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TRANSVERSE BEAM GROWTH DUE TO LONGITUDINAL COUPLING IN LINEAR ACCELERATORS[†]

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I. Introduction

The usual approximation for treating orbit dynamics in linear and circular accelerators is to consider the longitudinal and two transverse motions to be uncoupled and to treat the development of the corresponding 2 X 2 phase spaces independently. It has long been recognized that the orbit motions are coupled.[1,2] In linear accelerators the source of this coupling is the dependence of the rf defocusing term in the transverse motion on the longitudinal phase[2] and the dependence of the transit time factor in the longitudinal motion on the transverse position (these coupling terms arise from the same term in the Lagrangian). An effort to include the effect of these coupling terms to the two lowest orders was made[3-5] in order to determine their influence on

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1. See, for example, E.D. Courant and H.S. Snyder, Ann. Phys. 3, 1-48 (1958).

2. See, for example, L.S. Smith and R.L. Gluckstern, Rev. Sci. Instr. 26, 220 (1955).

3. R.L. Gluckstern, "Coupling of Longitudinal and Transverse Oscillations in Linear Accelerators," BNL Accelerator Dept. Internal Report AADD-31, April 2, 1964.

4. R.L. Gluckstern, "Note on Coupled Particle Motion with Errors," LASL Internal Report MP-4/RLG-5, March 14, 1966.

5. S. Ohnuma, "Coupled Motion in Low Energy Proton Linear Accelerators," Yale Internal Report Y-14, June 30, 1966.

transverse beam size. Comparison of these results with orbit computations[6,7] indicated agreement for the lowest order term, but not for the term which would be resonant if the longitudinal and transverse oscillation frequencies were equal.

The present report reviews the analysis, yielding a result for the second order term which takes into account higher order terms in the first coupling term. This result now appears to be in general agreement with the orbit computations[6] and can be used as the basis for predicting transverse beam growths due to this coupling.

II. Growth of Transverse Oscillations

The basic coupled equations for motion in the longitudinal and one transverse direction, including only the first two coupled terms, are

$$\frac{1}{\beta\gamma} \frac{d}{ds} \left(\beta\gamma \frac{dy}{ds} \right) + k_t^2 \, y = - \frac{k_\ell^2}{4} \, y \, (2 \, g \, \chi + \chi^2) \tag{1}$$

$$\frac{1}{(\beta\gamma)^3} \frac{d}{ds} \left(\beta^3\gamma^3 \frac{d\chi}{ds} \right) + k_\ell^2 \, \chi = g \, \frac{k_\ell^2}{2} \, \chi^2 - k_\ell^2 \left(\frac{\pi y}{\beta\gamma\lambda} \right)^2 (g + \chi) \tag{2}$$

where $g = \cot |\varphi_s|$.

These equations differ from those in References 3 and 4 by the inclusion of the χ^2 term in the equation for χ , and by treating y as an actual displacement. The notation is:

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6. Private communication from R. Chasman, BNL;   S. Ohnuma, Yale.

7. Private communication from D. Swenson, LASL.

λ = rf wavelength

φ_s = synchronous phase (< 0)

$\chi = \varphi - \varphi_s$

y = transverse displacement

$k_\ell = \left(\dfrac{2\pi\, e\, E_o\, T\, \sin |\varphi_s|}{mc^2\, \beta^3\, \gamma^3\, \lambda} \right)^{\frac{1}{2}}$, longitudinal oscillation wave number

k_t = smoothed transverse oscillation wave number

$\beta\gamma$ = longitudinal momentum of the synchronous particle in units of mc.[8]

Equations (1) and (2) contain several approximations. These are:

1) Only terms of order up to y^2 are included in the Lagrangian. This appears to be valid for beam sizes currently considered.

2) The transverse oscillation, which is usually of the strong focusing type, has been smoothed. This should be reasonable for the present coupling effects but not for any coupling effects arising at magnet boundaries.

3) Only terms of order up to χ^3 and $\chi^2 y^2$ are included in the Lagrangian. Stability limits are not correctly obtained in this approximation but the coupling effects will be consistent to the order shown.

4) Only that rf wave component traveling with the beam is included. Inclusion of other components is equivalent to taking into account the velocity dependence of the transit time factor, which has little relevance to the coupling effects.

