1 HEAT CONDUCTION

Contents

1.1 Introduction

Heat conduction is one of the three basic modes of thermal energy transport (convection and radiation being the other two) and is involved in virtually all process heat-transfer operations. In commercial heat exchange equipment, for example, heat is conducted through a solid wall (often a tube wall) that separates two fluids having different temperatures. Furthermore, the concept of thermal resistance, which follows from the fundamental equations of heat conduction, is widely used in the analysis of problems arising in the design and operation of industrial equipment. In addition, many routine process engineering problems can be solved with acceptable accuracy using simple solutions of the heat conduction equation for rectangular, cylindrical, and spherical geometries.

This chapter provides an introduction to the macroscopic theory of heat conduction and its engineering applications. The key concept of thermal resistance, used throughout the text, is developed here, and its utility in analyzing and solving problems of practical interest is illustrated.

1.2 Fourier's Law of Heat Conduction

The mathematical theory of heat conduction was developed early in the nineteenth century by Joseph Fourier [1]. The theory was based on the results of experiments similar to that illustrated in Figure 1.1 in which one side of a rectangular solid is held at temperature T_1, while the opposite side is held at a lower temperature, T_2. The other four sides are insulated so that heat can flow only in the x-direction. For a given material, it is found that the rate, q_x, at which heat (thermal energy) is transferred from the hot side to the cold side is proportional to the cross-sectional area, A, across which the heat flows; the temperature difference, $T_1 - T_2$; and inversely proportional to the thickness, B, of the material. That is:

$$q_x \propto \frac{A(T_1 - T_2)}{B}$$

Writing this relationship as an equality, we have:

$$q_x = \frac{kA(T_1 - T_2)}{B} \tag{1.1}$$

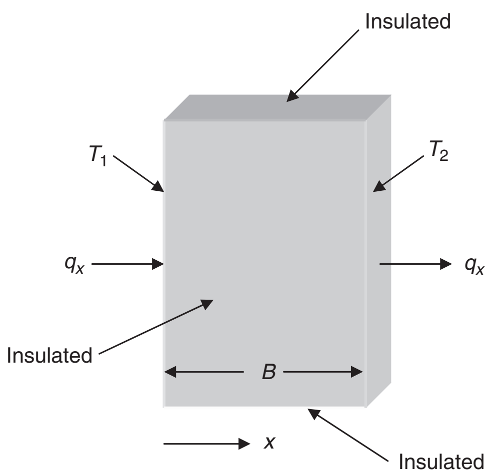


Figure 1.1 One-dimensional heat conduction in a solid.

The constant of proportionality, k, is called the thermal conductivity. Equation (1.1) is also applicable to heat conduction in liquids and gases. However, when temperature differences exist in fluids, convection currents tend to be set up, so that heat is generally not transferred solely by the mechanism of conduction.

The thermal conductivity is a property of the material and, as such, it is not really a constant, but rather it depends on the thermodynamic state of the material, i.e., on the temperature and pressure of the material. However, for solids, liquids, and low-pressure gases, the pressure dependence is usually negligible. The temperature dependence also tends to be fairly weak, so that it is often acceptable to treat k as a constant, particularly if the temperature difference is moderate. When the temperature dependence must be taken into account, a linear function is often adequate, particularly for solids. In this case,

$$k = a + bT \tag{1.2}$$

where a and b are constants.

Thermal conductivities of a number of materials are given in Appendices 1.A–1.E. Many other values may be found in various handbooks and compendiums of physical property data. Process simulation software is also an excellent source of physical property data. Methods for estimating thermal conductivities of fluids when data are unavailable can be found in the authoritative book by Poling et al. [2].

The form of Fourier's law given by Equation (1.1) is valid only when the thermal conductivity can be assumed constant. A more general result can be obtained by writing the equation for an element of differential thickness. Thus, let the thickness be Δx and let $\Delta T = T_2 - T_1$. Substituting in Equation (1.1) gives:

$$q_x = -kA\frac{\Delta T}{\Delta x} \tag{1.3}$$

Now in the limit as Δx approaches zero,

$$\frac{\Delta T}{\Delta x} \to \frac{dT}{dx}$$

and Equation (1.3) becomes:

$$q_x = -kA\frac{dT}{dx} \tag{1.4}$$

Equation (1.4) is not subject to the restriction of constant k. Furthermore, when k is constant, it can be integrated to yield Equation (1.1). Hence, Equation (1.4) is the general one-dimensional form of Fourier's law. The negative sign is necessary because heat flows in the positive x-direction when the temperature decreases in the x-direction. Thus, according to the standard sign convention that q_x is positive when the heat flow is in the positive x-direction, q_x must be positive when dT/dx is negative.

It is often convenient to divide Equation (1.4) by the area to give:

$$\hat{q}_x \equiv q_x/A = -k\frac{dT}{dx} \tag{1.5}$$

where $\hat{q}_x$ is the heat flux. It has units of $J/s \cdot m^2 = W/m^2$ or $Btu/h \cdot ft^2$. Thus, the units of k are $W/m \cdot K$ or $Btu/h \cdot ft \cdot {}^\circ F$.

Equations (1.1), (1.4), and (1.5) are restricted to the situation in which heat flows in the x-direction only. In the general case in which heat flows in all three coordinate directions, the total heat flux is obtained by adding vectorially the fluxes in the coordinate directions. Thus,

$$\vec{\hat{q}} = \hat{q}_x \vec{i} + \hat{q}_y \vec{j} + \hat{q}_z \vec{k} \tag{1.6}$$

where $\vec{\hat{q}}$ is the heat flux vector and $\vec{i}$, $\vec{j}$, $\vec{k}$ are unit vectors in the x-, y-, z-directions, respectively. Each of the component fluxes is given by a one-dimensional Fourier expression as follows:

$$\hat{q}_x = -k\frac{\partial T}{\partial x} \quad \hat{q}_y = -k\frac{\partial T}{\partial y} \quad \hat{q}_z = -k\frac{\partial T}{\partial z} \tag{1.7}$$

Partial derivatives are used here since the temperature now varies in all three directions. Substituting the above expressions for the fluxes into Equation (1.6) gives:

$$\vec{\hat{q}} = -k\left(\frac{\partial T}{\partial x}\vec{i} + \frac{\partial T}{\partial y}\vec{j} + \frac{\partial T}{\partial z}\vec{k}\right) \tag{1.8}$$

The vector in parenthesis is the temperature gradient vector, and is denoted by $\vec{\nabla} T$. Hence,

$$\vec{\hat{q}} = -k\vec{\nabla} T \tag{1.9}$$

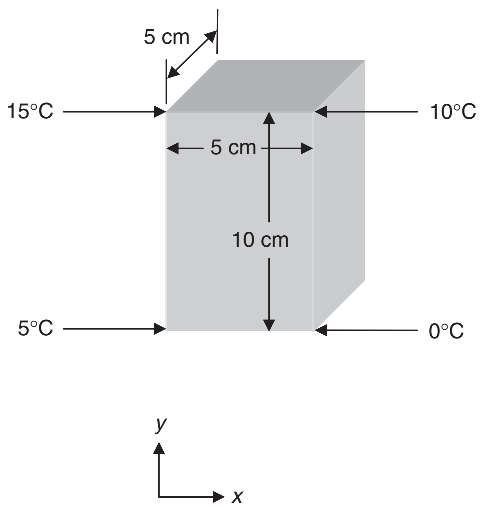
Equation (1.9) is the three-dimensional form of Fourier's law. It is valid for homogeneous, isotropic materials for which the thermal conductivity is the same in all directions.

Equation (1.9) states that the heat flux vector is proportional to the negative of the temperature gradient vector. Since the gradient direction is the direction of greatest temperature increase, the negative gradient direction is the direction of greatest temperature decrease. Hence, Fourier's law states that heat flows in the direction of greatest temperature decrease.

Example 1.1

The block of 304 stainless steel shown below is well insulated on the front and back surfaces, and the temperature in the block varies linearly in both the x- and y-directions, find:

(a) The heat fluxes and heat flows in the x- and y-directions.
(b) The magnitude and direction of the heat flux vector.



Solution

(a) From Table A.1, the thermal conductivity of 304 stainless steel is 14.4 W/m · K. The cross-sectional areas are:

$$A_x = 10 \times 5 = 50\,\text{cm}^2 = 0.0050\,\text{m}^2$$

$$A_y = 5 \times 5 = 25\,\text{cm}^2 = 0.0025\,\text{m}^2$$

Using Equation (1.7) and replacing the partial derivatives with finite differences (since the temperature variation is linear), the heat fluxes are:

$$\hat{q}_x = -k\frac{\partial T}{\partial x} = -k\frac{\Delta T}{\Delta x} = -14.4\left(\frac{-5}{0.05}\right) = 1440\,\text{W/m}^2$$

$$\hat{q}_y = -k\frac{\partial T}{\partial y} = -k\frac{\Delta T}{\Delta y} = -14.4\left(\frac{10}{0.1}\right) = -1440\,\text{W/m}^2$$

The heat flows are obtained by multiplying the fluxes by the corresponding cross-sectional areas:

$$q_x = \hat{q}_x A_x = 1440 \times 0.005 = 7.2\,\text{W}$$

$$q_y = \hat{q}_y A_y = -1440 \times 0.0025 = -3.6\,\text{W}$$

(b) From Equation (1.6):

$$\vec{\hat{q}} = \hat{q}_x \vec{i} + \hat{q}_y \vec{j}$$

$$\vec{\hat{q}} = 1440\,\vec{i} - 1440\,\vec{j}$$

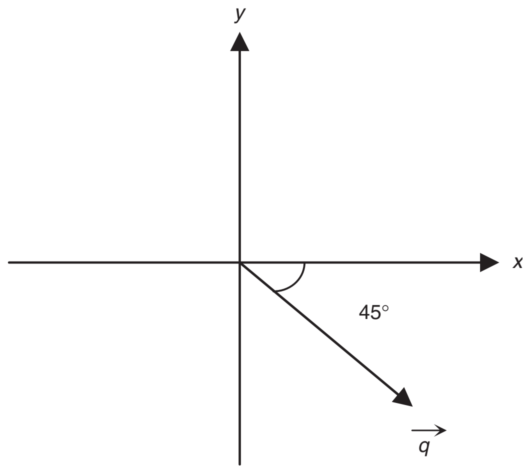
$$\left|\vec{\hat{q}}\right| = [(1440)^2 + (-1440)^2]^{0.5} = 2036.5\,\text{W/m}^2$$

The angle, θ, between the heat flux vector and the x-axis is calculated as follows:

$$\tan\theta = \hat{q}_y/\hat{q}_x = -1440/1440 = -1.0$$

$$\theta = -45^\circ$$

The direction of the heat flux vector, which is the direction in which heat flows, is indicated in the sketch below.



1.3 The Heat Conduction Equation

The solution of problems involving heat conduction in solids can, in principle, be reduced to the solution of a single differential equation, the heat conduction equation. The equation can be derived by making a thermal energy balance on a differential volume element in the solid. For the case of conduction only in the x-direction, such a volume element is illustrated in Figure 1.2. The balance equation for the volume element is:

$$\{\text{rate of thermal energy in}\} - \{\text{rate of thermal energy out}\} + \{\text{net rate of thermal energy generation}\} = \{\text{rate of accumulation of thermal energy}\} \tag{1.10}$$

The generation term appears in the equation because the balance is made on thermal energy, not total energy. For example, thermal energy may be generated within a solid by an electric current or by decay of a radioactive material.

