Convergence Properties of Curvature Scale Space Representations

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Abstract

Curvature Scale Space (CSS) representations have been shown to be very useful for recognition of noisy curves of arbitrary shapes at unknown orientations and scales [10,14]. This paper contains a number of important results on the convergence properties of CSS representations and on the *evolution* and *arc length evolution* of planar curves [6,12]. The processes which convolve arc length parametric representations of planar curves with Gaussian functions are referred to as the *evolution* or *arc length evolution* of those curves. It has been shown that every closed planar curve will eventually become simple and convex during evolution and arc length evolution and will remain in that state. This result is important since it demonstrates that the computation of a CSS image always has a clearly defined termination point.

1 Introduction

A multi-scale representation for 1-D functions was first proposed by Stansfield [18] and later developed by Witkin [19]. Function $f(x)$ is convolved with a Gaussian function as its width σ varies from a small to a large value. The zero-crossings of the second derivative of each convolved function are extracted and marked in the $x-\sigma$ plane. The result is the *scale space image* of $f(x)$.

The curvature of a planar curve represents that curve uniquely modulo a rigid motion [1,4]. The *Curvature Scale Space image* was introduced in [11,12] as a new shape representation for planar curves. The representation is computed by convolving an arc length parametric representation of the curve with a Gaussian function, as the width of the Gaussian varies from a small to a large value, and extracting the curvature zero-crossing points of the resulting curves. The representation is essentially invariant under rotation, uniform scaling and translation of the curve. It is also very robust with respect to noise and local distortions of shape. These properties make it suitable for recognition of a noisy curve of arbitrary shape at any scale or orientation [10].

The *Resampled curvature scale space image* is a substantial refinement of the CSS image which is based on the concept of *arc length evolution* [13]. The resampled CSS image is suitable for recognition of curves with non-uniform noise or local shape differences.

Given a planar curve

$$\Gamma = \{(x(w), y(w)\}$$

BMVC 1995 doi:10.5244/C.9.36

where w is the arc length parameter, an *evolved* version of Γ is defined by

$$\Gamma_\sigma = \{(X(u,\sigma), Y(u,\sigma)\}$$

where

$$X(u,\sigma) = x(u) * g(u,\sigma)$$
$$Y(u,\sigma) = y(u) * g(u,\sigma)$$

and $*$ represents the convolution operator. Function $g(u,\sigma)$ denotes a Gaussian of width σ [8]. The process of generating the ordered sequence of curves $\{\Gamma_\sigma | \sigma \geq 0\}$ is referred to as the *evolution* of Γ. The generalized evolution which maps Γ to Γ_σ is defined by:

$$\Gamma \rightarrow \Gamma_\sigma = \{(X(W,\sigma), Y(W,\sigma))\}$$

where

$$X(W,\sigma) = x(W) * g(W,\sigma)$$

and

$$Y(W,\sigma) = y(W) * g(W,\sigma).$$

Note that $W = W(w,\sigma)$, and $W(w,\sigma_0)$ where σ_0 is any value of σ, is a continuous and monotonic function of w. When W always remains the arc length parameter of the evolved curve, the evolution of Γ is referred to as *arc length evolution*. $W(w,\sigma)$ is given explicitly by [3]:

$$W(w,\sigma) = -\int_0^\sigma \int_0^W \kappa^2(W,\sigma)\, dW\, d\sigma + w.$$

Since $\kappa(W,\sigma)$ is unknown, the resampled CSS image is computed as follows: A Gaussian filter based on a small value of σ is computed. The curve Γ is parametrized by the normalized arc-length parameter and convolved with the filter. The resulting curve is reparametrized by the normalized arc-length parameter and convolved again with the same filter. This process is repeated until the curve becomes simple and convex.