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8. In Equations (1), (2), $\beta\gamma$ is taken to be the momentum of the synchronous particle. The derivation of these equations requires changing variables in the Lagrangian from y, $\dot{y}$, z, $\dot{z}$, t to y, y', χ, χ', s , where

$$t - \int_0^S \frac{ds}{v_s(s)} = - \frac{\chi}{\omega} \, , \quad z = s \ .$$

This transformation is similar to that discussed in Reference 1, and actually leads to additional terms on the right side of Equations (1) and (2), whose origin is the difference between the actual and synchronous momenta. These terms can safely be neglected in the present application, being of relative order the square of the ratio of cell length ($\beta\lambda$) to oscillation wavelength ($2\pi/k_\ell$).

The solutions of the uncoupled linearized part of Equations (1), (2) in the JWKB approximation are:[3]

$$y = A_t \, (\beta\gamma)^{-1/2} \, k_t^{-1/2} \, \sin \left(\int_0^s k_t \, ds + a_t \right) \tag{3}$$

$$\chi = A_\ell \, (\beta\gamma)^{-3/2} \, k_\ell^{-1/2} \, \sin \left(\int_0^s k_\ell \, ds + a_\ell \right) . \tag{4}$$

In order to compare with numerical results, we consider collections of points of fixed A_t , A_ℓ , and distributed values of a_t , a_ℓ . The most convenient coordinate systems for observing the motion of these phase points are the polar coordinates A_t , a_t and A_ℓ , a_ℓ which are equivalent to the Cartesian coordinates:

$$\bar{y} = y \, (\beta\gamma)^{1/2} \, (\beta_{SF})^{-1/2}$$

$$\bar{y}' = y' (\beta\gamma)^{1/2} \, (\beta_{SF})^{1/2}$$

$$\tag{5}$$

$$\bar{\chi} = \chi \, (\beta\gamma)^{3/2} \, k_\ell^{1/2}$$

$$\bar{\chi}' = \chi' (\beta\gamma)^{3/2} \, k_\ell^{-1/2} \quad ,$$

where β_{SF} is the β of Courant and Snyder.[1] The uncoupled oscillations correspond to motions of phase points in circles in the coordinate systems represented by Eq. (5). Amplitude increases then appear as (elliptical) distortions in these coordinate systems.

In the iterative scheme for solving Equations (1) and (2), the typical equation will contain oscillatory driving terms of decreasing amplitude. Integration of such an equation over the course of acceleration will then be done approximately and will only involve the starting amplitude, frequency and phase of the driving term. Specific damping variations will not appear explicitly. To simplify the presentation, therefore, we shall ignore damping, except where necessary to discard terms at the end of acceleration. With this understanding we then write

$$y = y^{(0)} + y^{(1)} + y^{(2)} + \ldots$$

$$\chi = \chi^{(0)} + \chi^{(1)} + \overline{\chi^{(1)}} + \chi^{(2)} + \ldots \tag{6}$$

with

$$y^{(0)} = y_0 \sin t \equiv y_0 \sin (k_t s + a_t) \tag{7}$$

$$\chi^{(0)} = \chi_0 \sin \ell \equiv \chi_0 \sin (k_\ell s + a_\ell) \tag{8}$$

$$y^{(1)''} + k_t^2 y^{(1)} = - g \frac{k_\ell^2}{2} y^{(0)} \chi^{(0)} \tag{9}$$

$$\chi^{(1)''} + k_\ell^2 \chi^{(1)} = - g k_\ell^2 \left(\frac{\pi y^{(0)}}{\beta \gamma \lambda} \right)^2 \tag{10}$$

$$\overline{\chi^{(1)}}'' + k_\ell^2 \overline{\chi^{(1)}} = g \frac{k_\ell^2}{2} \chi^{(0)2} \tag{11}$$

$$y^{(2)''} + k_t^2 y^{(2)} = - g \frac{k_\ell^2}{2} \left(y^{(0)} \overline{\chi^{(1)}} + y^{(1)} \chi^{(0)} \right) - \frac{k_\ell^2}{4} y^{(0)} \chi^{(0)2} \tag{12}$$

$$\chi^{(2)''} + k_\ell^2 \chi^{(2)} = - g k_\ell^2 \left(\frac{\pi}{\beta \gamma \lambda} \right)^2 2 y^{(0)} y^{(1)} - k_\ell^2 \left(\frac{\pi}{\beta \gamma \lambda} \right)^2 y^{(0)2} \chi^{(0)}$$

