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Minimisation of orbit deviations by the least square method taking into account the limited strength of the correctors

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1. Introduction

The SPS closed orbit correctors were designed to provide sufficient strength for correcting the orbit in a well-defined machine configuration. In these conditions the efficient beam bump method is usually preferred for its simplicity and numerical stability. The method is applied when the SPS operates without major modifications to the beta-function and with all the correctors working properly. However, for some low ß configurations the amplitude and phase of the betatronic function could lead to excessive intensity values in some correctors while using the classic beam bump method. The method could also be difficult to apply at places where the beta-functions are standard but some correctors are temporarily out of service , or when the closed orbit has to be corrected at high energies with pulsed CO dipoles.

For these reasons a method was worked out whereby the deviations between the desired closed orbit and those which could be obtained with the present correctors are minimised; this method takes into account the maximum kick attainable by each deflector. As 'minimisation' criterion one could choose either to minimise the maximum deviation or to minimise the sum of the squares of the deviations. The analysis of the former is more difficult to carry out, the algorithms being time consuming, while the later leads to well known techniques implemented with much faster algorithms.

Preference was given to minimising the sum of the squares of deviations from an ideal orbit. The present note describes a method which solves this problem and which takes into account the contraints imposed by the attainable kicks.

2. Description of the method

Let us consider a section of the ring, in which there are N position monitors and N correctors numbered from i = r to i = s (N = s - r + 1). Let be $\tilde{y}_i$ (i = r, ..., s) the closed orbit positions measured by the beam monitors (units in mm). Our aim is to use the N available correctors to correct (in the least square sense) the closed orbit inside this interval, while not perturbing it outside.

This is equivalent to saying that we have to find a set of N kicks created by the correctors such that the sum of the squares of the resulting closed orbit

$$S = \sum_{i=r}^{s} \left(y_i + \tilde{y}_i \right)^2 \tag{1a}$$

will be minimum and

$$y_i + \tilde{y}_i = \tilde{y}_i \qquad \text{for } i < r \text{ and } i > s \tag{1b}$$

y_i being given by the following expression:

$$y_i = \frac{1}{2 \sin \pi Q} \sum_{j=r}^{s} k_j \sqrt{\beta_i \beta_j} \cos \left[\pi Q - |\mu_i - \mu_j| \right] \tag{2}$$

Replacing the expression (2) in the conditions (1b), one obtains for i < r

$$\frac{\sqrt{\beta_i}}{2\sin\pi Q}\left\{\cos(\mu_i+\pi Q)\sum_{j=r}^{s} k_j\sqrt{\beta_j}\cos\mu_j + \sin(\mu_i+\pi Q)\sum_{j=r}^{s} k_j\sqrt{\beta_j}\sin\mu_j\right\} = \phi \qquad (3)$$

and for i > s

$$\frac{\sqrt{\beta_i}}{2\sin\pi Q}\left\{\cos(\pi Q-\mu_i)\sum_{j=r}^{s} k_j\sqrt{\beta_j}\cos\mu_j - \sin(\pi Q-\mu_i)\sum_{j=r}^{s} k_j\sqrt{\beta_j}\sin\mu_j\right\} = \phi \qquad (4)$$

Thus the conditions (1b) are satisfied if

$$\left\{\begin{array}{l} \displaystyle\sum_{j=r}^{s} k_j\sqrt{\beta_j}\cos\mu_j = \phi \\ \displaystyle\sum_{j=r}^{s} k_j\sqrt{\beta_j}\sin\mu_j = \phi \end{array}\right. \qquad (5)$$

This system shows that two kicks depend linearly on the other kicks. We choose the kicks to be k_r and k_s. The system (5) becomes

$$\left\{\begin{array}{l} k_r\sqrt{\beta_r}\cos\mu_r + k_s\sqrt{\beta_s}\cos\mu_s = -\displaystyle\sum_{j=r+1}^{s-1} k_j\sqrt{\beta_j}\cos\mu_j \\ k_r\sqrt{\beta_r}\sin\mu_r + k_s\sqrt{\beta_s}\sin\mu_s = -\displaystyle\sum_{j=r+1}^{s-1} k_j\sqrt{\beta_j}\sin\mu_j \end{array}\right. \qquad (6)$$

