the results of this experiment $R=\sigma_0 v/\pi b^3=(1.0\pm0.2)\times10^{15}$ sec^{-1} if the limits placed on $\beta_{\frac{1}{2}}{}^0-\alpha_{\frac{1}{2}}{}^0$ are taken seriously. This assumes the radius of the mesonic Bohr orbit $b=2.2\times10^{-11}$ cm and that $v_0=8\times10^9$ cm/sec; $\sigma_0=(8\pi/9k^2)(v_0/v)(\beta_{\frac{1}{2}}{}^0-\alpha_{\frac{1}{2}}{}^0)^2$. Panofsky's experiment shows that this capture rate should equal the capture rate for the competing process $p(\pi^-,\gamma)n$.

Bernardini[11] has discussed the cross sections for $p(\gamma,\pi^+)n$, $d(\gamma,\pi^+)2n$, and $d(\gamma,\pi^-)2p$ for E_γ between 170 and 190 Mev in the laboratory system. If it is assumed that the ratio of π^- to π^+ production obtained from the second reaction is the same as the photoproduction ratio between the free neutron and free proton then the principle of detailed balance and this ratio can be used to predict the corresponding cross sections for $p(\pi^-,\gamma)n$. If these cross sections are extrapolated to the energy of Panofsky's experiment Bernardini obtains a capture rate that requires the initial slope of $\beta_{\frac{1}{2}}{}^0-\alpha_{\frac{1}{2}}{}^0$ to be $\pm(9.2°)\eta'$ in contrast to the value of $-(16.5°)\eta'$ obtained from this experiment.

Bethe and Noyes[16] have given an argument to explain this discrepancy in terms of Marshak's[17] suggestion. In brief this argument assumes that the slope of $\beta_{\frac{1}{2}}{}^0-\alpha_{\frac{1}{2}}{}^0$ obtained from this experiment cannot be extrapolated to the energy of Panofsky's experiment, but that this initial slope is $\pm(9.2°)\eta'$. They then fit this initial slope and the data of this experiment with a smooth curve for $\beta_{\frac{1}{2}}{}^0-\alpha_{\frac{1}{2}}{}^0$ *versus* η'. When the values of $\beta_{\frac{1}{2}}{}^0$ and $\alpha_{\frac{1}{2}}{}^0$ from higher energies are extrapolated with this restriction on their difference, it is difficult to fit the data without assuming that $\alpha_{\frac{1}{2}}{}^0$ varies less rapidly than η' and that $\beta_{\frac{1}{2}}{}^0$ varies more rapidly than η' in the energy region between 20 and 42 Mev. For the most probable fit under these assumptions $\beta_{\frac{1}{2}}{}^0$ changes sign between 20 and 30 Mev. This energy dependence for $\beta_{\frac{1}{2}}{}^0$ suggests a Jastrow[18] potential for this phase shift.

ACKNOWLEDGMENTS

A number of people aided the author in this experiment. Thanks are due especially to Dr. A. Roberts and Dr. J. Tinlot for their help and advice throughout the experiment. The author also wishes to thank Dr. J. French and Dr. H. P. Noyes for their helpful discussions of the results. Others who assisted and who contributed numerous suggestions were Dr. E. Hafner, Dr. F. Tenney, R. Santirocco, and D. Nelson. W. Coombs, K. Enslein, L. Braun, H. VonThenen, and F. Palmer were responsible for construction of parts of the equipment and for the reliable operation of the cyclotron.

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PHYSICAL REVIEW VOLUME 95, NUMBER 5 SEPTEMBER 1, 1954

Quantum Electrodynamics at Small Distances*

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(Received April 1, 1954)

The renormalized propagation functions D_{FC} and S_{FC} for photons and electrons, respectively, are investigated for momenta much greater than the mass of the electron. It is found that in this region the individual terms of the perturbation series to all orders in the coupling constant take on very simple asymptotic forms. An attempt to sum the entire series is only partially successful. It is found that the series satisfy certain functional equations by virtue of the renormalizability of the theory. If photon self-energy parts are omitted from the series, so that $D_{FC}=D_F$, then S_{FC} has the asymptotic form $A[p^2/m^2]^n[i\gamma\cdot p]^{-1}$, where $A=A(e_1^2)$ and $n=n(e_1^2)$. When all diagrams are included, less specific results are found. One conclusion is that the *shape* of the charge distribution surrounding a test charge in the vacuum does not, at small distances, depend on the coupling constant except through a scale factor. The behavior of the propagation functions for large momenta is related to the magnitude of the renormalization constants in the theory. Thus it is shown that the unrenormalized coupling constant $e_0^2/4\pi\hbar c$, which appears in perturbation theory as a power series in the renormalized coupling constant $e_1^2/4\pi\hbar c$ with divergent coefficients, may behave in either of two ways:

(a) It may really be infinite as perturbation theory indicates;
(b) It may be a finite number independent of $e_1^2/4\pi\hbar c$.

1. INTRODUCTION

It is a well-known fact that according to quantum electrodynamics the electrostatic potential between two classical test charges in the vacuum is not given exactly by Coulomb's law. The deviations are due to vacuum polarization. They were calculated to first order in the coupling constant α by Serber[1] and Uehling[2] shortly after the first discussion of vacuum polarization by Dirac[3] and Heisenberg.[4] We may express their results by writing a formula for the potential energy be-

* This work was supported by grants from the U. S. Office of Naval Research and the U. S. Atomic Energy Commission.
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tween two heavy point test bodies, with renormalized charges q and q', separated by a distance r:

$$V(r)=\frac{qq'}{4\pi r}\left\{1+\frac{\alpha}{3\pi}\int_{(2m)^2}^{\infty}\exp\left[-r\left(\frac{M^2c^2}{\hbar^2}\right)^{\frac{1}{2}}\right]\times\left(1+\frac{2m^2}{M^2}\right)\left(1-\frac{4m^2}{M^2}\right)^{\frac{1}{2}}\frac{dM^2}{M^2}+O(\alpha^2)+\cdots\right\}. \quad (1.1)^5$$

Here $\alpha=e_1^2/4\pi\hbar c\cong 1/137$ is the renormalized fine structure constant and m is the renormalized (observed) rest mass of the electron.

If $r\ll\hbar/mc$, then (1.1) takes the simple asymptotic form,

$$V(r)=\frac{qq'}{4\pi r}\left\{1+\frac{2\alpha}{3\pi}\left[\ln\left(\frac{\hbar}{mcr}\right)-\tfrac{5}{6}-\ln\gamma\right]+O(\alpha^2)+\cdots\right\}, \quad (1.2)^5$$

where $\gamma\cong 1.781$. We shall discuss the behavior of the entire series (1.2), to all orders in the coupling constant, making use of certain simple properties that it possesses in virtue of the approximation $r\ll\hbar/mc$. These properties are intimately connected with the concept of charge renormalization. The relation between (1.2) and charge renormalization can be made clear by the following physical argument:

A test body of "bare charge" q_0 polarizes the vacuum, surrounding itself by a neutral cloud of electrons and positrons; some of these, with a net charge δq, of the same sign as q_0, escape to infinity, leaving a net charge $-\delta q$ in the part of the cloud which is closely bound to the test body (within a distance $\hbar/mc$). If we observe the body from a distance much greater than $\hbar/mc$, we see an effective charge q equal to $(q_0-\delta q)$, the renormalized charge. However, as we inspect more closely and penetrate through the cloud to the core of the test body, the charge that we see inside approaches the bare charge q_0, concentrated in a point at the center. It is clear, then, that the potential $V(r)$, in Eqs. (1.1) and (1.2), must approach $q_0q_0'/4\pi r$ as r approaches zero. Thus, using (1.1), we may write

$$q_0q_0'=qq'\left\{1+\frac{2\alpha}{3\pi}\left[\ln\left(\frac{h/mc}{0}\right)-\tfrac{5}{6}-\ln\gamma\right]+O(\alpha^2)\right\}, \quad (1.3)$$

where the individual terms in the series diverge logarithmically in a familiar way. The occurrence of these logarithmic divergences will play an important role in our work.

Such divergences occur in quantum electrodynamics whenever observable quantities are expressed in terms of the bare charge e_0 and the bare (or mechanical) mass m_0 of the electron. The renormalizability of the theory consists in the fact that, when the observable quantities are re-expressed in terms of the renormalized parameters e_1 and m, no divergences appear, at least when a power series expansion in $e_1^2/4\pi\hbar c$ is used. The proof of renormalizability has been given by Dyson,[6] Salam,[7] and Ward.[8] We shall make particular use in Secs. III and IV of the elegant techniques of Ward.

We shall show that the fact of renormalizability gives considerable information about the behavior of the complete series (1.2). It may be objected to an investigation of this sort that while (1.2) is valid for $r\ll\hbar/mc$, the first few terms should suffice for calculation unless r is as small as $e^{-137}\hbar/mc$, a ridiculously small distance. We have no reason, in fact, to believe that at such distances quantum electrodynamics has any validity whatever, particularly since interactions of the electromagnetic field with particles other than the electron are ignored. However, a study of the mathematical character of the theory at small distances may prove useful in constructing future theories. Moreover, in other field theories now being considered, such as the relativistic pseudoscalar meson theory, conclusions similar to ours may be reached, and the characteristic distance at which they become useful is much greater, on account of the largeness of the coupling constant.

