MURA DLJ-1
Internal

ANALYTICAL APPROXIMATION IN MARK V SCALLOPED ORBITS AND TO RADIAL BETATRON OSCILLATIONS ABOUT THEM

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As a check on the work of Laslett, and because of apparent discrepancies between (1) Ridge Runner (2) Feckless Five, and (3) Laslett's analytical values obtained with the aid of his tables, in determining frequencies of radial betatron oscillations, it was decided to compare analytically these various approaches, using an independent method of procedure, for motion in the median plane. At the conclusion of this work, correction of errors in Laslett's analysis produced satisfactory agreement, and all treatments of this problem now agree except for some discrepancies in the coefficients in x_s below. The present work is believed to be more accurate at this point.

I. Comparison of RR and FF.

The Ridge Runner (rigorous) equations are

$$x' = (1+x)\frac{P_x}{\sqrt{1-P_x^2}},$$

$$P_x' = \sqrt{1-P_x^2} - (1+x)^{k+1}\left[1 + f \sin\left\{\frac{1}{w}\ln(1+x) - N\phi^*\right\}\right],$$

for motion of a particle of mechanical momentum

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$p = e B_0 (r_1/r_0)^k r_1$ in the median plane of the field

$B = B_0 (r/r_0)^k [1 + f \sin \{\frac{1}{w} \ln (r/r_0) - N\phi\}]$. Here

$r \equiv r_1 (1+x)$; $N\phi^* \equiv N\phi + \frac{\ln r/r_0}{w}$

The Feckless Five equations, which are approximate, have been supplied by Laslett, and, when specialized to motion in the median plane, assume the form

$$x' = (1+x) p_x$$
$$p_x' = -(k+1)x - k(k+1)x^2/2 - k(k+1)(k-1)x^3/6$$
$$- f \cos \delta e^{(k+1)} [\sin (\tfrac{x}{w} - N\theta) - (k+1) w \cos (\tfrac{x}{w} - N\theta)]$$
$$+ f x^2 \cos (N\theta + 2\delta) \sec \delta / (2w) - \tfrac{1}{2} p_x^2.$$

Here $\delta \equiv \tan^{-1}(k+1)w$, $N\phi^* \equiv N\theta + \delta$. Straightforward expansion of the Ridge-Runner equations yields

$$x' = (1+x) p_x + \mathcal{O}(p_x^3)$$
$$p_x' = -(k+1)x - k(k+1)x^2/2 - k(k+1)(k-1)x^3/6$$
$$- f \cos \delta e^{(k+1)x} [\sin (\tfrac{x}{w} - N\theta) - (k+1) w \cos (\tfrac{x}{w} - N\theta)]$$
$$+ f x^2 (\cos N\theta + \delta)/(2w) - \tfrac{1}{2} p_x^2$$
$$+ \mathcal{O}(p_x^4, x^3),$$

where one term in x^3 has been explicitly kept to compare with Feckless Five, and where $(1+x)^{k+1}$ and $e^{(k+1)x}$ have been used interchangably since $x \ll 1$ and $k \gg 1$. Thus a discrepancy to this order in x exists in the terms in x^2 , since

$$\cos(N\theta+2\delta)\sec\delta = \cos(N\theta+\delta) - \tan\delta\sin(N\theta+\delta),$$

with the second term being erroneously present in FF and missing in the rigorous RR. The error is no doubt comparable with the neglect of terms of order x^3 for the small values of δ appropriate to the range of parameters of current interest. Validity of the neglect of x^3 terms is certainly open to question in the study of non-linear effects in betatron oscillations but does not appear to be significant in locating the closed orbits or in determining the betatron frequency in linear approximation, for the range of parameters of present interest.