$$+ g k_\ell^2 \chi^{(0)} \chi^{(1)} \quad . \tag{13}$$

We have here separated the $\chi^{(1)}$ contribution into a part proportional to y_0^2 and a part proportional to χ_0^2 and have treated the second order terms consistently. The solutions for $y^{(1)}$, $\chi^{(1)}$ and $\overline{\chi^{(1)}}$ (which are defined to vanish and have vanishing derivatives at s = 0) are:

$$y^{(1)} = \frac{g k_\ell}{8 k_t} y_0 \chi_0 \left[\frac{-2 k_t \cos (t - \ell) + k_\ell \cos (t - 2 a_t + a_\ell)}{2 k_t - k_\ell} \right.$$

$$\left. - \frac{2 k_t \cos (t + \ell) + k_\ell \cos (t - 2 a_t - a_\ell)}{2 k_t + k_\ell} + 2 \cos t \cos a_\ell \right] \tag{14}$$

$$\chi^{(1)} = -\frac{g}{2}\left(\frac{\pi\, y_o}{\beta\gamma\lambda}\right)^2 \left[\, 1 + \frac{k_\ell^2}{4k_t^2 - k_\ell^2}\cos 2t - \cos(\ell - a_\ell) + \right.$$

$$+ \frac{k_\ell}{2(2k_t + k_\ell)}\cos(\ell - 2a_t - a_\ell) -$$

$$\left. - \frac{k_\ell}{2(2k_t - k_\ell)}\cos(\ell - 2a_t + a_\ell)\, \right] \tag{15}$$

$$\overline{\chi^{(1)}} = \frac{g}{4}\chi_o^2 \left[\, 1 + \frac{\cos 2\ell}{3} - \cos(\ell - a_\ell) - \frac{\cos(\ell + a_\ell)}{2} + \frac{\cos(\ell - 3a_\ell)}{6}\, \right]. \tag{16}$$

The $\cos(t \pm \ell)$ terms in Eq. (14) and the 1 and $\cos 2t$ terms in Eq. (15) decrease with increasing s, whereas the remaining terms are present only to satisfy the boundary conditions at $s = 0$ and represent permanent deformations which persist at $s = \infty$. These latter terms give rise to a change in transverse amplitude which comes only from the coefficient of $\sin t$ in Eq. (14):

$$\frac{\delta A^{(1)}(\infty)}{A^{(0)}} \simeq \frac{g\, k_\ell^2}{8\, k_t}\chi_o \left[\, \frac{\sin(2a_t - a_\ell)}{2k_t - k_\ell} - \frac{\sin(2a_t + a_\ell)}{2k_t + k_\ell}\, \right]. \tag{17}$$

The contribution of $y^{(2)}$ to the transverse amplitude change can similarly be calculated from Eq. (12). We shall here only evaluate the effect of the term which dominates for k_t and k_ℓ almost equal. The result (which can also be obtained by the phase amplitude method of Reference 3) is

$$\frac{\delta A^{(2)}(\infty)}{A^{(0)}} \simeq \frac{k_\ell^2\, \chi_o^2\, \cos(2a_t - 2a_\ell)}{32\, k_t\, (2k_t - 2k_\ell)} \left[\, 1 - g^2\left(\frac{1}{3} + \frac{k_\ell}{2k_t - k_\ell}\right)\, \right]. \tag{18}$$

The factor proportional to g^2 represents the contribution of the first order terms in Eq. (12) which was not previously included.[3]

The dominant term in the sum of Equations (17) and (18) is expected to be the first term in Eq. (17) unless $k_t - k_\ell$ is particularly small. This term

then represents, for a given a_ℓ , an elliptical distortion of the transverse phase space by a factor

$$\frac{A(\infty)}{A(0)} \sim 1 + \frac{g \, k_\ell^2}{8 k_t \, (2k_t - k_\ell)} \, \chi_0 \qquad (19)$$

with orientation such that the maximum transverse amplitude occurs for

$$2a_t - a_\ell = \pi/2 \quad . \qquad (20)$$

If one now considers a collection of all values of a_ℓ , the occupied phase space will consist of the superposition of ellipses of distortion given by Eq. (19) with all orientations, and, for all intents and purposes, will appear to be a growth in the transverse amplitude by the factor given in (19).