In this paper we shall be mainly concerned with quantum electrodynamics, simply because gauge invariance and charge conservation simplify the calculations to a considerable extent. Actually, our considerations apply to any renormalizable field theory, and we shall from time to time indicate the form they would take in meson theory.

2. REPRESENTATIONS OF THE PROPAGATION FUNCTIONS

The modified Coulomb potential discussed in Sec. I can be expressed in terms of the finite modified photon propagation function $D_{FC}(p^2,e_1^2)$ that includes vacuum polarization effects to all orders in the coupling constant. (Here p^2 is the square of a four-vector momentum $p\mu$.) The function D_{FC} is calculated by summing all Feynman diagrams that begin and end with a single photon line, renormalizing to all orders. The potential is given by[9]

$$V(r)=\frac{qq'}{(2\pi)^3}\int d^3p e^{i\mathbf{p}\cdot\mathbf{r}}D_{FC}(p^2,e_1^2), \quad (2.1)$$

where $\mathbf{p}$ is a three-dimensional vector.

If we were to sum all the Feynman diagrams that make up D_{FC} without renormalizing the charge we

[5] J. Schwinger, Phys. Rev. **75**, 651 (1949).

[6] F. J. Dyson, Phys. Rev. **75**, 1756 (1949).

[7] A. Salam, Phys. Rev. **84**, 426 (1951).

[8] J. C. Ward, Proc. Phys. Soc. (London) **A64**, 54 (1951); see also Phys. Rev. **84**, 897 (1951).

[9] From this point on, we take $\hbar=c=1$.

would obtain the divergent function $D_F'(p^2,e_0^2)$, which is related to the finite propagation function by Dyson's equations

$$D_F'(p^2,e_0^2)=Z_3D_{FC}(p^2,e_1^2), \quad (2.2)$$

$$e_1^2=Z_3e_0^2, \quad (2.3)$$

where Z_3 is a power series in e_1^2 with divergent coefficients. The bare and renormalized charges of a test body satisfy a relation similar to (2.3):

$$q_1^2=Z_3q_0^2, \quad (2.4)$$

so that Z_3^{-1} is just the bracketed quantity in (1.3).

The function D_{FC} can be represented in the form

$$D_{FC}(p^2,e_1^2)=\frac{1}{p^2-i\epsilon}+\int_0^\infty f\left(\frac{M^2}{m^2},e_1^2\right)\frac{dM^2}{M^2}\frac{1}{p^2+M^2-i\epsilon}, \quad (2.5)$$

where f is real and positive; the quantity Z_3 may be expressed in terms of f through the relation

$$Z_3^{-1}=1+\int_0^\infty f\left(\frac{M^2}{m^2},e_1^2\right)\frac{dM^2}{M^2}. \quad (2.6)$$

These equations have been presented and derived, in a slightly different form by Källén.[10] Their derivation is completely analogous to the derivation given in Appendix A of Eqs. (2.8) and (2.9) for the propagation function of the electron, which is discussed below.

We see from (2.5) and (2.6) that a virtual photon propagates like a particle with a probability distribution of virtual masses. In D_{FC}, the distribution is not normalized, but in the unrenormalized propagation function $D_F'=Z_3D_{FC}$ the probabilities are normalized to 1. The normalization integral is just the formally divergent quantity Z_3^{-1}. In D_{FC}, it is the coefficient of $1/(p^2-i\epsilon)$ that is 1, corresponding to the fact that the potential $V(r)$ in (2.1) at large distances is simply $qq'/4\pi r$.

It has been remarked[10,11] that Z_3^{-1} must be greater than unity, a result that follows immediately from Eq. (2.6). To this property of the renormalization constant there corresponds a simple property of the finite function D_{FC}, to wit, that as $p^2\to\infty$, the quantity p^2D_{FC} approaches Z_3^{-1}. If Z_3^{-1} is in fact infinite, as it appears to be when expanded in a power series, then D_{FC} is more singular than the free photon propagation function $D_F=1/(p^2-i\epsilon)$. In any case, D_{FC} can never be less singular than D_F, nor even smaller asymptotically. This is a general property of existing field theories; it is of particular interest in connection with the hope often expressed that in meson theory the exact modified propagation functions are less singular, or smaller, for large momenta, than the corresponding free-particle propagation functions.

The propagation functions for the electron behave quite similarly to the photon functions we have been discussing. Analogous to D_F' is the divergent electron propagation function $S_F'(p,e_0^2)$. It is obtained by summing all Feynman diagrams beginning and ending in a single electron line, renormalizing the mass of the electron, but not its charge, to all orders. Corresponding to D_{FC} there is the finite function $S_{FC}(p,e_1^2)$, related to S_F' by an equation similar to (2.2):

$$S_F'(p,e_0^2)=Z_2S_{FC}(p,e_1^2). \quad (2.7)$$

The quantity Z_2, like Z_3, appears as a power series in e_1^2 with divergent coefficients. It does not, however, contribute to charge renormalization.

A parametric representation of S_{FC}, resembling Eq. (2.5) for D_{FC}, is derived in Appendix A and reproduced here:

$$S_{FC}(p,e_1^2)=\frac{1}{i\gamma p+m-i\epsilon}+\int_m^\infty\frac{g(M/m,e_1^2)}{i\gamma p+M-i\epsilon}\frac{dM}{M}+\int_m^\infty\frac{h(M/m,e_1^2)}{i\gamma p-M+i\epsilon}\frac{dM}{M}. \quad (2.8)$$

Both g and h are real; in meson theory they are positive, but in quantum electrodynamics they may assume negative values. Z_2 can be expressed in terms of g and h through the relation

$$Z_2^{-1}=1+\int_m^\infty g\left(\frac{M}{m},e_1^2\right)\frac{dM}{M}+\int_m^\infty h\left(\frac{M}{m},e_1^2\right)\frac{dM}{M}. \quad (2.9)$$

Again we have a sort of probability distribution of virtual masses with a formally divergent normalization integral. As before, the modified propagation function is more singular, or at least asymptotically greater, than the free-particle propagation function, since $Z_2^{-1}\geqslant 1$, except possibly in quantum electrodynamics.

Equations (2.8) and (2.9), like (2.5) and (2.6), are similar to ones derived by Källén.[10] However, our notation and approach are perhaps sufficiently different from his to warrant separate treatment.

It should be noted that Källén's paper contains a further equation, (70), which, in our notation expresses the mechanical mass m_0 of the electron in terms of g and h:

$$m_0=\left[m+\int_m^\infty Mg\frac{dM}{M}+\int_m^\infty(-M)h\frac{dM}{M}\right]\times\left[1+\int_m^\infty g\frac{dM}{M}+\int_m^\infty h\frac{dM}{M}\right]^{-1}. \quad (2.10)$$

We see that m_0 is simply the mean virtual mass of the electron.

[10] G. Källén, Helv. Phys. Acta 25, 417 (1952).
[11] J. Schwinger (private communication from R. Glauber).

It may be remarked that a quantity analogous to m_0 can be constructed for the photon field, that is, the mean squared virtual mass of the photon:[12]

$$\mu_0^2=\left[\int_0^\infty fM^2\frac{dM^2}{M^2}\right]\left[1+\int_0^\infty f\frac{dM^2}{M^2}\right]^{-1}. \quad (2.11)$$

While gauge invariance forbids the occurrence of a mechanical mass of the photon in the theory, it is well known that a quadratically divergent quantity that looks like the square of a mechanical mass frequently turns up in calculations and must be discarded. That quantity is just μ_0^2, as given by (2.11). An equation similar to (2.11) holds in pseudoscalar meson theory, where μ_0^2 is really the square of a mechanical mass:

$$\mu_0^2=\left[\mu^2+\int_{9\mu^2}^\infty fM^2\frac{dM^2}{M^2}\right]\left[1+\int_{9\mu^2}^\infty f\frac{dM^2}{M^2}\right]^{-1}. \quad (2.12)$$

Here μ^2 is the observed meson mass. Evidently (2.12) implies that $\mu_0^2>\mu^2$.

3. EXAMPLE: QUANTUM ELECTRODYNAMICS WITHOUT PHOTON SELF-ENERGY PARTS

Before examining the asymptotic forms of the singular functions in the full theory of quantum electrodynamics let us consider a simplified but still renormalizable form of the theory in which all photon self-energy parts are omitted. A photon self-energy part is a portion of a Feynman diagram which is connected to the remainder of the diagram by two and only two photon lines. By omitting such parts, we effectively set

$$D_{FC}(p^2)=D_F(p^2)=1/(p^2-i\epsilon). \quad (3.1)$$

Moreover, there is no charge renormalization left in the theory, so that

$$Z_3=1$$

and

$$e_0^2=e_1^2. \quad (3.2)$$

Although some finite effects of vacuum polarization (such as its contribution to the second-order Lamb shift) have been left out, others (such as the scattering of light by light) are still included.

If the mass of the electron is now renormalized the only divergence remaining in the theory is Z_2. It has been shown by Ward[8] that in the calculation of any observable quantity, such as a cross section, Z_2 cancels out. We shall nevertheless be concerned with Z_2 since it does appear in a calculation of the electron propagation function S_F'.

