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MAGNET ERRORS IN LINEAR ACCELERATORS

R. L. Gluckstern

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I INTRODUCTION

The effect of magnet imperfections on the transverse oscillations can be most conveniently treated in the formalism of Courant and Snyder.[1] The important elements of this formalism are reproduced here.

The transverse oscillations in a periodic focussing system with period length L obey the equation

$$\frac{d^2y}{ds^2} + K(s)y = 0 \tag{1}$$

where s is the axial length and K(s) is the periodic focussing function. A 2 x 2 matrix formalism can be used to describe the change in the "vector" (y,y') from one value of s to another. The matrix M which carries the vector through one repeat length L can be parametrized as

$$M = \begin{pmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\frac{(1+\alpha^2)}{\beta}\sin\mu & \cos\mu - \alpha\sin\mu \end{pmatrix} \tag{2}$$

where use has been made of the fact that the determinant is 1 and the trace is 2 cos μ. For stability the quantity μ must be real. This parameter is a constant related to the propagation constant of Floquet's theorem. The parameters β and α are periodic functions of s.

1. E. D. Courant and H. S. Snyder, Annals of Physics, 3, 1 (1958).

Courant and Snyder have shown that $\beta(s)$ satisfies the differential equation

$$2\beta\beta'' - \beta'^2 + 4K\beta^2 = 4 \tag{3}$$

which can be converted to the linear equation

$$\beta''' + 4K\beta' + 2K'\beta = 0 \tag{4}$$

The correct value of $\beta(s)$ is that linear combination of the three solutions of (4) which satisfies (3) and has $\beta(s + L) = \beta(s)$ and $\beta'(L + s) = \beta'(s)$. The function $\alpha(s)$ is given by

$$2\alpha = -\beta' \tag{5}$$

and the propagation constant is given by

$$\mu = \int_0^L \frac{ds}{\beta} \cong \frac{L}{\lambda\!\!\!^-} \tag{6}$$

where $2\pi\lambda\!\!\!^-$ is the wave length of the transverse oscillation.

Once $\beta(s)$ has been found from (3) and (4), the solutions of (1) can be written in the exact form

$$\begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} = \left[\beta(s)\right]^{\frac{1}{2}} \begin{Bmatrix} \sin \phi\ (s) \\ \cos \phi\ (s) \end{Bmatrix} \tag{7}$$

where

$$\phi(s) = \int_0^s \frac{ds}{\beta} \tag{8}$$

These solutions are normalized such that

$$y_1'y_2 - y_1y_2' = 1. \tag{9}$$

From (7) and (8), one can show that the quantity

$$W = \frac{Y^2}{\beta_{max}} = \frac{y^2}{\beta} + \frac{(\alpha y + \beta y')^2}{\beta} \tag{10}$$

is a constant of the motion. In fact, if the parameters vary slowly from period to period, W is an adiabatic invariant. The maximum displacement, $Y = \left[\beta_{max} W\right]^{1/2}$, occurs when β is at its maximum value β_{max} ($\alpha = 0$) and when $y' = 0$.

II MAGNET DISPLACEMENT

If the magnets (or other elements) are displaced according to the function $\Delta(s)$, the transverse oscillations obey the equation

$$\frac{d^2y}{ds^2} + K(s)y = K(s)\Delta(s). \tag{11}$$

The solution of this equation may be expressed in terms of y_1 and y_2 as

$$\begin{aligned} y(s) &= y_1(s)\left[W_o^{1/2} \cos\chi + \int_o^s ds'\; y_2(s')\Delta(s')K(s')\right] \\ &+ y_2(s)\left[W_o^{1/2} \sin\chi - \int_o^s ds'\; y_1(s')\Delta(s')K(s')\right] \end{aligned} \tag{12}$$

where W_o and χ determine the amplitude and phase of the oscillation in the absence of errors.* The adiabatic invariant now has the approximate value

$$W^{1/2} \cong W_o^{1/2} + \int_o^s ds'\; \Delta(s')K(s')\left[y_2(s')\cos\chi - y_1(s')\sin\chi\right]. \tag{13}$$

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* $y(o) = (W_o\beta)^{1/2} \sin\chi, \quad \alpha y(o) + \beta y'(o) = (W_o\beta)^{1/2}\cos\chi$

The task is now to obtain the expectation value of $(W^{1/2} - W_o^{1/2})^2$ for a particular method of adjusting the magnet positions, corresponding to a specific distribution of $\Delta(s)$. The result will also depend on the way in which the focussing elements are distributed within a period. Several cases will be discussed.

