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Joins and Meets in the Partial Orders of the Computably Enumerable ibT- and cl-Degrees

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Abstract

A bounded reducibility is a preorder $\leq_r$ on $2^{\mathbb{N}}$ which is obtained from Turing reducibility by the additional requirement that, for a reduction of A to B, for every input x the oracle B is only asked oracle queries $y \leq f(x)$, where f is from some given set F of total computable functions.

The most general example of a bounded reducibility is weak-truth-table reducibility, where F is just the set of all computable functions. In this thesis we study the so-called strongly bounded reducibilites $\leq_{\mathrm{ibT}}$ and $\leq_{\mathrm{cl}}$, which are obtained by choosing $F = \{id\}$ and $F = \{id + c : c \in \mathbb{N}\}$, respectively (where id is the identity function).

We start by giving a machine-independent characterisation of these reducibilities, define the degree structures $\mathcal{R}_{\mathrm{ibT}}$ and $\mathcal{R}_{\mathrm{cl}}$ of the computably enumerable ibT- and cl-degrees and review some important properties of $\leq_{\mathrm{ibT}}$ and $\leq_{\mathrm{cl}}$ concerning strictly increasing computable functions (called shifts) and the permitting method.

Then we turn to the degree structures mentioned above, and in particular to existence and nonexistence of joins and meets of a finite set of degrees. As Barmpalias [Barm 05] and independently Fan and Lu [Fan 05] have shown, $\mathcal{R}_{\mathrm{r}}$ is not an upper semi-lattice for $r \in \{\mathrm{ibT}, \mathrm{cl}\}$; it is also known that it is not a lower semi-lattice. We extend these results by showing that the existence of a join or meet of n degrees does in general not imply the existence of a join or meet, respectively, of any subset containining more than one element of these degrees. We also show that even if $deg_r(A)$ and $deg_r(B)$ have a join, there is no uniform way to compute a member of this join from A and B, contrasting the join in the Turing degrees. We conclude this part by looking at the substructure of $\mathcal{R}_r$ which consists of the degrees of simple sets and show that this structure is not closed with respect to the join operation. This is the dual of a theorem of Ambos-Spies [Amboa] stating that the simple degrees are not closed with respect to meets.

Next, we investigate lattice embeddings into $\mathcal{R}_r$. Due to an observation of Ambos-Spies, the proof that every finite distributive lattice can be embedded into the computably enumerable Turing degrees carries over to $\mathcal{R}_r$. We show that the smallest nondistributive lattices $\mathcal{N}_5$ and $\mathcal{M}_3$ can also be embedded into $\mathcal{R}_r$, but only the $\mathcal{N}_5$ can be embedded preserving the least element. Since every nondistributive lattice contains at least one of these two lattices as a sublattice, this motivates the

conjecture that every finite lattice can be embedded into $\mathcal{R}_r$. We show this for two other nondistributive lattices, the $\mathcal{S}_7$ und $\mathcal{S}_8$.

Finally, we compare $\mathcal{R}_{\mathrm{ibT}}$ and $\mathcal{R}_{\mathrm{cl}}$ and prove that these are not elementarily equivalent. To show this, we study under which conditions on two degrees $\mathbf{a}$ and $\mathbf{c}$ with $\mathbf{a} < \mathbf{c}$ it holds that there exists a degree $\mathbf{b} < \mathbf{c}$ such that $\mathbf{c}$ is the join of $\mathbf{a}$ and $\mathbf{b}$. In this context we also show that, while shifts provide a simple method to produce a lesser r-degree $\mathbf{a}$ to some given noncomputable r-degree $\mathbf{c}$, there is no computable shift which uniformly produces such an $\mathbf{a}$ with the additional property that no degree $\mathbf{b}$ as above exists.

Zusammenfassung

Eine beschränkte Reduzierbarkeit ist eine Quasiordnung $\leq_r$ auf $2^{\mathbb{N}}$, die man durch Einschränkung der Turing-Reduzierbarkeit erhält, indem man zusätzlich verlangt, dass für eine Reduktion von A auf B bei Eingabe x nur Anfragen $y \leq f(x)$ an das Orakel B gestellt werden dürfen, wobei f aus einer vorgegebenen Menge F total berechenbarer Funktionen stammt.

Das allgemeinste Beispiel einer beschränkten Reduzierbarkeit ist weak-truth-table-Reduzierbarkeit, hierbei besteht F gerade aus allen berechenbaren Funktionen. In der vorliegenden Arbeit werden Ergebnisse über die sogenannten stark beschränkten Reduzierbarkeiten $\leq_{\mathrm{ibT}}$ bzw. $\leq_{\mathrm{cl}}$ vorgestellt, die man bei Wahl von $F = \{id\}$ bzw. $F = \{id + c : c \in \mathbb{N}\}$ erhält (wobei id die Identitätsfunktion ist).

Wir geben zunächst eine maschinenunabhängige Charakterisierung dieser Reduzierbarkeiten an, definieren die zugehörigen Gradstrukturen $\mathcal{R}_{\mathrm{ibT}}$ und $\mathcal{R}_{\mathrm{cl}}$ der rekursiv aufzählbaren ibT- und cl-Grade und rekapitulieren einige wichtige Eigenschaften von $\leq_{\mathrm{ibT}}$ und $\leq_{\mathrm{cl}}$ im Zusammenhang mit streng monotonen berechenbaren Funktionen (Shifts) und mit der Permitting-Methode, die im späteren Verlauf von Nutzen sind.

Danach wenden wir uns den obigen Gradstrukturen zu, insbesondere dem Aspekt der Existenz von Suprema und Infima einer endlichen Menge von Graden. Von Barmpalias [Barm 05] und unabhängig von Fan und Lu [Fan 05] wurde gezeigt, dass $\mathcal{R}_{\mathrm{r}}$ für $r \in \{\mathrm{ibT}, \mathrm{cl}\}$ kein oberer Halbverband ist; ebenso ist bekannt, dass es sich um keinen unteren Halbverband handelt. Wir verallgemeinern diese Resultate dahingehend, dass aus der Existenz eines Supremums bzw. Infimums von n Graden im Allgemeinen noch nicht folgt, dass eine echte Teilmenge dieser Grade mit mehr als einem Element ein Supremum bzw. Infimum besitzt. Ferner zeigen wir, dass Suprema von Graden $deg_r(A)$ und $deg_r(B)$ selbst im Fall der Existenz nicht in der gleichen Weise uniform aus A und B berechnet werden können wie im Fall der Turing-Reduzierbarkeit. Wir beschließen diesen Teil mit einer Betrachtung der Teilstruktur von $\mathcal{R}_r$ der Grade einfacher Mengen und weisen nach, dass diese nicht unter Suprema abgeschlossen sind. Dies komplementiert ein entsprechendes Resultat von Ambos-Spies [Amboa] über die Nicht-Abgeschlossenheit unter Infima.

Das folgende Kapitel ist der Untersuchung von Verbandseinbettungen in $\mathcal{R}_r$ gewidmet. Nach einer Beobachtung von Ambos-Spies überträgt sich der Beweis, dass jeder endliche distributive Verband in die rekursiv aufzählbaren Turinggrade einbettbar

ist, auf $\mathcal{R}_r$. Wir zeigen, dass auch die kleinsten nichtdistributiven Verbände $\mathcal{N}_5$ und $\mathcal{M}_3$ in $\mathcal{R}_r$ eingebettet werden können, letzterer allerdings nicht unter Bewahrung des kleinsten Elements. Da jeder nichtdistributive Verband mindestens einen dieser beiden Verbände als Teilverband besitzt, gibt dies Anlass zu der Vermutung, dass jeder endliche Verband in $\mathcal{R}_r$ eingebettet werden kann. Wir weisen das für zwei weitere nichtdistributive Verbände $\mathcal{S}_7$ und $\mathcal{S}_8$ nach.

Zum Abschluss wenden wir uns dem Vergleich von $\mathcal{R}_{\mathrm{ibT}}$ und $\mathcal{R}_{\mathrm{cl}}$ zu und zeigen, dass diese nicht elementar äquivalent sind. Dazu untersuchen wir, wann für zwei Grade $\mathbf{a}$ und $\mathbf{c}$ mit $\mathbf{a} < \mathbf{c}$ gilt, dass ein Grad $\mathbf{b} < \mathbf{c}$ derart existiert, dass $\mathbf{c}$ das Supremum von $\mathbf{a}$ und $\mathbf{b}$ ist. In diesem Zusammenhang zeigen wir außerdem, dass, während Shifts eine einfache Methode liefern, zu einem nicht berechenbaren r-Grad $\mathbf{c}$ einen echt kleineren r-Grad $\mathbf{a}$ anzugeben, dieser nicht uniform so gewählt werden kann, dass kein $\mathbf{b}$ wie oben existiert.

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Chapter 1

Introduction

This thesis deals with the computably enumerable ibT- and cl-degrees.

Given some preorder (i.e. a reflexive and transitive 2-ary relation) $\leq_r$ on the power set $\{0,1\}^{\mathbb{N}}$ of the integers, one can define an equivalence relation $\equiv_r$ on $\{0,1\}^{\mathbb{N}}$ by letting $A \equiv_r B$ if and only if $A \leq_r B$ and $B \leq_r A$. The theme of degree theory is the study of the resulting equivalence classes, the *r-degrees*. In computability theory one also uses the term "degrees of unsolvability" ("unsolvable" meaning "not computable"), which reflects the fact that the relation $\leq_r$ is chosen such as to formalise a way to compare sets with respect to their computational power.

Degree theory in this sense is almost as old as computability theory itself. It was Turing himself, who, only a few years after inventing his now famous Turing machine [Turi 37], also brought up the notion of *oracle* Turing machines [Turi 39], thus giving a formal definition of what it meant for a set A to be computable from another set B and hence computationally not more powerful than B. The corresponding preorder $\leq_T$ is now known as Turing reducibility [Post 44]. There has been excessive research on the structure of the Turing degrees since then, and more and more complex techniques were developed – for an overview of the most important results (as of 1983) see the comprehensive textbook by Lerman [Lerm 83].

The subject of degree theory was continuously broadened in two different main directions. Firstly, instead of considering *all* degrees, one can study only a certain subset of the degrees. The subset which received the most interest is the subset of computably enumerable (c.e.) degrees, the degrees of c.e. sets. Post was the first to study these degrees and it turned out that already the most basic question on the c.e. degrees that he posed [Post 44], the question whether there are more than two such degrees, turned out to be all but trivial. It took more than ten years before Friedberg [Frie 57] and independently Muchnik [Mucn 56] could answer it to the positive by inventing the groundbreaking finite injury priority method. With this method, if one wants to show the existence of a c.e. set with some specified property, the desired global property is first split up into an infinite set of requirements which can be satisfied locally (i.e. whose satisfaction is determined by only a finite part of the set to be constructed); then a total order is fixed on the set of requirements and they are satisfied one by one in such a way that each requirement has to respect the action done by higher-priority requirements but may injure lower-priority requirements. The ideas of this method were refined and extended to the more complicated infinite injury priority method and beyond that. It also turned out that in the more elaborate proofs using these methods it is often more convenient to work with a *tree* of requirements instead of just a linearly ordered set (where each requirement appears along each infinite branch of the tree). A lot of difficult questions on the c.e. degrees could be tackled by exploiting these methods. For example, it is known that the c.e. Turing degrees form a dense partial order [Sack 64], that they do not constitute a lower semi-lattice [Lach 66, Yate 66] (though it is very easy to see that they are an upper semi-lattice), and that some but not all finite lattices can be embedded into the structure of the c.e. Turing degrees [Lach 80]. However, there are still some interesting open questions, for instance, whether there is a simple characterisation of the finite lattices which are embeddable into the c.e. degrees.

The second direction in which the subject of degree theory evolved was changing the notion of relative computability, that is the relation $\leq_r$. Instead of using the full power of oracle machines when computing one set from another, one can put restrictions on the number of oracle queries asked, the size of the oracle questions, or the way how the answers to oracle questions may correlate to the final result of the computation. Of course there are many other possibilities for interesting reducibility notions $\leq_r$. An example for a reducibility notion where the number of oracle questions is restricted to one and the final result must be the same as the answer to this oracle question, is *many-one-reducibility.* An example where both the number of oracle questions and the way how the answers to these questions correlate to the final output are arbitrary but must be fixed *before* checking the answer to any of these questions, is *truth-table reducibility.* Both notions were already defined by Post in his 1944 paper cited above. A weaker notion, where only the number of oracle questions asked has to be fixed in advance, is *weak truth-table reducibility*, first defined by Friedberg and Rogers [Frie 59]. More formally, A is weak truth-table reducible to B if there is a computable function f such that $f(x)$ is an upper bound to the oracle questions asked during the computation of $A(x)$ with oracle B (with respect to some fixed oracle machine). Weak truth-table reduciblity is thus a close variant of Turing reducibility.

What happens if we replace the condition that f from the definition above be computable by the condition that f be in some set $\mathcal{F}$ of non-decreasing computable functions, where – in order to make $\leq_r$ reflexive and transitive – we assume $\mathcal{F}$ to contain the identity function and be closed under composition? This has been the subject of more recent studies during the last decade. The most important cases apart from $\mathcal{F}$ being the class of all non-decreasing computable functions (which leads to weak truth-table reducbility) are that $\mathcal{F}$ contains only one element, the identity function, or that $\mathcal{F}$ consists of all functions f with $f(n) = n + c$ for some constant c. The former case was first considered by Soare [Soar 04] and gives rise to the notion of *identity-bounded Turing (*ibT-*) reducibility* $\leq_{\text{ibT}}$, the latter was looked at by Downey, Hirschfeldt and LaForte [Down 04] and the resulting reducibility notion $\leq_{\mathrm{cl}}$ is now called *computable Lipschitz (*cl-*) reducibility*. The notion of cl-reducibility is of particular interest also to the field of algorithmic randomness, because the class of Martin-Löf random sets is closed upwards with respect to $\leq_{\mathrm{cl}}$.

As we said at the beginning of this introduction, the focus in this thesis lies on results of degree theory with respect to the computably enumerable sets only, and with respect to ibT- and cl-reducibility. The outline is as follows.

In Chapter 2 a short summary of the basic ideas and notions from computability theory will be given. We will formally define ibT- and cl-reducibility. Since the ad-hoc definitions are not very robust with respect to slight variants of the oracle machine model (note that, for example, if an oracle machine with oracle B always were to ask at least the fixed oracle question $B(1)$, then no set would be ibT-reducible to any set, because the use function at input 0 would always be greater than 0), we will offer an alternative, machine-independent definition. We will also review some properties which are very helpful tools when working with ibT- and cl-reducibility. One is the observation that by shifting all the bits of a set by a certain amount to the left or right we obtain an easy way to find a computationally harder or simpler (in the sense of ibT- or cl-reducibility) set than a given noncomputable c.e. set. The other is a representation theorem stating that ibT- and cl-reducibility can in some sense be characterised as a reduction by permitting.

In Chapter 3 we study greatest lower and least upper bounds (or *meets* and *joins*) of finite sets of c.e. ibT- or cl-degrees. In the first two sections we review two more important tools. The ibT-cl-Join and -Meet Lemmas simplify considerations because they state that joins and meets in the c.e. ibT-degrees are preserved when we consider the corresponding cl-degrees. The Splitting Lemma is a positive result about the existence of joins, stating that if finitely many degrees can be represented by pairwise disjoint c.e. sets, then they have a join which is represented by the union of these sets. In contrast, as has been shown by Barmpalias [Barm 05] and by Fan and Lu [Fan 05], joins in the c.e. ibT- and cl-degrees do not exist in general. In Section 3.3 we generalise this result and show that for every $n \geq 0$ there exist c.e. ibT- or cl-degrees $\mathbf{a_0}, \ldots, \mathbf{a_n}$ which have a join but such that no k of these degrees have a join for $2 \leq k < n$. In Section 3.4 the corresponding result for meets in place of joins is proven, thereby also showing that the c.e. ibT- and cl-degrees are neither upper nor lower semi-lattices.

By the Splitting Theorem, a representative of the join of the degrees of two disjoint c.e. sets A and B can be computed from A and B in a simple and uniform way. Since for Turing degrees the join of two degrees $deg_{\mathrm{T}}(A)$ and $deg_{\mathrm{T}}(B)$ is always represented by $A \oplus B = \{2x : x \in A\} \cup \{2x+1 : x \in B\}$, the same can be said for general (not necessarily c.e.) sets A and B with respect to joins in the Turing degrees. In Section 3.5 we show that in contrast there is no such uniform procedure to compute a representative of the join of the ibT- or cl-degrees of c.e. sets A and B, though there are nonuniform procedures.

In Section 3.6 we look at joins of c.e. degrees of simple sets and show that these must not necessarily be simple, thus closing a gap in a paper of Ambos-Spies [Amboa].

While the focus of Chapter 3 is on existence and nonexistence results of joins and meets separate from each other, in Chapter 4 we look at joins and meets at the same time and study lattice embeddings. We sketch the proof that every distributive lattice can be embedded into the c.e. T-, wtt-, cl- and ibT-degrees before we turn to the more difficult area of embedding nondistributive lattices. Here we first sketch a proof that the nondistributive nonmodular 5-element lattice $\mathcal{N}_5$ can be embedded into the c.e. ibT- and cl-degrees. This result is due to Ambos-Spies, Bodewig, Kräling, and Yu [Amboc]. Then we extend the methods from this proof to show our main results that the nondistributive 7-element lattice $\mathcal{S}_7$ and the nondistributive modular 5-element lattice $\mathcal{M}_3$ can also be embedded, but only the former can be embedded preserving the least element. The proofs of these two results employ similar infinite injury tree constructions using rather technical combinatorial arguments. We also deduce an embeddability result of the 8-element lattice $\mathcal{S}_8$, contrasting a non-embeddability result of this lattice in the c.e. Turing degrees. The embeddability theorem for the lattice $\mathcal{M}_3$ is joint work with Ambos-Spies, Bodewig, and Wang (unpublished).

Throughout Chapters 3 and 4 all purely degree-theoretic results are equal, irrespective of the fact whether we consider ibT-reducibility or cl-reducibility. In Chapter 5 we show that this is not true in general, i.e. that the partial orders of the c.e. ibT-degrees and the c.e. cl-degrees are not elementarily equivalent. To this end we study the c.e. degrees **b** below a given noncomputable c.e. degree **a** which *cup* to **a**, that is, for which there exists a degree **c** such **a** is the join of **b** and **c**. Precisely, we show that these degrees have an upper bound less than **a** if we look at ibT-degrees but they do not have such an upper bound if we look at cl-degrees. We also prove some related results about cuppable and noncuppable degrees. Most of this chapter is based on joint work with Klaus Ambos-Spies, Philipp Bodewig and Yun Fan, and has been published in the Annals of Pure and Applied Logic in 2013 [Ambo 13a].