In order to deal with Z_2, a divergent quantity, we shall make use of the relativistic high-momentum cutoff procedure introduced by Feynman, which consists of replacing the photon function D_F by a modified function $D_{F\lambda}$ defined by

$$D_{F\lambda}(p^2)=D_F(p^2)\lambda^2/(\lambda^2+p^2-i\epsilon).$$

In a given calculation, if λ^2 is large enough, quantities that would be finite in the absence of a cutoff remain unchanged while logarithmically divergent quantities become finite logarithmic functions of λ^2.

Thus, if we calculate $S_F'(p)$ using a Feynman cutoff with $\lambda^2\gg|p^2|$ and $\lambda^2\gg m^2$, and drop terms that approach zero as λ^2 approaches infinity, we must find a relation similar to (2.7):

$$S_{F'\lambda}(p)=z_{2\lambda}S_{FC}(p), \quad (3.3)$$

where the finite function S_{FC} has remained unchanged by the cut-off process, while the infinite constant Z_2 has been converted to the finite quantity $z_{2\lambda}$, which is a function of λ^2/m^2. (The reader who is not impressed with the rigor of these arguments should refer to the next section, where a more satisfactory cut-off procedure is introduced.)

Calculation to the first few orders in the coupling constant indicates that $z_{2\lambda}$ has the form

$$z_{2\lambda}=1+e_1^2\left(a_1+b_1\ln\frac{\lambda^2}{m^2}\right)+e_1^4\left[a_2+b_2\ln\frac{\lambda^2}{m^2}+c_2\left(\ln\frac{\lambda^2}{m^2}\right)^2\right]+\cdots. \quad (3.4)$$

The propagation function S_{FC} may also be calculated to fourth order in e_1; for $|p^2|\gg m^2$ it has the form

$$|p^2|\gg m^2:\ S_{FC}(p)\ \frac{1}{i\gamma p}\{1+e_1^2[a_1'+b_1'\ln(p^2/m^2)]+e_1^4[a_2'+b_2'\ln(p^2/m^2)+c_2'(\ln(p^2/m^2))^2]+\cdots\}. \quad (3.5)$$

In order to obtain some understanding of the properties of Eq. (3.3), let us substitute these approximate expressions into that equation. Let us then examine what happens in the limit $m\to 0$. We see that in neither of the expressions (3.4) and (3.5) can m be set equal to zero with impunity; that is to say, both factors on the right-hand side of (3.3) contain logarithmic divergences as $m\to 0$. Thus we should naively expect that the left-hand side have no limit as $m\to 0$. Rather, we should expect to find logarithmic divergences to each order in e_1^2, unless fantastic cancellations, involving the constants a_1, a_1', etc., should happen to occur.

But such cancellations must indeed occur, since a direct calculation of $S_{F\lambda}'$ with m set equal to zero exhibits no divergences whatever. Instead, each Feynman diagram yields a term equal to $1/i\gamma p$ times a finite function of λ^2/p^2. It is clear that this must be so, since λ

[12] We are indebted to Dr. Källén for a discussion of this point.

provides an ultraviolet cutoff for every integral, while p provides an infrared cutoff.

Let us now make use of the remarkable cancellations that we have discussed, but in such a way that we do not rely on the specific forms of (3.4) and (3.5), for which we have so far claimed no validity beyond fourth order in e_1. We shall consider the asymptotic region $\lambda^2 \gg |p^2| \gg m^2$. We may write

$$S_{FC} = (1/i\gamma p) s_C(p^2/m^2) \tag{3.6}$$

and

$$z_{2\lambda} = z_2(\lambda^2/m^2). \tag{3.7}$$

Moreover, in the asymptotic region, we may drop m entirely in $S_{F'\lambda'}$, since a limit exists as m^2/λ^2 and m^2/p^2 approach zero, with $\lambda^2 \gg |p^2|$. Thus we have

$$S_{F'\lambda} \approx (1/i\gamma p) s(\lambda^2/p^2). \tag{3.8}$$

Equation (3.3) then implies the following functional equation:

$$s(\lambda^2/p^2) = z_2(\lambda^2/m^2) \cdot s_C(p^2/m^2). \tag{3.9}$$

The functional equation has the general solution[13]

$$z_2(\lambda^2/m^2) = A(\lambda^2/m^2)^{-n} = A \exp[-n \ln(\lambda^2/m^2)], \tag{3.10}$$

$$s_C(p^2/m^2) = B(p^2/m^2)^n = B \exp[n \ln(p^2/m^2)], \tag{3.11}$$

$$s(\lambda^2/p^2) = AB(\lambda^2/p^2)^{-n} = AB \exp[-n \ln(\lambda^2/p^2)]. \tag{3.12}$$

Here A, B, and n are functions of e_1^2 alone. If all three constants are expanded in power series in e_1^2, then formulas like (3.4) and (3.5) can be seen to be valid to all orders in e_1^2. The constants are given, to second order in e_1, by the equations

$$n = \frac{e_1^2}{16\pi^2} + \cdots, \tag{3.13}$$

$$A = 1 + \frac{e_1^2}{16\pi^2}\left[\tfrac{3}{2} + \int_{m^2}^{\infty} \frac{dM^2}{M^4(M^2-m^2)}(5m^2M^2 - m^4)\right], \tag{3.14}$$

$$B = 1 - \frac{e_1^2}{16\pi^2}\int_{m^2}^{\infty} \frac{dM^2}{M^4(M^2-m^2)}(5m^2M^2 - m^4) + \cdots. \tag{3.15}$$

It is now apparent that we have glossed over a difficulty, although it turns out to be a minor one. While AB is perfectly finite, A and B separately contain infrared divergences that must be cut off by the introduction of a small fictitious photon mass μ. These divergences are well-known and arise from the requirement that $(i\gamma p + m)S_{FC}(p)$ approach unity as $i\gamma p + m$ tends to zero, while the point $i\gamma p + m = 0$ is in fact a singularity of the function $(i\gamma p + m)S_F'(p)$.

From the asymptotic form for S_{FC},

$$S_{FC}(p) \approx \frac{B}{i\gamma p}\left(\frac{p^2}{m^2}\right)^n, \tag{3.16}$$

we may derive an asymptotic form for the vertex operator[6] $\Gamma_{\mu C}(p,p')$ for equal arguments.

We use Ward's[8] relation

$$\Gamma_{\mu C}(p,p) = \frac{1}{i}\frac{\partial}{\partial p_\mu}[S_{FC}(p)]^{-1}, \tag{3.17}$$

and obtain

$$\Gamma_{\mu C}(p,p) = B^{-1}(p^2/m^2)^{-n}(\gamma_\mu - 2n p_\mu \gamma p/p^2). \tag{3.18}$$

A result similar to this was found by Edwards,[14] who summed a small subset of the diagrams we consider here. We may note that corresponding to the increase in the singularity of S_F there is a decrease in that of Γ_μ. The two are obviously tied together by (3.17). It is therefore highly inadvisable to take seriously any calculation using a modified Γ_μ and unmodified S_F', or vice versa.

It is unfortunate that the inclusion of photon self-energy parts (omitted in this section) invalidates the simple results we have obtained here. In Sec. V we shall derive and solve the functional equations that replace (3.9) in the general case, but the solutions give much less detailed information than (3.16). In order to treat the general case, we must first develop (in Sec. IV) a more powerful cut-off technique than the one we have used so far.

4. WARD'S METHOD[8] USED AS A CUTOFF[15]

The starting point of Ward's method of renormalization is a set of four integral equations derived by summing Feynman diagrams. The equations involve four functions: $S_F'(p)$, $D_F'(k)$, the vertex operator $\Gamma_\mu(p_1,p_2)$, and a function $W_\mu(k)$ defined by

$$W_\mu = (\partial/\partial k_\mu)[D_F'(k)]^{-1}. \tag{4.1}$$

Two of the equations are trivial, following from (4.1) and (3.17), respectively:

$$[S_F'(p)]^{-1} = i\int_0^1 dx(p_\mu - p_\mu') \times \Gamma_\mu(px + p'(1-x),\, px + p'(1-x)) \tag{4.2}$$

and

$$[D_F'(k)]^{-1} = \int_0^1 dy k_\mu W_\mu(ky), \tag{4.3}$$

where p' is a free electron momentum, i.e., after integration p'^2 is to be replaced by $-m^2$ and $i\gamma p'$ by $-m$, where m is the experimental electron mass. No further mass renormalization is necessary.

[13] J. C. Maxwell, Phil. Mag. (Series 4) **19**, 19 (1860).

[14] S. F. Edwards, Phys. Rev. **90**, 284 (1953).

[15] The purpose of this section is to justify the use of a cutoff when photon self-energy parts are included. The reader who is willing to take this point for granted need devote only the briefest attention to the material between Eqs. (4.3) and (4.6). The remainder of the section contains some simple but important algebraic manipulation of the cut-off propagation functions.