Curve evolution and CSS representations have been used for a number of applications. Examples include organization of image edges at multiple scales [5], robust corner detection [16], natural scale selection [17], and constrained image segmentation [21]. A number of evolution and arc length evolution properties of planar curves have been studied in the past [7,9,15]. Some of those results have been used in this paper to prove new results. The existing results which have been used are as following:

Theorem 1.1 *The convolution of a sine function $f(u) = a\sin(bu)$ with a Gaussian of width σ is [15]:*

$$F(u,\sigma) = \frac{a}{e^{b^2\sigma^2/2}} \sin(bu).$$

Theorem 1.2 *The convolution of a cosine function $f(u) = a\cos(bu)$ with a Gaussian of width σ is [15]:*

$$F(u,\sigma) = \frac{a}{e^{b^2\sigma^2/2}} \cos(bu).$$

Theorem 1.3 *Simple planar curves remain simple during arc length evolution* [13].

Theorem 1.4 *Let Γ be a planar curve in C_2. If all evolved and arc length evolved curves Γ_σ are in C_2, then all extrema of contours in the regular and resampled CSS images of Γ are maxima* [7].

Theorem 1.5 *Let $\Gamma = (x(w), y(w))$ be a planar curve in C_1 and let $x(w)$ and $y(w)$ be polynomial functions of w. Let Γ_σ be an arc length evolved version of Γ with a cusp point at w_0. There is a $\delta > 0$ such that $\Gamma_{\sigma+\delta}$ has two new curvature zero-crossings in a neighborhood of w_0* [13].

Theorem 1.6 *Let $\Gamma = (x(w), y(w))$ be a planar curve in C_1 and let $x(w)$ and $y(w)$ be polynomial functions of w. Let Γ_σ be an arc length evolved version of Γ with a cusp point at w_0. There is a $\delta > 0$ such that $\Gamma_{\sigma-\delta}$ intersects itself in a neighborhood of point w_0* [7].

This paper contains a number of important results on the convergence properties of CSS representations. It has been shown that every closed planar curve will eventually become simple and convex during evolution and arc length evolution and will remain in that state. This result is very important since it shows that CSS images are well-behaved in the sense that we can always expect to find a scale level σ_0 at which the number of curvature zero-crossing points goes to zero and know that new curvature zero-crossing points will not be created beyond that scale level. σ_0 can be considered to be the high end of the CSS image. The results of this paper show that CSS computations always converge to stable states. The results presented in this paper complement earlier results and help promote a better understanding of CSS representations.

2 Convergence properties of 1-D functions

This section contains a number of results on the properties of 1-D functions when they are convolved with Gaussian functions with large standard deviations. The results of this section will be used in section 3 to prove important results about the convergence properties of planar curves. The first two theorems express facts about the convergence properties of an infinite sum of sine or cosine functions when convolved with Gaussians with large widths.

Theorem 2.1: *Let*

$$f_s(u) = \sum_{k=1}^{\infty} a_k \sin(2k\pi u)$$

where $u \in [0, 1]$ and let

$$F_s(u, t) = f_s(u) * g(u,t)$$

be the function obtained by convolving $f_s(u)$ with a Gaussian $g(u, t)$ such that $t = \sigma^2/2$. Each of $F_s(u)$ and its second derivative has only two zero-crossings for large t.