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Chapter 2

Strong Reducibilities

2.1 Sets, strings and trees

Computability theory usually deals with subsets of $\mathbb{N} = \{0, 1, 2 \ldots, \}$. Whenever we just use the term *set* without further explanation, it will refer to a set of this kind. Sets will be denoted by capital letters A, B, C, …, A_0, A_1, …. The power set of $\mathbb{N}$ is denoted by $\{0,1\}^{\mathbb{N}}$.

We identify a set A with its characteristic function $c_A : \mathbb{N} \to \{0, 1\}$,

$$c_A(n) = \begin{cases} 1 & \text{if } n \in A \\ 0 & \text{otherwise.} \end{cases}$$

From the viewpoint of this identification, a set is just a sequence of zeros and ones (in the formal mathematical sense). Hence we will occasionally write $A(0)A(1)\ldots$ to denote the set A. For $n \geq 0$, we call $A(n) = c_A(n)$ the *n-th bit* of A.

A *string* is an element from $\{0,1\}^*$, the language of all finite binary words. The length of a string α is denoted by $|\alpha|$. We say that α is an *initial segment* of a set A if $\alpha(i) = A(i)$ for $i < |\alpha|$, and we write $\alpha \sqsubset A$. The unique initial segment of A of length n is denoted by $A \upharpoonright n$.

Similarly, given two strings $\alpha = \alpha(0)\ldots\alpha(n)$ and $\beta = \beta(0)\ldots\beta(m)$, we say that α is a *prefix* or *initial segment* of β, denoted by $\alpha \sqsubseteq \beta$, if $n \leq m$ and for all $i \leq n$, $\alpha(i) = \beta(i)$; if, additionally, $n < m$, we call α a *proper initial segment* of β and write $\alpha \sqsubset \beta$. With respect to tree constructions we also say that α is below β (or β is above α) in this case. The *lexicographic order* $<_L$ on $\{0,1\}^*$ is defined by $\alpha <_L \beta$ if there exists some string γ such that $\gamma 0 \sqsubseteq \alpha$ and $\gamma 1 \sqsubseteq \beta$. In this case we say that α is to the left of β (or β is to the right of α). We can extend this partial order to a total order $<$ by defining $\alpha < \beta$ if $\alpha <_L \beta$ or $\alpha \sqsubseteq \beta$.

A *(binary) tree* is a set T of strings such that, for all $\alpha \in T$ and all $\beta \sqsubseteq \alpha$, it holds that $\beta \in T$ (i.e. T is closed under taking prefixes). When we talk about trees, we also call the elements of T *nodes* of T. A *path* in a tree T is a set A such that, for all $n \in \mathbb{N}$, $A \upharpoonright n \in T$. The lexicographic order extends to paths in a natural way.

2.2 Computable functions, coding functions, and computable enumerable sets

A *partial function* is a function $f : A \to \mathbb{N}$, where $A \subseteq \mathbb{N}$. Given some $n \in \mathbb{N}$, if $n \notin A$, we write $f(n) \uparrow$; if $n \in A$ and $f(n) = x$, we write $f(n) \downarrow= x$ or just $f(n) \downarrow$ to stress the fact that f is defined at n. In the special case that $A = \mathbb{N}$, we also call f *total*.

We assume that the reader is familiar with the notion of (one of the close variants of) the standard Turing machine model. For the remainder of this text, let $(N_e)_{e\in\mathbb{N}}$ be a standard enumeration of all Turing machines with input and output alphabet $\{0, 1\}$. A Turing machine M *computes* some partial function ψ if for all $n \in \mathbb{N}$, given the n-th binary word as input, the machine M outputs the $\psi(n)$-th binary word if $\psi(n)$ is defined and does not halt if $\psi(n) \uparrow$. We write φ_e for the partial function computed by the machine N_e. Then a partial function ψ is *partially computable* if $\psi = \varphi_e$ for some $e \in \mathbb{N}$; if a partially computable function ψ is total, we just call ψ *computable*.

The notion of partial functions $f : A \to \mathbb{N}$, where $A \subseteq \mathbb{N}^k$, $k > 1$, and the notion of partial computability of these functions are defined similar. In particular, for any fixed k we have a computable bijection $\langle\ \rangle_k : \mathbb{N}^k \to \mathbb{N}$ with computable inverse projections on each component, and it holds that a partial function $\psi : A \to \mathbb{N}$ with $A \subseteq \mathbb{N}^k$ is partially computable if and only if the partial function ψ_1 with $\psi_1(\langle x_1, \dots, x_k\rangle_k) = \psi(x_1, \dots, x_k)$ for all $x_1, \dots, x_k \in \mathbb{N}$ is partially computable. Since the index k will always be clear from the context, we usually omit it. Sometimes we will even let $\langle\rangle$ denote coding functions which code tuples of different arities from a finite set. We assume $\langle\rangle$ to be defined in such a way that $x_1, \dots, x_k \leq \langle x_1, \dots, x_k\rangle$.

A set A is *computable* if c_A is computable. It is *computably enumerable* (c.e. for short) if it is the domain of a partially computable function. We write W_e for the domain of the partial computable function φ_e.

We can extend the notions of partially computable functions, computable and c.e. sets etc. to sets and functions of strings, finite sets, pairs of a string and a number, or any other finitary codable objects by saying that a set of strings is computable if the set of its codes is computable etc. We will henceforward use the terms “computable function”, “computable set” and “c.e. set” in this wider sense. We also use expressions like $\langle D, e\rangle$ (where D is a finite set) without further ado, thus tacitly identifying finitary codable objects with their codes.

2.3 Relative computability

While the name may suggest otherwise, the focus of modern computability theory is not on the computable functions and sets but on the numerous relations between the *noncomputable* ones.

A key question is the following:

How much does the knowledge of a set A improve the notion of computability?

Here "knowledge of A" means that during a computation we may pick certain numbers and ask questions about whether these numbers are in A or not, and always get the correct answer; and "improve" means that we are able to compute more functions than in the standard model of computability.

One way to formalize this is the notion of Turing reducibility. An *oracle (Turing) machine* is defined like a Turing machine, but with an additional read-only tape with one cell marked by a special symbol #. To the right of this cell the bits $A(0), A(1), \ldots,$ of any set A, the so-called *oracle*, are written. Apart from the usual Turing machine instructions, the program of the machine may contain a new kind of instructions, which are of the type "when the machine is in state q and there are exactly k zeros written to the right of the reading head on its main tape, then check (by means of the oracle tape) whether the k-th bit of A is a 1; if yes, go into state q_1, otherwise go into state q_0". If such an instruction is carried out during some computation of the oracle machine M, we say that M asks the *oracle query* k.

A (partial) function f is *(partially) A-computable* or *Turing-reducible to A*, denoted by $f \leq_{\mathrm{T}} A$, if there exists an oracle machine M which, if A is written on the oracle tape of M, computes f. The notions of A-computability and A-computable enumerability of sets are defined analogously to the oracle-free case. As for oracle-free Turing machines, there is an effective enumeration $(M_e)_{e\in\mathbb{N}}$ of all oracle Turing machines. For the remainder of this text we fix such an enumeration. We write $\Phi_e(A)$ or Φ_e^A for the A-computable partial function computed by M_e with oracle A. Sometimes we also identify an oracle machine M with the partial function it computes and just write $M_e^A(n)$ instead of $\Phi_e^A(n)$ etc.

With each machine M_e and oracle A we associate the so-called *use function*, the partially A-computable function $u_e^A : \{n : \Phi_e^A(n)\downarrow\} \to \mathbb{N}$ such that $u_e^A(n)$ is the largest oracle query asked by M during the computation of M with oracle A and input n, and 0 if no oracle query is asked during this computation.[1]

For any set A and any $e, n, s \in \mathbb{N}$, let $\Phi_{e,s}^A(n) = \Phi_e^A(n)$ if M_e with input n and oracle A halts after at most s steps and $\max(e, n, u_e^A(n)) < s$. We define $\varphi_{e,s}$ analogously and let $W_{e,s}$ be the domain of $\varphi_{e,s}$.

For any $\sigma \in \{0,1\}^*$ we let $\Phi_e^\sigma(n)\downarrow= y$ if there is a set A such that $\sigma \sqsubset A$, $\Phi_e^A(n)\downarrow$ and $|\sigma| \leq u_e^A(n)$; otherwise we let $\Phi_e^\sigma(n)\uparrow$. For $s \in \mathbb{N}$, $\Phi_{e,s}^\sigma(n)$ is defined analogously.

[1]Usually the use function is defined slightly different as the largest oracle query asked +1, and 0 only if no oracle query is asked. For our purpose, however, the above definition seems more convenient.

2.4 Strong reducibilities

Looking at Turing reducibility is one way to answer the key question from the previous section, but not the only one. Other ways have been proposed, which in some way or the other restrict the oracle queries that an oracle Turing machine is allowed to ask. For example, if only one question may be asked to the oracle and must immediately halt and output the answer to this question (0 or 1), then we obtain the well-studied notion of *many-one reducibility* [Post 44].

Many-one reducibility is a strengthening of *weak-truth table reducibility* [Frie 59]. A set B is weak-truth table (wtt-)reducible to a set A if $B = \Phi_e^A$ for some e such that $u_e^A(n) \leq f(n)$ for all n, where f is a computable function, i. e. B is Turing reducible to A via an oracle machine whose use function at oracle A is computably bounded.

Many-one reducibility is more restrictive than weak truth-table reducibility only regarding the number of oracle queries asked and the way how the answers to these questions are evaluated. However, the *size* of the oracle queries can still be as big as any computable function. The next notion, which is the central definition of this thesis, proposes a different way of strengthening weak truth-table reducibility.

Definition 2.1. [Soar 04, Down 04] Let A and B be sets. Then B is called *identity-bounded Turing (ibT) reducible* to A, or $B \leq_{\mathrm{ibT}} A$ for short, if $B = \Phi_e^A$ for some $e \in \mathbb{N}$ such that $u_e^A(n) \leq n$ for all n.

More generally, for $c \geq 0$ we say that B is $(\mathrm{i}+\mathrm{c})\mathrm{bT}$-*reducible* to A ($B \leq_{(\mathrm{i+c})\mathrm{bT}} A$) if $B = \Phi_e^A$ for some $e \in \mathbb{N}$ such that $u_e^A(n) \leq n + c$ for all n. B is called *computable Lipschitz (cl) reducible* to A, or $B \leq_{\mathrm{cl}} A$ for short, if $B \leq_{(\mathrm{i+c})\mathrm{bT}} A$ for some $c \geq 0$.

It may be criticized that these definitions have little relevance for computability theory because they strongly depend on the model of relative computability we have chosen, namely standard oracle machines. Indeed, if instead of M_e we considered the oracle machine M_e' which for any input n first reads the first $2n$ bits of its oracle and then simulates machine M_e, then we would arrive at the same notion of relative computability as before; on the other hand, no set would be *cl*-reducible to any other with respect to this model, because the use of any reduction would be at least $2n$. Moreover, for some models of relative computability, like the abstract definition by recursive functions, there is no obvious substitute for the use function at all.

However, cl- and ibT- reducibility can be defined without mentioning relative computability, using only the notion of unrelativized computability, and hence, by Turing's thesis, are definable with respect to any model of computability.

Lemma 2.2. *Let A and B be sets. We call a c.e. set W consistent if for each $\alpha \in \{0,1\}^*$, there is at most one $y \in \mathbb{N}$ such that $n = \langle \alpha, y \rangle \in W$. Then the following are equivalent:*

(i) $B \leq_{(\mathrm{i+c})\mathrm{bT}} A$

(ii) there exists a partially computable function $\psi : \mathbb{N} \to \mathbb{N}$ such that, for all n,

$$\psi(\langle A(0) \dots A(n+c) \rangle) = B(n).$$

(iii) there is a consistent c.e. set W such that $\{\langle A \upharpoonright n+c+1, B(n)\rangle\} \subseteq W$.

Proof. $(i) \Rightarrow (ii)$: If $B \leq_{\mathrm{(i+c)bT}} A$ is witnessed by the oracle machine M_e, then a function ψ as above can be computed by a machine N with an additional tape which for any input $\langle x_0 \dots x_{n+c}\rangle$ with $x_0, \dots, x_{n+c} \in \{0,1\}$ writes $x_0, \dots, x_{n+c}$ on its additional tape. Then it deletes its input, writes n on the input tape and simulates M_e with this input, where the additional tape takes the role of the oracle tape as long as only oracle queries $\leq n+c$ are asked. If an oracle query $m > n+c$ is asked, then N goes into an infinite loop.

Thus $\psi(\langle A(0) \dots A(n+c)\rangle) = N(\langle A(0) \dots A(n+c)\rangle) = M_e^A(n) = B(n)$, because $u_e^A(n) \leq n+c$.

$(ii) \Rightarrow (iii)$: Let ψ be a partially computable function as in (ii). Then the graph of ψ is c.e. and consistent and contains $\langle A \upharpoonright n+c+1, B(n)\rangle$ for every $n \in \mathbb{N}$.

$(iii) \Rightarrow (i)$: Let W as in (iii) be given. Let M be the oracle machine which, for input n and oracle X, reads the string $X \upharpoonright n+c+1$ and then enumerates W until some element $\langle X \upharpoonright n+c+1, y\rangle \in W$ is enumerated (if this never happens, the machine does not halt). When this happens, it outputs y. For $M = M_e$, obviously $u_e^X(n) \leq n+c$ for every n. If $X = A$, then by (iii) $\langle X \upharpoonright n+c+1, B(n)\rangle \in W$, hence a number $\langle X \upharpoonright n+c+1, y\rangle$ is enumerated in W and for the first such number the machine outputs y. But $y = B(n)$ by consistency of W. Hence $B \leq_{\mathrm{(i+c)bT}} A$.

□

The consistent sets W appearing in Lemma 2.2(iii) are just the analogues of the consistent sets Rogers [Roge 67] uses for an alternative characterization of Turing reducibility. The main difference is that in the case of Turing reducibilty the elements $\langle \sigma, y\rangle$ of W (sometimes called *axioms*) need a third component refering to the input, while in our case the input n is already implicitly given by the length of σ.

It is easy to see that the uniform enumeration of all c.e. sets $(W_e)_{e\in\mathbb{N}}$ induces a uniform enumeration $(\hat{W}_e)_{e\in\mathbb{N}}$ of all consistent c.e. sets by just changing the enumeration of W_e in such a way that $\langle \sigma, y\rangle$ is enumerated only if no $\langle \sigma, y'\rangle$ with $y' \neq y$ has been enumerated before. Furthermore in the proof of Lemma 2.2 (for $n = 0$) from each $\hat{W}_e$ we have effectively obtained an oracle machine $M_{f(e)}$ which for any input x and irrespective of the oracle X asks only oracle queries $\leq x$. We call $\Phi_{f(e)}$ an ibT-*functional*. Letting $\hat{\Phi}_e = \Phi_{f(e)}$ we thus get an effective enumeration $(\hat{\Phi}_e)_{e\in\mathbb{N}}$ of all ibT-functionals, and Lemma 2.2 says that $B \leq_{\mathrm{ibT}} A$ if and only if $B = \hat{\Phi}_e^A$ for some $e \in \mathbb{N}$.

Similarly, the proof of Lemma 2.2 with each $\hat{W}_i$ and each $c \in \mathbb{N}$ effectively associates an oracle machine $M_{g(i,c)}$ which for any input x and irrespective of the oracle X asks only oracle queries $\leq x+c$. We call $\Phi_{g(i,c)}$ a cl-*functional*. Letting $\tilde{\Phi}_{\langle i,c\rangle} = \Phi_{g(i,c)}$ we thus get an effective enumeration $(\tilde{\Phi}_e)_{e\in\mathbb{N}}$ of all cl-functionals, and Lemma 2.2 says that $B \leq_{\mathrm{cl}} A$ if and only if $B = \tilde{\Phi}_e^A$ for some $e \in \mathbb{N}$. We define $\tilde{u}_{\langle i,c\rangle}^X = u_{g(i,c)}^X$. Since we assume that $i, c \leq \langle i,c\rangle$, for every oracle X, every x and every $e = \langle i, c\rangle$ such that $\tilde{u}_e^X(x)\downarrow$, it holds that

$$\tilde{u}_e^X(x) = u_{g(i,c)}^X(x) \leq x + c \leq x + \langle i, c\rangle = x + e.$$

2.5 Degree structures

Throughout this section, let $r \in \{\mathrm{ibT}, \mathrm{cl}, \mathrm{wtt}, \mathrm{T}\}$. We call two sets A and B *r-equivalent* and write $A \equiv_r B$ if $A \leq_r B$ and $B \leq_r A$. From the easily shown facts that $\leq_r$ is a reflexive and transitive relation it follows that $\equiv_r$ is indeed an equivalence relation. The equivalence classes with respect to $\equiv_r$ are called *r-degrees*. To be more precise, for any set $A \subseteq \mathbb{N}$, the set $deg_r(A) = \{B \subseteq \mathbb{N} : B \equiv_r A\}$ is called *the r-degree of A*. A *computably enumerable (c.e.) r-degree* is a r-degree which contains some c.e. set. Usually we denote r-degrees by lower-case boldface letters $\mathbf{a}$, $\mathbf{b}$, ...

By reflexivity and transitivity again, $\leq_r$ induces a partial order on the class of r-degrees. We are particularly interested in the restriction of this order to the class of the c.e. degrees.

Definition 2.3. For $r \in \{\mathrm{ibT}, \mathrm{cl}, \mathrm{wtt}, \mathrm{T}\}$, let $\mathbf{R}_r$ be the class of all c.e. r-degrees. The partial order $\mathcal{R}_r = (\mathbf{R}_r, \leq)$ with universe $\mathbf{R}_r$ is defined by letting $deg_r(B) \leq deg_r(A)$ if $B \leq_r A$.

Since the computable sets can be computed by an oracle Turing machine without any oracle queries, it holds that $B \leq_r A$ whenever B is computable. This implies that all computable sets are contained in the same r-degree, and that this r-degree $\mathbf{0}$ is below any other r-degree. In fact, $\mathbf{0}$ consists of *exactly* the computable sets, because if any set is r-reducible to a computable set then it is itself computable (since oracle queries can be answered by computations, not using the oracle).

Summarizing, $\mathcal{R}_r$ always has a least element, the r-degree $\mathbf{0}$ of the computable sets. With respect to other properties, however, the structures $\mathcal{R}_r$ behave differently, depending on which $r \in \{\mathrm{ibT}, \mathrm{cl}, \mathrm{wtt}, \mathrm{T}\}$ we choose. It is one of the main topics of computability theory to investigate such properties.