The two remaining equations are nonlinear power-series integral equations for Γ_μ and W_μ in which each term describes an "irreducible" Feynman diagram. An irreducible diagram, as defined by Dyson, is one which contains no vertex or self-energy parts inside itself. When the complete series of irreducible diagrams for Γ_μ or W_μ is written down, and in each one S_F' is substituted for S_F, D_F' for D_F, Γ_μ for γ_μ, and W_μ for $2k_\mu$, then the complete series of *all* diagrams for Γ_μ or W_μ is generated. We give below the first two terms of each of the integral equations. Equation (4.4) corresponds to Fig. 1 and Eq. (4.5) to Fig. 2.

$$\Gamma_\mu(p_1,p_2)\equiv\gamma_\mu+\Lambda_\mu(p_1,p_2)$$
$$=\gamma_\mu+\frac{ie_0^2}{(2\pi)^4}\int\Gamma_\lambda(p_1,p_1-k)$$
$$\times S_F'(p_1-k)\Gamma_\mu(p_1-k,p_2-k)\cdot S_F'(p_2-k)$$
$$\times\Gamma_\lambda(p_2-k,p_2)D_F'(k)d^4k+\cdots,\quad(4.4)$$

$$W_\mu(k)\equiv 2k_\mu+Tk_\mu$$
$$=2k_\mu+\frac{e_0^2}{(2\pi)^4}\cdot\tfrac{1}{3}\,\mathrm{Tr}\int\Gamma_\nu(p,p+k)$$
$$\times S_F'(p+k)\Gamma_\mu(p+k,p+k)S_F'(p+k)$$
$$\times\Gamma_\nu(p+k,p)S_F'(p)d^4p+\cdots.\quad(4.5)$$

The factor of one-third arises in (4.5) because we are interested only in the coefficient of $\delta_{\nu\lambda}$ in the tensor $[\delta_{\nu\lambda}-(k_\nu k_\lambda/k^2)]D_F'(k)$.

The heavy lines and dots have been drawn as a reminder that the complete S_F', D_F', and Γ_μ are to be inserted.

The symbols Λ_μ and T_μ are simply a convenient shorthand for the sum of all the integrals occurring on the right in (4.4) and (4.5), respectively.

The properties of (4.4) and (4.5) that are crucial for the possibility of renormalization are the following:

(i) All divergences that occur in the power series solution of the equations are logarithmic divergences. (W_μ is actually formally linearly divergent but on grounds of covariance the linear divergence will vanish.)

(ii) In (4.4), each terms with coefficient $(e_0^2)^n$ con-exactly nD_F' functions and contains one more Γ_μ function than S_F' functions.

(iii) In (4.5), each term with coefficient $(e_0^2)^n$ contains $(n-1)$ more D_F' functions than W_μ functions, and contains equal numbers of Γ_μ and S_F' functions.

At this point Ward introduces a subtraction procedure which alters Eqs. (4.4) and (4.5) so that the solutions of the new equations are finite functions S_{FC}, D_{FC}, $\Gamma_{\mu C}$, and $W_{\mu C}$. He then shows that the modi-

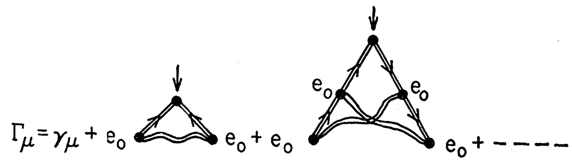

FIG. 1. The sequence of irreducible diagrams for Γ_μ.

fications introduced are equivalent to charge renormalization. The method that we shall present here is a slight generalization of Ward's involving two cut-off parameters λ and λ'. When $\lambda=0$ and $\lambda'=im$, our method reduces to Ward's.

From the right-hand side of (4.4) we subtract $\int dx\Lambda_\mu(l'x+p'(1-x),\, l'x+p'(1-x))$, where p' has the same significance as in (4.2) and l' is a vector parallel to p' but with $(l')^2$ set equal to $(\lambda')^2$ and $\gamma\cdot l'$ set equal to λ' after the integration. This choice of subtraction procedure may appear arbitrarily complicated, so that a remark about our motivation may be in order. Since Λ_μ consists of logarithmically divergent integrals, the quantity

$$\Lambda_\mu{}^x=\Lambda_\mu(p_1,p_2)-\Lambda_\mu(l'x+p'(1-x),\, l'x+p'(1-x))$$

is certainly finite. Therefore so is

$$\Lambda_{\mu l'}=\int_0^1 dx\Lambda_\mu{}^x=\Lambda_\mu(p_1,p_2)-\int\Lambda_\mu(l'x+p'(1-x),\, l'x+p'(1-x))dx,$$

which is the quantity of interest. However, referring to (4.2), we see that if we replace Γ_μ by $\gamma_\mu+\Lambda_{\mu l'}$, $S_F'(p)$ will have the value $1/(i\gamma p+m)$ at $p=l'$. This subtraction procedure therefore provides a convenient normalization for the cut-off functions.

Similarly, from the right-hand side of (4.5) we subtract $k_\mu\int_0^1 2ydyT(ly)$, where $l^2=\lambda^2$. (The motivation is the same here as before.) Let us denote the solutions of the modified equations by the symbols $S_{F\lambda'\lambda}$, etc. Like Ward's functions, they are finite to all orders in the coupling constant, the logarithmic divergences having disappeared in the course of the subtraction. In the modified equations, let us everywhere replace the coupling constant e_0^2 by another one, e_2^2. Then we may show that if e_2^2 is a properly chosen function of e_0^2, λ^2, and m^2 the modified functions are multiples of the

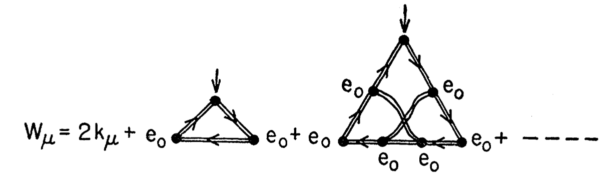

FIG. 2. The sequence of irreducible diagrams for W_μ. Notice that there is no irreducible diagram in fourth order.

original divergent functions. In fact, we have the relations:

$$\Gamma_\mu(p_1,p_2;e_0^2)=Z_2^{-1}(\lambda,\lambda';e_2^2)\Gamma_{\mu\lambda\lambda'}(p_1,p_2;e_2^2), \quad (4.6)$$

$$S_F'(p;e_0^2)=Z_2(\lambda,\lambda';e_2^2)S_{F\lambda\lambda'}(p,e_2^2), \quad (4.7)$$

$$W_\mu(k,e_0^2)=Z_3^{-1}(\lambda;e_2^2)W_{\mu\lambda}(k,e_2^2), \quad (4.8)$$

$$D_F'(k,e_0^2)=Z_3(\lambda;e_2^2)D_{F\lambda}(k,e_2^2), \quad (4.9)$$

$$e_2^2=Z_3(\lambda;e_2^2)e_0^2. \quad (4.10)$$

These relations can be proved by substituting them into the original integral Eqs. (4.2) through (4.5) and making use of the properties (ii) and (iii). Z_2 and Z_3 are then given by

$$\gamma_\mu Z_2(\lambda,\lambda';e_2^2)=\gamma_\mu-\int_0^1 dx\times$$
$$\Lambda_{\mu\lambda\lambda'}(l'x+p'(1-x),\,l'x+p'(1-x);e_2^2) \quad (4.11)$$

and

$$Z_3(\lambda,e_2^2)=1-\int_0^1 ydyT_\lambda(ly,e_2^2). \quad (4.12)$$

Here Λ and T stands for the series of integrals on the right-hand side of (4.4) and (4.5) calculated *with the cut-off functions.*

We note that the photon functions are independent of λ'. This is of course due to the fact that Z_2 cancels out in calculating these functions. Furthermore,

$$[(i\gamma p+m)S_{F\lambda\lambda'}(p)]_{\gamma p=\lambda'}=1, \quad (4.13)$$

and

$$[k^2D_{F\lambda}(k)]_{k^2=\lambda^2}=1. \quad (4.14)$$

It is clear, then, that for $\lambda'=im$ and $\lambda=0$ our modified functions reduce to the usual convergent functions as defined by Ward. Furthermore, when λ and λ' are both infinite our modified functions are (in some sense which we need not worry about) the original divergent functions S_F', D_F', etc. We may now relate our functions to the usual convergent functions very simply. (From now on we shall limit our discussion to S_F' and D_F', since Γ and W are obtained from them by differentiation.)

We rewrite (4.7), (4.9), and (4.10):

$$S_F'(p,e_0^2)=Z_2(\lambda,\lambda';e_2^2)S_{F\lambda\lambda'}(p,e_2^2), \quad (4.7)$$

$$D_F'(k,e_0^2)=Z_3(\lambda,e_2^2)D_{F\lambda}(k,e_2^2), \quad (4.9)$$

$$e_2^2=Z_3(\lambda,e_2^2)e_0^2. \quad (4.10)$$

We obtain the conventional renormalization theory by setting $\lambda=0$, $\lambda'=im$, so that

$$S_F'(p,e_0^2)=Z_2(0,im;e_1^2)S_{FC}(p,e_1^2), \quad (4.7)'$$

$$D_F'(k,e_0^2)=Z_3(0,e_1^2)D_{FC}(k,e_1^2), \quad (4.9)'$$

and

$$e_1^2=Z_3(0,e_1^2)e_0^2, \quad (4.10)'$$

where e_1 is the observed electronic charge.