Proof: It follows from theorem 1.1 that

$$F_s(u, t) = \sum_{k=1}^{\infty} \frac{a_k}{e^{4k^2\pi^2 t}} \sin(2k\pi u).$$

The first sine function in F_s (corresponding to $k = 1$) is the dominant one when t is large and determines the qualitative shape of F_s. The first sign function goes to zero only at $u = 0.5$ and at $u = 1$. All other sine functions in F_s also go to zero at $u = 0.5$ and at $u = 1$. Therefore F_s has at least two zero-crossing points at $u = 0.5$ and $u = 1$. If F_s has any additional zeros, they must be in small neighborhoods of $u = 0.5$ and $u = 1$ since at all other points, the first sine function which determines the shape of F_s is away from zero. Assume w.l.o.g. that the sign of the first sine function is positive. F_s can be approximated to any degree of accuracy by considering only the first N sine functions in the infinite sum (N can be arbitrarily large). Therefore there is a small neighborhood of point $u = 0.5$ in which all the first N sine functions in F_s are monotonically increasing or monotonically decreasing. Let F_s^+ be the sum of those sine functions which are monotonically decreasing and let F_s^- be the sum of those sine functions which are monotonically increasing. Note that in the neighborhood being considered, F_s^+ is positive when $u < 0.5$, is zero at $u = 0.5$ and is negative when $u > 0.5$. Note also that in the same neighborhood, F_s^- is negative when $u < 0.5$, is zero at $u = 0.5$ and is positive when $u > 0.5$. For any value of u in the neighborhood under consideration, $|F_s^+(u)| \gg |F_s^-(u)|$ since the first sine function belongs to F_s^+. It follows that $F_s^+ + F_s^-$ is also positive for $u < 0.5$ and negative for $u > 0.5$. Hence no additional zeros exist in a small neighborhood of $u = 0.5$. A similar argument shows that no additional zeros exist in a neighborhood of $u = 1$. Therefore F_s has only two zero-crossing points. Since the second derivative of F_s has the same form as F_s, the same proof also holds about the second derivative of F_s. It follows that the second derivative of F_s also has only two zero-crossing points. □

Theorem 2.2: *Let*

$$f_c(u) = \sum_{k=1}^{\infty} b_k \cos(2k\pi u)$$

where $u \in [0, 1]$ *and let*

$$F_c(u, t) = f_c(u) * g(u, t)$$

be the function obtained by convolving $f_c(u)$ *with a Gaussian* $g(u, t)$ *such that* $t = \sigma^2/2$. *Each of* $F_c(u)$ *and its second derivative has only two zero-crossings for large* t.

Proof: It follows from theorem **2.2** that

$$F_c(u, t) = \sum_{k=1}^{\infty} \frac{b_k}{e^{4k^2\pi^2 t}} \cos(2k\pi u).$$

The first cosine function in F_c (corresponding to $k = 1$) is dominant when t is large and determines the qualitative shape of F_c. The first cosine function goes to zero at $u = 0.25$ and $u = 0.75$. Therefore F_c has at least two zero-crossing points in small neighborhoods of those two points. If F_c has any additional zeros, they must also be in small neighborhoods of $u = 0.25$ and $u = 0.75$. Assume w.l.o.g. that the sign of the first cosine function is positive. F_c can be approximated to any degree of accuracy by considering only the first N cosine functions in the infinite sum (N can be arbitrarily large). Therefore there is a small neighborhood of point $u = 0.25$ in which all cosine functions are either monotonic or have only one extremum. All cosine functions in this neighborhood can be divided into the following four groups:

(1) Monotonically decreasing cosine functions which go to zero at $u = 0.25$. Denote their sum by F_1.
(2) Monotonically increasing cosine functions which go to zero at $u = 0.25$. Denote their sum by F_2.
(3) Cosine functions which have a maximum at $u = 0.25$. Denote their sum by F_3.
(4) Cosine functions which have a minimum at $u = 0.25$. Denote their sum by F_4.

Let $F = F_1 + F_2$. Since the first cosine function belongs to group (1), for any value of u in the neighborhood under consideration, $|F_1(u)| \gg |F_2(u)|$. It follows that F is also monotonically decreasing and has a zero at $u = 0.25$. Let ΔF be the resulting change in

F as u varies from u_1 to u_2, any two values of u in the neighborhood under consideration. Define ΔF_3 and ΔF_4 similarly. Since the first cosine function is dominant and F_3 and F_4 have extrema at $u = 0.25$, it follows that $|\Delta F| \gg |\Delta F_3|$ and $|\Delta F| \gg |\Delta F_4|$. Therefore $F_c = F + F_3 + F_4$ is also monotonically decreasing in the neighborhood under consideration. Therefore F_c goes to zero only once in that neighborhood. A similar argument shows that F_c goes to zero only once in a neighborhood of $u = 0.75$. Hence F_c goes to zero only twice. Since $\ddot{F}_c$ has the same form as F_c, the same proof also holds about $\ddot{F}_c$. It follows that the second derivative of F_c also goes to zero only twice. □

The following theorem expresses the main convergence result of section **2**.