We want to end this section with the remark that for $c > 0$, $(\mathrm{i}+\mathrm{c})\mathrm{bT}$-reducibility does not induce a degree structure as above. Indeed, $(\mathrm{i}+\mathrm{c})\mathrm{bT}$-reducibility is not transitive and hence $\equiv_{(\mathrm{i}+\mathrm{c})\mathrm{bT}}$ is not an equivalence relation. Since we are mainly interested in properties of degrees, we will not say much about $(\mathrm{i}+\mathrm{c})\mathrm{bT}$-reducibility for fixed c in this thesis.

2.6 Computable shifts

Given a noncomputable c.e. set $A \subseteq \mathbb{N}$ and a reducibility notion $\leq_r$ we may ask whether there is a noncomputable c.e. set $B <_r A$. For $r = \mathrm{T}$ and $A = \emptyset'$, this is Post's famous problem which took several years to resolve. In contrast, for $r \in \{\mathrm{ibT}, \mathrm{cl}\}$, such a set B can be obtained from A in a very simple and constructive way by just shifting the elements of A slightly to the right.

Definition 2.4 ([Ambo 13b])**.** (a) A *(computable) shift* is a strictly increasing (computable) function $f : \mathbb{N} \to \mathbb{N}$. A shift f is *nontrivial* if $f(n) > n$ for some (hence for almost all) n, and f is *unbounded* if for every number c there is a number n such that $f(n) - n > c$.

(b) For any set A and any shift f, the *f-shift* of A is defined by

$$A_f = \{f(n) : n \in A\}.$$

Note that, for any shift f, $n \leq f(n)$ and $f(n) - n$ is nondecreasing in n. So a shift f is unbounded if and only if

$$\lim_{n\to\infty} (f(n) - n) = \sup_{n\to\infty} (f(n) - n) = \infty.$$

Moreover note that for any bounded shift f there is a number $c \geq 0$ such that $f(n) = n + c$ for almost all n. We call $f(n) = n + c$ the *c-shift* and write $A + c$ in place of A_f.

Lemma 2.5 (Bounded Shift Lemma; [Ambo 13b, Ambob])**.** *Let A be a noncomputable c.e. set and let $c \geq 1$. Then $A + c$ is c.e., $A + c \equiv_{\mathrm{cl}} A$, and $A + c <_{\mathrm{ibT}} A$. Moreover, for any c.e. set B such that $A + c \cap B = \emptyset$ and such that $A \leq_{\mathrm{ibT}} A + c \cup B$, $A \leq_{\mathrm{ibT}} B$.*

Proof. It is easy to see that $A + c$ is c.e., $A + c \equiv_{\mathrm{cl}} A$, and $A + c \leq_{\mathrm{ibT}} A$. Assume that $A + c \cap B = \emptyset$ and that $A = \hat{\Phi}_e^{A+c \cup B}$ for some ibT-functional $\hat{\Phi}_e$. Then $A(n)$ can be recursively computed from $B \restriction (n+1)$ for $n = 0, 1, \ldots$ by the following algorithm. If $A(x)$ has already been computed for $x < n$, to compute $A(n)$ simulate the computation of $\hat{\Phi}_e^{A+c\cup B}(n)$, where each oracle query $y \leq n$ is answered by first checking whether $y \in B$ (with oracle $B \restriction (n+1)$) and then whether $y - c \in A$ (using the values $A(x)$ for $x < n$). This shows that $A \leq_{\mathrm{ibT}} B$.

For $B = \emptyset$ this also shows that $A \not\leq_{\mathrm{ibT}} A + c = A + c \cup B$, because otherwise $A \leq_{\mathrm{ibT}} \emptyset$ and A were computable. □

Note that $A \equiv_{\mathrm{ibT}} B$ implies that $A + c \equiv_{\mathrm{ibT}} B + c$. So, for a noncomputable c.e. set A and the ibT-degree $\mathbf{a}$ of A we may let $\mathbf{a} + c$ denote the ibT-degree of $A + c$. Then, by the Bounded Shift Lemma, $\cdots < \mathbf{a} + 2 < \mathbf{a} + 1 < \mathbf{a}$.

The following is the analogue of the Bounded Shift Lemma for cl-reducibility.

Lemma 2.6 (Computable Shift Lemma; [Ambo 13b, Ambob, Ambo 13a])**.** *Let A be a noncomputable c.e. set and let f be an unbounded computable shift. Then A_f is c.e., $A_f \equiv_{\mathrm{wtt}} A$, $A_f <_{\mathrm{ibT}} A$, and $A_f <_{\mathrm{cl}} A$. Moreover, for any c.e. set B such that $A_f \cap B = \emptyset$ and such that $A \leq_{\mathrm{cl}} A_f \cup B$, $A \leq_{\mathrm{cl}} B$. In fact, for any c.e. set B and any splitting of A into disjoint c.e. sets A_0 and A_1 such that $(A_0)_f \cap B = \emptyset$ and such that $A \leq_{\mathrm{cl}} (A_0)_f \cup B$, $A \leq_{\mathrm{cl}} B$.*

Proof. It is easy to see that A_f is c.e., $A_f \equiv_{\mathrm{wtt}} A$, and $A_f \leq_{\mathrm{cl}} A$. Moreover, $A_f \leq_{\mathrm{ibT}} A + 1$, implying $A_f <_{\mathrm{ibT}} A$ by the Bounded Shift Lemma. $A \not\leq_{\mathrm{cl}} A_f$ follows from the second part of the lemma by letting $B = \emptyset$, and from noncomputability of A. To prove the second part, in fact it suffices to prove the third part, which for $A_0 = A$, $A_1 = \emptyset$ is equivalent to the second part.

For a proof of the third part assume that $A = A_0 \cup A_1$ is a splitting of A into disjoint c.e. sets A_0 and A_1 such that $(A_0)_f \cap B = \emptyset$ and $A = \tilde{\Phi}_e^{(A_0)_f \cup B}$ for some cl-functional $\tilde{\Phi}_e$. Remember that by our assumption on $\tilde{\Phi}_e$ the largest oracle query of the computation of $\tilde{\Phi}_e^{(A_0)_f \cup B}(x)$ is bounded by $x + e$. Let $m = \min(\{x \in \mathbb{N} : f(x) > x + e\})$. Then $A(n)$ can be recursively computed from $B \upharpoonright (n + e + 1)$ for $n = 0, 1, \ldots$ by the following algorithm. For $n \leq m$ compute $A(n)$ using a finite table (not using the oracle at all). For $n > m$, assuming $A(x)$ has already been computed for all $x < n$, to compute $A(n)$ simulate the computation of $\tilde{\Phi}_e^{(A_0)_f \cup B}(n)$ with the following modification. To answer an oracle query "$y \in (A_0)_f \cup B$?", $y \leq n+e$, first check whether $y \in B$ (with oracle $B \upharpoonright (n + e + 1)$), and if so, give a positive answer. If $y \notin B$, using the values $A(x)$ for $x < n$ to check whether $x \in A$ and $f(x) = y$ for $x = \min\{z < n : f(z) \geq y\}$ (note that the set $\{z < n : f(z) \geq y\}$ is nonempty, since $f(n-1) > n - 1 + e$, hence $f(n-1) \geq n + e \geq y$). If this is the case and a simultaneous enumeration of A_0 and A_1 enumerates x into A_0, give a positive answer to the oracle query; otherwise give a negative answer. $\square$

2.7 The Permitting Lemma

In this section we introduce another very helpful tool in working with ibT- or cl-reducibility.

For a set A, being c.e. is equivalent to A being finite or the range of an injective computable function a. We call such a function an *enumeration function for A.*

If A and B are two infinite c.e. sets with enumeration functions a and b, respectively, and for all $s \in \mathbb{N}$ it holds that

$$a(s) \leq b(s) + c, \tag{1}$$

then B is (i+c)bT-reducible to A. More generally, if $(A_s)_{s \geq 0}$ and $(B_s)_{s \geq 0}$ are computable approximations of A and B, respectively, such that $A_s \subseteq A_{s+1}$ and $B_s \subseteq B_{s+1}$ for all $s \geq 0$ and

$$A_s \upharpoonright n + c = A \upharpoonright n + c \Rightarrow B_s \upharpoonright n = B \upharpoonright n \text{ for every } s, n \geq 0, \tag{2}$$

then B is (i+c)bT-reducible to A.

This is because to check whether $n \in B$ it suffices to check whether $n \in B_s$ for some s such that $B_s \upharpoonright n + 1 = B \upharpoonright n + 1$. But the least s such that $A_s \upharpoonright n + 1 + c = A \upharpoonright n + 1 + c$ satisfies this property and can be computed from $A \upharpoonright n + 1 + c$, i.e. by asking oracle queries $y \leq n + c$ to the oracle A.

Since Friedberg [Frie 57] and Yates [Yate 65] this property (usually with $c = 0$) has been used to effectively enumerate sets A and B satisfying 2 (and having additional properties as desired in the given context) to the end that B will be A-computable. We say that for sets A and B constructed in this fashion $B \leq_r A$ ($r \in \{(\mathrm{i}+\mathrm{c})\mathrm{bT}, \mathrm{cl}, \mathrm{wtt}, \mathrm{T}\}$) holds *by permitting.*

The interesting new property of ibT- and cl-reducibility is that as long as we are only interested in the *degrees* of sets, *every* reduction $B \leq_{\mathrm{cl}} A$ can be assumed to hold by permitting [Ambob]. We state this lemma in a stronger form for systems of simultaneous reductions.

Lemma 2.7 (Permitting Lemma)**.** *Let $c \geq 0$. Let $A_0, \ldots, A_k$ be noncomputable c.e. sets. Then there are sets $\hat{A}_0 \subseteq A_0, \ldots, \hat{A}_k \subseteq A_k$ with enumeration functions $\hat{a}_0, \ldots, \hat{a}_k$, respectively, such that for all $i, j \in \{0, \ldots, k\}$, $A_i \equiv_{\mathrm{ibT}} \hat{A}_i$ and if $A_i \leq_{\mathrm{(i+c)bT}} A_j$, then $\hat{a}_j(n) \leq \hat{a}_i(n) + c$ for all n.*

Proof. Let $a_0, \ldots, a_k$ be enumeration functions for $A_0, \ldots, A_k$, respectively, and let $A_{i,s} = \{a_i(0), \ldots, a_i(s-1)\}$ and $R = \{(i,j) : A_i \leq_{\mathrm{(i+c)bT}} A_j, 0 \leq i, j \leq k\}$. For $(i,j) \in R$, let $\tilde{\Phi}_{e(i,j)}$ be a cl-functional such that $\tilde{\Phi}^{A_j}_{e(i,j)} = A_i$ and for all x, $\tilde{u}^{A_j}_{e(i,j)}(x) \leq x + c$. Define

$$l_{i,j}(s) = \max\{x \leq s : (\forall y \leq x)\, (\tilde{\Phi}^{A_j,s}_{e(i,j),s}(y) = A_{i,s}(y)).\}$$

Note that for $(i,j) \in R$ it holds that

$$\lim_{s \to \infty} l_{i,j}(s) = \infty. \tag{3}$$

For all $(i,j) \in R$, given some computable sequence $t_0 < t_1 < \ldots$, there are infinitely many n such that

$$(\exists x < l_{i,j}(t_n))(x \in A_i - A_{i,t_n}), \tag{4}$$

because otherwise we could compute A_i as follows: To compute $A_i(x)$, find the least n such that $l_{i,j}(t_n) > x$; then $A_i(x) = A_{i,t_n}(x)$ up to finitely many exceptions x. This contradicts the noncomputability of A_i.

Now using 3 we can obtain a computable sequence $t_0 < t_1 \ldots$ such that, for all $(i,j) \in R$

$$(\forall n)(l_{i,j}(t_n) < l_{i,j}(t_{n+1})). \tag{5}$$

Furthermore, using 4, we can, by induction on the coded pairs $\langle i,j \rangle$ with $(i,j) \in R$, extract computable subsequences $t^{(i,j)}_0 < t^{(i,j)}_1 < \ldots$ of $(t_n)_{n \in \mathbb{N}}$ such that for all n

$$(\forall \langle i',j' \rangle \leq \langle i,j \rangle)((i',j') \in R \Rightarrow \exists x < l_{i',j'}(t^{(i,j)}_n))(x \in A_{i',t^{(i,j)}_{n+1}} - A_{i',t^{(i,j)}_n}). \tag{6}$$

Since R is finite, this induction consists of only finitely many steps and the outcome is a sequence $s_0 < s_1 < \ldots$ satisfying 5 (with s_n instead of t_n) and 6 (with s_n instead of $t^{(i,j)}_n$) for *all* $(i,j) \in R$ simultaneously.

For $0 \leq i \leq k$, define

$$\hat{a}_i(n) = \min\{x : x \in A_{s_{n+1}} - A_{s_n}\}.$$

From $(i,i) \in R$ and 6 (with s_n instead of $t^{(i,j)}_n$) it follows that this minimum exists for all n. The functions $\hat{a}_, \ldots, \hat{a}_k$ are obviously computable and one-to-one, hence the enumeration functions of c.e. sets $\hat{A}_0, \ldots, \hat{A}_k$.

If $(i,j) \in R$, then $\hat{a}_i(n) < l_{i,j}(s_n) < l_{i,j}(s_{n+1})$ by 5 and 6. Hence

$$\tilde{\Phi}_{e(i,j)}^{A_{j,s_n}} \restriction (\hat{a}_i(n)+1) = A_{i,s_n} \restriction (\hat{a}_i(n)+1) \neq A_{i,s_{n+1}} \restriction (\hat{a}_i(n)+1) = \tilde{\Phi}_{e(i,j)}^{A_{j,s_{n+1}}} \restriction (\hat{a}_i(n)+1).$$

Since $\tilde{u}_{e(i,j)}^{A_j}(x) \leq x+c$ for all x, this implies that $A_{j,s_n} \restriction (\hat{a}_i(n)+c+1) \neq A_{j,s_{n+1}} \restriction (\hat{a}_i(n)+c+1)$. But then $\hat{a}_j(n) \leq \hat{a}_i(n) + c$ as desired.

It remains to show that $\hat{A}_i$ and A_i are ibT-equivalent. For any x, with oracle A_i we can, asking only oracle queries $\leq x$, compute a stage s_n such that $A_{i,s_n} \restriction (x+1) = A_i \restriction (x+1)$. Then $x \in \hat{A}_i$ if and only if $x \in \{\hat{a}_i(0), \ldots, \hat{a}_i(n)\}$, showing that $\hat{A}_i \leq_{\mathrm{ibT}} A_i$. On the other hand, if $\{\hat{a}_i(0), \ldots, \hat{a}_i(n)\} \restriction (x+1) = \hat{A}_i \restriction (x+1)$, then x cannot enter A_i at any stage $s > s_n$, because otherwise, for $s_m < s \leq s_{m+1}$, it would follow that $\hat{a}_i(m) \leq x$ by definition of $\hat{a}_i$. This shows that $x \in A_i$ if and only if $x \in \{\hat{a}_i(0), \ldots, \hat{a}_i(n)\}$ for some n as above. Since such an n can be computed from x with oracle $\hat{A}_i$ asking only oracle questions $\leq x$, $A_i \leq_{\mathrm{ibT}} \hat{A}_i$. □

This lemma is the reason why in degree-theoretic constructions in the following chapters we can and will usually make all desired order-relations hold by permitting.

Chapter 3

Joins and Meets

In this chapter we start to look at the special aspect of greatest lower and least upper bounds of a finite number n of elements in a degree structure, the most important case being $n = 2$.

The central definition is the following.

Definition 3.1. Given a partial order $\mathcal{P} = (P, \leq)$, finitely many elements $b_0, b_1, \ldots, b_n \in P$ have *least upper bound* or *join* c (in $\mathcal{P}$) if $c \in P$, $b_0, b_1, \ldots, b_n \leq c$ and for every $d \in P$, if $b_i \leq d$ for all $i \in \{0, \ldots, n\}$, then $c \leq d$. In this case we write $b_0 \vee b_1 \vee \ldots \vee b_n = c$. If each two elements of P have a join in $\mathcal{P}$, then $\mathcal{P}$ is called an *upper semi-lattice*.

Finitely many elements $b_0, b_1, \ldots, b_n \in P$ have *greatest lower bound* or *meet* a in $\mathcal{P}$ if $a \in P$, $a \leq b_0, b_1, \ldots, b_n$ and for every $d \in P$, if $d \leq b_i$ for all $i \in \{0, \ldots, n\}$, then $d \leq a$. In this case we write $b_0 \wedge b_1 \wedge \ldots \wedge b_n = a$. If each two elements of P have a meet in $\mathcal{P}$, then $\mathcal{P}$ is called a *lower semi-lattice.*

$\mathcal{P}$ is a *lattice* if it is an upper semi-lattice and a lower semi-lattice.

A very nice property of the usual degree structures $\mathcal{R}_{\mathrm{T}}$, $\mathcal{R}_{\mathrm{wtt}}$ and others, like the structure $\mathcal{R}_{\mathrm{m}}$ of the c.e. m-degrees, is that they are upper semi-lattices. The proof is very simple: If $\mathbf{b_0}$ and $\mathbf{b_1}$ are r-degrees and B_0 and B_1 are sets in $\mathbf{b_0}$ and $\mathbf{b_1}$, respectively, then for $C = B_0 \oplus B_1 = \{2x : x \in B_0\} \cup \{2x + 1 : x \in B_1\}$ it holds that $deg_r(C) = \mathbf{b_0} \vee \mathbf{b_1}$.

In this chapter we look at a number of different topics related to the aspect of joins and meets in the $\mathcal{R}_{\mathrm{ibT}}$ and $\mathcal{R}_{\mathrm{cl}}$, but for now restrict ourselves to questions concerning either greatest lower or least upper bounds, but not both. The latter will be postponed until the next chapter.

3.1 The ibT-cl-Conversion Lemmas

When we look at questions concerning joins and meets in $\mathcal{R}_{\mathrm{ibT}}$ and $\mathcal{R}_{\mathrm{cl}}$, the following two lemmas often prove to be useful. In many cases they permit us to restrict certain considerations to the ibT-case and carry over results to the cl-case.

Lemma 3.2 (ibT-cl-Join Lemma [Ambo 13b]). *Let $B_0, \ldots, B_n, C$ be c.e. sets such that*

$$deg_{\mathrm{ibT}}(B_0) \vee \ldots \vee deg_{\mathrm{ibT}}(B_n) = \deg_{\mathrm{ibT}}(C).$$

Then

$$deg_{\mathrm{cl}}(B_0) \vee \ldots \vee deg_{\mathrm{cl}}(B_n) = \deg_{\mathrm{cl}}(C).$$

Proof. Since ibT-reducibility is stronger than cl-reducibility, $B_0, \ldots, B_n \leq_{\mathrm{cl}} C$.