Dividing the unprimed equations by the primed ones leads to the relations:

$$S_F(\lambda,\lambda',p;e_2^2)=z_2(\lambda,\lambda';e_1^2)S_{FC}(p,e_1^2), \quad (4.7)''$$

$$D_F(\lambda,k;e_2^2)=z_3(\lambda,e_1^2)D_{FC}(k,e_1^2), \quad (4.9)''$$

$$e_1^2=z_3(\lambda,e_1^2)e_2^2. \quad (4.10)''$$

In these last equations all the quantities involved are finite. As λ' and λ approach ∞, however, they approach their original divergent values. We have therefore established a cutoff which is useful in the presence of photon self-energy parts and which has the desired renormalization property for any values of the cut-off parameters. We may call attention to the essential simplicity of the cutoff. For example, if we had used it in Sec. III instead of the Feynman cutoff we would have found $s=(p^2/\lambda^2)^n$ to all orders in Eq. (3.12), i.e., $AB=1$.

For our purposes it is convenient to eliminate the trivial $1/(i\gamma p+m)$ dependence of S_{FC} and the $1/k^2$ dependence of D_{FC}. We therefore set, as in Sec. III:

$$S_{FC}(p)=\frac{1}{i\gamma p+m}s_C(p), \quad (4.15)$$

$$D_{FC}(k)=(1/k^2)d_C(k), \quad (4.16)$$

and

$$S_{F\lambda\lambda'}(p)=\frac{1}{i\gamma p+m}s(\lambda,\lambda',p) \quad (4.15)'$$

$$D_{F\lambda}(k)=(1/k^2)d(\lambda,k). \quad (4.16)'$$

Equations (4.13) and (4.14) are now very useful. Using them together with (4.7)″, (4.9)″, (4.10)″, and the definitions (4.15), (4.16), we find:

$$z_2^{-1}(\lambda,\lambda';e_1^2)=[s_C(p,e_1^2)]_{\gamma p=\lambda'}, \quad (4.17)$$

$$z_3^{-1}(\lambda,e_1^2)=[d_C(k,e_1^2)]_{k^2=\lambda^2}. \quad (4.18)$$

z_2 is thus independent of λ since the right-hand side of (4.17) is independent of λ.

Our final equations are therefore

$$s(\lambda,\lambda',p,e_2^2)=s_C(p,e_1^2)/s_C(\lambda',e_1^2), \quad (4.19)$$

$$d(\lambda,k,e_2^2)=d_C(k,e_1^2)/d_C(\lambda,e_1^2), \quad (4.20)$$

and

$$e_2^2=d_C(\lambda,e_1^2)e_1^2. \quad (4.21)$$

The renormalization constants are seen to be the convergent functions calculated at infinite values of their arguments. This confirms the results of Sec. II and Appendix A.

Before closing this section we might remark that the entire treatment presented here can be very easily transcribed to meson theory. The situation in that case is somewhat more complicated since both Z_2 and Z_5 (the renormalization constant for Γ_5) contribute to charge renormalization, which is present even in the absence of closed loops (as is well known). The renormalization of Γ_5 must be carried through somewhat differently from that of Γ_μ, since we shall want $\Gamma_{5\lambda\lambda'}(p,p;g_2^2)$ to equal γ_5 at $\gamma p=\lambda'$. The equations analogous to (4.19)–(4.21) for pseudoscalar meson theory are:

$$s(\lambda,\lambda',p,g_2^2)=s_C(p,g_1^2)/s_C(\lambda',g_1^2), \quad (4.19)'$$

$$\delta(\lambda,\lambda',k,g_2^2)=\delta_C(k,g_1^2)/\delta_C(\lambda,g_1^2), \quad (4.20)'$$

$$\gamma_5(\lambda,\lambda',p,g_2^2)=\gamma_{5C}(p,g_1^2)/\gamma_{5C}(\lambda',g_1^2), \quad (4.22)$$

$$g_2^2=g_1^2[s_C(\lambda',e_1^2)\gamma_{5C}(\lambda',g_1^2)]^2\delta_C(\lambda,g_1^2), \quad (4.21)'$$

where

$$\Gamma_{5C}(p,p,g_1^2)=\gamma_5\cdot\gamma_{5C}(p,g_1^2), \text{ etc.,}$$

and where $\Delta=\delta/(k^2+\mu^2)$ is the meson propagation function.

We shall not investigate these equations further; we shall confine our attention to the much simpler case of quantum electrodynamics.

5. ASYMPTOTIC BEHAVIOR OF THE PROPAGATION FUNCTIONS IN QUANTUM ELECTRODYNAMICS

With the aid of the cut-off procedure introduced in the previous section, we may return to the discussion, begun in Sec. III, of the behavior of the propagation functions in the asymptotic region ($|p^2|\gg m^2$). In Sec. III photon self-energy parts were omitted and it was sufficient to use a Feynman cutoff in order to find the functional equation (3.9) satisfied asymptotically by the electron propagation function S_F'. The new cutoff enables us to include all Feynman diagrams. We shall now find a new functional equation for S_F' and one for D_F' as well.

Our starting point is the set of Eqs. (4.19)–(4.21) that express the cut-off propagation functions in terms of renormalized quantities. [In Sec. III, we used Eq. (3.3) instead.] We must first observe, as in Sec. III, that in a power series calculation of the cut-off functions the results remain finite when the electron mass is set equal to zero. The quantities λ and λ' provide, of course, ultraviolet cutoffs, while p provides an infrared cutoff for all Feynman integrals. Thus in the asymptotic region we may drop the electron mass in a calculation of the cut-off functions, which then take the forms

$|p^2|,\ \lambda^2,\ \lambda'^2\gg m^2$:

$$s(p/\lambda,p/\lambda',m^2/p^2,e_2^2)\approx s(p/\lambda,p/\lambda',0,e_2^2), \quad (5.1)$$

$|k^2|,\ \lambda^2\gg m^2$:

$$d(k^2/\lambda^2,m^2/k^2,e_2^2)\approx d(k^2/\lambda^2,0,e_2^2). \quad (5.2)$$

It should be noted that the asymptotic form of s depends only on p^2 and not on $i\gamma p$.

Equations (4.19)–(4.21) now give us at once the required functional equations:

$$|p^2|,\ \lambda^2,\ \lambda'^2\gg m^2:\ s(p^2/\lambda^2,p^2/\lambda'^2,e_2^2)=\frac{s_C(p^2/m^2,e_1^2)}{s_C(\lambda'^2/m^2,e_1^2)} \quad (5.3)$$

$$|k^2|,\ \lambda^2\gg m^2:\qquad d(k^2/\lambda^2,e_2^2)=\frac{d_C(k^2/m^2,e_1^2)}{d_C(\lambda^2/m^2,e_1^2)} \quad (5.4)$$

$$e_2^2=e_1^2d_C(\lambda^2/m^2,e_1^2). \quad (5.5)$$

(We have omitted the argument $m^2/p^2=0$ in s and d.)

In Appendix B it is shown that the general nontrivial solution of these equations is given by:

$$e_1^2d_C(k^2/m^2,e_1^2)=F((k^2/m^2)\phi(e_1^2)), \quad (5.6)$$

$$s_C(p^2/m^2,e_1^2)=A(e_1^2)H((p^2/m^2)\phi(e_1^2)). \quad (5.7)$$

Here F, H, ϕ, and A are unknown functions of their arguments. (A contains an infrared divergent factor, which is always canceled, in calculations, by a similar factor in the vertex operator $\Gamma_{\mu C}$.) It is evident that we have obtained much less information here than we did in Sec. III. Also, the results of Sec. III are not correct in the general case, since Eq. (3.11) is not a special case of (5.7), but corresponds rather to a trivial solution of the functional equations (5.3)–(5.5), peculiar to the case of no charge renormalization.[16]

At least one striking result has emerged from the work in this section, however. The quantity on the left-hand side of Eq. (5.6) is, as remarked in the introduction, the Fourier transform of the Laplacian of the potential energy of two heavy point charges. It represents, therefore, the Fourier transform of a kind of effective charge density for the cloud of pairs surrounding a test body in the vacuum. Equation (5.6) states, in effect, that the *shape* of the charge distribution, at distances much smaller than $\hbar/mc$, is independent of the coupling constant $e_1^2/4\pi$, which enters only into the *scale factor* $\phi(e_1^2)$.

This result has an important consequence for the magnitude of e_0^2, the square of the bare charge, which is associated with the strength of the singularity at the center of the effective charge distribution. We have learned in Sec. II that e_0^2 is given by

$$e_0^2=e_1^2d_C(\infty,e_1^2) \quad (5.8)$$

[16] It should be noted that, in any simplified form of the theory in which a restricted class of diagrams is summed, our results are unchanged provided that the conditions for renormalizability, as discussed in Sec. IV, are fulfilled. If, in a renormalizable approximation to the theory, there is charge renormalization, then the results of this section apply; if not, then the results of Sec. III apply.

For example, if the full integral equations (4.4) and (4.5), when renormalized, should turn out not to have solutions, it may be that solutions will exist for the integral equations obtained by cutting off the sequences on the right-hand sides after a finite number of terms and renormalizing. Such a procedure would be renormalizable and would not affect our functional equations.

and that $e_1^2 d_C(k^2/m^2,e_1^2)$ is a positive increasing function of k^2/m^2. Equation (5.6) tells us, then, that $F(\phi(e_1^2)k^2/m^2)$ is an increasing function of k^2/m^2 and that:

(a) If as $k^2/m^2\rightarrow\infty$, $F(\phi k^2/m^2)\rightarrow\infty$, then e_0^2 is infinite, and the singularity at the center of the charge distribution is stronger than the δ function that corresponds to a finite point charge. This is the result indicated by perturbation theory.