Theorem 2.3: *Let $f(u)$ be a periodic function and let $F(u,t) = f(u) * g(u,t)$ be the function obtained by convolving $f(u)$ with a Gaussian function $g(u,t)$ such that $t = \sigma^2/2$. $\ddot{F}(u,t)$ has only two zero-crossing points for large t.*

Proof: Since $f(u)$ is periodic, it can be expressed as [2]:

$$f(u) = \sum_{k=1}^{\infty} a_k \sin(2k\pi u) + \sum_{k=1}^{\infty} b_k \cos(2k\pi u).$$

Therefore

$$f'(u) = \sum_{k=1}^{\infty} 2a_k k\pi \cos(2k\pi u) + \sum_{k=1}^{\infty} -2b_k k\pi \sin(2k\pi u)$$

and

$$f''(u) = \sum_{k=1}^{\infty} -4a_k k^2\pi^2 \sin(2k\pi u) + \sum_{k=1}^{\infty} -4b_k k^2\pi^2 \cos(2k\pi u).$$

In fact all derivatives of $f(u)$ have the same form: an infinite sum of sine functions plus an infinite sum of cosine functions. It follows from theorems **1.1** and **1.2** that:

$$F''(u,t) = \sum_{k=1}^{\infty} \frac{-4a_k k^2\pi^2}{e^{4k^2\pi^2 t}} \sin(2k\pi u) + \sum_{k=1}^{\infty} \frac{-4b_k k^2\pi^2}{e^{4k^2\pi^2 t}} \cos(2k\pi u).$$

Theorem **2.1** showed that, when t is large, the sum of the sine functions in $F''(u,t)$ is a function F_s with only two zero-crossing points. Theorem **2.2** showed that, when t is large, the sum of the cosine functions in $F''(u,t)$ is a function F_c with only two zero-crossing points. We will now show that $F'' = F_s + F_c$ also has only two zero-crossing points. Suppose that the signs of the first sine function in F_s and the first cosine function in F_c are positive. The arguments used in other cases are similar. F_s has two zero-crossing points at $u = 0.5$ and $u = 1$ and F_c has two zero-crossing points at $u = 0.25$ and $u = 0.75$. Now consider the range of values $[0, 0.25]$ for u. $F_s + F_c$ is positive in that range therefore there are no zero-crossings. Now consider the range $[0.25, 0.5]$. $F_s + F_c$ is positive at $u = 0.25$ and negative at $u = 0.5$ so there is at least one zero-crossing point between those two values of u. Since both F_s and F_c are monotonically decreasing in that range, there can be only one zero-crossing point between $u = 0.25$ and $u = 0.5$. There are no zero-crossing points between $u = 0.5$ and $u = 0.75$ since both F_s and F_c are negative in that range. Finally, consider the range $[0.75, 1]$. $F_s + F_c$ is negative at $u = 0.75$ and positive at $u = 1$ so there is at least one zero-crossing between those two values of u. Since both F_s and F_c are monotonically increasing in that range, there can be only one 0-crossing between $u = 0.75$ and $u = 1$. Hence $F'' = F_s + F_c$ has only two 0-crossing points. □

3 Convergence Properties of Planar Curves

This section contains important results on the convergence properties of evolution and arc length evolution of planar curves as defined in section 1. These results show that evolution and arc length evolution of planar curves are well-behaved processes. Theorems 3.1 through 3.4 examine the convergence properties of closed planar curves during evolution.