The situation is more complicated for the sum of Equations (17) and (18) since the distortion in general depends on a_ℓ . However, the maximum distortion occurs for a value of a_ℓ such that all terms are in phase. The maximum transverse amplitude growth is therefore

$$\left. \frac{\delta \, A(\infty)}{A_0} \right|_{max} = \chi_0 \, \frac{g \, k_\ell^2}{8 \, k_t} \left[\frac{1}{|2k_t - k_\ell|} + \frac{1}{2k_t + k_\ell} \right] +$$

$$+ \chi_0 \, \frac{k_\ell^2}{32 \, k_t} \left| \frac{g^2 \left(\frac{1}{3} + \frac{k_\ell}{2k_t - k_\ell} \right) - 1}{2k_t - 2k_\ell} \right| \quad . \qquad (21)$$

The result in Eq. (21) gives the transverse amplitude growth at the end of acceleration (after coupling terms are unimportant). One expects to see wider variations and oscillations during acceleration, and each term can be expected to reach a peak approximately twice that represented by the individual terms in Eq. (21). Although these peaks will not occur at the same energies, a rough upper limit to the maximum distortion is given by twice the result in Eq. (21), namely,

$$\left. \frac{\delta \, A(s)}{A_0} \right|_{max} \leq 2 \left. \frac{\delta \, A(\infty)}{A_0} \right|_{max} \quad . \qquad (22)$$

III. Comparison with Orbit Computations

Computations have been performed[6,7] to determine the seriousness of the transverse beam growth due to coupling with the longitudinal oscillations.[9] Comparisons have been made, essentially with the forms in Equations (17) and (18) with the following conclusions:

1) The main contribution to the effect comes from the first term in Eq. (17), as expressed in Eq. (19). For k_t and k_ℓ equal at the start of acceleration, $\varphi_s = -26°$ and $\chi_0 \sim 30°$, Eq. (19) predicts a 15% increase in amplitude at $s = \infty$ and a 30% increase at some intermediate value of s , according to Eq. (22). This is the order of magnitude of the effect observed.

2) A Fourier analysis of the distortion in the variable a_ℓ leads to excellent agreement with the two terms in Eq. (17). (The second term is roughly 1/3 of the first.) The agreement with Eq. (18) is not quite so good, primarily because of the difficulty in assigning an accurate value to $2k_t - 2k_\ell$ in the presence of the highly nonlinear and coupled oscillations. The order of magnitude and sign of this term, however, appear to be correctly given by Eq. (18).

3) The longitudinal wave number decreases more rapidly than the transverse wave number. In those designs where k_t starts out below k_ℓ , the two become equal at some intermediate value of s and the analysis in Section II is not applicable. Reference 3 contains a treatment of the resonant behavior for an assumed variation of k_ℓ and k_t and indicates that the magnitude of Eq. (18) should be increased by a large factor if resonance takes place shortly after the start of acceleration. Even in those cases where resonance does not occur, the rapid variation of $k_t - k_\ell$ with s makes the result in Eq. (18) only approximate. For these reasons the magnitude and phase of the term proportional to χ_0^2 are used as parameters in fitting the computations. Indications are that the term in Eq. (18) depends sensitively on the starting value of k_t in the design of the transverse focusing system, but that it is not as important as the first term in Eq. (17).

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9. These will be described in greater detail in reports at the LASL Linear Accelerator Conference, October 3-7, 1966.

IV. Application of Liouville's Theorem

In the course of numerical computations[6,7] to ascertain the seriousness of these transverse amplitude growths, it was noticed that the two-dimensional transverse projection of the four- or six-dimensional phase space did not exactly satisfy conservation of phase-space area. For a distribution in all phase-space directions, this is not surprising and is just the effect predicted in Equations (17) and (18). For a single starting point in longitudinal phase, however, the transverse area appeared to fluctuate by several percent.

The first order (in y_o^2) result in Equations (17) and (18) vanishes if one averages over all values of a_t for fixed a_ℓ . It now appears that higher order effects modify this conclusion. This section is therefore a re-examination of the conditions which apply in general and the extent to which the new notions affect the choice of beam aperture.

The basic coupled equations for the motion in the longitudinal and one transverse direction are given in Equations (1) and (2). The following conclusions are apparent from these equations:

1) For uncoupled motion all two-dimensional space trajectories are circles (ellipses) of constant area. The projected phase-space areas are obviously conserved.