The rate at which thermal energy enters the volume element across the face at x is given by the product of the heat flux and the cross-sectional area, $\hat{q}_x\big|_x A$. Similarly, the rate at which thermal energy leaves the element across the face at $x + \Delta x$ is $\hat{q}_x\big|_{x+\Delta x} A$. For a homogeneous heat source

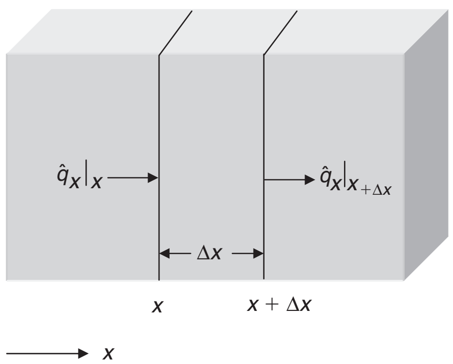


Figure 1.2 *Differential volume element used in derivation of conduction equation.*

of strength $\dot{q}$ per unit volume, the net rate of generation is $\dot{q}A\Delta x$. Finally, the rate of accumulation is given by the time derivative of the thermal energy content of the volume element, which is $\rho c(T - T_{ref})A\Delta x$, where T_{ref} is an arbitrary reference temperature. Thus, the balance equation becomes:

$$(\hat{q}_x|_x - \hat{q}_x|_{x+\Delta x})A + \dot{q}A\Delta x = \rho c \frac{\partial T}{\partial t} A\Delta x$$

It has been assumed here that the density, ρ, and heat capacity, c, are constant. Dividing by $A\Delta x$ and taking the limit as $\Delta x \to 0$ yields:

$$\rho c \frac{\partial T}{\partial t} = -\frac{\partial \hat{q}_x}{\partial x} + \dot{q}$$

Using Fourier's law as given by Equation (1.5), the balance equation becomes:

$$\rho c \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(\frac{k\,\partial T}{\partial x}\right) + \dot{q}$$

When conduction occurs in all three coordinate directions, the balance equation contains y- and z-derivatives analogous to the x-derivative. The balance equation then becomes:

$$\rho c \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(\frac{k\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{k\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{k\partial T}{\partial z}\right) + \dot{q} \tag{1.11}$$

Equation (1.11) is listed in Table 1.1 along with the corresponding forms that the equation takes in cylindrical and spherical coordinates. Also listed in Table 1.1 are the components of the heat flux vector in the three coordinate systems.

When k is constant, it can be taken outside the derivatives and Equation (1.11) can be written as:

$$\frac{\rho c}{k} \frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{q}}{k} \tag{1.12}$$

or

$$\frac{1}{\alpha} \frac{\partial T}{\partial t} = \nabla^2 T + \frac{\dot{q}}{k} \tag{1.13}$$

where $\alpha \equiv k/\rho c$ is the thermal diffusivity and ∇^2 is the Laplacian operator. The thermal diffusivity has units of m^2/s or ft^2/h.

Table 1.1 *The Heat Conduction Equation*

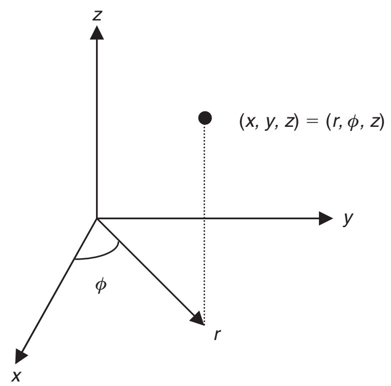
A. Cartesian coordinates

$$\rho c \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) + \dot{q}$$

The components of the heat flux vector, $\vec{\hat{q}}$, are:

$$\hat{q}_x = -k\frac{\partial T}{\partial x} \quad \hat{q}_y = -k\frac{\partial T}{\partial y} \quad \hat{q}_z = -k\frac{\partial T}{\partial z}$$

B. Cylindrical coordinates (r, ϕ, z)

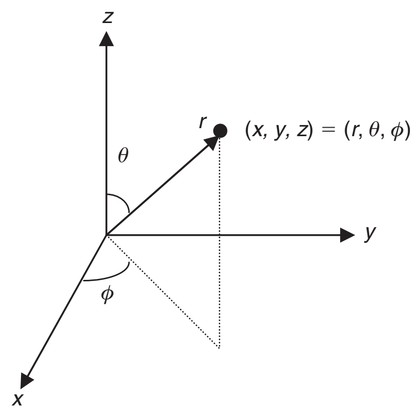


$$\rho c \frac{\partial T}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(k\, r\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \phi}\left(k\frac{\partial T}{\partial \phi}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) + \dot{q}$$

The components of $\vec{\hat{q}}$ are:

$$\hat{q}_r = -k\frac{\partial T}{\partial r}; \quad \hat{q}_\phi = \frac{-k}{r}\frac{\partial T}{\partial \phi}; \quad \hat{q}_z = -k\frac{\partial T}{\partial z}$$

C. Spherical coordinates (r, θ, ϕ)



$$\rho c \frac{\partial T}{\partial t} = \frac{1}{r^2} \frac{\partial}{\partial r}\left(k r^2 \frac{\partial T}{\partial r}\right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta}\left(k \sin\theta \frac{\partial T}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial}{\partial \phi}\left(k \frac{\partial T}{\partial \phi}\right) + \dot{q}$$

The components of $\vec{\hat{q}}$ are:

$$\hat{q}_r = -k\frac{\partial T}{\partial r}; \quad \hat{q}_\theta = -\frac{k}{r}\frac{\partial T}{\partial \theta}; \quad \hat{q}_\phi = -\frac{k}{r\sin\theta}\frac{\partial T}{\partial \phi}$$

The use of the conduction equation is illustrated in the following examples.

Example 1.2

Apply the conduction equation to the situation illustrated in Figure 1.1.

Solution

In order to make the mathematics conform to the physical situation, the following conditions are imposed:

(1) Conduction only in x-direction $\Rightarrow T = \mathrm{T(x)}$, so $\frac{\partial T}{\partial y} = \frac{\partial T}{\partial z} = 0$
(2) No heat source $\Rightarrow \dot{q} = 0$
(3) Steady state $\Rightarrow \frac{\partial T}{\partial t} = 0$
(4) Constant k

The conduction equation in Cartesian coordinates then becomes:

$$0 = k\frac{\partial^2 T}{\partial x^2} \quad \text{or} \quad \frac{d^2T}{dx^2} = 0$$

(The partial derivative is replaced by a total derivative because x is the only independent variable in the equation.) Integrating on both sides of the equation gives:

$$\frac{dT}{dx} = C_1$$

A second integration gives:

$$T = C_1 x + C_2$$

Thus, it is seen that the temperature varies linearly across the solid. The constants of integration can be found by applying the boundary conditions:

(1) At $x = 0$ $T = T_1$
(2) At $x = B$ $T = T_2$

The first boundary condition gives $T_1 = C_2$ and the second then gives:

$$T_2 = C_1 B + T_1$$

Solving for C_1 we find:

$$C_1 = \frac{T_2 - T_1}{B}$$

The heat flux is obtained from Fourier's law:

$$\hat{q}_x = -k\frac{dT}{dx} = -kC_1 = -k\frac{(T_2 - T_1)}{B} = k\frac{(T_1 - T_2)}{B}$$

Multiplying by the area gives the heat flow:

$$q_x = \hat{q}_x A = \frac{kA(T_1 - T_2)}{B}$$

Since this is the same as Equation (1.1), we conclude that the mathematics are consistent with the experimental results.

Example 1.3

Apply the conduction equation to the situation illustrated in Figure 1.1, but let $k = a + bT$, where a and b are constants.

Solution

Conditions 1–3 of the previous example are imposed. The conduction equation then becomes:

$$0 = \frac{d}{dx}\left(k\frac{dT}{dx}\right)$$

Integrating once gives:

$$k\frac{dT}{dx} = C_1$$

The variables can now be separated and a second integration performed. Substituting for k, we have:

$$(a + bT)dT = C_1\,dx$$

$$aT + \frac{bT^2}{2} = C_1 x + C_2$$

It is seen that in this case of variable k, the temperature profile is not linear across the solid.

The constants of integration can be evaluated by applying the same boundary conditions as in the previous example, although the algebra is a little more tedious. The results are:

$$C_2 = aT_1 + \frac{bT_1^2}{2}$$

$$C_1 = a\frac{(T_2 - T_1)}{B} + \frac{b}{2B}(T_2^2 - T_1^2)$$

As before, the heat flow is found using Fourier's law:

$$q_x = -kA\frac{dT}{dx} = -AC_1$$

$$q_x = \frac{A}{B}\left[a(T_1 - T_2) + \frac{b}{2}(T_1^2 - T_2^2)\right]$$

This equation is seldom used in practice. Instead, when k cannot be assumed constant, Equation (1.1) is used with an average value of k. Thus, taking the arithmetic average of the conductivities at the two sides of the block:

$$k_{ave} = \frac{k(T_1) + k(T_2)}{2}$$

$$= \frac{(a + bT_1) + (a + bT_2)}{2}$$

$$k_{ave} = a + \frac{b}{2}(T_1 + T_2)$$

Using this value of k in Equation (1.1) yields:

$$q_x = \frac{k_{ave}A(T_1 - T_2)}{B}$$

$$= \left[a + \frac{b(T_1 + T_2)}{2}\right]\frac{A}{B}(T_1 - T_2)$$

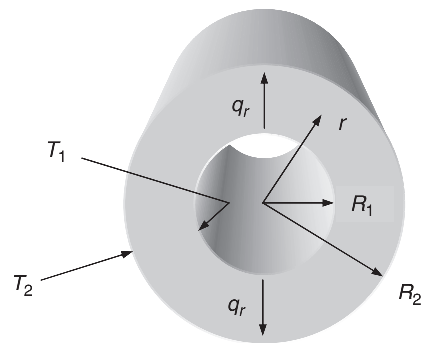
$$q_x = \frac{A}{B}\left[a(T_1 - T_2) + \frac{b}{2}(T_1^2 - T_2^2)\right]$$

This equation is exactly the same as the one obtained above by solving the conduction equation. Hence, using Equation (1.1) with an average value of k gives the correct result. This is a consequence of the assumed linear relationship between k and T.

Example 1.4

Use the conduction equation to find an expression for the rate of heat transfer for the cylindrical analog of the situation depicted in Figure 1.1.

Solution



As shown in the sketch, the solid is in the form of a hollow cylinder and the outer and inner surfaces are maintained at temperatures T_1 and T_2, respectively. The ends of the cylinder are insulated so that heat can flow only in the radial direction. There is no heat flow in the angular (ϕ) direction because the temperature is the same all the way around the circumference of the cylinder. The following conditions apply:

(1) No heat flow in z-direction $\Rightarrow \dfrac{\partial T}{\partial z} = 0$
(2) Uniform temperature in ϕ-direction $\Rightarrow \dfrac{\partial T}{\partial \phi} = 0$
(3) No heat generation $\Rightarrow \dot{q} = 0$
(4) Steady state $\Rightarrow \dfrac{\partial T}{\partial t} = 0$
(5) Constant k

With these conditions, the conduction equation in cylindrical coordinates becomes:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right) = 0$$

or

$$\frac{d}{dr}\left(r\frac{dT}{dr}\right) = 0$$

Integrating once gives:

$$r\frac{dT}{dr} = C_1$$

Separating variables and integrating again gives:

$$T = C_1 \ln r + C_2$$

It is seen that, even with constant k, the temperature profile in curvilinear systems is nonlinear.
The boundary conditions for this case are:

(1) At $r = R_1$ $T = T_1 \Rightarrow T_1 = C_1 \ln R_1 + C_2$
(2) At $r = R_2$ $T = T_2 \Rightarrow T_2 = C_1 \ln R_2 + C_2$

Solving for C_1 by subtracting the second equation from the first gives:

$$C_1 = \frac{T_1 - T_2}{\ln R_1 - \ln R_2} = -\frac{T_1 - T_2}{\ln(R_2/R_1)}$$

From Table 1.1, the appropriate form of Fourier's law is:

$$\hat{q}_r = -k\frac{dT}{dr} = -k\frac{C_1}{r} = \frac{k(T_1 - T_2)}{r\ln(R_2/R_1)}$$

The area across which the heat flows is:

$$A_r = 2\pi rL$$

where L is the length of the cylinder. Thus,

$$q_r = \hat{q}_r A_r = \frac{2\pi kL(T_1 - T_2)}{\ln(R_2/R_1)}$$

Notice that the heat-transfer rate is independent of radial position. The heat flux, however, depends on r because the cross-sectional area changes with radial position.