Theorem 3.1: *Let $\Gamma = (X(u,\sigma), Y(u,\sigma))$ be a simple and convex closed planar curve. Functions $\ddot{X}(u,\sigma)$ and $\ddot{Y}(u,\sigma)$ (which are periodic) each has two zero-crossing points.*

Proof: Since Γ is a simple and convex closed planar curve, the total change in the direction of its tangent vector as Γ is traversed one full cycle is equal to 2π. Let θ be the angle that the tangent vector makes with the positive x-axis. θ takes on each value in the range $[0, 360]$ only once. It follows that the tangent vector becomes horizontal only twice and becomes vertical only twice. Each time the tangent vector is horizontal, function $Y(u,\sigma)$ has an extremum and each time the tangent vector is vertical, function $X(u,\sigma)$ has an extremum. Hence each of $X(u,\sigma)$ and $Y(u,\sigma)$ has two extrema. Since these functions are periodic, each of $\ddot{X}(u,\sigma)$ and $\ddot{Y}(u,\sigma)$ has two zero-crossing points. □

Theorem 3.2: *Let $\Gamma = (x(u), y(u))$ be a closed planar curve such that functions $x(u)$ and $y(u)$ are periodic functions and each of $\ddot{x}(u)$ and $\ddot{y}(u)$ has two zero-crossing points. Γ is simple and convex.*

Proof: Reparametrize Γ by arc length s. Since $\ddot{x}(u)$ and $\ddot{y}(u)$ each intersect zero only twice and u is a monotonic function of s, $\ddot{x}(s)$ and $\ddot{y}(s)$ also intersect zero only twice. Note that on a curve with arc length parametrization, the magnitude of curvature κ is given by:

$$|\kappa| = \sqrt{(\ddot{x})^2 + (\ddot{y})^2}.$$

Therefore at each inflection point of the curve, $\ddot{x}(s) = \ddot{y}(s) = 0$.

Assume by contradiction that Γ is not simple and convex. Suppose that Γ is not convex. There can be only two inflection points on Γ. Let P and P' be those inflection points (Figure 3.1). Let T be the tangent line at point P and let T' be the tangent line at P'. Let L be the line going through P and P'. The curvature of Γ changes sign at P therefore Γ will turn to the right of line T. Since curvature of Γ does not change sign again, Γ will continue to turn until it intersects line L at point Q different from P. Similarly, Γ changes sign of curvature at P' and will intersect L at point Q' different from P'. Suppose line L is not vertical. Note that the x-coordinate of Q is larger than the x-coordinate of P and that the x-coordinate of Q' is smaller than the x-coordinate of P'. It follows that the maximum and minimum of $x(s)$ lie outside segment PP'. So there is at least one zero-crossing point of $\ddot{x}(s)$ which is not at P or P'. Hence $\ddot{x}(s)$ has more than two zero-crossing points which is a contradiction. If line L is vertical, a similar argument can be applied to $\ddot{y}(s)$ and a contradiction will be reached again. It follows that Γ must be convex.

Now suppose that Γ is not simple. Therefore Γ intersects itself in at least one point. Let P be the point of self-intersection (Figure 3.2) and let T and T' be the tangent vectors at P. Note that function $x(s)$ has at least one extremum inside the loop since the value of that function is equal to the x-coordinate of point P for two different values of s, s_1

and s_2, and takes on different values when s is between s_1 and s_2. Follow the curve in the direction of T. Since Γ is convex, it will continue to turn in the same direction until the tangent to the curve becomes vertical. Let that point be Q. Also follow the curve in the direction of T'. Again Γ will continue to turn in the same direction until its tangent becomes vertical at point Q'. There is an extremum of $x(s)$ at each of Q and Q'. Hence there are at least three extrema on $x(s)$. This is a contradiction since $\ddot{x}(s)$ is a periodic function with only two zero-crossing points and therefore $x(s)$ has only two extrema. It follows that Γ must also be simple. It was shown earlier that Γ is convex. It follows that Γ is simple and convex. □