II. <u>Analytical Determination of Closed Orbits</u>.

The rigorous (RR) equations may be expanded without the somewhat artificial introduction of the quantity δ . The result, to the order indicated, is

$$x' = (1+x)\,p_x + p_x^3/2 + \mathcal{O}(x p_x^3)$$

$$p_x' = -p_x^2/2 - f\sin\psi - (k+1)[1+f\sin\psi]\,x$$

$$-\left\{\frac{k(k+1)}{2}[1+f\sin\psi] - \frac{f}{2w}\cos\psi\right\}x^2$$

$$-\left\{\frac{k(k+1)(k-1)}{6}[1+f\sin\psi] - \frac{f}{2w}(k+1)\cos\psi\right\}x^3$$

$$+\mathcal{O}\left(k^4x^4,\ \frac{x^4}{w^2}\right)$$

Here $\Psi \equiv \frac{x}{w} - N\phi^*$. Neglected terms in the above equations are small with respect to the leading terms. In the first equation,

$$\frac{x\,p_x^{\,3}}{p_x} \sim N^2 x^3 \sim \frac{N^2 f^3}{N^6} \lesssim 5 \times 10^{-8} \text{ for } f \sim \tfrac{1}{4},\ N \gtrsim 25$$

In the second,

$$\frac{k^4 x^4}{f} \lesssim 10^{-5},\quad \frac{x^4}{w^2 f} \lesssim 10^{-6} \text{ for } f \sim \tfrac{1}{4},\ N \gtrsim 25,\ \tfrac{1}{w} \lesssim 3 \times 10^3,\ k \lesssim 100.$$

One may proceed to convert these highly accurate equations to a single second order differential equation:

$$x'' + (1+k)x = -f\sin\Psi - \left[(k+2) f \sin\Psi\right] x - \left[\frac{(k+1)(k+2)}{2}(1 + f\sin\Psi) - \frac{f}{2w}\cos\Psi\right] x^2$$

$$+ \tfrac{1}{2}(1 - 3f\sin\Psi)x'^2 - \left[\left\{\frac{k(k+1)(k-1)}{6} + \frac{k(k+1)}{2}\right\}(1 + f\sin\Psi) - \frac{k+2}{2w} f\cos\Psi\right] x^3$$

$$- \left[\tfrac{1}{2} + \tfrac{3}{2}(k+1)(1 + f\sin\Psi)\right] x x'^2 + O\left(x'^4,\ x x'^3,\ k^4 x^4,\ \frac{x^4}{w^2}\right).$$

In order to proceed further analytically, it is necessary to expand in powers of $\frac{x}{w}$, even though considerable inaccuracy may result for some values of x/w. The terms neglected in the degree of approximation used below are of order $\frac{1}{4!}\frac{x^4}{f w^4} \simeq \frac{f^3}{24 N^8 w^4}$ relative to the largest terms in the differential equation; this quantity is less than 2×10^{-5} for $f \lesssim \frac{1}{4}$, $N \gtrsim 50$, $1/w \lesssim 10^3$ but may be as large as $\frac{1}{2}$ for $N \gtrsim 25$, $1/w \lesssim 3 \times 10^3$! !

This expansion, when carried out, using the abbreviation $\xi \equiv N\phi^*$, yields

$$x'' + (k+1)x = f\sin\xi - \left[\frac{f}{w}\cos\xi - (k+2)f\sin\xi\right]x$$
$$+ \left[-(k+1)(k+2) + f\left\{\left[(k+1)(k+2) - \frac{1}{w^2}\right]\sin\xi - \frac{2k+3}{w}\cos\xi\right\}\right]\frac{x^2}{2}$$
$$+ \left[1 + 3f\sin\xi\right]\frac{x'^2}{2}$$
$$+ \left[-k(k+1)(k+2) + f\left\{\left[k(k+1)(k+2) - \frac{3(k+1)}{w^2}\right]\sin\xi - \left[\frac{(3k^2+6k+2) - w^2}{w}\right]\cos\xi\right\}\right]\frac{x^3}{6}$$
$$+ \left[-1 - 3(k+1) + 3f\left\{(k+3)\sin\xi - \frac{1}{w}\cos\xi\right\}\right]\frac{xx'^2}{2} + o\left(\frac{x^4}{4!\,w^4}, k^4x^4, x'^4\right).$$