Example 1.5

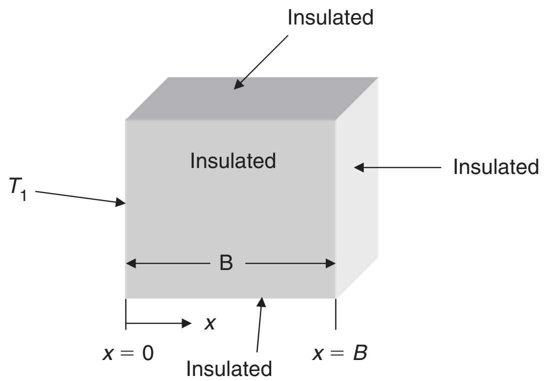
The block shown in the diagram below is insulated on the top, bottom, front, back, and the side at $x = \text{B}$. The side at $x = 0$ is maintained at a fixed temperature, T_1. Heat is generated within the block at a rate per unit volume given by:

$$\dot{q} = \Gamma e^{-\gamma x}$$

where $\Gamma, \gamma > 0$ are constants. Find the maximum steady-state temperature in the block. Data are as follows:

$\Gamma = 10\,\text{W/m}^3$ $\quad B = 1.0\,\text{m}$ $\quad k = 0.5\,\text{W/m} \cdot \text{K} = \text{block thermal conductivity}$

$\gamma = 0.1\,\text{m}^{-1}$ $\quad T_1 = 20°\text{C}$



Solution

The first step is to find the temperature profile in the block by solving the heat conduction equation. The applicable conditions are:

- Steady state
- Conduction only in x-direction
- Constant thermal conductivity

The appropriate form of the heat conduction equation is then:

$$\frac{d\,(kdT/dx)}{dx} + \dot{q} = 0$$

$$k\frac{d^2T}{dx^2} + \Gamma e^{-\gamma x} = 0$$

$$\frac{d^2T}{dx^2} = -\frac{\Gamma e^{-\gamma x}}{k}$$

Integrating once gives:

$$\frac{dT}{dx} = \frac{\Gamma e^{-\gamma x}}{k\gamma} + C_1$$

A second integration yields:

$$T = -\frac{\Gamma e^{-\gamma x}}{k\gamma^2} + C_1 x + C_2$$

The boundary conditions are:

(1) At $x = 0$ $T = T_1$

(2) At $x = \mathrm{B}$ $\dfrac{dT}{dx} = 0$

The second boundary condition results from assuming zero heat flow through the insulated boundary (perfect insulation). Thus, at $x = L$:

$$q_x = -kA\frac{dT}{dx} = 0 \quad \Rightarrow \quad \frac{dT}{dx} = 0$$

This condition is applied using the equation for dT/dx resulting from the first integration:

$$0 = \frac{\Gamma e^{-\gamma B}}{k\gamma} + C_1$$

Hence,

$$C_1 = -\frac{\Gamma e^{-\gamma B}}{k\gamma}$$

Applying the first boundary condition to the equation for T:

$$T_1 = -\frac{\Gamma e^{(0)}}{k\gamma^2} + C_1(0) + C_2$$

Hence,

$$C_2 = T_1 + \frac{\Gamma}{k\gamma^2}$$

With the above values for C_1 and C_2, the temperature profile becomes:

$$T = T_1 + \frac{\Gamma}{k\gamma^2}(1 - e^{-\gamma x}) - \frac{\Gamma e^{-\gamma B}}{k\gamma} x$$

Now at steady state, all the heat generated in the block must flow out through the un-insulated side at $x = 0$. Hence, the maximum temperature must occur at the insulated boundary, i.e., at $x = B$. (This intuitive result can be confirmed by setting the first derivative of T equal to zero and solving for x.) Thus, setting $x = B$ in the last equation gives:

$$T_{max} = T_1 + \frac{\Gamma}{k\gamma^2}(1 - e^{-\gamma B}) - \frac{\Gamma B L e^{-\gamma B}}{k\gamma}$$

Finally, the solution is obtained by substituting the numerical values of the parameters:

$$T_{max} = 20 + \frac{10}{0.5(0.1)^2}(1 - e^{-0.1}) - \frac{10 \times 1.0\, e^{-0.1}}{0.5 \times 0.1}$$

$$T_{max} \cong 29.4°\mathrm{C}$$

The procedure illustrated in the above examples can be summarized as follows:

(1) Write down the conduction equation in the appropriate coordinate system.
(2) Impose any restrictions dictated by the physical situation to eliminate terms that are zero or negligible.
(3) Integrate the resulting differential equation to obtain the temperature profile.
(4) Use the boundary conditions to evaluate the constants of integration.
(5) Use the appropriate form of Fourier's law to obtain the heat flux.
(6) Multiply the heat flux by the cross-sectional area to obtain the rate of heat transfer.

In each of the above examples there is only one independent variable so that an ordinary differential equation results. In unsteady-state problems and problems in which heat flows in more than one direction, a partial differential equation must be solved. Analytical solutions are often possible if the geometry is sufficiently simple. Otherwise, numerical solutions are obtained with the aid of a computer.

1.4 Thermal Resistance

The concept of thermal resistance is based on the observation that many diverse physical phenomena can be described by a general rate equation that may be stated as follows:

$$\text{Flow rate} = \frac{\text{Driving force}}{\text{Resistance}} \tag{1.14}$$

Ohm's Law of Electricity is one example:

$$I = \frac{E}{R} \tag{1.15}$$

In this case, the quantity that flows is electric charge, the driving force is the electrical potential difference, E, and the resistance is the electrical resistance, R, of the conductor.

In the case of heat transfer, the quantity that flows is heat (thermal energy) and the driving force is the temperature difference. The resistance to heat transfer is termed the thermal resistance, and is denoted by R_{th}. Thus, the general rate equation may be written as:

$$q = \frac{\Delta T}{R_{th}} \tag{1.16}$$

In this equation, all quantities take on positive values only, so that q and ΔT represent the absolute values of the heat-transfer rate and temperature difference, respectively.

An expression for the thermal resistance in a rectangular system can be obtained by comparing Equations (1.1) and (1.16):

$$q_x = \frac{kA(T_1 - T_2)}{B} = \frac{\Delta T}{R_{th}} = \frac{T_1 - T_2}{R_{th}} \tag{1.17}$$

$$R_{th} = \frac{B}{kA} \tag{1.18}$$

Similarly, using the equation derived in Example 1.4 for a cylindrical system gives:

$$q_r = \frac{2\pi kL(T_1 - T_2)}{\ln(R_2/R_1)} = \frac{T_1 - T_2}{R_{th}} \tag{1.19}$$

Table 1.2 *Expressions for Thermal Resistance*

Configuration	R_{th}
Conduction, Cartesian coordinates	B/kA
Conduction, radial direction, cylindrical coordinates	$\dfrac{\ln(R_2/R_1)}{2\pi kL}$
Conduction, radial direction, spherical coordinates	$\dfrac{R_2 - R_1}{4\pi k R_1 R_2}$
Conduction, shape factor	$1/kS$
Convection, un-finned surface	$1/hA$
Convection, finned surface	$\dfrac{1}{h\eta_w A}$

S = shape factor
h = heat-transfer coefficient
η_w = weighted efficiency of finned surface $= \dfrac{A_p + \eta_f A_f}{A_p + A_f}$
A_p = prime surface area
A_f = fin surface area
η_f = fin efficiency

$$R_{th} = \frac{\ln(R_2/R_1)}{2\pi kL} \tag{1.20}$$

These results, along with a number of others that will be considered subsequently, are summarized in Table 1.2. When k cannot be assumed constant, the average thermal conductivity, as defined in the previous section, should be used in the expressions for thermal resistance.

The thermal resistance concept permits some relatively complex heat-transfer problems to be solved in a very simple manner. The reason is that thermal resistances can be combined in the same way as electrical resistances. Thus, for resistances in series, the total resistance is the sum of the individual resistances:

$$R_{Tot} = \sum_i R_i \tag{1.21}$$

Likewise, for resistances in parallel:

$$R_{Tot} = \left(\sum_i 1/R_i\right)^{-1} \tag{1.22}$$

Thus, for the composite solid shown in Figure 1.3, the thermal resistance is given by:

$$R_{th} = R_A + R_{BC} + R_D \tag{1.23}$$

where R_{BC}, the resistance of materials B and C in parallel, is:

$$R_{BC} = (1/R_B + 1/R_C)^{-1} = \frac{R_B R_C}{R_B + R_C} \tag{1.24}$$

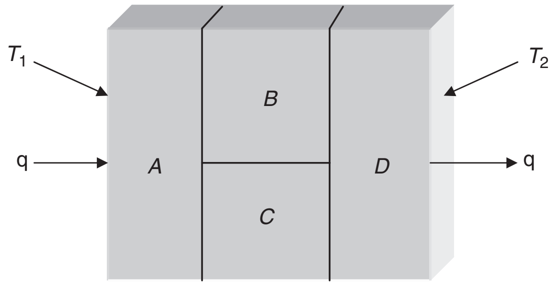


Figure 1.3 *Heat transfer through a composite material.*

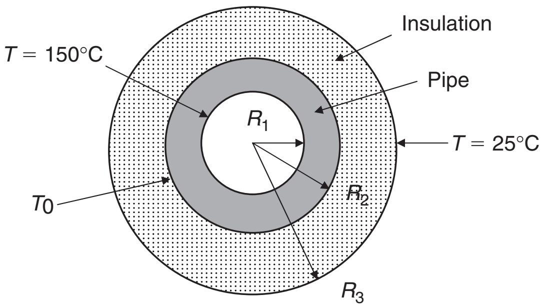
In general, when thermal resistances occur in parallel, heat will flow in more than one direction. In Figure 1.3, for example, heat will tend to flow between materials B and C, and this flow will be normal to the primary direction of heat transfer. In this case, the one-dimensional calculation of q using Equations (1.16) and (1.22) represents an approximation, albeit one that is generally quite acceptable for process engineering purposes.

Example 1.6

A 5-cm (2-in.) schedule 40 steel pipe carries a heat-transfer fluid and is covered with a 2-cm layer of calcium silicate insulation ($k = 0.06\ \text{W/m}\cdot\text{K}$) to reduce the heat loss. The inside and outside pipe diameters are 5.25 cm and 6.03 cm, respectively. If the inner pipe surface is at 150°C and the exterior surface of the insulation is at 25°C, calculate:

(a) The rate of heat loss per unit length of pipe.
(b) The temperature of the outer pipe surface.