By solving this system we obtain

$$K_r = -\frac{\sum_{j=r+1}^{s-1} k_j \sqrt{\beta_j}\,\sin(\mu_s-\mu_j)}{\sqrt{\beta_r}\,\sin(\mu_s-\mu_r)}$$

$$K_s = \frac{\sum_{j=r+1}^{s-1} k_j \sqrt{\beta_j}\,\sin(\mu_r-\mu_j)}{\sqrt{\beta_s}\,\sin(\mu_s-\mu_r)} \tag{7}$$

It is important to note that under the conditions (5) also y_r and y_s are null. In fact from expression (1) we have

$$y_r = \frac{\sqrt{\beta_r}}{2\sin\pi Q} \sum_{j=r}^{s} \sqrt{\beta_j}\, k_j \cos\left[\pi Q - \mu_j + \mu_r\right] = \phi$$

$$y_s = \frac{\sqrt{\beta_s}}{2\sin\pi Q} \sum_{j=r}^{s} \sqrt{\beta_j}\, k_j \cos\left[\pi Q + \mu_j - \mu_s\right] = \phi$$

Let us now call

$$\alpha_{ij} = \frac{\sqrt{\beta_i\beta_j}}{2\sin\pi Q} \cos\left[\pi Q - |\mu_i - \mu_j|\right]$$

The positions of the resulting closed orbit may then be written as

$$\tilde{y}_i + y_i = \tilde{y}_i + \sum_{j=r}^{s} \alpha_{ij} k_j \tag{8}$$

Bearing in mind that k_r and k_s depend linearly on the other k_j we could also write

$$\tilde{y}_i + y_i = \tilde{y}_i + \alpha_{ir} k_r + \alpha_{is} k_s + \sum_{j=r+1}^{s-1} \alpha_{ij} k_j =$$

$$= \tilde{y}_i + \sum_{j=r+1}^{s-1} k_j \left[\alpha_{ij} + \frac{\alpha_{is}\sqrt{\beta_r}\sin(\mu_r-\mu_j) - \alpha_{ir}\sqrt{\beta_s}\sin(\mu_s-\mu_j)}{\sqrt{\beta_r\beta_s}\,\sin(\mu_s-\mu_r)} \right] =$$

$$= \tilde{y}_i + \sum_{j=r+1}^{s-1} \gamma_{ij} k_j \qquad (9)$$

where

$$\gamma_{ij} = \alpha_{ij} + \frac{\alpha_{is}\sqrt{\beta_r}\sin(\mu_r-\mu_j) - \alpha_{ir}\sqrt{\beta_s}\sin(\mu_s-\mu_j)}{\sqrt{\beta_r\beta_s}\,\sin(\mu_s-\mu_r)} \qquad (10)$$

or, after some lengthy computations

$$\gamma_{ij} = \frac{\sqrt{\beta_i\beta_j}}{\sin(\mu_s-\mu_r)} \sin(\mu_s-\mu_j)\sin(\mu_r-\mu_i) \qquad i = r+1, \ldots, s-1 \quad j \geq i$$

$$\gamma_{ji} = \gamma_{ij}$$

Thus (1a) becomes

$$S = \sum_{i=r+1}^{s-1} (\tilde{y}_i + y_i)^2 + \tilde{y}_r^2 + \tilde{y}_s^2 = \qquad (11)$$

$$= \sum_{i=r+1}^{s-1} \left[\tilde{y}_i + \sum_{j=r+1}^{s-1} \gamma_{ij} k_j \right]^2 + \tilde{y}_r^2 + \tilde{y}_s^2$$

To minimise S with respect to k_j we have to solve the following linear system

$$\frac{\partial S}{\partial k_\ell} = 0 \qquad \ell = r+1, \ldots, s-1$$

that is

$$\sum_{j=r+1}^{s-1} \gamma_{ij} k_j = -\tilde{y}_i \qquad i = r+1, \ldots, s-1$$

$$\sum_{j=r+1}^{s-1} \Gamma_{ij} k_j = -\tilde{y}_i \qquad i = r+1, \ldots, s-1 \tag{12}$$

As one would have expected, the system (12) is the same as it would be obtained if we would have imposed the condition $\tilde{y}_i + y_i = 0$ for $i = r + 1, \ldots, s - 1$ instead of minimising S. One may easily recognise the classical beam bump method in (12) when $s = r + 2$.

So far we have not considered the limited strength of the correctors. Let us now suppose that the kicks should be bounded as follows:

$$|k_j| \leq \bar{k}_j \qquad j = r+1, \ldots, s-1 \tag{13}$$

Let us consider the case where some k_j, (the solution of the linear system (12)) do not satisfy the inequalities (13). We choose the largest kick and replace it by the corresponding $\bar{k}_j$. We then compute the remaining k_j by solving the linear system (as will be described later) and repeat the procedure until all the k_j's satisfy the inequalities (13). This is justified by the fact that S is a positive defined quadratic form.