(b) If, as $k^2/m^2\rightarrow\infty$, $F(\phi k^2/m^2)$ approaches a finite limit, then e_0^2 equals that limit, which is *independent of the value of* e_1^2. The singularity at the center of the effective charge distribution is then a δ function with a strength corresponding to a finite bare charge e_0.

We shall return, at the end of this section, to the discussion of cases (a) and (b). Meanwhile, let us look at the solution of the functional equations from another point of view.

While the functions F and ϕ are unknown, certain of their properties can be deduced from the perturbation expansion of d_C. In the asymptotic region, d_C appears as a double power series in e_1^2 and $\ln(k^2/m^2)$ with finite numerical coefficients. (Of course, the convergence properties of this series are unknown. We have assumed throughout, however, that it defines a function which satisfies the same functional equations that we have derived for the series.) To facilitate comparison of (5.6) with the series, let us make use of the alternative form derived in Appendix B:

$$|k^2|\gg m^2:\quad \ln\frac{k^2}{m^2}=\int_{q(e_1^2)}^{e_1^2 d_C(k^2/m^2,e_1^2)}\frac{dx}{\psi(x)}. \tag{5.9}$$

In Eq. (5.9) both unknown functions q and ψ have power series expansions in their arguments and the first few terms of these series may be determined from the first few orders of perturbation theory. (The functions F and ϕ do not have this property.)

Perturbation theory yields for d_C in the asymptotic region the expansion

$$|k^2|\gg m^2:\quad d_C\left(\frac{k^2}{m^2},e_1^2\right)=\left[1-\frac{e_1^2}{12\pi^2}\left(\ln\frac{k^2}{m^2}-\frac{5}{3}\right)-\frac{e_1^4}{64\pi^4}\left(\ln\frac{k^2}{m^2}+c\right)+\cdots\right]^{-1}. \tag{5.10}$$

The fourth-order calculation was performed by Jost and Luttinger,[17] who did not compute c. Comparison of (5.9) and (5.10) yields the expansions of q and ψ:

$$\psi(x)=\frac{1}{12\pi^2}\left[x^2+\frac{3}{16\pi^2}x^3+\cdots\right], \tag{5.11}$$

$$q(e_1^2)=e_1^2-\frac{5}{36\pi^2}e_1^4+\cdots. \tag{5.12}$$

[17] R. Jost and J. M. Luttinger, Helv. Phys. Acta **23**, 201 (1950).

For the actual value of the coupling constant in quantum electrodynamics, $q(e_1^2)$ is presumably well approximated by e_1^2 and need not concern us very much. The crucial function is $\psi(x)$, which is given by (5.11) for very small x but is needed for large x in order to determine the behavior of the propagation function at very high momenta and to resolve the question of the finiteness of the bare charge.

We can restate, in terms of the properties of ψ, the two possibilities (a) and (b) for the behavior of the theory at high momenta:

(a) The integral $\int dx/\psi(x)$ in (5.9) does not diverge until the upper limit reaches $+\infty$. In that case $\ln(k^2/m^2)=+\infty$ corresponds to $e_1^2 d_C(k^2/m^2,e_1^2)=+\infty$ and the bare coupling constant $e_0^2/4\pi$ is infinite.

(b) For some finite value x_0 of the upper limit, $\int^{x_0}dx/\psi(x)$ diverges; for this to happen, $\psi(x)$ must come down to zero at $x=x_0$. Then $e_1^2 d_C(k^2/m^2,e_1^2)\rightarrow x_0$ as $\ln(k^2/m^2)\rightarrow\infty$, so that $e_0^2=x_0$, a finite number independent of e_1^2. Since $e_1^2<e_0^2$, the theory can exist only for e_1^2 less than some critical value $e_c^2\leqslant e_0^2$, where $q(e_c^2)=e_0^2$. As $q(e_1^2)$ approaches its maximum value e_0^2, $\psi(q(e_1^2))\rightarrow 0$, so we learn from Eq. (B.26) in Appendix B that the asymptotic form of $e_1^2 d_C(k^2/m^2,e_1^2)$ reduces simply to the constant e_0^2. A constant asymptotic form of $e_1^2 d_C(k^2/m^2,e_1^2)$ means that the weighting function $f(M^2/m^2,e_1^2)$ in Eq. (2.5) must vanish in the asymptotic region to order $1/M^2$. If the bare charge is finite, then the effective coupling at high momenta varies in a strange way with $q(e_1^2)$, increasing at first with increasing $q(e_1^2)$ and then decreasing to zero at $q(e_1^2)=e_0^2$, beyond which point the theory is meaningless.

Since we cannot discriminate between cases (a) and (b), the methods we have developed have not really served to settle fully the question of the asymptotic character of the propagation function D_{FC}. However, it is to be hoped that these methods may be used in the future to obtain more powerful results.

Recently Källén[18] has investigated the question of the finiteness of the bare charge. His result is that of the three renormalization quantities Z_2^{-1}, e_0^2, and m_0, at least one is infinite. Unfortunately, it is not possible to conclude from Källén's work that case (b) must be rejected.

APPENDIX A. CONSTRUCTION OF PARAMETRIC REPRESENTATIONS FOR THE PROPAGATION FUNCTIONS

The function $S_{F'\alpha\gamma}(x-y)$ is given by the matrix element

$$S_{F'\alpha\gamma}(x-y)=\epsilon(x_0-y_0)(\Psi_0,P[\psi_\alpha(x),\bar{\psi}_\gamma(y)]\Psi_0). \tag{A.1}$$

Here $\psi(x)$ is the electron (or nucleon) field operator at the space-time point x in the Heisenberg representation; $\bar{\psi}(y)$ is the Dirac adjoint $\psi^*(y)\beta$ of $\psi(y)$; P is Dyson's

[18] G. Källén, Kgl. Danske Videnskab. Selskab, Mat.-fys.Medd. **27**, 12 (1953).

time-ordering operator; $\epsilon(t)$ is $t/|t|$; and Ψ_0 is the vacuum state of the coupled electron and photon (or nucleon and meson) fields.

In quantum electrodynamics S_F' is a gauge-variant function and for that reason one may easily be misled in dealing with it. To avoid such difficulties let us first discuss the function in meson theory (say the neutral pseudoscalar theory).

For $x_0 > y_0$ we may write (A.1) in the form

$$S_{F'\alpha\gamma}(x-y) = \sum_{\mathbf{p},M,s,\Pi,n} (\Psi_0, \psi_\alpha(x)\Psi_{\mathbf{p},M,s,\Pi,n}) \times (\Psi_{\mathbf{p},M,s,\Pi,n}, \bar{\psi}_\gamma(y)\Psi_0) \quad \text{(A.2)}$$

where the Ψ's are a complete set of eigenstates of the coupled-field Hamiltonian and momentum. The momentum eigenvalue is denoted by $\mathbf{p}$ and the energy eigenvalue is given by $(M^2+p^2)^{\frac{1}{2}}$, so that the system described by the state Ψ has the energy M in its rest frame. For those states which contribute nonvanishing matrix elements to (A.2), the angular momentum of the system in its rest frame is $\frac{1}{2}$ and may have z component $\pm\frac{1}{2}$, denoted by the values ± 1 of the index s. Π is the parity of the system in its rest frame and may be $+1$ or -1. The remaining index n labels all the other (invariant) quantum numbers necessary to specify the state.

We may list some of the simplest types of states that contribute:

(1) one nucleon: $M=m$, the renormalized nucleon mass, and $\Pi=+1$ by convention.

(2) one nucleon, one meson: $M=(m^2+k^2)^{\frac{1}{2}}+(\mu^2+k^2)^{\frac{1}{2}}$, where μ is the renormalized meson mass and k is the relative momentum in the rest frame. The parity Π is $+1$ for a meson in a p state and -1 for a meson in an s state relative to the nucleon. It should be emphasized that in the latter case the matrix element does not vanish. For more complicated systems, more quantum numbers n are needed.

The space-time dependence of the matrix elements in (A.2) is determined by the energy and momentum eigenvalues of the Ψ's and so we have, for $x_0 > y_0$,

$$S_{F'\alpha\gamma}(x-y) = \frac{1}{(2\pi)^3}\int d^3p \sum_M \times \exp\{i[\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})-(M^2+p^2)^{\frac{1}{2}}(x_0-y_0)]\} \times \sum_{n,s,\Pi} u_\alpha(\mathbf{p},M,s,\Pi,n)u_\delta^*(\mathbf{p},M,s,\Pi,n)\beta_{\delta\gamma}. \quad \text{(A.3)}$$

We may consider first only those states with $\mathbf{p}=0$ and later discuss the others by means of Lorentz transformations. For states of zero momentum the sum over spins must give simply

$$\sum_s u_\alpha(0,M,s,\Pi,n)u_\delta^*(0,M,s,\Pi,n) = \tfrac{1}{2}(1+\beta)_{\alpha\delta}U(M,n) \quad \text{if } \Pi=+1$$
$$= \tfrac{1}{2}(1-\beta)_{\alpha\delta}V(M,n) \quad \text{if } \Pi=-1, \quad \text{(A.4)}$$

since for such a state the parity operator is β. That all U's and V's are both real and positive follows from taking the trace of both sides of (A.4).