Let W be some c.e. set such that $B_i \leq_{\mathrm{cl}} W$ for every $i \in \{0, \ldots, n\}$. Choose $c \in \mathbb{N}$ sufficiently large such that $B_i \leq_{\mathrm{(i+c)bT}} W$ for every $i \in \{0, \ldots, n\}$. Then $B_i \leq_{\mathrm{ibT}} W - c$ for every $i \in \{0, \ldots, n\}$, where $W - c = \{x : x + c \in W\}$. Since $deg_{\mathrm{ibT}}(B_0), \ldots, deg_{\mathrm{ibT}}(B_n)$ have join $deg_{\mathrm{ibT}}(C)$, it follows that $C \leq_{\mathrm{ibT}} W - c$ and in particular $C \leq_{\mathrm{cl}} W - c$. But $W - c \equiv_{\mathrm{cl}} W$, hence $deg_{\mathrm{cl}}(C) \leq deg_{\mathrm{cl}}(W)$, proving the lemma. □

Lemma 3.3 (ibT-cl-Meet Lemma [Ambo 13b]). *Let $B_0, \ldots, B_n, C$ be c.e. sets such that*

$$deg_{\mathrm{ibT}}(B_0) \wedge \ldots \wedge deg_{\mathrm{ibT}}(B_n) = \deg_{\mathrm{ibT}}(C).$$

Then

$$deg_{\mathrm{cl}}(B_0) \wedge \ldots \wedge deg_{\mathrm{cl}}(B_n) = \deg_{\mathrm{cl}}(C).$$

Proof. Analogous to the proof of Lemma 3.2. □

3.2 The Splitting Lemma

When working with $r \in \{\mathrm{ibT}, \mathrm{cl}\}$, there is no longer any reason why a noncomputable c.e. set B_0 should be r-reducible to $B_0 \oplus B_1$ (for any c.e. set B_1). Indeed, the information whether $B_0(x) = 0$ or $B_0(x) = 1$ is coded into $B_0 \oplus B_1(2x)$, which for large x is not available with a use bound of $x + c$, where c is a constant. For this reason, $deg_r(B_0 \oplus B_1)$ will in general *not* be the join of $deg_r(B_0)$ and $deg_r(B_1)$. In fact, we will see that such a join does not even need to exist at all.

If the sets B_0 and B_1 are disjoint, however, then the join of the Turing degrees of B_0 and B_1 is not only represented by $B_0 \oplus B_1$ but also by the disjoint union $B_0 \cup B_1$. This latter result carries over to the r-degrees.

Lemma 3.4 (Splitting Lemma). *[Ambo 13b] Let $B_0, \ldots, B_n$ be pairwise disjoint c.e. sets. Then, for $r \in \{\mathrm{ibT}, \mathrm{cl}\}$,*

$$deg_r(B_0) \vee \ldots deg_r(B_n) = deg_r(B_0 \cup \ldots \cup B_n).$$

Proof. Let $i \in \{0, \ldots, n\}$. Then for $x \in \mathbb{N}$, to compute $B_i(x)$ with oracle $B_0 \cup \ldots \cup B_n$, check whether $x \in B_0 \cup \ldots \cup B_n$. If not, then $B_i(x) = 0$. Otherwise by enumerating $B_0, \ldots, B_n$ in parallel find the least $j \in \{0, \ldots, n\}$ such that $x \in B_j$; if $i = j$, then $B_i(x) = 1$, and otherwise $B_i(x) = 0$, because $B_i \cap B_j = \emptyset$. Note that this describes an ibT-reduction of B_i to $B_0 \cup \ldots \cup B_n$.

Furthermore, if $B_i \leq_r C$ for all $i \in \{0, \ldots, n\}$, then to compute $B_0 \cup \ldots \cup B_n(x)$ with oracle C it suffices to compute $B_0(x), \ldots, B_n(x)$ with oracle C and let $B_0 \cup \ldots \cup B_n(x)$ be the maximum of these values. This describes an r-reduction of $B_0 \cup \ldots \cup B_n$ to C. □

3.3 n-Tuples with and without Joins

A basic result concerning joins and meets in $\mathcal{R}_r$ for $r \in \{\text{ibT}, \text{cl}\}$, which was proven independently by Barmpalias [Barm 05] and by Fan and Lu [Fan 05], is that $\mathcal{R}_r$ is not an upper semi-lattice. Indeed, this is true in a strong sense, since $\deg_r(B_0)$ and $\deg_r(B_1)$ may have no upper bound *at all* (and not even no *least* upper bound):

Theorem 3.5 (Maximal Pair Theorem, [Barm 05, Fan 05])**.** *For $r \in \{\text{ibT}, \text{cl}\}$, there exist r-maximal pairs, i.e. there are c.e. sets B_0 and B_1 such that there is no c.e. set C with $B_0 \leq_r C$ and $B_1 \leq_r C$.*

The existence of maximal pairs, however, is not the only obstacle to the existence of joins as was shown by Ambos-Spies, Ding, Fan, and Merkle:

Lemma 3.6. *[Ambo 13b] For $r \in \{\text{ibT}, \text{cl}\}$, there exist c.e. sets B_0 and B_1 such that there is some c.e. set C with $B_0 \leq_r C$ and $B_1 \leq_r C$, but $deg_r(B_0)$ and $deg_r(B_1)$ do not have a join.*

This lemma can be immediately obtained as a corollary from the following new and stronger theorem, which states that the existence of a join of $n+1$ r-degrees does not necessarily imply the existence of a join of any nontrivial subset of these degrees.

Theorem 3.7. *For $r \in \{\text{ibT}, \text{cl}\}$ and for any $n \in \mathbb{N}$ there exist noncomputable c.e. sets $B_0, B_1, \ldots, B_n$ such that $deg_r(B_0) \vee deg_r(B_1) \vee \ldots \vee deg_r(B_n)$ exists but $\bigvee_{i \in D} deg_r(B_i)$ does not exist for any proper subset $D \subset \{0, 1, \ldots, n\}$ with $|D| \geq 2$.*

Proof. For $n = 0$ this is trivial and for $n = 1$ it suffices to let B_0 and B_1 be noncomputable c.e. sets such that $B_0 \cap B_1 = \emptyset$ (for example, let B_0 contain only even numbers and B_1 contain only odd numbers). Then $deg_r(B_0) \vee deg_r(B_1) = deg_r(B_0 \cup B_1)$ by the Splitting Lemma (Lemma 3.4).

Now let $n \geq 2$. For the proof we effectively enumerate c.e. sets $B_0, \ldots, B_n, C$ such that $deg_{\text{ibT}}(B_0) \vee deg_{\text{ibT}}(B_1) \vee \ldots \vee deg_{\text{ibT}}(B_n) = deg_{\text{ibT}}(C)$. Note that by the ibT-cl-Join Lemma this implies $deg_{\text{cl}}(B_0) \vee deg_{\text{cl}}(B_1) \vee \ldots \vee deg_{\text{cl}}(B_n) = deg_{\text{cl}}(C)$, too. The part of B_i enumerated up to stage s will be denoted by $B_{i,s}$ and the part of C enumerated up to stage s will be denoted by C_s.

To guarantee $B_i \leq_{\text{ibT}} C$ for $i \in \{0, \ldots, n\}$, whenever we enumerate a number x into B_i at stage s of the construction, then we enumerate a new number $y \leq x$ into C at stage s. Then equation (2) holds with C in place of A and B_i in place of B and with $c = 0$, whence $B_i \leq_{\text{ibT}} C$

by permitting. Additionally we enumerate y into B_j at stage s for some $j \leq n$. If W is a c.e. set such that $B_j \leq_{\mathrm{ibT}} W$ for every $j \in \{0, \ldots, n\}$, then for any x with oracle $W \upharpoonright x+1$ we can compute a stage s such that $B_{j,s} \upharpoonright x+1 = B_j \upharpoonright x+1$ for every $j \in \{0, \ldots, n\}$. By the strategy described above this will imply $C_s \upharpoonright x+1 = C \upharpoonright x+1$. Hence we can compute $C(x)$ with oracle $W \upharpoonright x+1$, that is $C \leq_{\mathrm{ibT}} W$.

To make sure that $\bigvee_{i \in D} deg_r(B_i)$ does not exist for some subset $D = \{i_0, \ldots, i_d\}$ of $\{0, 1, \ldots, n\}$ with $2 \leq d < n$ and $i_k < i_l$ for $k < l$, we enumerate sets $V = V^D_{\langle e_0, e_1, \ldots, e_d, e_{d+1}\rangle}$, which need to satisfy the following requirements.

$$\mathcal{N}_{\langle D, e_0, e_1, \ldots, e_d, e_{d+1}, e_{d+2}\rangle} : ((\forall k \leq d)(\tilde{\Phi}^{W_{e_{d+1}}}_{e_k} = B_{i_k})) \Rightarrow (\forall k \leq d)(B_{i_k} \leq_{\mathrm{ibT}} V) \text{ and } W_{e_{d+1}} \neq \tilde{\Phi}^V_{e_{d+2}}$$

Indeed, if there were some c.e. set $W = W_{e_{d+1}}$ such that $\bigvee_{i \in D} deg_r(B_i) = deg_r(W)$, then $\bigvee_{i \in D} deg_{\mathrm{cl}}(B_i) = deg_{\mathrm{cl}}(W)$ by the ibT-cl-Join Lemma. Hence there are $e_0, \ldots, e_d$ such that $\tilde{\Phi}^W_{e_k} = B_{i_k}$ for $0 \leq k \leq d$. But then, since the premise of requirement $\mathcal{N}_{\langle D, e_0, e_1, \ldots, e_d, e_{d+1}, e_{d+2}\rangle}$ is satisfied for every $e_{d+2} \in \mathbb{N}$, satisfaction of all these requirements implies that $B_{i_0}, \ldots, B_{i_d} \leq_{\mathrm{ibT}} V^D_{\langle e_0, e_1, \ldots, e_d, e_{d+1}\rangle}$ and $W \not\leq_{\mathrm{cl}} V^D_{\langle e_0, e_1, \ldots, e_d, e_{d+1}\rangle}$, a contradiction.

We define the *length of agreement* $l_s(\mathcal{N}_e)$ of requirement $\mathcal{N}_e = \mathcal{N}_{\langle D, e_0, e_1, \ldots, e_d, e_{d+1}, e_{d+2}\rangle}$ at stage s by

$$l_s(\mathcal{N}_e) = \max(x \leq s : (\forall k \leq d)(\tilde{\Phi}^{W_{e_{d+1},s}}_{e_k,s} \upharpoonright x = B_{i_k,s} \upharpoonright x)).$$

Then the premise of $\mathcal{N}_e$ is true if and only if $\liminf_{s \geq 0} l_s(\mathcal{N}_e) = \infty$.

The strategy to satisfy requirement $\mathcal{N}_{\langle D, e_0, e_1, \ldots, e_d, e_{d+1}, e_{d+2}\rangle}$ is as follows. We assign to each $\mathcal{N}_e$ an interval I such that $|I| \geq \min(I) \cdot e$. We will try to satisfy $W \neq \tilde{\Phi}^V_{e_{d+2}}$ by enumerating numbers from I into B_{i_0} and B_{i_1} and into V, but we only do this at stages $s+1$ such such that $l_s(\mathcal{N}_e) > \max(I)$.

More precisely, we say that $\mathcal{N}_e$ *requires attention at stage* $s+1$ if $e \geq 1$ and

(Case 1) $\mathcal{N}_e$ has no interval assigned at stage s

(Case 2) $\mathcal{N}_e$ has an interval I but no diagonalisation witness assigned at stage s, $l_s(\mathcal{N}_e) > \max(I)$ and $I \not\subseteq B_{i_0,s} \cup C_s$

(Case 3) $\mathcal{N}_e$ has an interval I and a diagonalisation witness $x \in I$ assigned at stage s, $l_s(\mathcal{N}_e) > \max(I)$, $\min(I) \notin C_s$ and $W_s \upharpoonright x+1 = \tilde{\Phi}^{V_s}_{e_{d+2},s} \upharpoonright x+1$.

Now the construction is as follows:

Let $B_{i,0} = C_0 = V^D_{e,0} = \emptyset$ for all $i \in \{0, \ldots, n\}$, all $e \in \mathbb{N}$ and all sets D as above. No requirement has an interval or a diagonalisation witness assigned at stage 0.

At stage $s+1$, let e be minimal such that $\mathcal{N}_e$ requires attention at stage $s+1$ (note that there is such an e, because at every stage only finitely many requirements have an interval assigned). We say that $\mathcal{N}_e$ *is active* due to Case 1, Case 2 or Case 3, respectively, at stage $s+1$, depending on which Case gave rise to $\mathcal{N}_e$ requiring attention.

If $\mathcal{N}_e$ is active due to Case 1, then let x be the least number that is not contained in any interval assigned to any requirement up to stage s. Assign the interval $I = [x, x + (x + e + 2) \cdot (2e + 1)]$ to $\mathcal{N}_e$.

If $\mathcal{N}_{\langle D,e_0,e_1,\dots,e_d,e_{d+1},e_{d+2}\rangle}$ is active due to Case 2 and has the interval I assigned, then

(a) if there is some $y \in I$ such that $[y, y+2e] \subseteq I \cap W_{e_{d+1},s}$, then assign $y+e$ as diagonalisation witness to $\mathcal{N}_e$; let $B_{i,s+1} = B_{i,s}$ for every $i \in \{0,\dots,n\}$, $C_{s+1} = C_s$ and enumerate $\min(I)$ into $V^D_{\langle e_0,\dots,e_{d+1}\rangle}$;

(b) if there is no $y \in I$ such that $[y, y+2e] \subseteq W_{e_{d+1},s}$, then for $b_s = \max(\{x \in I : x \notin B_{i_0,s} \cup C_s\})$ set $B_{i_0,s+1} = B_{i_0,s} \cup \{b_s\}$, $B_{j,s+1} = B_{j,s}$ for $j \neq i_0$, $C_{s+1} = C_s \cup \{b_s\}$ and enumerate b_s into $V^{\tilde{D}}_{\tilde{e}}$ for every $\tilde{e}$ and every $\tilde{D}$ with $(\tilde{D}, \tilde{e}) \neq (D, \langle e_0,\dots,e_{d+1}\rangle)$;

If $\mathcal{N}_e$ is active due to Case 3, then let I be the interval assigned to $\mathcal{N}_e$ at stage s and let $x \in I$ be the diagonalisation witness of $\mathcal{N}_e$ in I. Set $B_{i_1,s+1} = B_{i_1,s} \cup \{x\}$, $B_{i,s+1} = B_{i,s} \cup \{\min(I)\}$ for the least $i \in \{0,\dots,n\} - D$ and $B_{j,s+1} = B_{j,s}$ for $j \neq i_0, i$, $C_{s+1} = C_s \cup \{\min(I)\}$, enumerate x into $V^D_{\langle e_0,\dots,e_{d+1}\rangle}$ and enumerate $\min(I)$ into $V^{\tilde{D}}_{\tilde{e}}$ for every $\tilde{e}$ and every $\tilde{D}$ with $(\tilde{D}, \tilde{e}) \neq (D, \langle e_0,\dots,e_{d+1}\rangle)$.

In either case, for every $\tilde{e} \geq e$, *initialise* $\mathcal{N}_{\tilde{e}}$: If $\mathcal{N}_{\tilde{e}}$ has an interval $\tilde{I}$ but no diagonalisation witness assigned at stage s and $\tilde{e} = \langle D, \tilde{e}_0, \tilde{e}_1, \dots, \tilde{e}_{d+2}\rangle$, then enumerate $\min(\tilde{I})$ into $V^D_{\langle \tilde{e}_0,\tilde{e}_1,\dots,\tilde{e}_{d+1}\rangle}$. Cancel the assignment of $\tilde{I}$ to $\mathcal{N}_{\tilde{e}}$.

This finishes the construction. Obviously the construction is effective up to the fact that at some stages we enumerate numbers into infinitely many sets; but since each stage depends on only finitely many of the actions of previous stages, by a dovetailing method one can easily define an effective equivalent construction.

To prove the theorem, we first prove the following claim:

Claim: Each requirement $\mathcal{N}_e$ is active at only finitely many stages and is satisfied.

We show this by induction on e. Without loss of generality assume that the claim holds for $e = 0$ (otherwise rearrange the requirements such that $l_s(\mathcal{N}_0) = 0$ for every $s \geq 0$). Fix $e = \langle D, e_0, e_1, \dots, e_d, e_{d+1}, e_{d+2}\rangle \geq 1$ and let the claim be true for all $e' < e$. Since $\mathcal{N}_e$ is initialised at some stage s only if some requirement $\mathcal{N}_{e'}$ with $e' < e$ is active at stage s, there is a maximal stage s_0 such that $\mathcal{N}_e$ is initialised at stage s_0 (if $\mathcal{N}_e$ is never initialised, let $s_0 = 0$). Then $\mathcal{N}_e$ requires attention and is active due to Case 1 at stage $s_0 + 1$, because if any $\mathcal{N}_{e'}$ with $e' < e$ would require attention at stage $s_0 + 1$, then some such requirement were active and would initialise $\mathcal{N}_e$ at stage $s_0 + 1$, contradicting the choice of s_0. So $\mathcal{N}_e$ gets an interval I assigned which is assigned to $\mathcal{N}_e$ at every stage $s \geq s_0 + 1$.

Since each time that $\mathcal{N}_e$ is active after stage $s_0 + 1$ some new number from I is enumerated into C or $\mathcal{N}_e$ gets a diagonalisation witness $x \in I$, but the latter can happen at most once, $\mathcal{N}_e$ can be active at most finitely often. It remains to show that $\mathcal{N}_e$ is satisfied.

If $\tilde{\Phi}_{e_k}^{W_{e_{d+1}}} \neq B_{i_k}$ for some $k \leq d$, where $D = \{i_0, \dots, i_d\}$ and $i_0 < \dots < i_d$, then $\mathcal{N}_e$ is trivially satisfied. Hence we may assume that $\tilde{\Phi}_{e_k}^{W_{e_{d+1}}} = B_{i_k}$ for all $k \leq d$. In particular, $\lim_{s\to\infty} l_s(\mathcal{N}_e) = \infty$.