The largest neglect is that of the terms in $\frac{x^4}{4!\,w^4}$. This equation may be solved by iteration for the closed orbit, which is that solution having period $2\pi/N$ in θ.

$$x = x_0 + x_1 + x_2 + x_3 + \cdots,$$
$$x_0'' + (k+1)x_0 = f\sin N\phi^*$$
$$x_0 = -(f/D_1)\sin\xi, \quad D_1 \equiv N^2 - (k+1).$$
$$x_1'' + (k+1)x_1 = -f\left[\frac{1}{w}\cos\xi - (k+2)\sin\xi\right]x_0 = \frac{f^2}{2D_1}\left[\frac{1}{w}\sin 2N\phi^* - (k+2)(1-\cos 2N\phi^*)\right]$$
$$x_1 = -\frac{k+2}{2(k+1)}\frac{f^2}{D_1} - \frac{f^2}{2D_1D_2}\left[\frac{1}{w}\sin 2N\phi^* + (k+2)\cos 2N\phi^*\right]$$
$$x_2'' + (k+1)x_2 = -f\left[\frac{1}{w}\cos\xi - (k+2)\sin\xi\right]x_1 + \left[-(k+1)(k+2) + f\left\{\left[(k+1)(k+2) - \frac{1}{w^2}\right]\sin\xi - \frac{2k+3}{w}\cos\xi\right\}\right]\frac{x_0^2}{2}$$

$$+ \left[1 + 3 f \sin \xi\right] \frac{x_0'^2}{2} \; ; \quad D_2 \equiv 4N^2 - (k+1)$$

$$= \alpha_0 + \alpha_1 \cos N\phi^* + \alpha_2 \cos 2N\phi^* + \alpha_3 \cos 3N\phi^* + \beta_1 \sin N\phi^*$$

$$+ \beta_3 \sin 3N\phi^*.$$

$$\alpha_0 = \frac{1}{4}\left(\frac{f}{D_1}\right)^2 \left[N^2 - (k+1)(k+2)\right]$$

$$\alpha_1 = \frac{f^3}{2\omega D_1}\left[\frac{k+2}{k+1} - \frac{2k+3}{4D_1}\right]$$

$$\alpha_2 = \frac{f^2}{4D_1^2}\left[N^2 + (k+1)(k+2)\right]$$

$$\alpha_3 = \frac{f^3}{2\omega D_1}\left[\frac{k+2}{D_2} + \frac{2k+3}{4D_1}\right]$$

$$\beta_1 = \frac{f^3}{2D_1}\left[-\frac{(k+2)^2}{k+1} + \frac{(k+2)^2}{2D_2} + \frac{1}{2\omega^2 D_2} + \frac{3}{4}\frac{(k+1)(k+2)}{D_1} - \frac{3}{4}\frac{1}{\omega^2 D_1} + \frac{3}{4}\frac{N^2}{D_1}\right]$$

$$\beta_2 = 0$$

$$\beta_3 = \frac{f^3}{2D_1}\left[-\frac{(k+2)^2}{2D_2} + \frac{1}{2\omega^2 D_2} - \frac{1}{4}\frac{(k+1)(k+2)}{D_1} + \frac{1}{4}\frac{1}{\omega^2 D_1} + \frac{3}{4}\frac{N^2}{D_1}\right]$$

$$x_2 = \frac{\alpha_0}{k+1} - \frac{\alpha_1}{D_1}\cos N\phi^* - \frac{\beta_1}{D_1}\sin N\phi^* - \frac{\alpha_2}{D_2}\cos 2N\phi^* - \frac{\alpha_3}{D_3}\cos 3N\phi^* - \frac{\beta_3}{D_3}\sin 3N\phi^* ;$$

$$D_3 \equiv 9N^2 - (k+1)$$

$$x_3'' + (k+1)x_3 = -Ax_2 + Bx_0x_1 + Cx_0'x_1' + D\frac{x_0^3}{6} + E\frac{x_0 x_0'^2}{2}.$$

Here A, B,.....E are the square brackets on the right side of the differential equation being solved.