2) If the right sides of Equations (1) and (2) are approximated by using the uncoupled values of y and χ , the individual phase-space areas are distorted but retain their original values. However, the distortions depend on the initial values of the oscillations and for a distribution of initial values there will be an apparent increase in the projected phase-space area. This is the situation discussed previously, and treated in Section II.

3) If the uncoupled solution for χ is used in Eq. (1), one obtains the equivalent homogeneous equation

$$\frac{d}{ds} \left(\beta\gamma \frac{dy}{ds} \right) + \beta\gamma \, q^2 \, y = 0 \tag{23}$$

where q^2 is a known function of s . Once again phase-space area must be conserved in a particular two-dimensional projection for a point in the other projections. (The determinant of the infinitesimal transformation is still 1.)

4) If one includes in the χ on the right side of Eq. (1) that part of the solution of Eq. (2) which depends on y , the arguments which lead to conservation of area no longer apply. It is this case which is explored now in greater detail.

It is clear from Equations (17) and (18) that, to first order in y_o^2 , the change in transverse phase-space area vanishes. In order to see whether this is true to second order in y_o^2 , it is necessary to perform a more careful integration of Equations (1), (2). We shall, however, accomplish this same goal by using the fact that the __sum__ of the phase-space projections in the transverse and longitudinal projections remains constant.[10] The longitudinal projection can more easily be calculated to second order since it starts as a point at s = 0. In fact one can just solve Eq. (13) to the order shown to obtain the desired area change.

As in Section II, we shall include all first order (in χ_o) terms and only those second order terms which are important for k_t almost equal to k_ℓ . The terms in Eq. (15) which persist at s = ∞ , and which lead to a distortion from a point to an ellipse, are the last two. The terms on the right side of Eq. (13) which contribute for k_t near k_ℓ include the cos (t - ℓ) term of $y^{(1)}$ from Eq. (14) and the cos 2t term of $\chi^{(1)}$ from Eq. (15). After considerable algebra, one finds that the point in longitudinal phase space has become an ellipse of major and minor axes given by

$$\left(\frac{\pi y_o}{\beta\gamma\lambda} \right)^2 \frac{k_\ell}{4} \left\{ \left| \frac{g}{2k_t - k_\ell} - \frac{i \chi_o e^{-ia\ell}}{2k_t - 2k_\ell} \left(\frac{g^2 k_\ell(k_\ell + k_t)}{4k_t^2 - k_\ell^2} - \frac{1}{2} \right) \right| \pm \frac{g}{2k_t + k_\ell} \right\} .$$

$$(24)$$

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10. See, for example, L. Goldstein, "Classical Mechanics," p. 247ff, 1st edition, Addison-Wesley Publishing Co. By use of the Poincaré Invariants, Goldstein shows that, in a Hamiltonian system, the quantity

$$\sum_i \iint_s dp_i \, dq_i$$

is an invariant if the surface S, bounded by a closed curve, is allowed to move with the points in phase space. The individual terms in the sum over i are the projections, if one takes into account the sign and multiplicity of the contributions.

This ellipse is actually traversed twice as a_t goes from 0 to 2π. One therefore uses the Poincaré Invariant[10] to obtain

$$\delta \text{ Area}^{(t)} = - 2 \delta \text{ Area}^{(\ell)} \tag{25}$$

and finally

$$\frac{\delta \text{ Area}^{(t)}}{\text{Area}^{(t)}} = - \frac{k_\ell^3}{32 \, k_t} \left(\frac{\pi \, y_o}{\beta\gamma\lambda} \right)^2 \left\{ \left| \frac{g}{2k_t - k_\ell} - \frac{i \, \chi_o \, e^{-ia\ell}}{2k_t - 2k_\ell} \left(\frac{g^2 \, k_\ell(k_\ell + k_t)}{4k_t^2 - k_\ell^2} - \frac{1}{2} \right) \right|^2 - \left(\frac{g}{2k_t + k_\ell} \right)^2 \right\} . \tag{26}$$

The result in Eq. (26) gives the area change at the end of acceleration (after coupling terms are unimportant). One expects to see wider variations and oscillations during acceleration. If a particular term in Eq. (26) dominates, one will actually see oscillations which reach approximately double this value. If several terms are important, however, each will reach intermediate peaks at different energies and the situation will be more complicated. As a guide, one can assume that fluctuations will carry the area change to between -1 and +2 times the value given by Eq. (26). This is indicated by the numerical calculations.