For $m > 0$, let s_m be the m-th stage greater than s_0 such that $l_{s_m}(\mathcal{N}_e) > \max(I)$. For a contradiction assume that $\mathcal{N}_e$ never gets a diagonalisation witness $x \in I$ assigned. Since $I \cap (B_{i_0,s_0+1} \cup C_{s_0+1}) = \emptyset$ and since no requirement $\mathcal{N}_{\tilde{e}}$ with $\tilde{e} \neq e$ enumerates any numbers from I into B_{i_0} or C during the construction, it follows that for $1 \leq m \leq |I|$ requirement $\mathcal{N}_e$ requires attention and is active due to Case 2(b) at stage $s_m + 1$ and enumerates $b_{s_m} = \max(I) - m + 1$ into $B_{i_0,s_m+1} - B_{i_0,s_m}$ and $C_{s_m+1} - C_{s_m}$. Since $l_{s_m}(\mathcal{N}_e) > \max(I)$ and $l_{s_{m+1}}(\mathcal{N}_e) > \max(I)$, it follows that

$$\tilde{\Phi}_{e_0,s_m}^{W_{e_{d+1},s_m}} \restriction b_{s_m} + 1 = B_{i_0,s_m} \restriction b_{s_m} + 1 \neq B_{i_0,s_{m+1}} \restriction b_{s_m} + 1 = \tilde{\Phi}_{e_0,s_{m+1}}^{W_{e_{d+1},s_{m+1}}} \restriction b_{s_m} + 1. \quad (7)$$

Since $\tilde{\Phi}_{e_0}$ is an $(\mathrm{i} + \mathrm{e}_0) - \mathrm{bt}$-functional by our convention, this implies that there is some $z_m \in W_{e_{d+1},s_{m+1}} - W_{e_{d+1},s_m}$ such that $z_m \leq b_{s_m} + e_0 \leq \max(I) + e$. Hence $|W_{e_{d+1},s_{|I|}} \cap [0, \max(I) + e]| \geq |I| - 1 = (\min(I) + e + 2) \cdot (2e + 1)$. In particular,

$$\begin{aligned}
&|W_{e_{d+1},s_{|I|}} \cap I| \\
=&|W_{e_{d+1},s_{|I|}} \cap [0, \max(I) + e]| - |W_{e_{d+1},s_{|I|}} \cap [0, \min(I))| - |W_{e_{d+1},s_{|I|}} \cap (\max(I), \max(I) + e]| \\
\geq&|I| - 1 - \min(I) - e.
\end{aligned}$$

This means that I is contained in $W_{e_{d+1},s_{|I|}}$, except for up to $\min(I) + e + 1$ numbers. Since I contains $\min(I)+e+2$ pairwise disjoint subintervals of length $2e+1$, there must be at least one such interval $[y, y + 2e]$ which is contained in $W_{e_{d+1},s_{|I|}}$, and $\mathcal{N}_e$ gets a diagonalisation witness assigned at stage $s_{|I|} + 1$.

This proves that $\mathcal{N}_e$ gets a diagonalisation witness $x \in I$ assigned at some stage $s_m + 1$, $1 \leq m \leq |I|$. Note that $\min(I) \notin C_{s_m+1}$, since the elements from I were enumerated into C in decreasing order at stages $s_1, \dots, s_{m-1}$ and no number is enumerated into C at stage $s_m + 1$.

Let $V = V^D_{\langle e_0,\dots,e_{d+1}\rangle}$. Assume that there is some stage $s_{m'}$, $m' > m$, such that $W_{e_{d+1},s_{m'}} \restriction x + 1 = \tilde{\Phi}_{e_{d+2},s_{m'}}^{V_{s_{m'}}} \restriction x + 1$. Then the diagonalisation witness x is enumerated into $B_{i_1,s_{m'}+1} - B_{i_1,s_{m'}}$, where $i_1 \in D$ (note that no numbers from I have been enumerated into B_{i_1} before stage $s_{m'}$). Since $l_{s_{m'}}(\mathcal{N}_e) > \max(I)$ and $l_{s_{m'+1}}(\mathcal{N}_e) > \max(I)$, we conclude that

$$\tilde{\Phi}_{e_1,s_{m'}}^{W_{e_{d+1},s_{m'}}} \restriction x + 1 = B_{i_1,s_{m'}} \restriction x + 1 \neq B_{i_1,s_{m'+1}} \restriction x + 1 = \tilde{\Phi}_{e_1,s_{m'+1}}^{W_{e_{d+1},s_{m'+1}}} \restriction x + 1. \quad (8)$$

As before, this implies that there must be some $z \leq x + e$ in $W_{e_{d+1}} - W_{e_{d+1},s_{m'}}$. But by the choice of diagonalisation witnesses, $[x - e, x + e] \subseteq W_{e_{d+1},s_m} \subseteq W_{e_{d+1},s_{m'}}$. Hence $z < x - e$. Moreover, no number less than x is enumerated into V at any stage $s \geq s_{m'} + 1$, because requirements $\mathcal{N}_{\tilde{e}}$ with $\tilde{e} \leq e$ are not active or initialised at any such stage, requirement $\mathcal{N}_e$ is active only at stage $s_{m'}+1$, when it enumerates only x into V, and is not active or initialised thereafter, and requirements $\mathcal{N}_{\tilde{e}}$ with $\tilde{e}>e$ were initialised at stage s_0 and only enumerate numbers into intervals $\tilde{I}$ with $\min(\tilde{I})>\max(I)$ when they become active or initialised after stage s_0.

Since we assume for the use function $\tilde{u}^X_{e_1}$ of $\tilde{\Phi}^X_{e_1}$ that $\tilde{u}^X_{e_1}(z)\leq z+e_1\leq z+e$ (for any oracle X), it follows that

$$\tilde{\Phi}^V_{e_{d+2}}(z)=\tilde{\Phi}^{V_{s_{m'}}}_{e_{d+2},s_{m'}}(z)=W_{e_{d+1},s_{m'}}(z)\neq W_{e_{d+1}}(z).$$

Hence to show that $\mathcal{N}_e$ is satisfied it only remains to prove that $B_{i_k}\leq_{\mathrm{ibT}}V$ for $0\leq k\leq d$. But indeed, $B_{i_k}\leq_{\mathrm{ibT}}V$ by permitting. To see this, first note that the proof just given actually showed that if some requirement $\mathcal{N}_{\tilde{e}}$ gets an interval assigned at some stage s_0, then $\mathcal{N}_{\tilde{e}}$ is active for at most $|I|-1$ times due to Case 2(b) while I is assigned to $\mathcal{N}_{\tilde{e}}$. Hence if some $\mathcal{N}_{\tilde{e}}=\mathcal{N}_{\langle\tilde{D},\tilde{e_0},\ldots,\tilde{e}_{\tilde{d}+2}\rangle}$ is active due to Case 2(b) at some stage $s+1$ for the n-th time since the current interval $\tilde{I}$ became assigned to $\mathcal{N}_{\tilde{e}}$, then $b_s=\max(\tilde{I})-n+1$ is enumerated into $V_{s+1}-V_s$ if $\langle D,e_0,\ldots,e_{d+1}\rangle\neq\left\langle\tilde{D},\tilde{e_0},\ldots,\tilde{e}_{\tilde{d}+1}\right\rangle$. Since the only number possibly enumerated into B_{i_k} at stage $s+1$ is b_s, this enumeration is permitted by V. Note that no number is enumerated into B_{i_k} or V at stage $s+1$ if $\mathcal{N}_{\tilde{e}}$ is active due to Case 1 or Case 2(a) at stage $s+1$. Moreover, if it happens for the first time that some number from $\tilde{I}$ is enumerated into B_{i_k} ($i_k\in D$) by $\mathcal{N}_{\tilde{e}}$ being active due to Case 3 or initialised at stage $s+1$, then $\min(\tilde{I})$ is enumerated into $V_{s+1}-V_s$ and no number from $\tilde{I}$ is enumerated into B_{i_k}, $i_k\in D$ at any stage $t>s+1$. Hence every enumeration into B_{i_k} by $\mathcal{N}_{\tilde{e}}$ being active due to Case 3 or initialised at stage $s+1$ is permitted by V.

On the other hand, if some number from an interval I assigned to $\mathcal{N}_e$ is enumerated into B_{i_k} by $\mathcal{N}_e$ being active due to Case 2(b) at stage $s+1$, then, as we have shown, $\mathcal{N}_e$ is initialised at some stage $t+1>s+1$ or gets a diagonalisation witness $x\in I$ assigned at some stage $t+1>s+1$. In either case, $\min(I)$ is enumerated into $V_{t+1}-V_t$ for the least such t. Hence the enumeration into B_{i_k} at stage $s+1$ is permitted by V.

Finally, if some number from I is enumerated into B_{i_k} by $\mathcal{N}_e$ being active due to Case 3, then the only number enumerated into any B_{i_k} with $i_k\in D$ at stage $s+1$ is the diagonalisation witness x of $\mathcal{N}_e$. Since $x>\min(I)$ (by $e\geq 1$ and $[x-e,x+e]\subseteq I$), $x\notin V_s$. But x is enumerated into V_{s+1}; hence the enumeration of x into B_{i_k} is permitted by V.

This completes the proof of the claim.

It is easy to check that C permits every enumeration into any B_i, $i\leq n$ (note that $C_s\cap I=V^{\tilde{D}}_{\langle\tilde{e_0},\ldots,\tilde{e}_{\tilde{d}+1}\rangle,s}\cap I$ for every s, whenever I is assigned to some $\mathcal{N}_{\langle D,e_0,\ldots,e_{d+2}\rangle}$ with $\langle D,e_0,\ldots,e_{d+1}\rangle\neq\left\langle\tilde{D},\tilde{e_0},\ldots,\tilde{e}_{\tilde{d}+1}\right\rangle$) and that the construction obeyed the strategy described at the beginning of the proof to make $deg_{\mathrm{ibT}}(C)$ the join of $deg_{\mathrm{ibT}}(B_0),\ldots,deg_{\mathrm{ibT}}(B_n)$.

This completes the proof of Theorem 3.7. $\square$

For $r = \text{cl}$ the dual (with respect to existence and nonexistence of joins) of Theorem 3.7 is also true. In fact, it is possible to find a maximal n-tuple of cl-degrees, i.e. a set of n cl-degrees which do not have a common upper bound, such that every proper subset of the given degrees has a least upper bound.

Theorem 3.8. *For any $n \in \mathbb{N}$ there exist c.e. sets $B_0, B_1, \ldots, B_n$ such that $\bigvee_{i\in D} deg_{\text{cl}}(B_i)$ exists for any proper subset D of $\{0, 1, \ldots, n\}$, but $deg_r(B_0)$, $deg_r(B_1), \ldots, deg_r(B_n)$ do not have an upper bound.*

Proof. We effectively enumerate sets $B_0, \ldots, B_n$ that will satisfy the theorem. To ensure that $\bigvee_{i\in D} deg_r(B_i)$ exists for each $D = \{i_0, \ldots, i_d\} \subset \{0, \ldots, n\}$ with $i_0 < \ldots < i_d$, we simultaneously enumerate a c.e. set C_D such that $B_i \leq_{\text{ibT}} C_D$ for every $i \in D$ holds by permitting. To ensure that $deg_{\text{ibT}}(B_0), \ldots, deg_{\text{ibT}}(B_n)$ are a maximal $(n + 1)$-tuple, we satisfy the *diagonalisation requirements*

$$\mathcal{D}_{\langle e_0,\ldots,e_{n+1}\rangle} : \tilde{\Phi}_{e_0}^{W_{e_{n+1}}} \neq B_0 \text{ or } \ldots \text{ or } \tilde{\Phi}_{e_n}^{W_{e_{n+1}}} \neq B_n$$

for every $e = \langle e_0, \ldots, e_{n+1}\rangle \in \mathbb{N}$. This guarantees that there exists no c.e. set $W = W_{e_{n+1}}$ such that B_i is cl-reducible to W for $i \in \{0, \ldots, n\}$.

To satisfy a diagonalisation requirement $\mathcal{D}_e = \mathcal{D}_{\langle e_0,\ldots,e_{n+1}\rangle}$, we adapt the standard strategy for maximal pair constructions, as described in [Ambo 13b]. Let

$$l_s(\mathcal{D}_e) = \max(\{x \leq s : (\forall i \leq n)(\tilde{\Phi}_{e_i,s}^{W_{e_{n+1},s}} \upharpoonright x = B_{i,s} \upharpoonright x\})$$

be the *length of agreement* of $\mathcal{D}_e$ at stage s.

Define a computable sequence of disjoint intervals $I_e = (p_e \cdot (n + 1), q_e \cdot (n + 1)]$ for $e \in \mathbb{N}$ such that

$$|I_e| > (n + 1) \cdot (\min(I_e) + e) = (n + 1) \cdot (p_e \cdot (n + 1) + e + 1).$$

We assign the interval I_e to requirement $\mathcal{D}_e$.

Then we consider a sequence of stages $s_1 < s_2 < \ldots < s_{\max(I_e)+e+3}$ such that $l_{s_m}(\mathcal{D}_e) > \max(I_e)$ for $m \in \{0, \ldots, \max(I_e) + e + 3\}$. (If no such sequence exists, then $\liminf_{s\geq 0} l_s(\mathcal{D}_e)$ is finite and $\mathcal{D}_e$ is satisfied.) For each such stage s_m with $1 \leq m \leq \max(I) + e + 2$ we choose some $i_m \in \{0, \ldots, n\}$ and some $b_m \in I_e - B_{i,s_m}$ and enumerate b_m into B_{i,s_m+1}. Since no other requirement will enumerate any numbers from I_e into any set B_i, there are indeed $(n+1)\cdot|I_e| \geq |I_e| + \frac{|I_e|}{n} > |I_e| + \min(I_e) + e = \max(I_e) + e + 1$ enumerations possible, hence the strategy is feasible. Since $l_{s_m}(\mathcal{D}_e) > \max(I_e)$ and $l_{s_{m+1}}(\mathcal{D}_e) > \max(I_e)$ for $1 \leq m \leq \max(I_e) + e + 2$, it holds that

$$\tilde{\Phi}_{e_{i_m},s_m}^{W_{e_{n+1}},s_m}(b_m) = B_{i_m,s_m}(b_m) \neq B_{i_m,s_{m+1}}(b_m) = \tilde{\Phi}_{e_{i_m},s_{m+1}}^{W_{e_{n+1}},s_{m+1}}(b_m).$$

Since $\tilde{\Phi}_{e_{i_m}}$ is a cl-functional with $\tilde{u}_{e_{i_m}}^{W_{e_{n+1}},s_m}(b_m) \leq b_m + e_{i_m} \leq b_m + e$, this implies that there must be some $z_m \in W_{e_{n+1},s_{m+1}} - W_{e_{n+1},s_m}$ such that $z_m \leq b_m + e$, in particular $z_m \leq \max(I_e)+e$. But there are only $\max(I_e)+e+1$ such numbers z_m, while we follow the strategy for $\max(I_e)+e+2$ many stages. Since this is a contradiction, the sequence $s_1 < \ldots < s_{\max(I)+e+3}$ as above does not exist and $\mathcal{D}_e$ is satisfied.

We say that a requirement $\mathcal{D}_e$ *requires attention at stage* $s+1$ if $l_s(\mathcal{D}_e) > \max(I_e)$ and

(Case 1) $I_e \not\subseteq (B_{0,s} \cup \ldots \cup B_{n,s} \cup \bigcup_{D \subset \{0,\ldots,n\}} C_{D,s})$ or

(Case 2) $I_e \subseteq B_{0,s} \cup \ldots \cup B_{n,s} \cup \bigcup_{D \subset \{0,\ldots,n\}} C_{D,s}$ and there is some $k \in \mathbb{N}$ such that

$$(k \cdot (n+1), (k+1) \cdot (n+1)] \subseteq I_e,$$

$(k+1) \cdot (n+1) \notin B_{1,s}$ and for every $D \subset \{0,\ldots,n\}$ there is some $y \in (k \cdot (n+1), (k+1) \cdot (n+1)]$ such that $y \notin C_{D,s}$.

The formal construction is now as follows.

Stage 0: Let $B_{i,0} = C_{D,0} = \emptyset$ for every $i \in \{0,\ldots n\}$ and every $D \subset \{0,\ldots,n\}$.

Stage $s+1$: If no requirement requires attention at stage $s+1$, let $B_{i,s+1} = B_{i,s}$ and $C_{D,s+1} = C_{D,s}$ for every $i \in \{0,\ldots n\}$ and every $D \subset \{0,\ldots,n\}$.

Otherwise let e be minimal such that $\mathcal{D}_e$ requires attention at stage $s+1$. We say that $\mathcal{D}_e$ *is active at stage* $s+1$.

If $\mathcal{D}_e$ requires attention due to Case 1, then let b_s be the maximal number in I_e that is not in $B_{0,s} \cup \ldots \cup B_{n,s} \cup \bigcup_{D \subset \{0,\ldots,n\}} C_{D,s}$. Let $k \in \mathbb{N}$ and $i \in \{0,\ldots,n\}$ be such that $b_s = k \cdot (n+1) + i$. Let $B_{i,s+1} = B_{i,s} \cup \{b_s\}$ and $B_{j,s+1} = B_{j,s}$ for $j \neq i$. Also let $C_{D,s+1} = C_{D,s} \cup \{b_s\}$ for every $D \subset \{0,\ldots,n\}$ with $i \in D$ and $C_{D,s+1} = C_{D,s}$ for all D with $i \notin D$.

If $\mathcal{D}_e$ requires attention due to Case 2, then let k be maximal such that the conditions described above are true and for every $D \subset \{0,\ldots,n\}$ let $y_{D,s}$ be the maximal number in $(k \cdot (n+1), (k+1) \cdot (n+1)]$ such that $y_{D,s} \notin C_{D,s}$. For $b_s = (k+1) \cdot (n+1)$ let $B_{1,s+1} = B_{1,s} \cup \{b_s\}$ and $B_{j,s+1} = B_{j,s}$ for $j \neq 1$. Also let $C_{D,s+1} = C_{D,s} \cup \{y_{D,s}\}$ for every $D \subset \{0,\ldots,n\}$ with $1 \in D$ and $C_{D,s+1} = C_{D,s}$ for all D with $1 \notin D$.

To show that the construction is correct, first note that indeed $B_i \leq_{\mathrm{ibT}} C_D$ by permitting for every $D \subset \{0,\ldots,n\}$ with $i \in D$.

Let $S_1 = \{s :$ some $\mathcal{D}_e$ is active due to Case 1 at stage $s+1\}$ and $S_2 = \mathbb{N} - S^1$. For $i \in \{0,\ldots,n\}$ define

$$B_i^{S_1} = \bigcup_{s \in S_1} (B_{i,s+1} - B_{i,s}) \text{ and } B_i^{S_2} = \bigcup_{s \in S_2} (B_{i,s+1} - B_{i,s}),$$

and for $D \subset \{0,\ldots,n\}$ define

$$C_D^{S_1} = \bigcup_{s \in S_1} (C_{D,s+1} - C_{D,s}) \text{ and } C_D^{S_2} = \bigcup_{s \in S_2} (C_{D,s+1} - C_{D,s}).$$

Fix $D \subset \{0, \ldots, n\}$. We claim that $deg_{\mathrm{cl}}(C_D) = \bigvee_{i \in D} deg_{\mathrm{cl}}(B_i)$.

