots\Big]$$

$$\cong \frac{N}{2}\left[-\frac{3f^4}{2\omega D_1^2} + \frac{3f^6}{4\omega D_1^3}\left(\frac{k+2}{k+1}\right)^2(k+3) + \cdots\right],$$

which is zero except for the small term of order f^6; this term would be cancelled by higher order terms not retained here. The important oscillating terms in F can be written down:

$$(C+E x_s)\,x_s' \simeq N\left[\frac{f}{D_1}\cos\xi + \frac{3}{2}\frac{f^2}{D_1}\sin 2\xi - \frac{f^3}{2w D_1{}^2}\left(\frac{k+2}{k+1}\right)\sin\xi + \cdots\right]$$

$$\sim N\left[\frac{1}{4N^2}\cos\xi\ \frac{3}{32N^2}\sin 2\xi - \frac{1}{128N^2}\sin\xi + \cdots\right]$$

(order of magnitude)

$$F = \exp\theta'\left[\frac{f}{D_1}\sin\xi - \frac{3}{4}\frac{f^2}{D_1}\cos 2\xi\ \frac{f^3}{2w D_1{}^2}\left(\frac{k+2}{k+1}\right)\cos\xi + \cdots\right]$$

$$\sim \exp\left[\frac{1}{8N^2}\sin\xi - \frac{3}{128N^2}\cos 2\xi - \frac{1}{256N^2}\cos\xi + \cdots\right]$$

(order of magnitude)

The exponential function can therefore be expanded as

$$F \simeq 1 + \frac{1}{8N^2}\sin N\phi^* - \frac{3}{128N^2}\cos 2N\phi^* + \cdots$$

, and, as was shown in the body of this report, can be taken as equal to unity with adequate accuracy for practical calculations.