To minimize the labor from here on, we note that, according to Symon's smooth approximation,

$$\left(\frac{f}{N^2\omega}\right)^2 \simeq \frac{1}{4}\left[\left(\frac{\sigma_r}{\pi}\right)^2 + \left(\frac{\sigma_z}{\pi}\right)^2\right] - \frac{1}{N^2}, \quad k \simeq \left(\frac{N\sigma_r}{2\pi}\right)^2 - 1;$$

so that for reasonable values $\sigma_r \sim \pi/2$, $\sigma_2 \sim \pi/4$, we have, approximately, $\frac{\chi}{\omega} \simeq \frac{1}{4}$, $k+1 \simeq \frac{N^2}{16}$; since $\chi \sim f/D_1$, and $D_1 \sim N^2$, $D_2 \sim 4N^2$. Thus the first terms in α_0, α_1, and α_2 are the largest, $1/\omega$ is an order of magnitude larger than k, and the largest terms in β_1 and β_3 are those in $f^3/(w^2\ D_1^2)$ and $f^3 k/D_1$.

The largest terms on the right side are those coming from the large part of A $\left(-\frac{f}{\omega}\cos\xi\right)$ times the largest terms in $x_2\left(-\frac{\alpha_1}{D_1}\cos N\phi^* - \frac{\beta_1}{D_1}\sin N\phi^*\right)$, from the largest term in B $\left(-\frac{f}{\omega^2}\sin\xi\right)$ times x_0 times the two largest parts of x_1 $\left(-\frac{k+2}{2(k+1)}\frac{f^2}{D_1} - \frac{f^2}{2D_1D_2\omega}\sin 2\xi\right)$, and from the largest term in D $\left(f/\omega^3 \cos\xi\right)$ times $1/6\ x_0^3$.
We thus obtain the approximate equation

$$x_3'' + (k+1)x_3 \simeq \frac{f\alpha_1}{2\omega D_1}(1+\cos 2\xi) + \frac{f\beta_1}{2\omega D_1}\sin 2\xi$$

$$-\frac{f^4}{4\omega^2 D_1^2}\left[\left(\frac{k+2}{k+1}\right)(1-\cos 2\xi) + \frac{1}{\omega D_2}\left(\sin 2\xi - \frac{\sin 4\xi}{2}\right)\right]$$

$$-\frac{f^4}{48\omega^3 D_1^3}(2\sin 2\xi - \sin 4\xi)$$

$$\simeq -\frac{f^4(2k+3)}{16\omega^2 D_1^3} + \frac{f^4}{2\omega^2 D_1^2}\left(\frac{k+2}{k+1}\right)\cos 2\xi - \frac{7f^4}{24\omega^3 D_1^3}\sin 2\xi + \frac{f^4}{20\omega^3 D_1^3}\sin 4\xi$$

where only the largest term in β_1 $\left(\propto \frac{1}{\omega^2}\right)$ has been kept, and $D_1/D_2 \sim 1/4$. We thus obtain

$$x_3 \simeq -\frac{f^4}{16\omega^2 D_1^3}\,\frac{2k+3}{k+1} - \frac{f^4}{2\omega^2 D_1^2 D_2}\,\frac{k+2}{k+1}\cos 2N\phi^* + \frac{7f^4}{24\omega^3 D_1^3 D_2}\sin 2N\phi^*$$
$$-\frac{f^4}{20\omega^3 D_1^3 D_4}\sin 4\xi$$

Finally we may collect all our terms below and evaluate the orders of magnitude of the various terms, expressing all quantities in terms of N through use of Symon's smooth approximation expressions with the choices of σ's made above.