Case 1. Uncorrelated Magnet Errors -- Focussing Doublets

For individual magnets of length ℓ, one finds

$$W^{1/2} - W_o^{1/2} = \ell \sum_j K_j \Delta_j \left[y_2(s_j) \cos\chi - y_1(s_j) \sin\chi \right] . \quad (14)$$

Setting $\langle \Delta_i \Delta_j \rangle = \Delta^2 \delta_{ij}$, one obtains

$$\langle (W^{1/2} - W_o^{1/2})^2 \rangle = \ell^2 \Delta^2 \sum_m K_m^2 (\beta_m^+ + \beta_m^-) \cos^2(\phi_m + \chi) \quad (15)$$

where β_m^+ and β_m^- are the values of β at the focussing and defocussing magnets respectively, and each value of m corresponds to one magnet period. If Y_{fo} is the value of Y expected at the exit of the focussing system in the absence of magnet errors and the subscript f stands for the exit, one has

$$\Delta Y_{rms}^2 \equiv \langle (Y_f - Y_{fo})^2 \rangle = \beta_{max,f} \frac{\ell^2 \Delta^2}{2} \sum_{m=1}^{N} K_m^2 (\beta_m^+ + \beta_m^-) \quad (16)$$

where the average value of $\cos^2$ has been taken to be 1/2. For a design with constant values of K_m and $\beta_m^{\pm}$ from cell to cell, one has approximately

$$\Delta Y_{rms}^2 \simeq N \beta^2 K^2 \ell^2 \Delta^2 \quad (16a)$$

where N is the total number of magnet periods. It should be remembered that any alignment procedure which relies on angular measurement will lead to a length-dependent value of Δ, in which case ΔY^2_{rms} will be proportional to $N^{3/2}$.

For doublet magnets each of length ℓ, separated by a distance ℓ', in a period L, one can show in the smooth approximation (neglecting rf defocussing forces) that*

$$\frac{1}{\lambda^2} \simeq K^2 \ell^2 \, \frac{\ell'(L - \ell')}{L^2} \qquad (16b)$$

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*An expansion in powers of the focussing strength is carried out in Courant and Snyder who show that (Eq. 3.39 with $\bar{g} = 0$)

$$\frac{\mu^2}{L^2} = \frac{1}{\lambda^2} \simeq \left\langle \left(\frac{\epsilon f_1}{2}\right)^2 \right\rangle$$

where $\frac{\epsilon f_1''}{2}(s) = - K(s)$. If $\ell \ll \ell'$, one can write $K(s) =$

$K\ell \left[\delta(s) - \delta(s - \ell')\right]$. This leads to

$$\frac{\epsilon f_1'}{2} = \begin{cases} \dfrac{-K(L - \ell')}{L} & o < s < \ell' \\[2ex] +K\ell \, \dfrac{\ell'}{L} & \ell' < s < L \end{cases}$$

which in turn leads directly to (16b).

In this approximation $\beta \sim \lambda$ and (16a) becomes

$$\Delta Y_{rms}^{2} \sim \frac{N\Delta^2 L^2}{\ell' (L - \ell')} \qquad (16c)$$

Case 2. Uncorrelated Magnet Errors -- Focussing Triplets

The analysis for triplets is parallel to that for doublets, with (16) becoming

$$\Delta Y_{rms}^{2} = \beta_{max,\ f} \frac{\ell^2 \Delta^2}{2} \sum_{m=1}^{N} K_m^{2} (2\beta_m^{+} + 4\beta_m^{-}) \qquad (17)$$