Note that by the construction $\bigcup_{i \in D} B_i^{S_1} = C_D^{S_1}$. Moreover $B_i^{S_1} \subseteq (n+1) \cdot \mathbb{N} + i$, hence $B_0^{S_1}, \ldots, B_n^{S_1}$ are pairwise disjoint. By the Splitting Theorem it follows that $\bigvee_{i \in D} deg_{\mathrm{cl}}(B_i^{S_1}) = deg_{\mathrm{cl}}(C_D^{S_1})$.

Furthermore, $B_i^{S_2} = \emptyset$ for $i \neq 1$ and $C_D^{S^2} = \emptyset$ if $1 \notin D$. Hence if $1 \notin D$, then $deg_{\mathrm{cl}}(C_D) = \bigvee_{i \in D} deg_{\mathrm{cl}}(B_i)$. On the other hand, if $1 \in D$, then $C_D^{S_2} \equiv_{\mathrm{(i+n)bT}} B_1^{S_2}$ by permitting, because each enumeration of a number x into $C_{D,s+1}^{S_2} - C_{D,s}^{S_2}$ is of the form $x = y_{D,s}$ where $b_s \in B_{1,s+1}^{S_2} - B_{1,s}^{S_2}$ and $y_{D,s} \leq b_s \leq y_{D,s} + n$. Hence $\bigvee_{i \in D} deg_{\mathrm{cl}}(B_i^{S_2}) = deg_{\mathrm{cl}}(B_1^{S_2}) = deg_{\mathrm{cl}}(C_D^{S_2})$.

Now

$$\begin{aligned}
\bigvee_{i \in D} deg_{\mathrm{cl}}(B_i) &= \bigvee_{i \in D} deg_{\mathrm{cl}}(B_i^{S_1} \dot{\cup} B_i^{S_2}) \\
&= \bigvee_{i \in D} (deg_{\mathrm{cl}}(B_i^{S_1}) \vee deg_{\mathrm{cl}}(B_i^{S_2})) \quad \text{[by the Splitting Lemma]} \\
&= \bigvee_{i \in D} deg_{\mathrm{cl}}(B_i^{S_1}) \vee \bigvee_{i \in D} deg_{\mathrm{cl}}(B_i^{S_2}) \\
&= deg_{\mathrm{cl}}(C_D^{S_1}) \vee deg_{\mathrm{cl}}(C_D^{S_2}) \\
&= deg_{\mathrm{cl}}(C_D^{S_1} \cup C_D^{S_2}) \quad \text{[by the Splitting Lemma]} \\
&= deg_{\mathrm{cl}}(C_D).
\end{aligned}$$

This proves the claim.

To prove the theorem, it hence suffices to show that every requirement $\mathcal{D}_e$ is satisfied. By induction on e we see that every requirement $\mathcal{D}_e$ requires attention only finitely often: If this holds for all $e' < e$, then let s_0 be a stage such that no $e' < e$ requires attention after stage s_0. Then whenever $\mathcal{D}_e$ requires attention at a stage $s+1 > s_0$, then $\mathcal{D}_e$ is active at stage $s+1$ and a new number from I_e is enumerated into some B_i with $i \in \{0, \ldots, n\}$. But this can happen at most $|I_e| \cdot (n+1)$ times. Hence $\mathcal{D}_e$ requires attention only finitely often.

For a contradiction assume that some $\mathcal{D}_e$ is not satisfied. Then $\liminf_{s \to \infty} l_s(\mathcal{D}_e) = \infty$. By a straightforward induction it follows that there are stages $s_1, \ldots, s_{|I_e|}$ such that, for $m \in \{1, \ldots, |I_e|\}$, requirement $\mathcal{D}_e$ is active due to Case 1 at stage $s_m + 1$ and the number $b_{s_m} = \max(I_e) - m + 1 = q_e \cdot (n+1) - m + 1$ is enumerated into B_i at stage $s_m + 1$, where $i \in \{0, \ldots, n\}$ and $b_{s_m} = k \cdot (n+1) + i$. Since there can be no other stages $s \leq s_{|I_e|}$ such that $\mathcal{D}_e$ is active at stage $s+1$, and since no other requirement enumerates any numbers from I_e into B_1, it holds that $(n+1) \cdot \mathbb{N} \cap I \cap B_{1, s_{|I_e|}+1} = \emptyset$. Moreover, for every $D \subset \{0, \ldots, n\}$ and every $r \in [p_e, q_e)$ there is a number $y_D^r \in (r \cdot (n+1), (r+1) \cdot (n+1)]$ such that $y_D^r \notin C_{D, s_{|I_e|}+1}$: There exists some least number $i \in \{0, \ldots, n\} - D$. If $i = 0$, then let $y_D^r = (r+1) \cdot (n+1)$; otherwise let $y_D^r = r \cdot (n+1) + i$.

By another easy induction it now follows that there are stages $s_{|I_e|+1}, s_{|I_e|+2}, \ldots, s_{|I_e|+(q_e - p_e)}$ such that, for $m \in \{1, \ldots, q_e - p_e\}$ requirement $\mathcal{D}_e$ is active due to Case 2 at stage $s_{|I_e|+m} + 1$ and the number $(q_e - m + 1) \cdot (n+1)$ is enumerated into B_1. By the choice of I_e, $|I_e| >$

$(n+1)\cdot(\min(I_e)+e)$, hence $q_e - p_e > \min(I_e)+e$. But this means that $|I_e| + (q_e - p_e) > \min(I_e) + e + |I_e| = \max(I_e) + e + 1$.

By the argument that we have given before the construction, the existence of these stages $s_1, \ldots, s_{\max(I_e)+e+3}$ (where we let $s_{\max(I)+e+3}$ be the least stage $s > s_{\max(I_e)+e+2}$ with $l_s(\mathcal{D}_e) > \max(I_e)$ if $s_{\max(I)+e+3}$ is not yet defined), is a contradiction. This shows that all requirements are satisfied and completes the proof. □

3.4 n-Tuples with and without Meets

The dual of Theorem 3.7 with respect to joins and meets also holds; in fact, it is possible to choose the sets $B_0, \ldots, B_n$ in such a way that they form a minimal n-tuple.

Theorem 3.9. *For $r \in \{\mathrm{ibT}, \mathrm{cl}\}$ and for any $n \in \mathbb{N}$ there exist noncomputable c.e. sets $B_0, B_1, \ldots, B_n$ such that $deg_r(B_0) \wedge deg_r(B_1) \wedge \ldots \wedge deg_r(B_n) = \mathbf{0}$ but $\bigwedge_{i \in D} deg_r(B_i)$ does not exist for any proper subset $D \subset \{0, 1, \ldots, n\}$ with $|D| \geq 2$.*

Proof. For $n = 0$ this is trivial and for $n = 1$ it suffices to take c.e. sets B_0 and B_1 such that $deg_T(B_0)$ and $deg_T(B_1)$ form a minimal pair in $\mathcal{R}_\mathrm{T}$ (the existence of such pairs was proven by Lachlan [Lach 66] and independently by Yates [Yate 66]).

Let $n \geq 2$. For the proof we effectively enumerate c.e. sets $B_0, \ldots, B_n$ such that $deg_{\mathrm{ibT}}(B_0) \wedge deg_{\mathrm{ibT}}(B_1) \wedge \ldots \wedge deg_{\mathrm{ibT}}(B_n) = \mathbf{0}$. Note that by the ibT-cl-Join Lemma this implies $deg_{\mathrm{cl}}(B_0) \wedge deg_{\mathrm{cl}}(B_1) \wedge \ldots \wedge deg_{\mathrm{cl}}(B_n) = \mathbf{0}$, too. The part of B_i enumerated up to stage s will be denoted by $B_{i,s}$.

To make sure that $\bigwedge_{i \in D} deg_r(B_i)$ does not exist for some subset $D = \{i_0, \ldots, i_d\}$ of $\{0, 1, \ldots, n\}$ with $2 \leq d < n$ and $i_0 < \ldots < i_d$, we enumerate sets $V = V^D_{\langle e_0, e_1, \ldots, e_d, e_{d+1}\rangle}$, which need to satisfy the following requirements.

$$\mathcal{N}_{\langle D, e_0, e_1, \ldots, e_d, e_{d+1}, e_{d+2}\rangle} : ((\forall k \leq d)(\tilde{\Phi}^{B_{i_k}}_{e_k} = W_{e_{d+1}})) \Rightarrow (\forall k \leq d)(V \leq_{\mathrm{ibT}} B_{i_k}) \text{ and } V \neq \tilde{\Phi}^{W_{e_{d+1}}}_{e_{d+2}}$$

Indeed, if there were some c.e. set $W = W_{e_{d+1}}$ such that $\bigwedge_{i \in D} deg_r(B_i) = deg_r(W)$, then $\bigwedge_{i \in D} deg_{\mathrm{cl}}(B_i) = deg_{\mathrm{cl}}(W)$ by the ibT-cl-Join Lemma. Hence there are $e_0, \ldots, e_d$ such that $\tilde{\Phi}^{B_{i_k}}_{e_k} = W$ for $0 \leq k \leq d$. But then, since the premise of requirement $\mathcal{N}_{\langle D, e_0, e_1, \ldots, e_d, e_{d+1}, e_{d+2}\rangle}$ is satisfied for every $e_{d+2} \in \mathbb{N}$, satisfaction of all these requirements implies that $V^D_{\langle e_0, e_1, \ldots, e_d, e_{d+1}\rangle} \leq_{\mathrm{ibT}} B_{i_0}, \ldots, B_{i_d}$ and $V^D_{\langle e_0, e_1, \ldots, e_d, e_{d+1}\rangle} \not\leq_{\mathrm{cl}} W$, a contradiction.

We define the *length of agreement* $l_s(\mathcal{N}_e)$ of requirement $\mathcal{N}_e = \mathcal{N}_{\langle D, e_0, e_1, \ldots, e_d, e_{d+1}, e_{d+2}\rangle}$ at stage s by

$$l_s(\mathcal{N}_e) = \max(x \leq s : (\forall k \leq d)(\tilde{\Phi}^{B_{i_k,s}}_{e_k,s} \upharpoonright x = W_{e_{d+1},s} \upharpoonright x)).$$

Then the premise of $\mathcal{N}_e$ is true if and only if $\liminf_{s \geq 0} l_s(\mathcal{N}_e) = \infty$.

To guarantee that $deg_{\mathrm{ibT}}(B_0) \wedge deg_{\mathrm{ibT}}(B_1) \wedge \ldots \wedge deg_{\mathrm{ibT}}(B_n) = \mathbf{0}$, we need to satisfy the *meet requirements*

$$\mathcal{M}_{\langle e_0,\ldots,e_n,e_{n+1}\rangle} : \hat{\Phi}^{B_0}_{e_0} = \ldots = \hat{\Phi}^{B_n}_{e_n} = W_{e_{n+1}} \Rightarrow W_{e_{n+1}} \text{ is computable.}$$

Similarly as above, we define the *length of agreement* $l_s(\mathcal{M}_e)$ of requirement $\mathcal{M}_e = \mathcal{M}_{\langle e_0,\ldots,e_n\rangle}$ at stage s by

$$l_s(\mathcal{M}_e) = \max(x \leq s : \hat{\Phi}^{B_{0,s}}_{e_0,s} \upharpoonright x = \ldots = \hat{\Phi}^{B_{n,s}}_{e_n,s} \upharpoonright x = W_{e_{n+1}} \upharpoonright x).$$

Then the premise of $\mathcal{M}_e$ is true if and only if $\liminf_{s\geq 0} l_s(\mathcal{M}_e) = \infty$.

Assuming that the premise of requirement $\mathcal{M}_e$ is true, our basic strategy to satisfy $\mathcal{M}_e$ is simple: If s is a stage such that $l_s(\mathcal{M}_e) > x$ and s' is the least stage after stage s such that $l_{s'}(\mathcal{M}_e) > x$, then there will be some set B_j such that $B_{j,s} \upharpoonright x+1 = B_{j,s'} \upharpoonright x+1$. This implies that

$$W_{e_{n+1},s} \upharpoonright x+1 = \hat{\Phi}^{B_{j,s}}_{e_j,s} \upharpoonright x+1 = \hat{\Phi}^{B_{j,s'}}_{e_j,s'} \upharpoonright x+1 = W_{e_{n+1},s'} \upharpoonright x+1.$$

By induction, it follows that $W_{e_{n+1},t} \upharpoonright x+1 = W_{e_{n+1},s} \upharpoonright x+1$ for all $t \geq s$, hence $W_{e_{n+1},s} \upharpoonright x+1 = W_{e_{n+1}} \upharpoonright x+1$. Consequently, to compute $W_{e_{n+1}}(x)$, it suffices to wait for a stage s such that $l_s(\mathcal{M}_e) > x$ and compute $W_{e_{n+1},s}(x)$.

Since it is not effectively decidable whether the premise of requirement $\mathcal{M}_e$ is true, we implement the full construction on a tree $T = \{0,1\}^*$. Each node $\alpha \in T$ represents a guess about the premises of which meet requirements $\mathcal{M}_e$ with $e < |\alpha|$ are true; to be more precise, if $\alpha(e) = 0$ for some $e < |\alpha|$, then α represents the guess that the premise of $\mathcal{M}_e$ is true, while otherwise α represents the guess that the premise of $\mathcal{M}_e$ is false.

The concept of guessing after stage s that α represents the true outcomes of the first s meet requirements (with respect to the underlying coding) is formalised by so-called *α-stages*. Every stage $s \geq 0$ is a λ-stage. Stage s is an $\alpha 0$-stage if s is an α-stage which is α-expansionary, i.e. for $|\alpha| = e$,

$$l_s(\mathcal{M}_e) > \max(\{l_t(\mathcal{M}_e) : t < s \text{ and } t \text{ is an } \alpha\text{-stage}\});$$

and s is an $\alpha 1$-stage if s is an α-stage but not *α-expansionary.* Then for each $s \geq 0$ there exists a unique node $\delta_s \in T$ such that $|\delta_s| = s$ and s is a δ_s-stage.

We define a path $\mathrm{TP} \in 2^{\mathbb{N}}$ in T by $\mathrm{TP}(e) = 0$ if the hypothesis of $\mathcal{M}_e$ is true, that is $\hat{\Phi}^{B_0}_{e_0} = \ldots = \hat{\Phi}^{B_n}_{e_n}$, where $e = \langle e_0,\ldots,e_n\rangle$, and $\mathrm{TP}(e) = 1$ otherwise. TP is called the *true path of T*.

Lemma 3.10 (True Path Lemma)**.** *It holds that* $\mathrm{TP} = \liminf_{s\geq 0} \delta_s$*, i.e.* $\alpha \sqsubset \mathrm{TP}$ *if and only if α is the leftmost string of length $|\alpha|$ in T such that $\alpha \sqsubseteq \delta_s$ for infinitely many s.*

Proof. The proof is by induction on $|\alpha|$. Since λ is the only string of length 0, the claim is true for $|\alpha| = 0$. Let the claim be true for $|\alpha| = e$ and consider $\alpha = \mathrm{TP} \upharpoonright e$. If $\mathrm{TP}(e) = 1$, then the hypothesis of $\mathcal{M}_e$ is not true. In this case $l_s(\mathcal{M}_e)$ is bounded in s and hence there are only finitely many α-expansionary stages. In particular, there are only finitely many $\alpha 0$-stages and finitely many s such that $\alpha 0 \sqsubseteq \delta_s$, while there must be infinitely many s such that $\alpha 1 \sqsubseteq \delta_s$.

Consider the case that $\mathrm{TP}(e) = 0$. By the inductive hypothesis, there are infinitely many s such that $\mathrm{TP} \restriction e \sqsubseteq \delta_s$, hence infinitely many $\mathrm{TP} \restriction e$-stages. Since the hypothesis of $\mathcal{M}_e$ is true, $\liminf_{s \geq 0} l_s(\mathcal{M}_e) = \infty$, so there must indeed be infinitely many $\mathrm{TP} \restriction e$-expansionary $\mathrm{TP} \restriction e$-stages. Consequently, there are infinitely many $\mathrm{TP} \restriction e+1$-stages, that is, $\mathrm{TP} \restriction e+1 \sqsubseteq \delta_s$ for infinitely many s. Moreover, if β is a string of length $e+1$ such that $\beta <_L \mathrm{TP} \restriction e+1$, then $\beta \restriction e <_L \mathrm{TP} \restriction e$; by the inductive hypothesis then $\beta \restriction e \sqsubseteq \delta_s$ for only finitely many s and a fortiori $\beta \sqsubseteq \delta_s$ for only finitely many s.

Hence the claim is true for $|\alpha| = e + 1$. $\square$

We are now ready to give the construction. At Stage 0 let $B_{i,0} = V_e^D = \emptyset$ for every $i \in \{0, \ldots, n\}$, every $e \in \mathbb{N}$ and every $D \subset \{0, \ldots, n\}$.

Stage $s + 1$ consists of two phases.

Phase 1: For every node α that has an interval I assigned and is ready for diagonalisation at stage s, check whether $l_s(\mathcal{N}_e) > \max(I) + e$, where $e = |\alpha| = \langle D, e_0, e_1, \ldots, e_d, e_{d+1}, e_{d+2}\rangle$. In this case do the following: Say that α is not ready for diagonalisation any more; additionally, if $V^D_{\langle e_0,e_1,\ldots,e_d,e_{d+1}\rangle,s}(\max(I)) = \tilde{\Phi}^{W_{e_{d+1},s}}_{e_{d+2},s}(\max(I)) = 0$, enumerate $\max(I)$ into $V^D_{\langle e_0,e_1,\ldots,e_d,e_{d+1}\rangle,s+1}$ and assign $\max(I)$ as diagonalisation witness to α.

We say that a node α with $|\alpha| = e = \langle D, e_0, e_1, \ldots, e_d, e_{d+1}, e_{d+2}\rangle$ *requires attention at stage* $s+1$ if $\alpha \sqsubseteq \delta_s$ and

(Case 1) α has no interval assigned at stage s or

(Case 2) α has an interval I but no diagonalisation witness assigned after Phase 1 of stage $s + 1$, and there is some $x \in I$ such that $x + 3e \in I$, $l_s(\mathcal{N}_e) > \max(I) + e$, $V^D_{\langle e_0,e_1,\ldots,e_d,e_{d+1}\rangle,s}(\max(I)) = \tilde{\Phi}^{W_{e_{d+1},s}}_{e_{d+2},s}(\max(I)) = 0$,

$$x \notin \bigcup_{i \in D - \{i_0\}} B_{i,s}, \text{ where } i_0 = \min(D), \tag{9}$$

and

$$x + 3e \notin B_{i_0,s}. \tag{10}$$

Phase 2: Let α be the least node that requires attention at stage $s+1$ (such a node exists, because δ_s has no interval assigned at stage s and hence requires attention). We say that α *is active* at stage $s + 1$.