$$\chi = -\left[\frac{k+2}{(2k+1)}\frac{f^2}{D_1} + \frac{1}{4}\frac{f^2}{D_1^2}\left\{\frac{(k+1)(k+2)-N^2}{k+1}\right\} + \frac{f^4}{16\omega^2 D_1^3}\frac{2k+3}{k+1} + \cdots\right]$$

$$-\sin N\phi^*\left[\frac{f}{D_1} + \frac{f^3}{2D_1^3}\left\{-\frac{3}{4\omega^2} + \frac{3}{4}N^2 + \frac{3}{4}(k+1)(k+2) + \frac{1}{2\omega^2}\frac{D_1}{D_2} + \frac{(k+3)^2}{2}\frac{D_1}{D_2}\right.\right.$$

$$\left.\left. - \frac{(k+2)^2 D_1}{k+1}\right\} + \cdots\right]$$

$$-\cos N\phi^*\left[\frac{f^3}{2\omega D_1^2}\left(\frac{k+2}{k+1} - \frac{3k+3}{4D_1} + \cdots\right]$$

$$-\sin 2N\phi^*\left[\frac{f^2}{2\omega D_1 D_2} - \frac{7f^4}{24\omega^3 D_1^3 D_2} + \cdots\right]$$

$$-\cos 2N\phi^*\left[\frac{f^2}{2D_1D_2}(k+2) + \frac{f^2}{4D_1^2D_2}\left\{N^2 + (k+1)(k+2)\right\} + \frac{f^4}{2\omega^2 D_1^3 D_2}\frac{k+2}{k+1} + \cdots\right]$$

$$-\sin 3N\phi^*\left[\frac{f^3}{2D_1^2D_3}\left\{+\frac{1}{4\omega^2} + \frac{3}{4}N^2 - \frac{1}{4}(k+1)(k+2) + \frac{1}{2\omega^2}\frac{D_1}{D_2} - \frac{(k+2)^2}{2}\frac{D_1}{D_2}\right\} + \cdots\right]$$

$$-\cos 3N\phi^*\left[\frac{f^3}{2\omega D_1^2 D_3}\left(\frac{2k+3}{4} + (k+2)\frac{D_1}{D_2}\right) + \cdots\right]$$

$$-\sin 4N\phi^*\left[\frac{f^4}{20\omega^3 D_1^3 D_4} + \cdots\right]$$

$$+ \cdots\cdots$$

To get orders of magnitude: take $k \simeq k+1 \simeq k+2 \simeq \frac{N^2}{16}$

$$\chi \simeq \frac{1}{4N^2}\Bigg[\!\Bigg[-\sin N\phi^* \Big[1 + \Big(-.024 + \frac{.072}{N^2} + .0016 + 0.004 + 0.0003 - 0.002\Big) + \cdots \Big]$$

$$- \Big[0.125 + 0.004 \Big(1 - \frac{256}{N^2} + 0.002 + \cdots \Big]$$

$$- \cos N\phi^* \Big[0.03 + \cdots \Big]$$

$$- \sin 2N\phi^* \Big[0.03 - .001 + \cdots \Big]$$

$$- \cos 2N\phi^* \Big[0.002 + \cdots \Big]$$

$$- \sin 3N\phi^* \Big[\Big(+ 0.01 + \frac{.0026}{N^2} - 0.001 + 0.005 - 0.0005 \Big) + \cdots \Big]$$

$$- \cos 3N\phi^* \Big[0.0002 + \cdots \Big]$$

$$- \sin 4N\phi^* \Big[0.00005 + \cdots \Big] \cdots \Bigg]\!\Bigg]$$

It is felt unlikely that for any reasonable design the numerical relationships used above will differ by more than about a factor of 2, or that the orders of magnitude of the various terms arrived at will be changed by more than a factor of 10.