Here the focussing magnets are each of length ℓ and strength K, and the defocussing magnet of length 2ℓ and strength K, the + - + triplet having inter-magnet spacing $\frac{\ell'}{2}$. The equivalent relations to (16a), (16b) and (16c) are

$$\Delta Y_{rms}^{2} \simeq 3N\beta^2 K^2 \ell^2 \Delta^2 \ , \qquad (17a)$$

$$\frac{1}{\lambda^2} \simeq K^2 \ell^2 \frac{\ell'}{L} , \qquad (17b)$$

$$\Delta Y_{rms}^{2} \sim 3N\Delta^2 \frac{L}{\ell'} , \qquad (17c)$$

Case 2a. Triplet Units

If the triplets are each perfectly bench-aligned and the ends of the triplets are then aligned relative to the machine axis, the main component of the error is removed. In this case (17) is replaced in the smooth approximation by

$$\Delta Y^2_{rms} = \beta_{max,f}\,\ell^2\Delta^2 \sum_{m=1}^{N} K_m{}^2 \left(\frac{\ell' m}{2\lambda}\right)^2 \beta_m^+ \qquad (18)$$

The equivalent of (17c) is

$$\Delta Y^2_{rms} \sim N\Delta^2 \frac{L}{\ell'} \left(\frac{\ell'}{2\lambda}\right)^2 \qquad (18c)$$

which represents a significant reduction from (17c), since ℓ' is usually small compared to λ .

Case 3. Alignment via Monuments

As pointed out in the discussion following (16a), alignment of long machines will lead to prohibitively large displacements if each magnet is aligned to some predetermined curve with an accuracy proportional to the length. For this reason Courant[2] has analyzed a procedure of aligning monuments and subsequently aligning the magnets with respect to these monuments for a circular accelerator. The procedure works as well for a linear accelerator and leads to a reduction in the amplitude growth by a factor of order 1/M, where M is the number of monuments.

This can be seen as follows:

There are two sets of uncorrelated errors in such a procedure. The first is the set of monument errors and the second is the set of errors in the adjustment of the magnets to the line joining the nearest pair of monuments. The second error leads of course to formulas (16) - (18), but in this case the value of $\langle \Delta^2 \rangle^{1/2}$ is consistent with a length NL/M instead of NL.

2. E. D. Courant, Internal Report IA-3, EDC-43 (1961).

The set of monument errors leads to an increase in amplitude which can be viewed as being due to a change in angle as one passes a monument. If the line joining the mth and (m + 1)st monuments makes an angle θ_m with the accelerator axis, one finds from (10)

$$Y\Delta Y = \beta_{max} \sum_m (\alpha y + \beta y')_m \Delta y'_m \qquad (19)$$

with $\Delta y'_m = \theta_{m+1} - \theta_m$. Writing

$$\alpha y + \beta y' = Y\left(\frac{\beta}{\beta_{max}}\right)^{1/2} \cos\psi, \quad y = Y\left(\frac{\beta}{\beta_{max}}\right)^{1/2} \sin\psi \qquad (20)$$

with $\psi' = \frac{1}{\beta}$, one finds

$$\Delta Y = \beta_{max}^{1/2} \sum_m \beta_m^{1/2} (\theta_{m+1} - \theta_m) \cos\psi_m \qquad (21)$$

The value of ΔY_{rms}^2 now depends on how the angular errors are correlated.

Case 3a. First Difference Monument Errors

If the monument errors are of the first difference type (as in the case of alignment of the vertical position with a level or plumb line), each θ_m is uncorrelated and one has

$$\Delta Y_{rms}^2 \simeq \beta_{max,f} \left\langle \theta^2 \right\rangle \sum_{m=1}^{M} \beta_m . \qquad (22)$$

For a constant β design, in the smooth approximation one finds

$$\Delta Y_{rms}^2 \simeq M\beta^2 \left\langle \theta^2 \right\rangle \qquad (22a)$$

Case 3b. Second Difference Monument Errors

For second difference type errors (as in the alignment of horizontal position), each value of $\Delta\theta_m = \theta_{m+1} - \theta_m$ is uncorrelated and one has

$$\Delta Y^2_{rms} \simeq \frac{\beta_{max,f}}{2} \left\langle (\Delta\theta)^2 \right\rangle \sum_{m=1}^{M} \beta_m \tag{23}$$

and

$$\Delta Y^2_{rms} \simeq \frac{M\beta^2}{2} \left\langle (\Delta\theta)^2 \right\rangle \tag{23a}$$