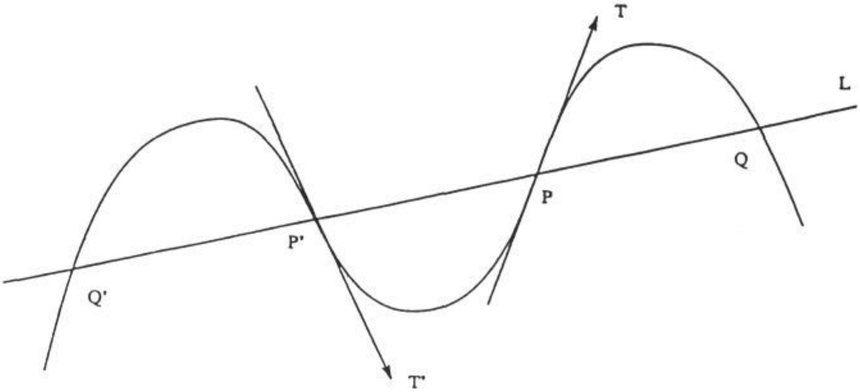

Figure 3.1

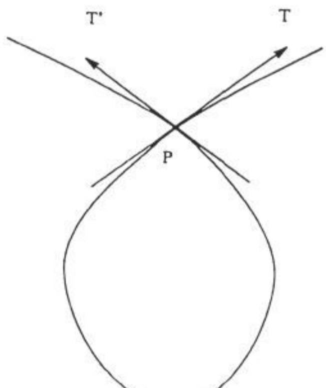

Figure 3.2

Theorem 3.3: *Simple and convex planar curves remain simple and convex during evolution.*

Proof: Let $\Gamma = (x(u), y(u))$ be a simple and convex closed planar curve. It follows from theorem **3.1** that functions $\ddot{x}(u)$ and $\ddot{y}(u)$ each has only two zero-crossing points. It is known that the number of zero-crossing points on $\ddot{x}(u)$ and $\ddot{y}(u)$ will not increase when $x(u)$ and $y(u)$ are convolved with Gaussian functions [20]. Therefore each of the coordinate functions of evolved versions of Γ also has only two zero-crossing points of the second derivative. It follows from theorem **3.2** that evolved versions of Γ are also simple and convex. Hence Γ remains simple and convex during evolution. □

The following theorem expresses the main result of this section on evolution. It follows from this theorem that the correct stopping criterion when computing the CSS image of a planar curve is when the curve becomes simple and convex.

Theorem 3.4: *Let Γ be a closed planar curve. Γ becomes simple and convex during evolution and remains simple and convex.*

Proof: Let $\Gamma_\sigma = (X(u,\sigma), Y(u,\sigma))$ be an evolved version of Γ and let σ be large. Theorem **2.3** showed that $\ddot{X}(u,\sigma)$ and $\ddot{Y}(u,\sigma)$ each has only two zero-crossing points. It follows from theorem **3.2** that Γ_σ must be simple and convex. It follows from theorem **3.3** that Γ remains simple and convex. □

The remaining theorems in this section explore the convergence properties of planar curves during arc length evolution.

Theorem 3.5: *Simple and convex planar curves remain simple and convex during arc length evolution.*

Proof: Suppose Γ is a simple and convex planar curve. Theorem **1.3** states that simple curves remain simple during arc length evolution. Therefore Γ remains simple during arc length evolution. To become non-convex, Γ must form new curvature zero-crossings. It follows from theorems **1.4** and **1.5** that every planar curve must form a cusp point just before formation of new curvature zero-crossings during arc length evolution and theorem **1.6** states that every planar curve must intersect itself just before formation of a cusp point during arc length evolution. It follows that every planar curve must intersect itself just before formation of new curvature zero-crossings during arc length evolution. Since Γ remains simple during arc length evolution, formation of new curvature zero-crossings on Γ is not possible. Therefore Γ also remains convex. □

The next theorem expresses the main result of this section on arc length evolution of planar curves.