III. Linearized Equations for Radial Betatron Oscillations.

The periodic solution just obtained will henceforth be denoted by x_s. If we set $x = x_s + \rho$, insert this in the differential equation, and retain only linear terms in ρ, we obtain

$$\rho'' + (k+1)\rho \simeq A\rho + Bx_s\rho + Cx_s'\rho' + \frac{Dx_s^2}{2}\rho + E\left(\frac{x_s'^2}{2}\rho + x_s x_s'\rho'\right) + O(\rho^2; \rho'^2).$$

Rearranging,

$$\rho'' - (C + Ex_s)x_s'\rho' + \left[(k+1) + A - Bx_s - \frac{Dx_s^2}{2} - \frac{Ex_s'^2}{2}\right]\rho = O(\rho^2 - \rho'^2)$$

Transforming to eliminate the first derivative term, set $\rho = v \exp\left[\frac{1}{2}\int (C + Ex_s)x_s'\, d\phi^*\right]$, obtaining

$$v'' + \left[(k+1) + A - Bx_s - \frac{Dx_s^2}{2} - E\frac{x_s'^2}{2} - \frac{1}{4}\left\{(C + Ex_s)x_s'\right\}^2 - \frac{1}{2}\frac{d}{d\phi^*}\left\{(C+Ex_s)x_s'\right\}\right]v = O(v^2; v'^2).$$

We will now try to evaluate the square bracket in this equation. By use of the same interrelationships based on the smooth approximation as were applied above to pick out large terms and reject small ones, we may verify that the terms to be kept are

$$k+1 + \frac{f}{\omega}\cos\xi - (k+2)f\sin\xi + \left[(k+1)(k+2) + \frac{f}{\omega^2}\sin\xi + \frac{f(2k+3)}{\omega}\cos\xi\right]x_s - \frac{f}{\omega^3}\cos\xi\,\frac{x_s^2}{2}.$$

After inserting x_s as obtained earlier and dropping small terms, we obtain the square bracket as

$$(k+1) - \frac{f^2}{2\omega^2 D_1}\left\{1 - \frac{3}{8}\left(\frac{f}{\omega D_1}\right)^2\right\} + \cdots$$

$$+\frac{f}{\omega}\cos\xi\left[1-\frac{f^2}{8\omega^2 D_1^2}(1-3k^2\omega^2)\left\{1-\frac{3}{8}\left(\frac{f}{\omega D_1}\right)^2\right\}-\frac{(k+2)(2k+3)f^2}{2(k+1)D_1}+---\right]$$

$$-(k+2)f\sin\xi\left[1+\frac{f^2}{2(k+1)\omega^2 D_1}+\frac{k+1}{D_1}+---\right]$$

$$+\frac{f^2}{2\omega^2 D_1}\cos 2\xi\left\{1-\frac{3}{8}\left(\frac{f}{\omega D_1}\right)^2+---\right\}$$

$$-\frac{(2k+3)f^2}{2\omega D_1}\sin\xi\left[1+\frac{(k+2)}{(k+1)(2k+3)}\ \frac{f^2}{\omega^2 D_1}+--\right]$$

$$+\frac{f^3}{8\omega^3 D_1^2}\cos 3\xi\left\{1-\frac{3}{8}\left(\frac{f}{\omega D_1}\right)^2\right\}^2+---$$

$$+---=\frac{N^2}{4}\left[A+B_c\cos\xi-B_s\sin\xi+C_c\cos 2\xi-C_s\sin 2\xi+D_c\cos 3\xi+\cdots\right]$$

Making the change of variable $\xi \equiv N\phi^* = 2t$, we may write the linearized equation for v as

$$\frac{d^2v}{dt^2}+\left[A+B\cos(2t+\delta_1)+C\cos(4t+\delta_2)+D\cos(6t+\delta_3)+\cdots\right]v\simeq 0,$$

with $B=+\sqrt{B_c^2+B_s^2}$, $\tan\delta_1\ B_s/B_c$, $C=\sqrt{C_c^2+C_s^2}$, $\tan\delta_2=C_s/C_c$,