Solution



(a)

$$q_r = \frac{\Delta T}{R_{th}} = \frac{150 - 25}{R_{th}}$$

$$R_{th} = R_{pipe} + R_{insulation}$$

$$R_{th} = \frac{\ln(R_2/R_1)}{2\pi k_{steel} L} + \frac{\ln(R_3/R_2)}{2\pi k_{ins} L}$$

$$R_1 = 5.25/2 = 2.625\,\text{cm}$$
$$R_2 = 6.03/2 = 3.015\,\text{cm}$$
$$R_3 = 3.015 + 2 = 5.015\,\text{cm}$$
$$k_{steel} = 43\,\text{W/m}\cdot\text{K (Table A.1)}$$
$$k_{ins} = 0.06\,\text{W/m}\cdot\text{K (given)}$$
$$L = 1\,\text{m}$$

$$R_{th} = \frac{\ln\left(\frac{3.015}{2.625}\right)}{2\pi \times 43} + \frac{\ln\left(\frac{5.015}{3.015}\right)}{2\pi \times 0.06} = 0.000513 + 1.349723$$
$$= 1.350236\,\text{K/W}$$
$$q_r = \frac{125}{1.350236} \cong 92.6\,\text{W/m of pipe}$$

(b) Writing Equation (1.16) for the pipe wall only:

$$q_r = \frac{150 - T_0}{R_{pipe}}$$
$$92.6 = \frac{150 - T_0}{0.000513}$$
$$T_0 = 150 - 0.0475 \cong 149.95^\circ\text{C}$$

Clearly, the resistance of the pipe wall is negligible compared with that of the insulation, and the temperature difference across the pipe wall is a correspondingly small fraction of the total temperature difference in the system.

It should be pointed out that the calculation in Example 1.6 tends to overestimate the rate of heat transfer because it assumes that the insulation is in perfect thermal contact with the pipe wall. Since solid surfaces are not perfectly smooth, there will generally be air gaps between two adjacent solid materials. Since air is a very poor conductor of heat, even a thin layer of air can result in a substantial thermal resistance. This additional resistance at the interface between two materials is called the contact resistance. Thus, the thermal resistance in Example 1.5 should really be written as:

$$R_{th} = R_{pipe} + R_{insulation} + R_{contact} \tag{1.25}$$

The effect of the additional resistance is to decrease the rate of heat transfer according to Equation (1.16). Since the contact resistance is difficult to determine, it is often neglected or a rough approximation is used. For example, a value equivalent to an additional 5 mm of material thickness is sometimes used for the contact resistance between two pieces of the same material [3]. A more rigorous method for estimating contact resistance can be found in Ref. [4].

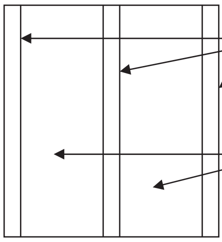
A slightly modified form of the thermal resistance, the *R*-value, is commonly used for insulations and other building materials. The *R*-value is defined as:

$$R\text{-value} = \frac{B\,(\text{ft})}{k\,(\text{Btu/h}\cdot\text{ft}\cdot{}^\circ\text{F})} \tag{1.26}$$

where B is the thickness of the material and k is its thermal conductivity. Comparison with Equation (1.18) shows that the R-value is the thermal resistance, in English units, of a slab of material having a cross-sectional area of 1 ft^2. Since the R-value is always given for a specified thickness, the thermal conductivity of a material can be obtained from its R-value using Equation (1.26). Also, since R-values are essentially thermal resistances, they are additive for materials arranged in series.

Example 1.7

Triple-glazed windows like the one shown in the sketch below are often used in very cold climates. Calculate the R-value for the window shown. The thermal conductivity of air at normal room temperature is approximately 0.015 Btu/h · ft · °F.



Triple-pane window

Solution

From Table A.3, the thermal conductivity of window glass is 0.78 W/m · K. Converting to English units gives:

$$k_{glass} = 0.78 \times 0.57782 = 0.45\ \text{Btu/h} \cdot \text{ft} \cdot {}^\circ\text{F}$$

The R-values for one pane of glass and one air gap are calculated from Equation (1.26):

$$R_{glass} = \frac{0.08/12}{0.45} \cong 0.0148$$

$$R_{air} = \frac{0.25/12}{0.015} \cong 1.3889$$

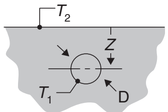
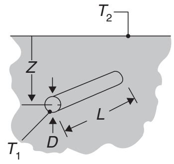
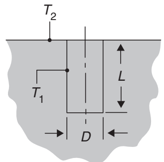
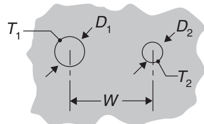
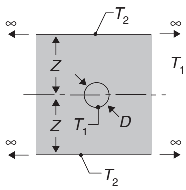
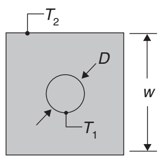
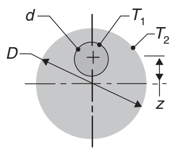
The R-value for the window is obtained using the additive property for materials in series:

$$\begin{aligned} R\text{-value} &= 3R_{glass} + 2R_{air} \\ &= 3 \times 0.0148 + 2 \times 1.3889 \\ R\text{-value} &\cong 2.8 \end{aligned}$$

1.5 The Conduction Shape Factor

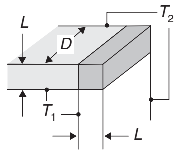
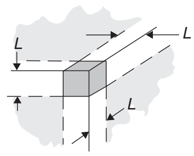
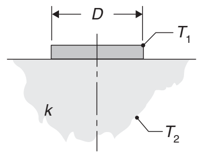
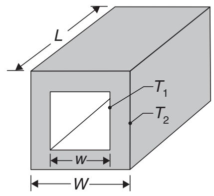
The conduction shape factor is a device whereby analytical solutions to multi-dimensional heat conduction problems are cast into the form of one-dimensional solutions. Although quite restricted

Table 1.3 *Conduction Shape Factors (Source: Ref. [5])*

Case 1 Isothermal sphere buried in a semi-infinite medium		$z > D/2$	$S = \dfrac{2\pi D}{1 - D/4z}$
Case 2 Horizontal isothermal cylinder of length L buried in a semi-infinite medium		$L >> D$	$S = \dfrac{2\pi L}{\cosh^{-1}(2z/D)}$
Case 3 Vertical cylinder in a semi-infinite medium		$L >> D$	$S = \dfrac{2\pi L}{\ln(4L/D)}$
Case 4 Conduction between two cylinders of length L in infinite medium		$L >> D_1, D_2$ $L >> w$	$S = \dfrac{2\pi L}{\cosh^{-1}\left(\dfrac{4w^2 - D_1^2 - D_2^2}{2D_1D_1}\right)}$
Case 5 Horizontal circular cylinder of length L midway between parallel planes of equal length and infinite width		$z >> D/2$ $L >> z$	$S = \dfrac{2\pi L}{\ln(8z/\pi D)}$
Case 6 Circular cylinder of length L centered in a square solid of equal length		$w > D$ $L >> w$	$S = \dfrac{2\pi L}{\ln(1.08\, w/D)}$
Case 7 Eccentric circular cylinder of length L in a cylinder of equal length		$D > d$ $L >> D$	$S = \dfrac{2\pi L}{\cosh^{-1}\left(\dfrac{D^2 + d^2 - 4z^2}{2Dd}\right)}$

(Continued)

Table 1.3 *(Continued)*

Case 8 Conduction through the edge of adjoining walls		$D > L/5$	$S = 0.54\,D$
Case 9 Conduction through corner of three walls with a temperature difference $\Delta T_1 - T_2$ across the walls		$L << $ length and width of wall	$S = 0.15\,L$
Case 10 Disk of diameter D and T_1 on a semi-finite medium of thermal conductivity k and T_2			$S = 2D$
Case 11 Square channel of length L		$\frac{W}{w} < 1.4$ $\frac{W}{w} > 1.4$	$S = \frac{2\pi L}{0.785\ \ln(W/w)}$ $S = \frac{2\pi L}{0.930\ \ln(W/w) - 0.050}$

in scope, the shape factor method permits rapid and easy solution of multi-dimensional heat-transfer problems when it is applicable. The conduction shape factor, S, is defined by the relation:

$$q = kS\Delta T \tag{1.27}$$

where ΔT is a specified temperature difference. Notice that S has the dimension of length. Shape factors for a number of geometrical configurations are given in Table 1.3. The solution of a problem involving one of these configurations is thus reduced to the calculation of S by the appropriate formula listed in the table.

The thermal resistance corresponding to the shape factor can be found by comparing Equation (1.16) with Equation (1.27). The result is:

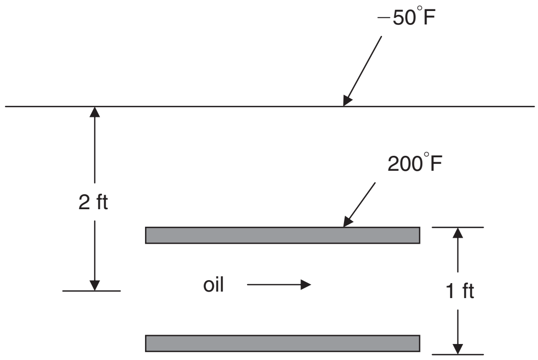
$$R_{th} = 1/kS \tag{1.28}$$

This is one of the thermal resistance formulas listed in Table 1.2. Since shape-factor problems are inherently multi-dimensional, however, use of the thermal resistance concept in such cases will, in general, yield only approximate solutions. Nevertheless, these solutions are usually entirely adequate for process engineering calculations.

Example 1.8

An underground pipeline transporting hot oil has an outside diameter of 1 ft and its centerline is 2 ft below the surface of the earth. If the pipe wall is at 200°F and the earth's surface is at −50°F, what is the rate of heat loss per foot of pipe? Assume $k_{earth} = 0.5$ Btu/h · ft ·°F.

Solution



From Table 1.3, the shape factor for a buried horizontal cylinder is:

$$S = \frac{2\pi L}{\cosh^{-1}(2z/D)}$$

In this case, $z = 2$ ft and $r = 0.5$ ft. Taking $L = 1$ ft we have:

$$S = \frac{2\pi L}{\cosh^{-1}(4)} = 3.045\ \text{ft}$$

$$q = k_{earth} S\, \Delta T$$

$$= 0.5 \times 3.045 \times [200 - (-50)]$$

$$q \cong 380\ \text{Btu/h} \cdot \text{ft of pipe}$$

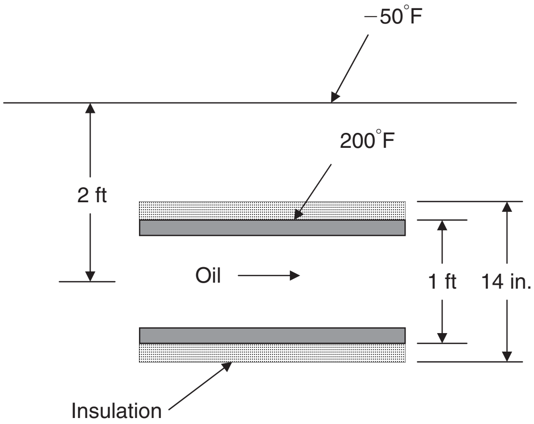
Note: If necessary, the following mathematical identity can be used to evaluate $\cosh^{-1}(x)$:

$$\cosh^{-1}(x) = \ln\left(x + \sqrt{x^2 - 1}\right)$$

Example 1.9

Suppose the pipeline of the previous example is covered with 1 in. of magnesia insulation ($k = 0.07$ W/m · K). What is the rate of heat loss per foot of pipe?