III ERRORS IN MAGNET FIELDS

If the magnetic field gradients have uncorrelated errors, the transverse amplitude will be stimulated as it is by misalignments. In fact, the result in (15) can be converted to the case of gradient errors by replacing the displacement by its equivalent.

$$\Delta \longrightarrow \frac{\Delta K}{K} y = \frac{\Delta K}{K} Y \sin (\phi_m + \chi) \tag{24}$$

where $\frac{\Delta K}{K}$ is the fractional gradient error. This leads to

$$\left\langle (W^{1/2} - W_o^{1/2})^2 \right\rangle = Y^2 \ell^2 \left\langle \left(\frac{\Delta K}{K}\right)^2 \right\rangle \sum_m K_m^2 (\beta_m^+ + \beta_m^-) \cos^2(\varphi_m + \chi) \sin^2(\varphi_m + \chi), \tag{25}$$

Since the average of $\sin^2\varphi \cos^2\varphi$ is $\frac{1}{8}$, one finds

$$\Delta Y^2_{rms} = \beta_{max,f} \frac{Y^2 \ell^2}{8} \left(\frac{\Delta K}{K}\right)^2 \sum_{m=1}^{N} K_m^2 (\beta_m^+ + \beta_m^-) \qquad (26)$$

which, in the smooth approximation, becomes

$$\Delta Y^2_{rms} \simeq \frac{NY^2\beta^2K^2\ell^2}{4} \left\langle \left(\frac{\Delta K}{K}\right)^2 \right\rangle \qquad (26a)$$

or

$$\Delta Y^2_{rms} \sim \frac{NY^2}{4} \frac{L^2}{\ell'(L - \ell')} \left\langle \left(\frac{\Delta K}{K}\right)^2 \right\rangle \qquad (26b)$$

Clearly, any correlation of field errors in adjacent magnets will lead to a reduction of the estimate in (26) similar to that in (18c).

IV ERRORS IN ORIENTATION OF TRANSVERSE MAGNETIC AXES

Any random error in rotation of individual magnets about their null axes will lead to a coupling of both transverse oscillations and to an increase in each. An analysis similar to that in Section III leads to the result

$$\Delta Y^2_{rms} \simeq 4NX^2\beta^2K^2\ell^2 \left\langle (\Delta\varphi)^2 \right\rangle \qquad (27)$$

$$\Delta X^2_{rms} \simeq 4NY^2\beta^2K^2\ell^2 \left\langle (\Delta\varphi)^2 \right\rangle \qquad (28)$$

where X and Y are the two transverse amplitudes. Here $\left\langle (\Delta\varphi)^2 \right\rangle$ is the average of the square of the angular error. Once again, if the errors in adjacent magnet orientations are correlated, these results are greatly reduced.

V SUMMARY

In general all types of errors may be present -- individual magnet positions, monument positions, position errors for groups of magnets, field errors, orientation errors, etc. As long as these are uncorrelated, one just adds values of $\Delta Y_{rms}^{\ 2}$ for each source of error.

As an example, let us consider the case of imperfectly "bench-aligned" triplets, which are then aligned relative to monuments. The contribution due to the monument (2nd difference alignment) errors is given in (23a). The contribution due to the errors in the alignment of the triplets to the line joining monuments is given in (18c). The error due to the inaccuracy of bench-alignment within a triplet is given by (17a), where Δ^2 is the rms displacement error, which is one-third of the averaged square second difference error of the triplet.

It is also clear from the analysis, and from the treatment of circular accelerators, that the essential requirement is the alignment of the transverse focussing system to any "smooth" curve. The quantitative definition of "smooth" is contained in a Fourier analysis of the displacement error. The use of monuments provides a method of minimizing those Fourier components which cause large buildup of the transverse amplitude.

It should be mentioned that the difficulties caused by alignment errors are relatively less important in linear accelerators than in circular accelerators where the particles traverse the focussing system many times.