(Case 1) If α has no interval assigned at stage s, let x be the least number which is larger than s and larger than $\max(I') + 2e'$ for any interval I' assigned to any node of length e' during the construction up to stage s. Assign the interval $I = [x, x + 6e]$ to α. Let $B_{i,s+1} = B_{i,s}$ for every $i \in \{0, \ldots, n\}$.

(Case 2) If α has the interval I assigned at stage s, let x be the least number in I satisfying (9) and (10).

If $x \notin B_{i_0,s}$ and $x + 3e \notin \bigcup_{i \in D} B_i$, then set $B_{i_0,s+1} = B_{i_0,s} \cup \{x\}$, $B_{i,s+1} = B_{i,s} \cup \{x + 3e\}$ for $i \in D - \{i_0\}$ and $B_{j,s+1} = B_{j,s}$ for $j \notin D$.

Otherwise set $B_{i_0,s+1} = B_{i_0,s} \cup \{x + 3e\}$, $B_{i,s+1} = B_{i,s} \cup \{x\}$ for $i \in D - \{i_0\}$ and $B_{j,s+1} = B_{j,s}$ for $j \notin D$.

Say that α is ready for diagonalisation at stage $s + 1$.

In either case, *initialise* all nodes $\beta > \alpha$, that is, cancel the assignment of all intervals and diagonalisation witnesses to these nodes and say that they are not ready for diagonalisation at stage $s + 1$.

The assignment of all intervals, diagonalisation candidates and diagonalisation witnesses, and all aproximations to sets V_e^D not mentioned so far remain the same as after stage s. This ends the construction.

Lemma 3.11. *Every node $\alpha \sqsubset \mathrm{TP}$ is initialised, requires attention and is active at only finitely many stages.*

Proof. The proof is by induction on $|\alpha|$. Let the claim be true for $\mathrm{TP} \upharpoonright k$ and let $\alpha = \mathrm{TP} \upharpoonright k{+}1$. By the True Path Lemma and by the inductive hypothesis, there is a stage s_0 such that $\delta_s \not<_L \alpha$ and no node $\alpha' \sqsubset \alpha$ requires attention at stage s, for every $s \geq s_0$. Then α is not initialised after stage s_0, and if α requires attention at some stage $s > s_0$, then α is active at stage s.

If α does not require attention after stage s_0, then the claim is true. Otherwise, let s_1 be the least stage after stage s_0 such that α requires attention at stage s_1. Then α is active at stage s_1 and has an interval I assigned after stage s_1. Since α is not initialised after stage s_1, this interval is assigned to α at all stages $s > s_1$. Now whenever α requires attention at some stage $s + 1 > s_1$, α is active at stage $s + 1$ and enumerates some number from I into $B_{i,s+1} - B_{i,s}$, for some $i \in \{0, \dots, n\}$. Since I is finite, this can happen at most finitely often, thus proving that α requires attention and is active at most finitely many times.

□

Lemma 3.12. *Every meet requirement $\mathcal{M}_e$, $e \in \mathbb{N}$, is satisfied.*

Proof. For $e = \langle e_0, \dots, e_n, e_{n+1} \rangle$, assume that $\hat{\Phi}_{e_0}^{B_0} = \dots = \hat{\Phi}_{e_n}^{B_n} = W_{e_{n+1}}$ (otherwise $\mathcal{M}_e$ is trivially satisfied). Let $\alpha \sqsubset \mathrm{TP}$ be the unique node of length e on the true path. By Lemma 3.11 and by the True Path Lemma there is a least stage s_0 such that for all $s > s_0$, nodes $\alpha' \sqsubseteq \alpha$ are not active and $\delta_s \not< \alpha 0$. Now $W_{e_{n+1}}$ can be computed as follows. To determine $W_{e_{n+1}}(x)$, compute the least stage $s_1 > \max(\{s_0, x\})$ such that $\alpha 0 \sqsubseteq \delta_{s_1}$ and $l_{s_1}(\mathcal{M}_e) > x$; such a stage exists by the True Path Lemma, because $\alpha 0 \sqsubseteq \mathrm{TP}$, too. We claim that $x \in W_{e_{n+1}}$ if and only if $x \in W_{e_{n+1},s_1}$.

To prove this claim, note that, by the choice of s_0, no node $\alpha'0 < \alpha 0$ enumerates any numbers into any B_i after stage s_1. Moreover, nodes $\alpha' >_L \alpha 0$ are initialised at stage s_1 and only get intervals I' with $\min(I') > s_1 > x$ assigned after stage s_1; hence they enumerate only numbers greater than x into any set B_i after stage s_1. It follows that every enumeration of a number $y \leq x$ into any set B_i after stage s_1 is caused by some node α' with $\alpha 0 \sqsubseteq \alpha'$. In particular, if such an enumeration occurs at stage $t+1 > s_1$, then t is α-expansionary, that is $l_t(\mathcal{M}_e) > l_{s_1}(\mathcal{M}_e) > x$.

Let $s_1 < s_2 < \ldots$ be the sequence of $\alpha 0$-stages, starting with s_1 (by the True Path Lemma, this sequence is infinite). At every stage $s_k + 1$, $k \geq 1$, there is at least one $i \in \{0, \ldots, n\}$ such that $B_{i,s_k+1} = B_{i,s_k}$. By the above observation, $B_{i,s_{k+1}} \restriction x+1 = B_{i,s_k+1} \restriction x+1$, too. Together, this amounts to $B_{i,s_k} \restriction x+1 = B_{i,s_{k+1}} \restriction x+1$ and hence by $l_{s_k}(\mathcal{M}_e) > x$ and $l_{s_{k+1}}(\mathcal{M}_e) > x$ to

$$W_{e_{n+1},s_k} \restriction x+1 = \hat{\Phi}_{e_i,s_k}^{B_{i,s_k}} \restriction x+1 = \hat{\Phi}_{e_i,s_{k+1}}^{B_{i,s_{k+1}}} \restriction x+1 = W_{e_{n+1},s_{k+1}} \restriction x+1,$$

because $\hat{\Phi}_{e_i}$ is an ibT-functional. It follows that $W_{e_{n+1}}(x) = \lim_{k\to\infty} W_{e_{n+1},s_k}(x) = W_{e_{n+1},s_1}(x)$, as claimed. $\square$

Lemma 3.13. *Let $D = \{i_0, \ldots, i_d\} \subset \{0, \ldots, n\}$ with $i_0 < \ldots < i_d$ and let $e_0, \ldots, e_{d+1}$ be such that $\tilde{\Phi}_{e_k}^{B_{i_k}} = W_{e_{d+1}}$ for all $k \leq d$. Then $V = V^D_{\langle e_0, e_1, \ldots, e_d, e_{d+1}\rangle} \leq_{\mathrm{ibT}} B_i$ for all $i \in D$.*

Proof. Let $i \in D$. Then V is B_i-computable as follows. Check whether $x = \max(I)$ for some interval I assigned to some node α with $|\alpha| = e = \langle D, e_0, \ldots, e_{d+1}, e_{d+2}\rangle$ until stage x. If not, then $x \neq \max(I')$ for every interval I' assigned to any such node during the construction, because after stage x only intervals I' with $\min(I') > x$ become assigned. In this case, $x \notin V$. Otherwise, let I and α be as above, and using the oracle $B_i \restriction x+1$ compute the least stage s such that I has been assigned to α before stage s, such that $B_{i,s} \restriction x+1 = B_i \restriction x+1$ and such that $l_s(\mathcal{N}_e) > x + e$. Such a stage exists because $\tilde{\Phi}_{e_k}^{B_{i_k}} = W_{e_{d+1}}$ for all $k \leq d$. Then $x \in V$ if and only if $x \in V_{s+1}$. Indeed, if I is not assigned to α any more or if α is not ready for diagonalisation at stage s, then it is never declared ready for diagonalisation after stage s while I is assigned to α, because there are no enumerations into $B_i \cap I$ after stage s. On the other hand, if I is assigned to α and α is ready for diagonalisation at stage s, then during Phase 1 of stage $s+1$ it is declared not ready for diagonalisation any more. Since $x = \max(I)$ can be enumerated into V only at a stage $s'+1$ such that I is assigned to α and α is ready for diagonalisation at stage s', the claim follows. $\square$

Lemma 3.14. *Every requirement $\mathcal{N}_e$, $e \in \mathbb{N}$, is satisfied.*

Proof. Let $e = \langle D, e_0, \ldots, e_{d+2}\rangle$, where $D = \{i_0, \ldots, i_d\}$ with $i_0 < \ldots, i_d$. By padding, we may assume that $e \geq 4$. Assume that $\tilde{\Phi}_{e_k}^{B_{i_k}} = W_{e_{d+1}}$ for all $k \leq d$ (otherwise $\mathcal{N}_e$ is trivially satisfied). By the previous lemma, $V = V^D_{\langle e_0, e_1, \ldots, e_d, e_{d+1}\rangle} \leq_{\mathrm{ibT}} B_i$ for all $i \in D$. It remains to show that $V \neq \tilde{\Phi}_{e_{d+2}}^{W_{e_{d+1}}}$.

Let $\alpha \sqsubset \mathrm{TP}$ be the unique node of length e on the true path, and let s_0 be the last stage such that α is initialised at stage s_0, or $s_0 = 0$ if α is never initialised; such a stage exists by Lemma 3.11. Let s_1 be the least α-stage with $s_1 \geq s_0$; such a stage exists by the True Path Lemma. Since α had no interval assigned at the end of stage s_0 and did not get an interval assigned at stages s with $s_0 \leq s \leq s_1$, α requires attention at stage $s_1 + 1$. By the choice of s_0, no node $\alpha' < \alpha$ requires attention at stage $s_1 + 1$, because otherwise the least such node would become active and initialise α. It follows that α is active at stage $s_1 + 1$ and gets an interval I assigned; by the choice of $s_1 \geq s_0$ again, this interval is permanently assigned to α after stage s_1.

If there is any stage $s > s_1$ such that α gets a diagonalisation witness assigned at stage $s + 1$, then $\tilde{\Phi}^{W_{e_{d+1},s}}_{e_{d+2},s}(\max(I)) = 0 \neq 1 = V(\max(I))$. In this case $\mathcal{N}_e$ is satisfied unless $\tilde{\Phi}^{W_{e_{d+1}}}_{e_{d+2}}(\max(I)) \neq \tilde{\Phi}^{W_{e_{d+1},s}}_{e_{d+2},s}(\max(I))$. Since $\tilde{u}^{W_{e_{d+1},s}}_{e_{d+2}}(\max(I)) \leq \max(I) + e_{d+2} \leq \max(I) + e$, this is only possible if $W_{e_{d+1}} \restriction \max(I)+e+1 \neq W_{e_{d+1},s} \restriction \max(I)+e+1$. But by the conditions for α getting a diagonalisation witness assigned, $l_s(\mathcal{N}_e) > \max(I) + e$, hence

$$\tilde{\Phi}^{B_{i_0,s}}_{e_0,s} \restriction \max(I)+e+1 = W_{e_{d+1},s} \restriction \max(I)+e+1 \neq W_{e_{d+1}} \restriction \max(I)+e+1 = \tilde{\Phi}^{B_{i_0}}_{e_0} \restriction \max(I)+e+1.$$

Since $\tilde{u}^{B_{i_0,s}}_{e_0}(x) \leq x + e_0 \leq x + e$ for every x, it follows that $B_{i_0,s} \restriction \max(I) + 2e + 1 \neq B_{i_0} \restriction \max(I) + 2e + 1$. But no node $\alpha' < \alpha$ enumerates any numbers into any set B_i after stage s_0, node α is never active after stage s, and nodes $\alpha' > \alpha$ were initialised at stage $s_1 + 1$ and only get intervals I' with $\min(I') > \max(I) + 2e$ assigned after stage $s_1 + 1$. Hence no node enumerates any number less than $\max(I)+2e+1$ into any set B_i after stage s_1, a contradiction. This shows that in this case $\mathcal{N}_e$ must be satisfied.

For a contradiction assume that α never gets a diagonalisation witness assigned after stage s_1 and that $V = \tilde{\Phi}^{W_{e_{d+1}}}_{e_{d+2}}$. Note that $\max(I) \notin V$ in this case. Inductively define stages $s_2 < \ldots < s_{6e}$ such that s_{k+1} is the least stage $s > s_k$ with $l_s(\mathcal{N}_e) > \max(I) + e$ and $\tilde{\Phi}^{W_{e_{d+1},s}}_{e_{d+2},s}(\max(I)) = V_s(\max(I))$. By our assumptions such stages exist for $k = 1, \ldots, 6e - 1$. A simple induction on $k \geq 2$ shows that at stage $s_k + 1$ the node α requires attention and equations (9) and (10) are satisfied for $x_m = \min(I) + m - 1$ if $k = 2m$ or $k = 2m + 1$; moreover, since no $\alpha' < \alpha$ is active after stage s_0, α is active at stage $s_k + 1$ and enumerates x_m into B_{i_0,s_k+1} and $x_m + 3e$ into $B_{i_1,s_k+1}, \ldots, B_{i_d,s_k+1}$ if $k = 2m$ while it enumerates x_m into $B_{i_1,s_k+1}, \ldots, B_{i_d,s_k+1}$ and $x_m + 3e$ into B_{i_0,s_k+1} if $k = 2m + 1$.

For $k = 2, \ldots, 6e$, let t_k be the least stage $t > s_k$ such that $l_t(\mathcal{N}_e) > \max(I) + e$. Since α was active and defined ready for diagonalisation at stage $s_k + 1$ (and has not been initialised after stage s_0), Phase 1 of stage $t_k + 1$ applies to α. But since α does not get a diagonalisation witness at stage $t_k + 1$, it must hold that

$$\tilde{\Phi}^{W_{e_{d+1},t_k}}_{e_{d+2},t_k}(\max(I)) \neq 0 = \tilde{\Phi}^{W_{e_{d+1},s_k}}_{e_{d+2},s_k}(\max(I)).$$

By $\tilde{u}^{W_{e_{d+1},s_k}}_{e_{d+2}}(\max(I)) \leq \max(I)+e_{d+2} \leq \max(I)+e$, it follows that $W_{e_{d+1},t_k} \restriction \max(I)+e+1 \neq$

$W_{e_{d+1},s_k} \restriction \max(I) + e + 1$. On the other hand, if $k = 2m$ is even, then $B_{i_1,t_k} \restriction \min(I) + 3e = B_{i_1,s_k} \restriction \min(I) + 3e$, because no node $\alpha' < \alpha$ enumerates any numbers into any set B_i after stage s_0, node α only enumerated $x_m \geq \min(I) + 3e$ into B_{i_1} at stage $s_k + 1$ and did not enumerate any numbers from I at stages $s \in \{s_k + 2, \ldots, t_k\}$, and nodes $\alpha' > \alpha$ were initialised at stage $s_1 + 1$ and only get intervals I' with $\min(I') > \max(I) + 2e$ assigned after stage $s_1 + 1$. By $\tilde{u}_{e_1}^{B_{i_1},s_k}(x) \leq x + e_1 \leq x + e$ it follows that

$$W_{e_{d+1},t_k} \restriction \min(I)+2e = \tilde{\Phi}_{e_1,t_k}^{B_{i_1,t_k}} \restriction \min(I)+2e = \tilde{\Phi}_{e_1,s_k}^{B_{i_1,s_k}} \restriction \min(I)+2e = W_{e_{d+1},s_k} \restriction \min(I)+2e.$$

Similarly, if $k = 2m + 1$, then $B_{i_0,t_k} \restriction \min(I) + 3e = B_{i_0,s_k} \restriction \min(I) + 3e$ and hence

$$W_{e_{d+1},t_k} \restriction \min(I)+2e = \tilde{\Phi}_{e_0,t_k}^{B_{i_0,t_k}} \restriction \min(I)+2e = \tilde{\Phi}_{e_0,s_k}^{B_{i_0,s_k}} \restriction \min(I)+2e = W_{e_{d+1},s_k} \restriction \min(I)+2e.$$

Altogether, this shows that

$$\begin{aligned} |W_{e_{d+1},s_{k+1}} \cap [\min(I) + 2e, \max(I) + e]| &\geq |W_{e_{d+1},t_k} \cap [\min(I) + 2e, \max(I) + e]| \\ &> |W_{e_{d+1},s_k} \cap [\min(I) + 2e, \max(I) + e]| \end{aligned}$$

for $k \in \{2, \ldots, 6e\}$. Inductively it follows that $|W_{e_{d+1},s_k} \cap [\min(I) + 2e, \max(I) + e]| \geq k - 2$, in particular $|W_{e_{d+1},s_{6e}} \cap [\min(I) + 2e, \max(I) + e]| \geq 6e - 2$. But

$$\begin{aligned} |[\min(I) + 2e, \max(I) + e]| &= \max(I) + e - \min(I) - 2e + 1 \\ &= \min(I) + 6e + e - \min(I) - 2e + 1 \\ &= 5e + 1 \\ &< 6e - 2 \end{aligned}$$

for $e \geq 4$, a contradiction.

□

□

Corollary 3.15. *[Barm 05, Fan 05, Down 04] For $r \in \{\mathrm{ibT}, \mathrm{cl}\}$, $\mathcal{R}_r$ is neither an upper semi-lattice nor a lower semi-lattice.*

Proof. By Theorem 3.7 there exist r-degrees $\mathbf{b_0}, \mathbf{b_1}$ which have no join in $\mathcal{R}_r$ and by Theorem 3.9 there exist r-degrees $\mathbf{b_0'}, \mathbf{b_1'}$ which have no meet in $\mathcal{R}_r$. □

3.5 Noneffectivity of the Join

As remarked above, for c.e. sets B_0 and B_1 it holds that $deg_{\mathrm{T}}(B_0) \vee deg_{\mathrm{T}}(B_1) = deg_{\mathrm{T}}(B_0 \oplus B_1)$. Hence there exists a simple and effective procedure to obtain a c.e. set C representing the join of two c.e. Turing degrees $deg_{\mathrm{T}}(B_0)$ and $deg_{\mathrm{T}}(B_1)$ from the c.e. sets B_0 and B_1. One way to formalise what "effective" means here is the following.

Definition 3.16. Let $\leq_r$ be a reducibility notion and $\mathcal{R}_r = (\mathbf{R}_r, \leq)$ be the corresponding degree structure of c.e. r-degrees.