Solution



This problem can be solved by treating the earth and the insulation as two resistances in series. Thus,

$$q = \frac{\Delta T}{R_{th}} = \frac{200 - (-50)}{R_{earth} + R_{insulation}}$$

The resistance of the earth is obtained by means of the shape factor for a buried horizontal cylinder. In this case, however, the diameter of the cylinder is the diameter of the exterior surface of the insulation. Thus,

$$z = 2\,\text{ft} = 24\,\text{in.}$$

$$D = 12 + 2 = 14\,\text{in.}$$

$$2z/D = \frac{48}{14} = 3.4286$$

Therefore,

$$S = \frac{2\pi L}{\cosh^{-1}(2z/D)} = \frac{2\pi \times 1}{\cosh^{-1}(3.4286)} = 3.3012\,\text{ft}$$

$$R_{earth} = \frac{1}{k_{earth}S} = \frac{1}{0.5 \times 3.3012} = 0.6058\,\text{h} \cdot {}^\circ\text{F/Btu}$$

Converting the thermal conductivity of the insulation to English units gives:

$$k_{ins} = 0.07 \times 0.57782 = 0.0404\,\text{Btu/h} \cdot \text{ft} \cdot {}^\circ\text{F}$$

Hence,

$$R_{insulation} = \ln\frac{(R_2/R_1)}{2\pi k_{ins}L} = \frac{\ln(7/6)}{2\pi \times 0.0404 \times 1}$$
$$= 0.6073\,\text{h} \cdot {}^\circ\text{F/Btu}$$

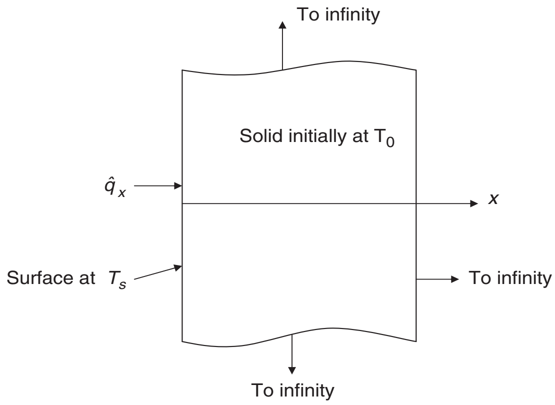


Figure 1.4 *Semi-infinite solid.*

Then

$$q = \frac{250}{0.6058 + 0.6073} = 206\,\text{Btu/h} \cdot \text{ft of pipe}$$

1.6 Unsteady-State Conduction

The heat conduction problems considered thus far have all been steady state, i.e., time-independent, problems. In this section, solutions of a few unsteady-state problems are presented. Solutions to many other unsteady-state problems can be found in heat-transfer textbooks and monographs, e.g., Refs. [5–10].

We consider first the case of a semi-infinite solid illustrated in Figure 1.4. The rectangular solid occupies the region from $x = 0$ to $x = \infty$. The solid is initially at a uniform temperature, T_0. At time $t = 0$, the temperature of the surface at $x = 0$ is changed to T_s and held at that value. The temperature within the solid is assumed to be uniform in the y- and z-directions at all times, so that heat flows only in the x-direction. This condition can be achieved mathematically by allowing the solid to extend to infinity in the $\pm y$- and $\pm z$-directions. If T_s is greater that T_0, heat will begin to penetrate into the solid, so that the temperature at any point within the solid will gradually increase with time. That is, $T = T(x, t)$, and the problem is to determine the temperature as a function of position and time.

Assuming no internal heat generation and constant thermal conductivity, the conduction equation for this situation is:

$$\frac{1}{\alpha}\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} \tag{1.29}$$

The boundary conditions are:

(1) At $t = 0$, $T = T_0$ for all $x \geq 0$
(2) At $x = 0$, $T = T_s$ for all $t > 0$
(3) As $x \to \infty$, $T \to T_0$ for all $t \geq 0$

The last condition follows because it takes an infinite time for heat to penetrate an infinite distance into the solid.

The solution of Equation (1.29) subject to these boundary conditions can be obtained by the method of combination of variables [11]. The result is:

$$\frac{T(x, t) - T_s}{T_0 - T_s} = \text{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right) \tag{1.30}$$

The error function, erf, is defined by:

$$\operatorname{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right) = \frac{2}{\sqrt{\pi}} \int_{0}^{\frac{x}{2\sqrt{\alpha t}}} e^{-z^2} dz \qquad (1.31)$$

This function, which occurs in many diverse applications in engineering and applied science, can be evaluated by numerical integration. Values are listed in Table 1.4.

Table 1.4 *The Error Function*

x	erf x	x	erf x	x	erf x
0.00	0.00000	0.76	0.71754	1.52	0.96841
0.02	0.02256	0.78	0.73001	1.54	0.97059
0.04	0.04511	0.80	0.74210	1.56	0.97263
0.06	0.06762	0.82	0.75381	1.58	0.97455
0.08	0.09008	0.84	0.76514	1.60	0.97635
0.10	0.11246	0.86	0.77610	1.62	0.97804
0.12	0.13476	0.88	0.78669	1.64	0.97962
0.14	0.15695	0.90	0.79691	1.66	0.98110
0.16	0.17901	0.92	0.80677	1.68	0.98249
0.18	0.20094	0.94	0.81627	1.70	0.98379
0.20	0.22270	0.96	0.82542	1.72	0.98500
0.22	0.24430	0.98	0.83423	1.74	0.98613
0.24	0.26570	1.00	0.84270	1.76	0.98719
0.26	0.28690	1.02	0.85084	1.78	0.98817
0.28	0.30788	1.04	0.85865	1.80	0.98909
0.30	0.32863	1.06	0.86614	1.82	0.98994
0.32	0.34913	1.08	0.87333	1.84	0.99074
0.34	0.36936	1.10	0.88020	1.86	0.99147
0.36	0.38933	1.12	0.88079	1.88	0.99216
0.38	0.40901	1.14	0.89308	1.90	0.99279
0.40	0.42839	1.16	0.89910	1.92	0.99338
0.42	0.44749	1.18	0.90484	1.94	0.99392
0.44	0.46622	1.20	0.91031	1.96	0.99443
0.46	0.48466	1.22	0.91553	1.98	0.99489
0.48	0.50275	1.24	0.92050	2.00	0.995322
0.50	0.52050	1.26	0.92524	2.10	0.997020
0.52	0.53790	1.28	0.92973	2.20	0.998137
0.54	0.55494	1.30	0.93401	2.30	0.998857
0.56	0.57162	1.32	0.93806	2.40	0.999311
0.58	0.58792	1.34	0.94191	2.50	0.999593
0.60	0.60386	1.36	0.94556	2.60	0.999764
0.62	0.61941	1.38	0.94902	2.70	0.999866
0.64	0.63459	1.40	0.95228	2.80	0.999925
0.66	0.64938	1.42	0.95538	2.90	0.999959
0.68	0.66278	1.44	0.95830	3.00	0.999978
0.70	0.67780	1.46	0.96105	3.20	0.999994
0.72	0.69143	1.48	0.96365	3.40	0.999998
0.74	0.70468	1.50	0.96610	3.60	1.000000

The heat flux is given by:

$$\hat{q}_x = \frac{k(T_s - T_0)}{\sqrt{\pi\alpha t}}\exp(-x^2/4\alpha t) \tag{1.32}$$

The total amount of heat transferred per unit area across the surface at $x = 0$ in time t is given by:

$$\frac{Q}{A} = 2k(T_s - T_0)\sqrt{\frac{t}{\pi\alpha}} \tag{1.33}$$

Although the semi-infinite solid may appear to be a purely academic construct, it has a number of practical applications. For example, the earth behaves essentially as a semi-infinite solid. A solid of any finite thickness can be considered a semi-infinite solid if the time interval of interest is sufficiently short that heat penetrates only a small distance into the solid. The approximation is generally acceptable if the following inequality is satisfied:

$$\frac{\alpha t}{L^2} < 0.1 \tag{1.34}$$

where L is the thickness of the solid. The dimensionless group $\alpha t/L^2$ is called the Fourier number and is designated *Fo*.

Example 1.10

The steel panel of a firewall is 5-cm thick and is initially at 25°C. The exterior surface of the panel is suddenly exposed to a temperature of 250°C. Estimate the temperature at the center and at the interior surface of the panel after 20 s of exposure to this temperature. The thermal diffusivity of the panel is $0.97 \times 10^{-5}\ \text{m}^2/\text{s}$.

Solution

To determine if the panel can be approximated by a semi-infinite solid, we calculate the Fourier number:

$$Fo = \frac{\alpha t}{L^2} = \frac{0.97 \times 10^{-5} \times 20}{(0.05)^2} \cong 0.0776$$

Since $Fo < 0.1$, the approximation should be acceptable. Thus, using Equation (1.30) with $x = 0.025$ for the temperature at the center,

$$\frac{T - T_s}{T_0 - T_s} = \text{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right)$$

$$\frac{T - 250}{25 - 250} = \text{erf}\left(\frac{0.025}{2\sqrt{0.97 \times 10^{-5} \times 20}}\right) = \text{erf}(0.8974)$$

$$\frac{T - 250}{-225} = 0.7969 \text{ (from Table 1.4)}$$

$$T \cong 70.7^\circ\text{C}$$

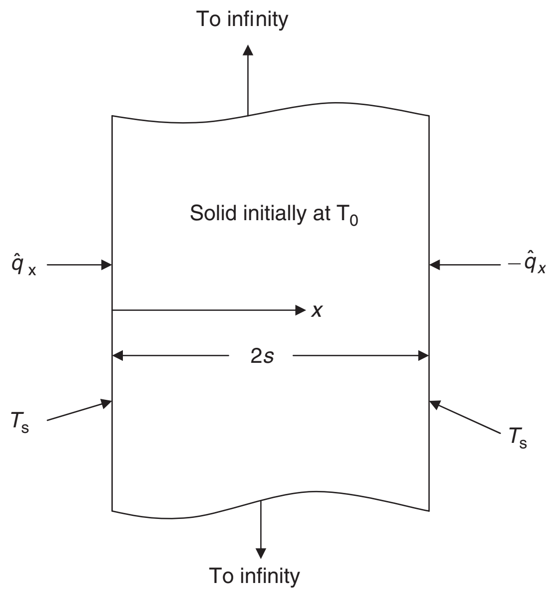


Figure 1.5 *Infinite solid of finite thickness.*

For the interior surface, $x = 0.05$ and Equation (1.30) gives:

$$\frac{T - 250}{-225} = \text{erf}\left(\frac{0.05}{2\sqrt{0.97 \times 10^{-5} \times 20}}\right) = \text{erf}(1.795)$$
$$= 0.9891$$
$$T \cong 27.5°\text{C}$$

Thus, the temperature of the interior surface has not changed greatly from its initial value of 25°C, and treating the panel as a semi-infinite solid is therefore a reasonable approximation.

Consider now the rectangular solid of finite thickness illustrated in Figure 1.5. The configuration is the same as that for the semi-infinite solid except that the solid now occupies the region from $x = 0$ to $x = 2s$. The solid is initially at uniform temperature T_0 and at time $t = 0$ the temperature of the surfaces at $x = 0$ and $x = 2s$ are changed to T_s. If $T_s > T_0$, then heat will flow into the solid from both sides. It is assumed that heat flows only in the x-direction, which again can be achieved mathematically by making the solid of infinite extent in the $\pm y$- and $\pm z$-directions. This condition will be approximated in practice when the areas of the surfaces normal to the y- and z-directions are much smaller than the area of the surface normal to the x-direction, or when the former surfaces are insulated.