Theorem 3.6: *Let Γ be a closed planar curve. Γ becomes simple and convex during arc length evolution and remains simple and convex.*

Proof: Let $\Gamma_\sigma = (X(W,\sigma), Y(W,\sigma))$ be an arc length evolved version of Γ and let σ be large. The proof of theorem **2.3** also applies to each of $X(W,\sigma)$ and $Y(W,\sigma)$ and therefore each of $\ddot{X}(W,\sigma)$ and $\ddot{Y}(W,\sigma)$ has only two zero-crossing points. To see this, note that it was shown in [15] that the process of arc length evolution of Γ is equivalent to convolving an arc length parametric representation of Γ with a Gaussian filter based on a small width σ_0, resampling the output curve by the arc length parameter, reconvolving the resampled curve with the same filter and repeating this procedure many times. It follows from theorem **2.3** that when Γ is convolved with a Gaussian, $X(W, t_0)$ can be represented as:

$$X(W, t_0) = \sum_{k=1}^{\infty} \frac{a_k}{e^{4k^2\pi^2 t_0}} \sin(2k\pi W) + \sum_{k=1}^{\infty} \frac{b_k}{e^{4k^2\pi^2 t_0}} \cos(2k\pi W).$$

where $t_0 = \sigma_0^2/2$. $Y(W, t_0)$ can be represented in a similar way. When the output curve is resampled by the arc length parameter, function $X(W, t_0)$ is mapped to a new function $X_r(W, t_0)$:

$$X_r(W, t_0) = \sum_{k=1}^{\infty} \frac{a_k c_k}{e^{4k^2\pi^2 t_0}} \sin(2k\pi W) + \sum_{k=1}^{\infty} \frac{b_k d_k}{e^{4k^2\pi^2 t_0}} \cos(2k\pi W)$$

where c_k and d_k are multiplied by the old constants to obtain the new ones. When the resampled curve is reconvolved with the same filter, we obtain:

$$X_r(W, t_1) = \sum_{k=1}^{\infty} \frac{a_k c_k}{e^{4k^2\pi^2 t_1}} \sin(2k\pi W) + \sum_{k=1}^{\infty} \frac{b_k d_k}{e^{4k^2\pi^2 t_1}} \cos(2k\pi W).$$

where $t_1 > t_0$. Observe that as this procedure is repeated many times, the t in the denominator becomes larger and larger but the coefficients in the numerator remain bounded by upper and lower bounds determined by the original function (since resampling does not alter the shape of the curve). Therefore, as stated earlier, theorem **2.3** applies to each of $X(W, \sigma)$ and $Y(W, \sigma)$. From theorem **3.2**, it follows that Γ_σ must be simple and convex, and from theorem **3.5**, it follows that Γ remains simple and convex. □

4 Conclusions

A number of important results on the convergence properties of curvature scale space representations were presented in this paper. CSS representations have been shown to be very useful for recognition of noisy curves of arbitrary shapes at unknown orientations and scales. The results of this paper show that the computation of these representations always converges to stable states. It was shown that every closed planar curve eventually becomes simple and convex during evolution and arc length evolution and will remain in that state. These results shows that evolution and arc length evolution of planar curves are well-behaved processes, and that the computation of a CSS image always has a clearly defined termination point.

Acknowledgments

This work was supported by the Laboratory for Computational Intelligence at the University of British Columbia, Basic Research Laboratories of the Nippon Telegraph and Telephone Corporation, and the Vision, Speech, and Signal Processing Laboratory at the University of Surrey.

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