$D\simeq D_c$, $\tan\delta_3$ was not evaluated since D_s is less than 0.1% of B. From the smooth approximation, orders of magnitude are:

$$A \sim \tfrac{1}{4} - 1/8 + \cdots$$

$$B_c \sim 1 - 1/128 - 1/256 + \cdots$$

$$B_s \sim 1/16 + 1/32 + 1/256 + \cdots$$

$$C_c \sim 1/8 + \cdots$$

$$C_s \sim 1/64 + 1/28 + \cdots$$

$$D_c \sim 1/128 + \cdots$$

$$D_s \sim 1/2000$$

The largest terms calculated from the original equation accurate through third order terms in x, but not written above, are of order 1/1000 or of order $1/N^2$ when compared with the orders of magnitude given immediately above. [Neglect of all of the terms of order $1/N^2$ is not justified for model parameters with small N; this point is investigated separately below.] We finally then obtain:

$$A \doteq \frac{4}{N^2}\left[(k+1) - \frac{f^2}{2\omega^2 D_1}\left\{1 - \frac{3}{8}\left(\frac{f}{\omega D_1}\right)^2\right\}\right]$$

$$B \doteq \frac{4}{N^2}\frac{f}{\omega}\left[\left\{1 - \frac{f^2}{8\omega^2 D_1^2}(1 - 3k^2\omega^2)\left[1 - \frac{3}{8}\left(\frac{f}{\omega D_1}\right)^2\right] - \frac{(k+2)(2k+3)f^2}{2(k+1)D_1}\right\}^2 + (k+2)^2\omega^2\left\{1 + \frac{f^2}{2(k+1)\omega^2 D_1} + \frac{k+1}{D_1}\right\}^2\right]^{1/2}$$

$$C \doteq \frac{4}{N^2}\frac{f^2}{2\omega^2 D_1}\left[\left\{1 - \frac{3}{8}\left(\frac{f}{\omega D_1}\right)^2\right\}^2 + (2k+3)^2\omega^2\left\{1 + \frac{(k+2)}{(k+1)(2k+3)}\frac{f^2}{\omega^2 D_1}\right\}^2\right]^{1/2}$$

$$D \doteq \frac{4}{N^2}\frac{f^3}{8\omega^3 D_1^2}\left[1 - \frac{3}{8}\left(\frac{f}{\omega D_1}\right)^2\right]^2$$

One way shift the origin of t an amount δ_1 so as to make the new $\delta_1' = 0$; then δ_2' can be obtained from the equation

$$\tan \delta_2' = \frac{\tan \delta_2 - \tan \delta_1}{1 + \tan \delta_2 \tan \delta_1} .$$

There appears to be little point in evaluating this expression further at present since tables do not exist which cover values of δ_2' other than 0 or π. For the cases studied this far numerically, δ_1 is less than about 0.15 radian, and δ_2 is less than twice δ_1, so that δ_2' does not exceed 0.1 radian.

For <u>model parameters</u> use the following formulas:

A = same as for large machines

$$B = \sqrt{B_s^2 + B_c^2}$$

$$C = \sqrt{C_s^2 + C_c^2}$$

where

$$B_s = \frac{2(2k+3)f^2}{N^2 \omega D_1} \left\{ 1 + \frac{k+2}{(k+1)(2k+3)} \frac{f^2}{\omega^2 D_1} \right\} + \underline{\frac{2f}{D_1}}$$

$$B_c = \frac{4f}{N^2 W} \left\{ 1 - \frac{f^2}{8\omega^2 D_1^2} (1 - 3k^2\omega^2) \left(1 - \frac{3}{8} \left(\frac{f}{\omega D_1}\right)^2\right) - \frac{(k+2)(2k+3)f^2}{2(k+1) D_1} \right\} - \underline{\frac{f^3}{\omega D_1^2} \left(\frac{k+2}{k+1}\right)}$$

$$C_s = \frac{4f^2}{2 D_1 N^2} (2k+3) \left\{ 1 + \frac{(k+2)}{(k+1)(2k+3)} \frac{f^3}{\omega^2 D_1} \right\} + \underline{\frac{3}{2} \frac{f^3}{\omega D_1^2} \left(\frac{k+2}{k+1}\right)}$$

$$C_c = \frac{2f^2}{N^2 \omega^2 D_1} \left\{ 1 - \frac{3}{8} \left(\frac{f}{\omega D_1}\right)^2 \right\} + \underline{\frac{6f^2}{D_1}}$$

$$D_1 = N^2 - (k+1)$$

(The underlined terms may be omitted in large scale machines)