Equation (26) expresses the decrease in transverse phase-space area for a fixed starting point in longitudinal phase space and a "matched" beam in transverse phase space. This must be included with any apparent increase in phase-space area due to a distribution of points in longitudinal phase space. It should be noted that Eq. (26) predicts an area change even in the absence of longitudinal oscillation, a phenomenon observed in the numerical results. However, this area change appears to be substantially smaller than that due to the distortions discussed in Section II, and is primarily of academic interest only.

Numerical computations[6,7,9] illustrate the area changes discussed here, although precise agreement has not been obtained for the term proportional to χ_o. Nevertheless the transverse area does decrease by the order of magnitude specified in Eq. (26), and the effect is clearly proportional to the square of the transverse displacement. Precise agreement is not expected because of:

1) The possibility of contribution of terms for $j \geq 3$, and of terms of higher order in χ_o .

2) The fact that the values of k_ℓ , k_t used in the denominators of Eq. (26) should be those appropriate to the actual longitudinal and transverse oscillations, not the linearized version appropriate to $X_o = 0$, $y_o = 0$.

3) The possibility of contributions from resonances of the form $2k_t = j \, k_\ell$.

4) The limited accuracy of the numerical calculations.

One last point should be discussed. Courant[11] has shown that a coupled dynamical system satisfying certain conditions[12] conserves the area of its two-dimensional projections along particular axes. In the uncoupled case these are the normal axes for the separate coordinates. In the coupled case these special axes are linear combinations of the longitudinal and transverse coordinates. The conclusions in this note, applying only to the simple coordinate axes, are therefore not in violation of the theorem proved by Courant.

V.   Summary and Conclusions

1) Coupling of the longitudinal and transverse motion takes place through the phase dependence of the rf defocusing force and the radial dependence of the longitudinal transit time factor. Both of these effects arise from the same term in the Lagrangian.

2) The coupling forces decrease rapidly with increasing β . Consequently the seriousness of the effect depends primarily on the parameters at the start of acceleration.

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11.   E.D. Courant, private communication. The author is grateful to Dr. Courant for calling his attention to this point.

12.   Hamiltonian system, with the motion given by a linear transformation matrix which is symplectic.

3) The main contribution to the growth in transverse dimension (and phase-space area) comes from the first coupling term in Equations (1) and (2). The prediction is for an amplitude increase

$$\frac{\delta \ A(\infty)}{A(0)} = \frac{k_\ell^2 \ \cot \ |\varphi_s|}{2(4k_t^2 - k_\ell^2)} \ \chi_o \tag{27}$$

at the end of acceleration (actually this increase takes place within the first 10 MeV of acceleration). The growth will be approximately twice as large in the course of reaching the final value in Eq. (27).

4) For comparable k_ℓ and k_t at the start of acceleration, Eq. (27) predicts a 15 - 20% transverse amplitude increase, and a 30 - 45% increase in transverse emittance. The effect increases for lower k_t (weaker transverse focusing) and suggests transverse focusing designs which make k_t as large as possible at injection.

5) The effect of that part of the second coupling term which dominates for small $2k_t$ - $2k_\ell$ has also been taken into account, and the result is given in Eq. (18). Numerical computations of orbits indicate that Eq. (18) overestimates the effect. The following considerations are relevant, however:

a) For small $2k_t$ - $2k_\ell$ one should use a resonant treatment,[3] which leads to a finite result at $k_t = k_\ell$.

b) Equation (18) is sensitive to the actual values of k_t and k_ℓ used. For nonlinear coupled oscillations these will be different enough from the linear uncoupled values to make the prediction of Eq. (18) uncertain.

c) There will be also contributions of order χ_o^3 and higher which have been omitted here, but which can be important.

In spite of these uncertainties, Eq. (18) does predict a contribution to the distortion which has been observed in the numerical computations to be of the same sign and order of magnitude as predicted.

6) If one considers a collection of starting points in phase space, each having the same χ_o , a_ℓ , y_o , but with different a_t , the transverse phase-space area can change. This change (usually a decrease in transverse area)

is relatively small and unimportant compared to the distortion, but it does not violate Liouville's Theorem. Analytic results for this area change are given in Eq. (26).

VI. Acknowledgments

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