The mathematical statement of this problem is the same as that of the semi-infinite solid except that the third boundary condition is replaced by:

$$(3') \quad \text{At}\, x = 2s \;\; T = T_s$$

The solution for $T(x, t)$ can be found in the textbooks cited at the beginning of this section. Frequently, however, one is interested in determining the average temperature, $\overline{T}$, of the solid as a function of time, where:

$$\overline{T}(t) = \frac{1}{2s}\int_0^{2s} T(x,t)\,dx \tag{1.35}$$

That is, $\overline{T}$ is the temperature averaged over the thickness of the solid at a given instant of time. The solution for $\overline{T}$ is in the form of an infinite series [12]:

$$\frac{T_s - \overline{T}}{T_s - T_0} = \frac{8}{\pi^2}\left(e^{-aFo} + \frac{1}{9}e^{-9aFo} + \frac{1}{25}e^{-25aFo} + \cdots\right) \tag{1.36}$$

where $a = (\pi/2)^2 \cong 2.4674$ and $Fo = \alpha t/s^2$.

The solution given by Equation (1.36) is shown graphically in Figure 1.6. When the Fourier number, *Fo*, is greater than about 0.1, the series converges very rapidly so that only the first term is significant. Under these conditions, Equation (1.36) can be solved for the time to give:

$$t = \frac{1}{\alpha}\left(\frac{2s}{\pi}\right)^2 \ln\left[\frac{8(T_s - T_0)}{\pi^2(T_s - \overline{T})}\right] \tag{1.37}$$

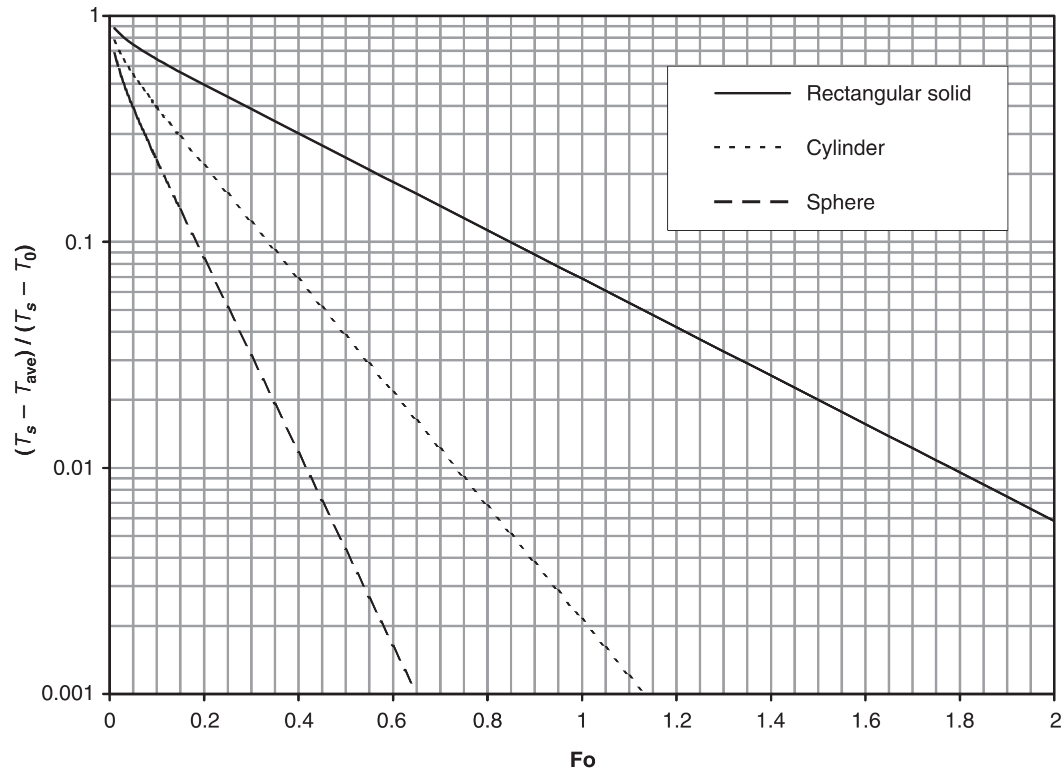


Figure 1.6 *Average temperatures during unsteady-state heating or cooling of a rectangular solid, an infinitely long cylinder, and a sphere.*

The total amount of heat, Q, transferred to the solid per unit area, A, in time t is:

$$\frac{Q(t)}{A} = \frac{m\,c}{A}\left[\overline{T}(t) - T_0\right] \tag{1.38}$$

where m, the mass of solid, is equal to $2\rho sA$. Thus,

$$\frac{Q(t)}{A} = 2\rho cs\left[\overline{T}(t) - T_0\right] \tag{1.39}$$

The analogous problem in cylindrical geometry is that of an infinitely long solid cylinder of radius, R, initially at uniform temperature, T_0. At time $t = 0$ the temperature of the surface is changed to T_s. This situation will be approximated in practice by a finite cylinder whose length is much greater than its diameter, or whose ends are insulated. The solutions corresponding to Equations (1.36), (1.37), and (1.39) are [12]:

$$\frac{T_s - \overline{T}}{T_s - T_0} = 0.692e^{-5.78Fo} + 0.131e^{-30.5Fo} + 0.0534e^{-74.9Fo} + \cdots \tag{1.40}$$

$$t = \frac{R^2}{5.78\alpha}\ln\left[\frac{0.692(T_s - T_0)}{T_s - \overline{T}}\right] \tag{1.41}$$

$$\frac{Q(t)}{A} = \frac{\rho cR}{2}\left[\overline{T}(t) - T_0\right] \tag{1.42}$$

where

$$Fo = \frac{\alpha t}{R^2} \tag{1.43}$$

Here A is the circumferential area, which is equal to $2\pi R$ times the length of the cylinder. Equation (1.40) is shown graphically in Figure 1.6.

The corresponding equations for a solid sphere of radius R are [12]:

$$\frac{T_s - \overline{T}}{T_s - T_0} = 0.608e^{-9.87Fo} + 0.152e^{-39.5Fo} + 0.0676e^{-88.8Fo} + \cdots \tag{1.44}$$

$$t = \frac{R^2}{9.87\alpha}\ln\left[\frac{0.608(T_s - T_0)}{T_s - \overline{T}}\right] \tag{1.45}$$

$$Q(t) = \frac{4}{3}\pi R^3\rho c[T(t) - T_0] \tag{1.46}$$

The Fourier number for this case is also given by Equation (1.43). Equation (1.44) is shown graphically in Figure 1.6.

Example 1.11

A 12-ounce can of beer initially at 80°F is placed in a refrigerator, which is at 36°F. Estimate the time required for the beer to reach 40°F.

Solution

Application to this problem of the equations presented in this section requires a considerable amount of approximation, a situation that is not uncommon in practice. Since a 12-ounce beer can has a diameter of 2.5 in. and a length of 4.75 in., we have:

$$L/D = \frac{4.75}{2.5} = 1.9$$

Hence, the assumption of an infinite cylinder will not be a particularly good one. In effect, we will be neglecting the heat transfer through the ends of the can. The effect of this approximation will be to overestimate the required time.

Next, we must assume that the temperature of the surface of the can suddenly drops to 36°F when it is placed in the refrigerator. That is, we neglect the resistance to heat transfer between the air in the refrigerator and the surface of the can. The effect of this approximation will be to underestimate the required time. Hence, there will be at least a partial cancellation of errors.

We must also neglect the heat transfer due to convection currents set up in the liquid inside the can by the cooling process. The effect of this approximation will be to overestimate the required time.

Finally, we will neglect the resistance of the aluminum can and will approximate the physical properties of beer by those of water. We thus take:

$$k = 0.341\ \text{Btu/h}\cdot\text{ft}\cdot{}^\circ\text{F} \qquad T_s = 36^\circ\text{F}$$

$$\rho = 62.4\ \text{lbm/ft}^3 \qquad T_0 = 80^\circ\text{F}$$

$$c = 1.0\ \text{Btu/lbm}\cdot{}^\circ\text{F} \qquad \overline{T} = 40^\circ\text{F}$$

With these values we have:

$$\alpha = \frac{k}{\rho c} = 0.0055\ \text{ft}^2/\text{h}$$

$$\frac{T_s - \overline{T}}{T_s - T_0} = \frac{36 - 40}{36 - 80} = 0.0909$$

From Figure 1.6, we find a Fourier number of about 0.35. Thus,

$$Fo = \frac{\alpha t}{R^2} = 0.35$$

$$t = \frac{0.35\,R^2}{\alpha} = \frac{0.35(1.25/12)^2}{0.0055} \cong 0.69\ \text{h}$$

Alternatively, since $Fo > 0.1$, we can use Equation (1.41):

$$t = \frac{R^2}{5.78\alpha}\ln\left[\frac{0.692(T_s - T_0)}{T_s - \overline{T}}\right]$$

$$= \frac{(1.25/12)^2}{5.78 \times 0.0055} \quad \ln\left[\frac{0.629}{0.0909}\right]$$

$$t = 0.66\,\text{h}$$

This agrees with the previous calculation to within the accuracy with which one can read the graph of Figure 1.6. Experience suggests that this estimate is somewhat optimistic and, hence, that the error introduced by neglecting the thermal resistance between the air and the can is predominant. Nevertheless, if the answer is rounded to the nearest hour (a reasonable thing to do considering the many approximations that were made), the result is a cooling time of 1 h, which is essentially correct. In any event, the calculations show that the time required is more than a few minutes but less than a day, and in many practical situations this level of detail is all that is needed.

1.7 Mechanisms of Heat Conduction

This chapter has dealt with the computational aspects of heat conduction. In this concluding section we briefly discuss the mechanisms of heat conduction in solids and fluids. Although Fourier's law accurately describes heat conduction in both solids and fluids, the underlying mechanisms differ. In all media, however, the processes responsible for conduction take place at the molecular or atomic level.

Heat conduction in fluids is the result of random molecular motion. Thermal energy is the energy associated with translational, vibrational, and rotational motions of the molecules comprising a substance. When a high-energy molecule moves from a high-temperature region of a fluid toward a region of lower temperature (and, hence, lower thermal energy), it carries its thermal energy along with it. Likewise, when a high-energy molecule collides with one of lower energy, there is a partial transfer of energy to the lower-energy molecule. The result of these molecular motions and interactions is a net transfer of thermal energy from regions of higher temperature to regions of lower temperature.

Heat conduction in solids is the result of vibrations of the solid lattice and of the motion of free electrons in the material. In metals, where free electrons are plentiful, thermal energy transport by electrons predominates. Thus, good electrical conductors, such as copper and aluminum, are also good conductors of heat. Metal alloys, however, generally have lower (often much lower) thermal and electrical conductivities than the corresponding pure metals due to disruption of free electron movement by the alloying atoms, which act as impurities.

Thermal energy transport in non-metallic solids occurs primarily by lattice vibrations. In general, the more regular the lattice structure of a material is, the higher its thermal conductivity. For example, quartz, which is a crystalline solid, is a better heat conductor than glass, which is an amorphous solid. Also, materials that are poor electrical conductors may nevertheless be good heat conductors. Diamond, for instance, is an excellent conductor of heat due to transport by lattice vibrations.

Most common insulating materials, both natural and man-made, owe their effectiveness to air or other gases trapped in small compartments formed by fibers, feathers, hairs, pores, or rigid foam. Isolation of the air in these small spaces prevents convection currents from forming within the material, and the relatively low thermal conductivity of air (and other gases) thereby imparts a low effective thermal conductivity to the material as a whole. Insulating materials with effective thermal conductivities much less than that of air are available; they are made by incorporating evacuated layers within the material.