Let k_n be the largest kick and $|k_n| > \bar{k}_n$. Replacing k_n by $\bar{k}_n \operatorname{sgn}(k_n)$ in (11) we obtain the new sum

$$S_1 = \sum_{i=r+1}^{s-1} \left[\tilde{y}_i + \gamma_{in} \bar{k}_n \operatorname{sgn}(k_n) + \sum_j{}' \gamma_{ij} k_j \right]^2 + \tilde{y}_r^2 + \tilde{y}_s^2 \tag{14}$$

where $\sum_j{}'$ stands for $\sum_{\substack{j=r+1 \\ j \neq n}}^{s-1}$

The minimum of S_1 is found by solving the system

$$\frac{\partial S_1}{\partial k_\ell} = \phi \qquad \ell = r+1, \ldots, n-1, n+1, \ldots, s-1 \tag{15}$$

Let us now compute

$$\frac{\partial S}{\partial k_\ell} = 2 \sum_{i=r+1}^{s-1} \left\{ \tilde{y}_i + \gamma_{in} \bar{k}_n \, \mathrm{sgn}(k_n) + \sum_j{}' \gamma_{ij} k_j \right\} \gamma_{i\ell} =$$

$$= 2 \left\{ \sum_{i=r+1}^{s-1} \left[\gamma_{i\ell} \left(\tilde{y}_i + \gamma_{in} \bar{k}_n \, \mathrm{sgn}(k_n) \right) \right] + \sum_j{}' \left[\sum_{i=r+1}^{s-1} \left(\gamma_{i\ell} \gamma_{ij} \right) \right] k_j \right\}$$

The linear system (15) becomes

$$\sum_j{}' \left[\sum_{i=r+1}^{s-1} \gamma_{ij} \gamma_{i\ell} \right] k_j = - \sum_{i=r+1}^{s-1} \left\{ \gamma_{i\ell} \left[\tilde{y}_i + \gamma_{in} \bar{k}_n \, \mathrm{sgn}(k_n) \right] \right\} \tag{16}$$

Solving this system we obtain a new set of $k_j^{(1)}$. If some kicks are larger than $\bar{k}_j$ we again choose the largest one, say $k_m^{(1)}$, replace it by the corresponding $\bar{k}_m(\mathrm{sgn}(k_m)$ and form the new sum:

$$S_2 = \tilde{y}_r^2 + \tilde{y}_s^2 + \sum_{i=r+1}^{s-1} \left\{ \tilde{y}_i + \gamma_{in} \bar{k}_n \, \mathrm{sgn}(k_n) + \gamma_{im} \bar{k}_m \, \mathrm{sgn}(k_m) + \sum_{\substack{j=r+1 \\ j \neq n \\ j \neq m}}^{s-1} \gamma_{ij} k_j \right\}^2$$

We then find a new set of $k_j^{(2)}$ which minimises S_2 and continue the procedure until no new k_j exceeds its corresponding $\bar{k}_j$.

This method has been successfully tested with a real closed orbit. It is now implemented in a FORTRAN subroutine called CORINS and stored in the Library file < 497> LB:FOR. Table 1 shows the results of the correction at the energy of 270 GeV.

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Vertical Plane

Beam Monitor	Pos. Before correction (in mm)	Pos. After correction (in mm)
3-11	.8	- .4
3-13	5.	- 1
3-15	- 1.2	- .4
4-13	- 1.	- .6
4-15	- 2.60	.8
4-17	6.1	- 1
4-19	- 4.6	.3
4-21	5.8	1.2
4-23	- 4.6	1.6
5-13	- 5.3	- .4
5-15	2.3	- .6
5-17	1.7	.6
5-19	- .5	- .2
5-21	1.1	.4
5-23	- 4.2	.6

Horizontal Plane

Beam Monitor	Pos. Before correction (in mm)	Pos. After correction (in mm)
3-12	7.3	- 2.4
3-14	- 3.2	- 1.2
3-16	- 4.8	- .3
4-16	2.3	- .4
4-18	3.8	- 1.2
4-20	- 5.5	.3
5-16	- 3.4	+ .6
5-18	- 4.8	1
5-20	9.4	- .8

TABLE I