IV. Numerical Comparisons.

The closed orbit results may be put in the form

$$x_s = -\alpha - \beta \sin N\phi^* - \gamma \cos N\phi^* - \delta \sin 2N\phi^* + \text{terms} < 1\% \text{ of } \beta .$$

The results of Laslett may be written as:

$$\alpha = \frac{1}{2}\left(\frac{k+2}{k+1}\right)\left(\frac{f}{N}\right)^2$$

$$\beta = \frac{f}{N^2-(k+1)+\frac{3}{8}\left(\frac{f}{wN}\right)^2}$$

$$\gamma = -\frac{kw}{4}\left[\frac{f}{N^2-(k+1)+\frac{3}{8}\left(\frac{f}{wN}\right)^2}\right]^3$$

$$\delta = 0$$

The results of the present investigation have been quoted earlier. A numerical comparison has been made for the representative parameters $f = .25$, $N = 50$, $k = 100$, $w^{-1} = 3 \times 10^3$.

	α	β	γ	δ
Laslett	1.25×10^{-5}	1.01×10^{-4}	$\sim 10^{-14}$	0
Judd	1.33×10^{-5}	1.00×10^{-4}	4.0×10^{-6}	3.9×10^{-6}

The discrepancies indicate that the values calculated here may more accurately predict fixed points.

Results for A, B, and C in the linearlized radial betatron oscillation equation have been computed for six cases, and one tabulated below, together with values of σ_x/π obtained from the Laslett tables and from the Illiac. Earlier discrepancies, largely due to errors in Laslett's earlier work, have disappeared upon introduction of his corrected values, which are essentially the same as those obtained here. Results for σ_z/π using his corrected values are also tabulated for completeness; good agreement is again obtained.

								σ_x/π		σ_z/π	
k	f	$1/\omega$	N	A_L	A_J	$B_L \simeq B_J$	$C_L \simeq C_J$	Tables	Illiac	Tables	Illiac
150	1/4	2094	37	.131	.135	1.50	.309	$.8_7$*	.86	$.4_1$	.39
75	1/4	1047	27	.144	.146	1.41	.272	$.8_0$	.79	$.3_3$	.32
150	1/4	2094	40	.151	.153	1.29	.227	$.7_2$	.71	$.2_3$	.22
150	1/4	2200	42	.138	.139	1.23	.206	$.6_7$	~2/3	$.2_1$	.21
150	1/4	3142	60	.070	.070	0.868	.0975	$.4_2$	.43	$.1_5$	.15
150	1/4	2620	50	.099	.100	1.04	.143	$.5_2$	.53	$.2_0$	.18

* This value obtained by linear interpolation in C from plots of levies of constant B on A vs cos σ graphs. Linear interpolation is not too well justified here, and is especially bad from plots of lines of constant σ on A vs B graphs in this case lying near a stability boundary; the value obtained in this way is .92.

In the table above it should be noted that the reading of values of σ from graphs made from the Laslett tables is uncertain to at least one and probably two digits in the second decimal place in many places.

V. Conclusions.

It is concluded (1) that the Ridge Runner and Feckless Five equations agree in the median plane through terms of second order in x and x' except for a small discrepancy which will be unimportant for small kev; (2) that a reliable expression, accurate to better than 1% of the largest term, for the closed orbits has been obtained; (3) that the coefficients A, B, and C for the radial linearized betatron oscillation as obtained here and independently by Laslett are now in satisfactory agreement; and (4) that the frequencies obtained from inserting these into the Laslett tables are in satisfactory agreement with the frequencies denied from Illiac computations. The principal recommendation arising from this work is that the section of the tables from C = -0.5 to C = +0.5 be extended at intervals of 0.1 if it is anticipated that extensive use of these tables is expected in the future. It may also be repeated that the coefficients of x_s obtained here are believed to be more accurate than those of Laslett and may have some utility in more precise location of fixed points in the machine calculations.