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Notations

A	Area
A_f	Fin surface area (Table 1.2)
A_p	Prime surface area (Table (1.2)
A_r	$2\pi rL$
A_x, A_y	Cross-sectional area perpendicular to x- or y-direction
a	Constant in Equation (1.2); constant equal to $(\pi/2)^2$ in Equation (1.36)
B	Thickness of solid in direction of heat flow
b	Constant in Equation (1.2)
c	specific heat of solid
C_1, C_2	Constants of integration
D	Diameter; distance between adjoining walls (Table 1.3)
d	diameter of eccentric cylinder (Table 1.3)
E	Voltage difference in Ohm's law
erf	Gaussian error function defined by Equation (1.31)
Fo	Fourier number
h	Heat-transfer coefficient (Table 1.2)
I	Electrical current in Ohm's law
$\vec{i}$	Unit vector in x-direction
$\vec{j}$	Unit vector in y-direction
k	Thermal conductivity
$\vec{k}$	Unit vector in z-direction
L	Length; thickness of edge or corner of wall (Table 1.3)
Q	Total amount of heat transferred
q	Rate of heat transfer
q_x, q_y, q_r	Rate of heat transfer in x-, y-, or r-direction
$\hat{q}=q/A$	Heat flux
$\dot{q}$	Rate of heat generation per unit volume
$\vec{q}$	Heat flow vector

$\vec{\hat{q}}$	Heat flux vector
R	Resistance; radius of cylinder or sphere
R_{th}	Thermal resistance
R-value	Ratio of a material's thickness to its thermal conductivity, in English units
r	Radial coordinate in cylindrical or spherical coordinate system
S	Conduction shape factor defined by Equation (1.27)
s	Half-width of solid in Figure 1.5
T	Temperature
$\overline{T}$	Average temperature
t	Time
W	Width
w	Width or displacement (Table 1.3)
x	Coordinate in Cartesian system
y	Coordinate in Cartesian system
z	Coordinate in Cartesian or cylindrical system; depth or displacement (Table 1.3)

Greek Letters

$\alpha = k/\rho c$	Thermal diffusivity
Γ	Constant in Example 1.5
γ	Constant in Example 1.5
ΔT, Δx, etc.	Difference in T, x, etc.
η	Efficiency
η_f	Fin efficiency (Table 1.2)
η_w	Weighted efficiency of a finned surface (Table 1.2)
θ	Angular coordinate in spherical system; angle between heat flux vector and x-axis (Example 1.1)
ρ	Density
ϕ	Angular coordinate in cylindrical or spherical system

Other Symbols

$\vec{\nabla} T$	Temperature gradient vector
∇^2	Laplacian operator $= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ in Cartesian coordinates
$\rightarrow$	Overstrike to denote a vector
$\vert_x$	Evaluated at x

Problems

(1.1) The temperature distribution in a bakelite block ($k = 0.233\,\mathrm{W/m\cdot k}$) is given by:

$$T(x, y, z) = x^2 - 2y^2 + z^2 - xy + 2yz$$

where $T \propto {}^\circ\mathrm{C}$ and $x, y, z \propto \mathrm{m}$. Find the magnitude of the heat flux vector at the point $(x, y, z) = (0.5,\ 0,\ 0.2)$.

Ans. $0.252\,\mathrm{W/m^2}$.

(1.2) The temperature distribution in a Teflon rod ($k = 0.35\,\mathrm{W/m\cdot k}$) is:

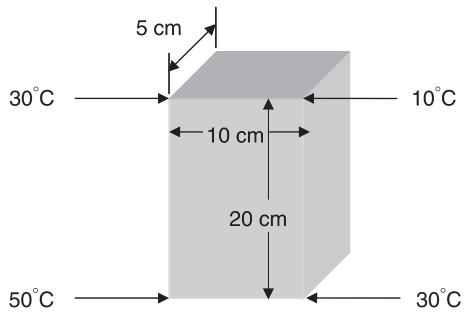
$$T(r, \phi, z) = r\sin\phi + 2z$$

where
$T \propto °C$
$r =$ radial position (m)
$\phi =$ circumferential position (rad)
$z =$ axial position (m)

Find the magnitude of the heat flux vector at the position $(r, \phi, z) = (0.1, 0, 0.5)$.

Ans. 0.78 W/m^2.

(1.3) The rectangular block shown below has a thermal conductivity of 1.4 W/m · k. The block is well insulated on the front and back surfaces, and the temperature in the block varies linearly from left to right and from top to bottom. Determine the magnitude and direction of the heat flux vector. What are the heat flows in the horizontal and vertical directions?



Ans. 313 W/m^2at an angle of 26.6° with the horizontal; 1.4 W and 5.76 W.

(1.4) The temperature on one side of a 6-in. thick solid wall is 200°F and the temperature on the other side is 100°F. The thermal conductivity of the wall can be represented by:

$$k(\text{Btu/h} \cdot \text{ft} \cdot °\text{F}) = 0.1 + 0.001\, T(°\text{F})$$

(a) Calculate the heat flux through the wall under steady-state conditions.
(b) Calculate the thermal resistance for a 1 ft^2 cross-section of the wall.

Ans. (a) 50 Btu/h · ft^2. (b) 2 h · ft^2 · °F/Btu

(1.5) A long hollow cylinder has an inner radius of 1.5 in. and an outer radius of 2.5 in. The temperature of the inner surface is 150°F and the outer surface is at 110°F. The thermal conductivity of the material can be represented by:

$$k(\text{Btu/h} \cdot \text{ft} \cdot °\text{F}) = 0.1 + 0.001\, T(°\text{F})$$

(a) Find the steady-state heat flux in the radial direction:
  (i) At the inner surface
  (ii) At the outer surface

(b) Calculate the thermal resistance for a 1 ft length of the cylinder.

Ans: (a) 144.1 Btu/h · ft^2, 86.4 Btu/h ·ft^2. (b) 0.3535 h · ft^2 ·°F/Btu.

(1.6) A rectangular block has thickness B in the x-direction. The side at $x=0$ is held at temperature T_1 while the side at $x=B$ is held at T_2. The other four sides are well insulated. Heat is generated in the block at a uniform rate per unit volume of Γ.

(a) Use the conduction equation to derive an expression for the steady-state temperature profile, $T(x)$. Assume constant thermal conductivity.
(b) Use the result of part (a) to calculate the maximum temperature in the block for the following values of the parameters:

$$T_1 = 100°\text{C} \quad k = 0.2\,\text{W/m} \cdot \text{k} \quad B = 1.0\,\text{m}$$

$$T_2 = 0°\text{C} \qquad \Gamma = 100\,\text{W/m}^3$$

Ans. (a) $T(x) = T_1 + \left(\dfrac{T_2 - T_1}{B} + \dfrac{\Gamma L}{2k}\right)x - \dfrac{\Gamma x^2}{2k}$. (b) $T_{max} = 122.5°C$ at $x = 0.3$ m

(1.7) Repeat Problem 1.6 for the situation in which the side of the block at $x=0$ is well insulated.

Ans. (a) $T(x) = T_2 + \dfrac{\Gamma}{2k}(B^2 - x^2)$. (b) $T_{max} = 250°$C

(1.8) Repeat Problem 1.6 for the situation in which the side of the block at $x=0$ is exposed to an external heat flux, $\hat{q}_o$, of 20 W/m^2. Note that the boundary condition at $x=0$ for this case becomes

$$\frac{dT}{dx} = -\frac{\hat{q}_o}{k}.$$

Ans. (a) $T(x) = T_2 + \dfrac{\hat{q}_o}{k}(B-x) + \dfrac{\Gamma}{2k}(B^2 - x^2)$. (b) $T_{max} = 350°$C

(1.9) A long hollow cylinder has inner and outer radii R_1 and R_2, respectively. The temperature of the inner surface at radius R_1 is held at a constant value, T_1, while that of the outer surface at radius R_2 is held constant at a value of T_2. Heat is generated in the wall of the cylinder at a rate per unit volume given by $\dot{q} = \Gamma r$, where r is radial position and Γ is a constant. Assuming constant thermal conductivity and heat flow only in the radial direction, derive expressions for:

(a) The steady-state temperature profile, $T(r)$, in the cylinder wall.
(b) The heat flux at the outer surface of the cylinder.

Ans.

$$(a)\, T(r) = T_1 + (\Gamma/9k)(R_1^3 - r^3) + \frac{\left\{T_2 - T_1 + \dfrac{\Gamma}{9k}(R_2^3 - R_1^3)\right\}\ln(r/R_1)}{\ln(R_2/R_1)}.$$

$$\text{(b)}\ \hat{q}_r|_{r=R_2} = \frac{\Gamma R_2^3}{3} + \frac{k\{T_1 - T_2 - (\Gamma/9k)(R_2^3 - R_1^3)\}}{R_2 \ln(R_2/R_1)}.$$

(1.10) Repeat Problem 1.9 for the situation in which the inner surface of the cylinder at R_1 is well insulated.

Ans. (a) $T(r) = T_2 - (\Gamma/9k)(R_2^3 - r^3) + \dfrac{\Gamma R_1^3 \ln(r/R_2)}{3k}$. (b) $\hat{q}_r|_{r=R_2} = \dfrac{\Gamma(R_2^3 - R_1^3)}{3R_2}$.

(1.11) A hollow sphere has inner and outer radii R_1 and R_2, respectively. The inner surface at radius R_1 is held at a uniform temperature T_1, while the outer surface at radius R_2 is held at temperature T_2. Assuming constant thermal conductivity, no heat generation and steady-state conditions, use the conduction equation to derive expressions for:

(a) The temperature profile, $T(r)$.
(b) The rate of heat transfer, q_r, in the radial direction.
(c) The thermal resistance.

Ans.

(a) $T(r) = T_1 + \dfrac{R_1 R_2 (T_1 - T_2)\left(\dfrac{1}{r} - \dfrac{1}{R_1}\right)}{R_2 - R_1}$.

(b) $q_r = \dfrac{4\pi k R_1 R_2 (T_1 - T_2)}{R_2 - R_1}$.

(c) See Table 1.2.

(1.12) A hollow sphere with inner and outer radii R_1 and R_2has fixed uniform temperatures of T_1 on the inner surface at radius R_1 and T_2 on the outer surface at radius R_2. Heat is generated in the wall of the sphere at a rate per unit volume given by $\dot{q} = \Gamma r$, where r is radial position and Γ is a constant. Assuming constant thermal conductivity, use the conduction equation to derive expressions for:

(a) The steady-state temperature profile, $T(r)$, in the wall.
(b) The heat flux at the outer surface of the sphere.

Ans.

(a) $T(r) = T_1 + (\Gamma/12k)(R_1^3 - r^3) + \dfrac{R_1 R_2 \{T_1 - T_2 - (\Gamma/12k)(R_2^3 - R_1^3)\}\left(\dfrac{1}{r} - \dfrac{1}{R_1}\right)}{R_2 - R_1}$.

(b) $\hat{q}_r|_{r=R_2} = \dfrac{\Gamma R_2^2}{4} + \{kR_1/R_2(R_2 - R_1)\}\{T_1 - T_2 - (\Gamma/12k)(R_2^3 - R_1^3)\}$.

(1.13) Repeat Problem 1.12 for the situation in which the inner surface at radial position R_1 is well insulated.

Ans.

(a) $T(r) = T_2 + (\Gamma/12k)(R_2^3 - r^3) + \left(\dfrac{\Gamma R_1^4}{4k}\right)\left(\dfrac{1}{R_2} - \dfrac{1}{r}\right)$.

(b) $\hat{q}_r|_{r=R_2} = \dfrac{\Gamma(R_2^4 - R_1^4)}{4R_2^2}$

(1.14) When conduction occurs in the radial direction in a solid rod or sphere, the heat flux must be zero at the center ($r = 0$) in order for a finite temperature to exist there. Hence, an appropriate boundary condition is:

$$\frac{dT}{dr} = 0 \quad \text{at } r = 0$$

Consider a solid sphere of radius R with a fixed surface temperature, T_R. Heat is generated within the solid at a rate per unit volume given by $\dot{q} = \Gamma_1 + \Gamma_2 r$, where Γ_1 and Γ_2 are constants.