The generalization of (A.4) to the case of $\mathbf{p}\neq 0$ is easily calculated by transforming both sides with the Lorentz transformation matrix for velocity $-\mathbf{p}/(p^2+M^2)^{\frac{1}{2}}$. We obtain

$$\sum_s u_\alpha(\mathbf{p},M,s,\Pi,n)u_\delta^*(\mathbf{p},M,s,\Pi,n) = \left(\frac{\boldsymbol{\alpha}\cdot\mathbf{p}+\beta M+(p^2+M^2)^{\frac{1}{2}}}{2(p^2+M^2)^{\frac{1}{2}}}\right)_{\alpha\delta} U(M,n) \quad \text{if } \Pi=+1,$$
$$= \left(\frac{\boldsymbol{\alpha}\cdot\mathbf{p}-\beta M+(p^2+M^2)^{\frac{1}{2}}}{2(p^2+M^2)^{\frac{1}{2}}}\right)_{\alpha\delta} V(M,n) \quad \text{if } \Pi=-1. \quad \text{(A.5)}$$

Before substituting these expressions into the formula for S_F', let us make use of the relations

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{dp_0 \exp(-ip_0t)}{i\gamma_\mu p_\mu+M-i\epsilon} = \frac{\boldsymbol{\alpha}\cdot\mathbf{p}+\beta M+(p^2+M^2)^{\frac{1}{2}}}{2(p^2+M^2)^{\frac{1}{2}}}\beta \times \exp[-i(p^2+M^2)^{\frac{1}{2}}t], \quad (t>0), \quad \text{(A.6)}$$

and

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{dp_0 \exp(-ip_0t)}{i\gamma_\mu p_\mu-M+i\epsilon} = \frac{\boldsymbol{\alpha}\cdot\mathbf{p}-\beta M+(p^2+M^2)^{\frac{1}{2}}}{2(p^2+M^2)^{\frac{1}{2}}}\beta \times \exp[-i(p^2+M^2)^{\frac{1}{2}}t] \quad (t>0). \quad \text{(A.6')}$$

Here ϵ is a positive infinitesimal quantity.

We may now rewrite (A.3) as follows:

$$S_{F'}(x) = \frac{1}{(2\pi)^4 i}\int d^4p \exp(ip_\mu x_\mu)\sum_M \times\left[\frac{1}{i\gamma_\lambda p_\lambda+M-i\epsilon}\sum_{\substack{n\\ \Pi=1}} U(M,n) + \frac{1}{i\gamma_\lambda p_\lambda-M+i\epsilon}\sum_{\substack{n\\ \Pi=-1}} V(M,n)\right]. \quad \text{(A.7)}$$

Let us separate off the contribution to S_F' of one-nucleon states, which are associated with $M=m$; the coefficient U for that case is the formally divergent constant called Z_2. For all other values of M, let us put

$$\sum_{\substack{n\\ \Pi=+1}} U(M,n) = Z_2 g(M)/M \quad \text{(A.8)}$$

and

$$\sum_{\substack{n\\ \Pi=-1}} V(M,n) = Z_2 h(M)/M. \quad \text{(A.9)}$$

Evidently Z_2, g, and h are all real and positive. We now have

$$S_{F'}(x)=\frac{Z_2}{(2\pi)^4 i}\int d^4p\,\exp(ip_\mu x_\mu)\times\left\{\frac{1}{i\gamma p+m-i\epsilon}+\int_{m+\mu}^{\infty}\frac{g(M)}{i\gamma p+M-i\epsilon}\frac{dM}{M}+\int_{m+\mu}^{\infty}\frac{h(M)}{i\gamma p-M+i\epsilon}\frac{dM}{M}\right\}\quad (x_0>0). \tag{A.10}$$

It follows from the invariance of the theory under charge conjugation that the Eq. (A.10), which we have derived for $x_0>0$, holds also for $x_0<0$.

We must still find an expression for Z_2 in terms of g and h. So far, we have used nothing but relativistic invariance; now we must make use of the anticommutation rules for ψ and ψ^*. Let us calculate the quantity

$$\text{disc}\,S_F'\equiv\lim_{t\to0^+}[S_F'(\mathbf{x},t)-S_F'(\mathbf{x},-t)]. \tag{A.11}$$

We utilize the relation

$$\text{disc}\,\frac{1}{(2\pi)^4 i}\int\frac{d^4p\,\exp(ip_\mu x_\mu)}{i\gamma p+M-i\epsilon}=\text{disc}\,\frac{1}{(2\pi)^4 i}\int\frac{d^4p\,\exp(ip_\mu x_\mu)}{i\gamma p-M+i\epsilon}=\beta\delta(\mathbf{x}). \tag{A.12}$$

Equation (A.10) then yields

$$\text{disc}\,S_F'=Z_2\beta\delta(\mathbf{x})\times\left(1+\int_{m+\mu}^{\infty}\frac{dM}{M}[g(M)+h(M)]\right). \tag{A.13}$$

Another expression for disc S_F' can be obtained from Eq. (A.1):

$$\text{disc}\,S_F'=(\Psi_0,\{\psi(\mathbf{x},t),\bar{\psi}(0,t)\}\Psi_0)=\beta\delta(\mathbf{x}). \tag{A.14}$$

Comparison of (A.13) and (A.14) yields

$$Z_2^{-1}=1+\int_{m+\mu}^{\infty}\frac{dM}{M}[g(M)+h(M)]. \tag{A.15}$$

With the aid of (A.10) and Dyson's Eq. (2.7) we have for the Fourier transform of the renormalized propagation function the representation

$$S_{FC}(p)=\frac{1}{i\gamma p+m-i\epsilon}+\int_{m+\mu}^{\infty}\frac{g(M)dM/M}{i\gamma p+M-i\epsilon}+\int_{m+\mu}^{\infty}\frac{h(M)dM/M}{i\gamma p-M+i\epsilon}. \tag{A.16}$$

The meson propagation functions Δ_F' and Δ_{FC} can be dealt with by methods entirely analogous to those we have used for the nucleon functions. In place of (A.1) we have

$$\Delta_F'(x-y)=(\Psi_0,P[\phi(x),\phi(y)]\Psi_0). \tag{A.17}$$

Considerations of relativistic invariance lead us to the result

$$\Delta_F'(x)=Z_3\frac{1}{(2\pi)^4 i}\int d^4k\,\exp(ik_\mu x_\mu)\times\left[\frac{1}{k^2+\mu^2-i\epsilon}+\int_{(3\mu)^2}^{\infty}\frac{dM^2}{M^2}\frac{f(M^2)}{k^2+M^2-i\epsilon}\right] \tag{A.18}$$

corresponding to (A.10). Utilizing the canonical commutation rule for ϕ and $\partial\phi/\partial t$ at equal times, we obtain, in complete analogy with (A.15),

$$Z_3^{-1}=1+\int_{(3\mu)^2}^{\infty}\frac{dM^2 f(M^2)}{M^2}. \tag{A.19}$$

In quantum electrodynamics, S_F' must be calculated, according to Gupta[19] and Bleuler,[20] from the equation

$$S_F'(x-y)=\epsilon(x_0-y_0)(\Psi_0,\eta P[\psi(x),\bar{\psi}(y)]\Psi_0) \tag{A.20}$$

rather than (A.1). Here $\eta=(-1)^{N_4}$ and N_4 is the operator describing the number of "temporal photons." The only effect of the introduction of η into our previous work is to remove the requirement that g and h be positive.

The photon propagation function D_F' satisfies the relation

$$\langle\Psi_0,\eta P[A_\mu(x),A_\nu(y)]\Psi_0\rangle=\delta_{\mu\nu}D_F'(x-y)+\frac{\partial^2}{\partial x_\mu\partial x_\nu}G(x-y), \tag{A.21}$$

analogous to (A.17). Here G is gauge-variant but D_F' is not. We may determine D_F', moreover, from the transverse part of $A_\mu(x)$ alone, so that the operator η does not disturb us. The results are then identical with (A.18) and (A.19) with $\mu=0$; f is positive as in meson theory.

The functions S_{FC} and D_{FC} in quantum electrodynamics are given, to first order in the coupling constant $\alpha=e_1^2/4\pi$, by

$$S_{FC}(p)=\frac{1}{i\gamma p+m-i\epsilon}+\frac{e_1^2}{16\pi^2}\int_m^{\infty}\frac{dM}{M^3(M^2-m^2)}\times\left[\frac{(M+m)^2(M^2+m^2-4mM)}{i\gamma p+M-i\epsilon}+\frac{(M-m)^2(M^2+m^2+4mM)}{i\gamma p-M+i\epsilon}\right] \tag{A.22}$$

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[19] S. W. Gupta, Proc. Phys. Soc. (London) **63**, 681 (1950).

[20] K. Bleuler, Helv. Phys. Acta **23**, 567 (1950).

and

$$D_{FC}(k)=\frac{1}{k^2}+\frac{e_1^2}{12\pi^2}\int_{4m^2}^{\infty}\frac{dM^2}{M^2}\times\frac{(1+2m^2/M^2)(1-4m^2/M^2)^{\frac{1}{2}}}{k^2+M^2-i\epsilon}. \quad (A.23)$$

It will be seen that the function $g(M)$ in (A.22) is not always positive. This is in contrast with the situation in meson theory. In the scalar symmetric theory with $\mu=0$ we have

$$S_{FC}(p)=\frac{1}{i\gamma p+m-i\epsilon}+\frac{3g_1^2}{16\pi^2}\int_m^{\infty}\times\frac{dM}{M^3(M^2-m^2)}\left[\frac{(M+m)^2(M^2+m^2)}{i\gamma p+M-i\epsilon}+\frac{(M-m)^2(M^2+m^2)}{i\gamma p-M+i\epsilon}\right], \quad (A.24)$$

while in the pseudoscalar symmetric theory we have

$$S_{FC}(p)=\frac{1}{i\gamma p+m-i\epsilon}+\frac{3g_1^2}{16\pi^2}\int_m^{\infty}\frac{dM}{2M^3}(M^2-m^2)\times\left[\frac{1}{i\gamma p+M-i\epsilon}+\frac{1}{i\gamma p-M+i\epsilon}\right]. \quad (A.25)$$

APPENDIX B. SOLUTION OF THE FUNCTIONAL EQUATIONS[21]

We may solve Eqs. (5.4) and (5.5) for the photon propagation function without reference to Eq. (5.3). For convenience we reproduce the equations:

$$d(k^2/\lambda^2,e_2^2)=d_C(k^2/m^2,e_1^2)/d_C(\lambda^2/m^2,e_1^2), \quad (5.4)$$

$$e_2^2=e_1^2d_C(\lambda^2/m^2,e_1^2). \quad (5.5)$$

Combining the equations, we obtain

$$e_1^2d_C(k^2/m^2,e_1^2)=e_1^2d_C(\lambda^2/m^2,e_1^2)\times d(k^2/\lambda^2,e_1^2d_C(\lambda^2/m^2,e_1^2)). \quad (B.1)$$

Giving new names to the left- and right-hand sides of (B.1), we may write

$$g(k^2/m^2,e_1^2)=Q(k^2/\lambda^2,g(\lambda^2/m^2,e_1^2)) \quad (B.2)$$

or

$$g(x,e_1^2)=Q(x/y,g(y,e_1^2)). \quad (B.3)$$

Except in trivial cases we may invert the function g and put

$$x=h(g,e_1^2),\quad y=h(g',e_1^2). \quad (B.4)$$

[21] We would like to thank Dr. T. D. Lee for suggesting the form of the solution to us.

Then (B.3) becomes

$$g=Q(h(g,e_1^2)/h(g',e_1^2),g'). \quad (B.5)$$

If the functional form of Q is nontrivial, then $h(g,e_1^2)/h(g',e_1^2)$ must be independent of e_1^2. Thus

$$h(g,e_1^2)=G(g)/\phi(e_1^2), \quad (B.6)$$

which implies that

$$g(x,e_1^2)=F(x\phi(e_1^2)), \quad (B.7)$$

where F is the inverse function to G. In terms of the original labels, we have

$$e_1^2d_C(k^2/m^2,e_1^2)=F((k^2/m^2)\phi(e_1^2)). \quad (B.8)$$

Substituting (B.8) into (5.5), we obtain

$$e_2^2=F((\lambda^2/m^2)\phi(e_1^2)). \quad (B.9)$$

Using (B.8) and (B.9) we find

$$F\left(\frac{k^2}{m^2}\phi(e_1^2)\right)=F\left(\frac{k^2}{\lambda^2}\frac{\lambda^2}{m^2}\phi(e_1^2)\right)=F((k^2/\lambda^2)G(e_2^2)). \quad (B.10)$$

Since F and G are inverse functions, there is only one arbitrary function in (B.10). It is now evident that (B.1) is indeed satisfied, with

$$e_2^2d(k^2/\lambda^2,e_2^2)=F((k^2/\lambda^2)G(e_2^2)). \quad (B.11)$$

In a power series calculation, e_2^2d appears as a double series in e_2^2 and $\ln(k^2/\lambda^2)$. Let us transform (B.11) so that comparison with the series solution becomes possible:

$$G(e_2^2d)=(k^2/\lambda^2)G(e_2^2), \quad (B.12)$$

$$\ln k^2/\lambda^2=\ln G(e_2^2d)-\ln G(e_2^2), \quad (B.13)$$

$$\ln\frac{k^2}{\lambda^2}=\int_{e_2^2}^{e_2^2d}\frac{dx}{\psi(x)}. \quad (B.14)$$

Here

$$\psi(x)=G(x)(dG/dx)^{-1}. \quad (B.15)$$

Differentiating both sides of (B.14) with respect to $\ln(k^2/\lambda^2)$, we have

$$1=\frac{1}{\psi(e_2^2d)}\frac{\partial(e_2^2d)}{\partial(\ln(k^2/\lambda^2))}, \quad (B.16)$$

or

$$\psi(e_2^2d)=\frac{\partial(e_2^2d)}{\partial(\ln(k^2/\lambda^2))}. \quad (B.17)$$

If in (B.17) we put $\ln(k^2/\lambda^2)=0$, we obtain

$$\psi(e_2^2)=\left.\frac{\partial(e_2^2d)}{\partial(\ln(k^2/\lambda^2))}\right|_{\ln(k^2/\lambda^2)=0}. \tag{B.18}$$

Evidently the double series expansion of e_2^2d yields a power series expansion of $\psi(e_2^2)$. In fact the entire double series can be rewritten in terms of $\psi(e_2^2)$. If we differentiate both sides of (B.14) with respect to e_2^2, we get

$$0=\frac{1}{\psi(e_2^2d)}\frac{\partial(e_2^2d)}{\partial(e_2^2)}-\frac{1}{\psi(e_2^2)}. \tag{B.19}$$

Combining this result with (B.17) yields

$$\psi(e_2^2)=\frac{\partial(e_2^2d)}{\partial(\ln(k^2/\lambda^2))}\bigg/\frac{\partial(e_2^2d)}{\partial e_2^2}, \tag{B.20}$$

whence

$$e_2^2d\left(\frac{k^2}{\lambda^2},e_2^2\right)=\sum_{n=0}^{\infty}\frac{[\ln(k^2/\lambda^2)]^n}{n!}\left[\psi(e_2^2)\frac{d}{de_2^2}\right]^n e_2^2. \tag{B.21}$$

Representations similar to (B.14) and (B.21) can be found for $e_1^2d_C(k^2/m^2,e_1^2)$. We transform (B.8) as follows:

$$G(e_1^2d_C)=(k^2/m^2)\phi(e_1^2), \tag{B.22}$$

$$\ln(k^2/m^2)=\ln G(e_1^2d_C)-\ln\phi(e_1^2), \tag{B.23}$$

$$\ln(k^2/m^2)=\int_{q(e_1^2)}^{e_1^2d_C}dx/\psi(x) \tag{B.24}$$

where ψ is the same function as before and

$$q(e_1^2)=F(\phi(e_1^2)). \tag{B.25}$$

A comparison of (B.24) and (B.14) shows that the functional dependence of $e_1^2d_C$ on k^2/m^2 and $q(e_1^2)$ is the same as the dependence of e_2^2d on k^2/λ^2 and e_2^2. Therefore, we have, from Eq. (B.21), the series expansion

$$e_1^2d_C\left(\frac{k^2}{m^2}e_1^2\right)=\sum_{n=0}^{\infty}\frac{[\ln(k^2/m^2)]^n}{n!}\times\left\{\left[\psi(y)\frac{d}{dy}\right]^n y\right\}_{y=q(e_1^2)}. \tag{B.26}$$

The representation (B.26) may easily be compared with the double series in e_1^2 and $\ln(k^2/m^2)$ obtained from perturbation theory. We see that when $\ln(k^2/m^2)$ is set equal to 0 we obtain for the right-hand side just $q(e_1^2)$, so that the perturbation theory gives a power series expansion for $q(e_1^2)$.

So far our discussion has been confined to the photon propagation function. We must now solve the functional equation (5.3) for the electron propagation function:

$$s(p^2/\lambda^2,p^2/\lambda'^2,e_2^2)=\frac{s_C(p^2/m^2,e_1^2)}{s_C(\lambda'^2/m^2,e_1^2)}. \tag{5.3}$$

Using (B.9), we can write the left-hand side as

$$s(p^2/\lambda^2,p^2/\lambda'^2,e_2^2)=s(p^2/\lambda^2,p^2/\lambda'^2,F(\phi(e_1^2)\lambda^2/m^2)). \tag{B.27}$$

By virtue of (5.3) this must be independent of λ^2 and may be written

$$s(p^2/\lambda^2,p^2/\lambda'^2,e_2^2)=R((p^2/m^2)\phi(e_1^2),(\lambda'^2/m^2)\phi(e_1^2)). \tag{B.28}$$

Since the quotient on the right-hand side of (5.3) depends on its arguments only through $(p^2/m^2)\phi(e_1^2)$ and $(\lambda'^2/m^2)\phi(e_1^2)$, we must have

$$s_C(p^2/m^2,e_1^2)=A(e_1^2)H(p^2/m^2\phi(e_1^2)). \tag{B.29}$$