(a) Assuming constant thermal conductivity, use the conduction equation to derive an expression for the steady-state temperature profile, $T(r)$, in the sphere.
(b) Calculate the temperature at the center of the sphere for the following parameter values:

$$R = 1.5\,\text{m} \qquad \Gamma_1 = 20\,\text{W/m}^3 \qquad T_R = 20°\text{C}$$
$$k = 0.5\,\text{W/m}\cdot\text{K} \qquad \Gamma_2 = 10\,\text{W/m}^4$$

Ans. (a) $T(r) = T_R + (\Gamma_1/6k)(R^2 - r^2) + (\Gamma_2/12k)(R^3 - r^3)$. (b) 40.625°C.

(1.15) A solid cylinder of radius R is well insulated at both ends, and its exterior surface at $r = R$ is held at a fixed temperature, T_R. Heat is generated in the solid at a rate per unit volume given by $\dot{q} = \Gamma(1 - r/R)$, where $\Gamma =$ constant. The thermal conductivity of the solid may be assumed constant. Use the conduction equation together with an appropriate set of boundary conditions to derive an expression for the steady-state temperature profile, $T(r)$, in the solid.

Ans. $T(r) = T_R + (\Gamma/36k)(5R^2 + 4r^3/R - 9r^2)$.

(1.16) A rectangular wall has thickness B in the x-direction and is insulated on all sides except the one at $x = B$, which is held at a constant temperature, T_w. Heat is generated in the wall at a rate per unit volume given by $\dot{q} = \Gamma(B - x)$, where Γ is a constant.

(a) Assuming constant thermal conductivity, derive an expression for the steady-state temperature profile, $T(x)$, in the wall.
(b) Calculate the temperature of the block at the side $x = 0$ for the following parameter values:

$$\Gamma = 0.3 \times 10^6\,\text{W/m}^4 \quad B = 0.1\,\text{m}$$

$$T_w = 90°\text{C} \quad k = 25\,\text{W/m}\cdot\text{K}$$

Ans. (a) $T(x) = T_w + (\Gamma/k)\left(\dfrac{x^3}{6} - \dfrac{x^2 B}{2} + \dfrac{B^3}{3}\right)$. (b) 94°C.

(1.17) The exterior wall of an industrial furnace is to be covered with a 2-in. thick layer of high-temperature insulation having an R-value of 2.8, followed by a layer of magnesia (85%) insulation. The furnace wall may reach 1200°F, and for safety reasons, the exterior of the magnesia insulation should not exceed 120°F. At this temperature, the heat flux from the insulation to the surrounding air has been estimated for design purposes to be 200 Btu/h · ft^2.

(a) What is the thermal conductivity of the high-temperature insulation?
(b) What thickness of magnesia insulation should be used?
(c) Estimate the temperature at the interface between the high-temperature insulation and the magnesia insulation.

Ans. (a) $k = 0.0595$ Btu/h · ft · °F. (c) 640°F.

(1.18) A storage tank to be used in a chemical process is spherical in shape and is covered with a 3-in. thick layer of insulation having an *R*-value of 12. The tank will hold a chemical intermediate that must be maintained at 150°F. A heating unit is required to maintain this temperature in the tank.

(a) What is the thermal conductivity of the insulation?
(b) Determine the duty for the heating unit assuming as a worst-case scenario that the exterior surface of the insulation reaches a temperature of 20°F.
(c) What thermal resistances were neglected in your calculation?

Ans. (a) $k = 0.02083$ Btu/h · ft · °F. (b) $q \cong 7900$ Btu/h.

(1.19) A 4-in. schedule 80 steel pipe (ID = 3.826 in., OD = 4.5 in.) carries a heat-transfer fluid at 600°F and is covered with a ½-in. thick layer of pipe insulation. The pipe is surrounded by air at 80°F. The vendor's literature states that a 1-in. thick layer of the pipe insulation has an *R*-value of 3. Neglecting convective resistances, the resistance of the pipe wall, and thermal radiation, estimate the rate of heat loss from the pipe per foot of length.

Ans. 453 Btu/h · ft pipe

(1.20) A pipe with an OD of 6.03 cm and an ID of 4.93 cm carries steam at 250°C. The pipe is covered with 2.5 cm of magnesia (85%) insulation followed by 2.5 cm of polystyrene insulation ($k = 0.025$ W/m · K). The temperature of the exterior surface of the polystyrene is 25°C. The thermal resistance of the pipe wall may be neglected in this problem. Also neglect the convective and contact resistances.

(a) Calculate the rate of heat loss per meter of pipe length.
(b) Calculate the temperature at the interface between the two types of insulation.

Ans. (a) 63 W/m of pipe. (b) 174.5°C.

(1.21) It is desired to reduce the heat loss from the storage tank of problem 1.18 by 90%. What additional thickness of insulation will be required?

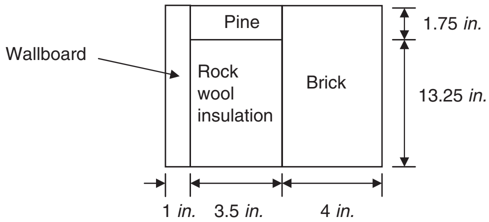
(1.22) A steel pipe with an OD of 2.375 in. is covered with a ½-in. thick layer of asbestos insulation ($k = 0.048$ Btu/h · ft · °F) followed by a 1-in. thick layer of fiberglass insulation ($k = 0.022$ Btu/h · ft · °F). The temperature of the pipe wall is 600°F and the exterior surface of the fiberglass insulation is at 100°F. Calculate:

(a) The rate of heat loss per foot of pipe length.
(b) The temperature at the interface between the asbestos and fiberglass insulations.

Ans. (a) 110 Btu/h · ft pipe. (b) 471°F.

(1.23) A building contains 6000 ft^2 of wall surface area constructed of panels shown in the sketch below. The interior sheathing is gypsum wallboard and the wood is yellow pine. Calculate the rate of heat loss through the walls if the interior wall surface is at 70°F and the exterior surface is at 30°F.

Ans. 22,300 Btu/h



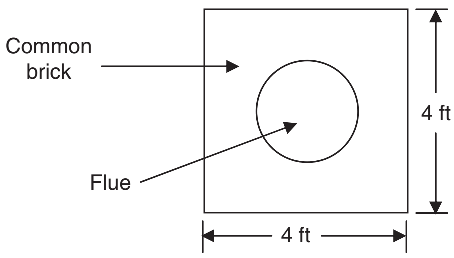
(1.24) A 6 in. schedule 80 steel pipe (OD = 6.62 in.) will be used to transport 450°F steam from a boiler house to a new process unit. The pipe will be buried at a depth of 3 ft (to the pipe centerline). The soil at the plant site has an average thermal conductivity of 0.4 Btu/h · ft · °F and the minimum expected ground surface temperature is 20°F. Estimate the rate of heat loss per foot of pipe length for the following cases:

(a) The pipe is not insulated.
(b) The pipe is covered with a 2-in. thick layer of magnesia insulation.

Neglect the thermal resistance of the pipe wall and the contact resistance between the insulation and pipe wall.

Ans. (a) 350 Btu/h · ft of pipe. (b) 160 Btu/h · ft of pipe.

(1.25) The cross-section of an industrial chimney is shown in the sketch below. The flue has a diameter of 2 ft and the process waste gas flowing through it is at 400°F. If the exterior surface of the brick is at 120°F, calculate the rate of heat loss from the waste gas per foot of chimney height. Neglect the convective resistance between the waste gas and interior surface of the flue for this calculation.



Ans. 910 Btu/h · ft of chimney height.

(1.26) A new underground pipeline at a chemical complex is to be placed parallel to an existing underground pipeline. The existing line has an OD of 8.9 cm, carries a fluid at 283 K and is not insulated. The new line will have an OD of 11.4 cm and will carry a fluid at 335 K. The center-to-center distance between the two pipelines will be 0.76 m. The ground at the plant site has an average thermal conductivity of 0.7 W/m · K. In order to determine whether the new line will need to be insulated, calculate the rate of heat transfer between the two pipelines per meter of pipe length if the new line is not insulated. For the purpose of this calculation, neglect the resistances of the pipe walls and the convective resistances between the fluids and pipe walls.

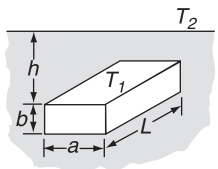
Ans. 42 W/m of pipe length.

(1.27) Hot waste gas at 350°F will be transported from a new process unit to a pollution control device via an underground duct. The duct will be rectangular in cross-section with a height of 3 ft and a width of 5 ft. The top of the duct will be 1.25 ft below the ground surface, which for design purposes has been assigned a temperature of 40°F. The average thermal conductivity of the ground at the plant site is 0.4 Btu/h · ft ·°F. Calculate the rate of heat loss from the waste gas per foot of duct length. What thermal resistances are neglected in your calculation?

The following shape factor for a buried rectangular solid is available in the literature:

$$S = 2.756\,L\left[\ln\left(1+\frac{h}{a}\right)\right]^{-0.59}\left(\frac{h}{b}\right)^{-0.078}$$

$$L >> h, a, b$$



(1.28) An industrial furnace wall will be made of diatomaceous refractory brick ($\alpha = 1.3228 \times 10^{-7}$ m²/s) and is to be designed so that the exterior surface will remain cool enough for safety purposes. The design criterion is that the mid-plane temperature in the wall will not exceed 400 K after 8 h of operation with an interior wall surface temperature of 1100 K.

(a) Assume that the furnace wall can be approximated as a semi-infinite solid. Calculate the wall thickness required to meet the design specification assuming that the wall is initially at a uniform temperature of 300 K.
(b) Using the wall thickness obtained in part (a), calculate the exterior wall surface temperature after 8 h of operation.
(c) Based on the above results, is the assumption that the furnace wall can be approximated as a semi-infinite solid justified, i.e., is the wall thickness calculated in part (a) acceptable for design purposes? Explain why or why not.

Ans. (a) 26.8 cm. (b) 301.7 K.

(1.29) The steel panel ($\alpha = 0.97 \times 10^{-5}$ m²/s) of a firewall is 5 cm thick and its interior surface is insulated. The panel is initially at 25°C when its exterior surface is suddenly exposed to a temperature of 250°C. Calculate the average temperature of the panel after 2 min of exposure to this temperature.

Note: A wall of width s with the temperature of one side suddenly raised to T_s and the opposite side insulated is mathematically equivalent to a wall of width 2 s with the temperature of both sides suddenly raised to T_s. In the latter case, $dT/dx = 0$ at the mid-plane due to symmetry, which is the same condition that exists at a perfectly insulated boundary.

Ans. 192°C.

(1.30) An un-insulated metal storage tank at a chemical plant is cylindrical in shape with a diameter of 4 ft and a length of 25 ft. The liquid in the tank, which has properties similar to those of water, is at a temperature of 70°F when a frontal passage rapidly drops the ambient temperature to 40°F. Assuming that ambient conditions remain constant for an extended period of time, estimate:

(a) The average temperature of the liquid in the tank 12 h after the frontal passage.
(b) The time required for the average temperature of the liquid to reach 50°F.

Ans. (a) 59°F. (b) 92 h.

(1.31) Repeat Problem 1.30 for the situation in which the fluid in the tank is

(a) Methyl alcohol.
(b) Aniline.

(1.32) Repeat Problem 1.30 for the situation in which the tank is spherical in shape with a diameter of 4.2 ft.

Ans. (a) 63°F. (b) 207 h (From Equation (1.44). Note that $Fo < 0.1$.).