(i) We say that the join is *weakly effective in* $\mathcal{R}_r$ if for all degrees $\mathbf{b_0}, \mathbf{b_1}, \mathbf{c} \in \mathbf{R}_r$, there are c.e. sets $B_0 \in \mathbf{b_0}$, $B_1 \in \mathbf{b_1}$ and $C \in \mathbf{c}$, a computable function $g : \mathbb{N} \to \mathbb{N}$ and a partial computable function $\psi : A \to \mathbb{N}$, $A \subseteq \mathbb{N} \times \{0,1\}^* \times \{0,1\}^*$, with

$$\mathbf{b_0} \vee \mathbf{b_1} = \mathbf{c} \Rightarrow C(n) = \psi(n, B_0 \upharpoonright g(n), B_1 \upharpoonright g(n))$$

for every n.

(ii) We say that the join is *strongly effective in* $\mathcal{R}_r$ if there is a computable function $g : \mathbb{N} \to \mathbb{N}$ such that for all degrees $\mathbf{b_0}, \mathbf{b_1}, \mathbf{c} \in \mathbf{R}_r$, there are c.e. sets $B_0 \in \mathbf{b_0}$, $B_1 \in \mathbf{b_1}$ and $C \in \mathbf{c}$ and a partial computable function $\psi : A \to \mathbb{N}$, $A \subseteq \mathbb{N} \times \{0,1\}^* \times \{0,1\}^*$ with

$$\mathbf{b_0} \vee \mathbf{b_1} = \mathbf{c} \Rightarrow C(n) = \psi(n, B_0 \upharpoonright g(n), B_1 \upharpoonright g(n)) \tag{11}$$

for every n.

Hence the join is weakly effective in $\mathcal{R}_r$ if, whenever two r-degrees $\mathbf{b_0}$ and $\mathbf{b_1}$ have a join $\mathbf{c}$, then it is possible to compute a finite prefix of a member C of $\mathbf{c}$ from sufficiently long finite prefixes of members of $\mathbf{b_0}$ and $\mathbf{b_1}$, where we can compute how long these prefixes have to be. If the length of the prefixes only depends on the length of the prefix of C we want to compute (but not on $\mathbf{b_0}$, $\mathbf{b_1}$ or $\mathbf{c}$), then the join is strongly effective in $\mathcal{R}_r$. In particular, every strongly effective join is weakly effective.

The join is strongly effective in $\mathcal{R}_r$ for $r \in \{\mathrm{T}, \mathrm{wtt}, \mathrm{m}\}$, because we can just let $g(n) = n+1$ and

$$\psi(n, \sigma, \tau) = \begin{cases} \sigma(m) & \text{if } n = 2m \text{ and } |\sigma| > m \\ \tau(m) & \text{if } n = 2m+1 \text{ and } |\tau| > m \\ 0 & \text{otherwise} \end{cases}$$

in the above definition.

For $r \in \{\mathrm{ibT}, \mathrm{cl}\}$, on the other hand, the situation looks different. First let us make the following easy observation.

Lemma 3.17. *For $r \in \{\mathrm{ibT}, \mathrm{cl}\}$, the join is strongly effective in $\mathcal{R}_r$ if and only if there is a strictly increasing computable function g with $g(x) > x$ for all x such that for all degrees $\mathbf{b_0}, \mathbf{b_1}, \mathbf{c} \in \mathbf{R}_r$ and for all c.e. sets $B_0 \in \mathbf{b_0}$, $B_1 \in \mathbf{b_1}$ and $C \in \mathbf{c}$ there is a partial computable function $\psi : A \to \mathbb{N}$, $A \subseteq \mathbb{N} \times \{0,1\}^* \times \{0,1\}^*$ satisfying* (11).

Proof. Assume that the join is strongly effective in $\mathcal{R}_r$ and let g be as in Definition 3.16 (ii). If we substitue g by any computable function $\hat{g}$ with $\hat{g}(x) > g(x)$ for all x, and a partial computable function ψ satisfying (11) by $\hat{\psi}$ with

$$\hat{\psi}(n,\sigma,\tau) = \psi(n, \sigma \restriction \min(\{|\sigma|, g(n)\}), \tau \restriction \min(\{|\tau|, g(n)\})),$$

then (11) is still true. In particular, we may assume that g is strictly increasing and satisfies $g(x) > x$ for all x.

Define $g'(x) = 2 \cdot g(2x)$ for $x \in \mathbb{N}$. Let $\mathbf{b_0}, \mathbf{b_1}$ and $\mathbf{c}$ be c.e. r-degrees. By the assumption, there exist c.e. sets $B_0 \in \mathbf{b_0}$, $B_1 \in \mathbf{b_1}$, $C \in \mathbf{c}$ and a partial computable function ψ such that (11) is satisfied. Let $B_0' \in \mathbf{b_0}$, $B_1' \in \mathbf{b_1}$ and $C' \in \mathbf{c}$ be c.e. sets. Then there are cl-functionals $\tilde{\Phi}_{e_0}$, $\tilde{\Phi}_{e_1}$ and $\tilde{\Phi}_{e_2}$ such that $\tilde{\Phi}_{e_0}^{\hat{B}_0} = B_0$, $\tilde{\Phi}_{e_1}^{\hat{B}_1} = B_1$ and $\tilde{\Phi}_{e_2}^{C} = \hat{C}$.

We define

$$\psi'(n,\sigma,\tau) = \tilde{\Phi}_{e_2}^{\rho}(n),$$

where ρ is the string of length $n + e_2 + 1$ or less defined by

$$\rho(m) = \begin{cases} C(m) & \text{if } |\sigma| = |\tau| = 2 \cdot g(2n) \leq g(m) + \max(\{e_0, e_1\}) \\ \psi(m, \tilde{\Phi}_{e_0}^{\sigma} \restriction g(m), \tilde{\Phi}_{e_1}^{\tau} \restriction g(m)) & \text{if } |\sigma| = |\tau| = 2 \cdot g(2n) > g(m) + \max(\{e_0, e_1\}) \\ \uparrow & \text{otherwise} \end{cases}$$

for $0 \leq m \leq n + e_2$. Note that for the definition of ρ the first case is only used if $2 \cdot g(2n) \leq g(n + e_2) + \max(\{e_0, e_1\})$, because we assume that g is increasing. But for almost every n, $2n > n + e_2$ and hence $g(2n) > g(n+e_2)$; and then clearly $2 \cdot g(2n) > g(n+e_2) + \max(\{e_0, e_1\})$ for almost every n. Hence the first case is only used for finitely many n and only those finitely many σ and τ which have length n. This shows that ψ' is partially computable, since only finitely many values $C(m)$ (which can be hard-coded into a machine) are needed.

If $\mathbf{b_0} \vee \mathbf{b_1} = \mathbf{c}$, then for every $n \in \mathbb{N}$ and for $\sigma = B_0' \restriction g'(n)$, $\tau = B_1' \restriction g'(n)$ we obtain $\rho(m) = C(m)$ if $2 \cdot g(2n) \leq g(m) + \max(\{e_0, e_1\})$, while for $2 \cdot g(2n) > g(m) + \max(\{e_0, e_1\})$ we obtain

$$\begin{aligned} \rho(m) &= \psi(m, \tilde{\Phi}_{e_0}^{\sigma} \restriction g(m), \tilde{\Phi}_{e_1}^{\tau} \restriction g(m)) \\ &= \psi(m, \tilde{\Phi}_{e_0}^{B_0' \restriction 2 \cdot g(2n)} \restriction g(m), \tilde{\Phi}_{e_1}^{B_1' \restriction 2 \cdot g(2n)} \restriction g(m)) \\ &= \psi(m, \tilde{\Phi}_{e_0}^{B_0'} \restriction g(m), \tilde{\Phi}_{e_1}^{B_1'} \restriction g(m)) \\ &= \psi(m, B_0 \restriction g(m), B_1 \restriction g(m)) \\ &= C(m). \end{aligned}$$

Here for the third equality we use the fact that $\tilde{u}_{e_0}^{B_0'}(x) \leq x + e_0 \leq g(m) + e_0 < 2 \cdot g(2n)$ for all $x \leq g(m)$, and similarly for $\tilde{u}_{e_1}^{B_1'}(x)$.

Hence $\rho = C \restriction n + e_2 + 1$ and $\psi'(n,\sigma,\tau) = \tilde{\Phi}_{e_2}^{\rho}(n) = \tilde{\Phi}_{e_2}^{C \restriction n+e_2+1}(n) = C'(n)$, because $\tilde{u}_{e_2}^{C}(n) \leq n + e_2$.

$\square$

Lemma 3.18. *The join is weakly effective in* $\mathcal{R}_{\text{ibT}}$ *and* $\mathcal{R}_{\text{cl}}$*.*

The proof follows from the cl-wtt-Join Lemma:

Lemma 3.19 (cl-wtt-Join Lemma [Ambob])**.** *Let* $B_0, \ldots, B_n, C$ *be c.e. sets such that*

$$deg_{\text{cl}}(B_0) \vee \ldots \vee deg_{\text{cl}}(B_n) = deg_{\text{cl}}(C).$$

Then

$$deg_{\text{wtt}}(B_0) \vee \ldots \vee deg_{\text{wtt}}(B_n) = deg_{\text{wtt}}(C).$$

Proof. Since wtt-reducibility is stronger than cl-reducibility, $B_i \leq_{\text{cl}} C$ implies that $B_i \leq_{\text{wtt}} C$ for $i \in \{0, \ldots, n\}$. Since $deg_{\text{wtt}}(B_0) \vee \ldots \vee deg_{\text{wtt}}(B_n) = deg_{\text{wtt}}(B_0 \oplus \ldots \oplus B_n)$ it follows that $B_0 \oplus \ldots \oplus B_n \leq_{\text{wtt}} C$.

Hence it suffices to show that $C \leq_{\text{wtt}} B_0 \oplus \ldots \oplus B_n$. Let $k \in \mathbb{N}$ be such that all B_i are $(\mathrm{i}+\mathrm{k})$bT-reducible to C. We may assume that all B_i are noncomputable, because if all of them are computable, then C is computable, too, and the lemma is clear; and if only some of the B_i are computable then leaving them out does not affect the join. Since $B_0 \leq_{\text{cl}} C$, C must also be noncomputable. Hence there is an infinite computable subset $R \subset C$, and $C' = C - R$ is ibT-equivalent to C. By Lemma 2.7 there are sets $\hat{B}_0 \subseteq B_0, \ldots, \hat{B}_n \subseteq B_n, \hat{C} \subseteq C'$ with respective enumeration functions $b_0, \ldots, b_n, c$ such that each $\hat{B}_i$ is ibT-equivalent to B_i, $\hat{C}$ is ibT-equivalent to C and $c(s) \leq b_i(s) + k$ for $i \in \{0, \ldots, n\}$.

Let f be the computable function such that $f(s)$ is the s-th element of R according to the natural order. Define

$$C'' = \hat{C}_f \cup \{c(s) : (\exists i \leq n)(b_i(s) < f(c(s)))\}.$$

Note that f is a computable unbounded shift and that $\hat{C}_f \subseteq R$ and $\{c(s) : (\exists i \leq n)(b_i(s) < f(c(s)))\} \subseteq \hat{C} \subseteq C' = C - R$, hence the two sets are disjoint.

For

$$C''_s = \{f(c(t)) : t < s\} \cup \{c(t) : (\exists i \leq n)(b_i(t) < f(c(t))) \text{ and } t < s\},$$

we get a computable approximation $(C''_s)_{s \geq 0}$ of C'', and for $\hat{B}_{i,s} = \{b_i(t) : t < s\}$ we get a computable approximation $(\hat{B}_{i,s})_{s \geq 0}$ of $\hat{B}_i$.

For $i \leq n$ it holds that $\hat{B}_i \leq_{\text{cl}} C''$ by permitting. Indeed, if $y = b_i(s)$ enters $\hat{B}_{i,s+1} - \hat{B}_{i,s}$, then either $f(c(s)) \leq y$ and $f(c(s)) \in C''_{s+1} - C''_s$, or $b_i(s) < f(c(s))$ and $c(s) \in C''_{s+1} - C''_s$, where $c(s) \leq b_i(s) + k$. Hence (2) holds with C'' in place of A and $\hat{B}_i$ in place of B.

It follows that $B_i \leq_{\text{cl}} C''$ for $i \in \{0, \ldots, n\}$, and hence, by definition of the join, $\hat{C} \equiv_{\text{cl}} C \leq_{\text{cl}} C''$. Using the Computable Shift Lemma (Lemma 2.6), it follows that

$$C \leq_{\text{cl}} \{c(s) : (\exists i \leq n)(b_i(s) < f(c(s)))\} \leq_{\text{wtt}} B_0 \oplus \ldots \oplus B_n,$$

as claimed. $\square$

We can now give the short proof of Lemma 3.18.

Proof of Lemma 3.18. Let $r \in \{\mathrm{ibT}, \mathrm{cl}\}$. Let $\mathbf{b_0}$, $\mathbf{b_1}$, $\mathbf{c} \in \mathbf{R}_r$. Let $B_0 \in \mathbf{b_0}$, $B_1 \in \mathbf{B_1}$ and $C \in \mathbf{C}$ be arbitrary c.e. sets.

If $\mathbf{b_0} \vee \mathbf{b_1}$ does not exist or is different from $\mathbf{c}$, take arbitrary computable functions f and g. Otherwise by Lemma 3.19 (and in the case of $r = \mathrm{ibT}$, Lemma 3.2) it holds that $deg_{\mathrm{wtt}}(B_0 \oplus B_1) = deg_{\mathrm{wtt}}(B_0) \vee deg_{\mathrm{wtt}}(B_1) = deg_{\mathrm{wtt}}(C)$. Hence $C \leq_{\mathrm{wtt}} B_0 \oplus B_1$. That means, $C = \Phi_e^{B_0 \oplus B_1}$ for some e, where $u_e(x) \leq g(x)$ for a computable function g. Now

$$\psi(n, \sigma, \tau) = \begin{cases} \Phi_e^{\sigma \oplus \tau}(n) & \text{if } |\sigma| = |\tau| = g(n) + 1 \\ 0 & \text{otherwise} \end{cases},$$

(where $\sigma(0)\ldots\sigma(k) \oplus \tau(0)\ldots\tau(k) = \sigma(0)\tau(0)\ldots\sigma(k)\tau(k)$) witnesses that equation (11) is satisfied. □

Turning to strong effectivity, however, the join is not effective in $\mathcal{R}_r$ for $r \in \{\mathrm{ibT}, \mathrm{cl}\}$.

Theorem 3.20. *The join is not strongly effective in $\mathcal{R}_{\mathrm{ibT}}$ or $\mathcal{R}_{\mathrm{cl}}$.*

Proof. Let g be a stricly increasing computable function with $g(x) > x$ for all x. We will construct c.e. sets B_0, B_1 and C such that $deg_r(B_0) \vee deg_r(B_1) = deg_r(C)$ but for every partial computable function φ_e there exists some $n \in \mathbb{N}$ such that $C(n) \neq \varphi_e(n, B_0 \upharpoonright g(n), B_1 \upharpoonright g(n))$. Then it follows by Lemma 3.17 that the join cannot be strongly effective in $\mathcal{R}_r$.

By the ibT-cl-Join Lemma (Lemma 3.2) it suffices to consider the case $r = \mathrm{ibT}$.

We will effectively enumerate the sets B_0, B_1 and C satisfying the following requirements for all $e = \langle e_0, e_1, e_2 \rangle$.

$$B_0, B_1 \leq_{\mathrm{ibT}} C$$

$$\mathcal{D}_e : (\exists n)(C(n) \neq \varphi_e(n, B_0 \upharpoonright g(n), B_1 \upharpoonright g(n)))$$

$$\mathcal{J}_e : (B_0 = \hat{\Phi}_{e_0}^{W_{e_2}} \text{ and } B_1 = \hat{\Phi}_{e_1}^{W_{e_2}}) \Rightarrow C \leq_{\mathrm{ibT}} W_{e_2}.$$

Satisfaction of the *join requirements* $\mathcal{J}_e$ ensures that for every c.e. ibT-degree $\mathbf{d}$, if $\mathbf{d} \geq deg_{\mathrm{ibT}}(B_0)$ and $\mathbf{d} \geq deg_{\mathrm{ibT}}(B_1)$, then $\mathbf{d} \geq deg_{\mathrm{ibT}}(C)$. Namely, if e_0, e_1 and e_2 are chosen in such a way that W_{e_2} is some c.e. set with $\mathbf{d} = deg_{\mathrm{ibT}}(W_{e_2})$, $B_0 = \hat{\Phi}_{e_0}^{W_{e_2}}$ and $B_1 = \hat{\Phi}_{e_1}^{W_{e_2}}$, then with $e = \langle e_0, e_1, e_2 \rangle$ the satisfaction of $\mathcal{J}_e$ implies that $deg_{\mathrm{ibT}}(C) \leq deg_{\mathrm{ibT}}(W_{e_2}) = \mathbf{d}$.

Let $B_{0,s}$, $B_{1,s}$ and C_s denote the finite parts of B_0, B_1 and C enumerated by the end of stage s of the construction. $B_0 \leq_{\mathrm{ibT}} C$ will be satisfied by direct coding. That is, if a new number x is enumerated into B_0 at stage $s+1$ then $x \notin C_s$ and x is simultaneously enumerated into C at stage $s+1$. Similarly, $B_1 \leq_{\mathrm{ibT}} C$ will hold by permitting. The requirements $\mathcal{D}_e$ will be satisfied by the usual diagonalisation method, i.e. by waiting for a stage s such that $\varphi_{e,s}(n, B_{0,s} \upharpoonright g(n), B_{1,s} \upharpoonright g(n)) = 0$ for some appropriate witness $n \notin C_s$ and then putting n into C_{s+1} and restraining $B_0 \upharpoonright g(n)$ and $B_1 \upharpoonright g(n)$.

In order to guarantee that putting n into C at stage $s+1$ is compatible with the higher priority join requirements, however, n will be appropriate only if for any requirement $\mathcal{J}_{e'}$ ($e' = \langle e'_0, e'_1, e'_2 \rangle < e$) with correct hypothesis $B_0 = \hat{\Phi}_{e'_0}^{W_{e'_2}}$ and $B_1 = \hat{\Phi}_{e'_1}^{W_{e'_2}}$ there is a number $y \leq n$ in $W_{e'_2} - W_{e'_2,s}$, thereby implying that $C \leq_{\mathrm{ibT}} W_{e'_2}$ by permitting. We will ensure the existence of such a number y by guaranteeing that $B_{1,s} \upharpoonright g(n)+1 = \hat{\Phi}_{e'_1,s}^{W_{e'_2,s}} \upharpoonright g(n)+1$ and $[n+1, g(n)] \subseteq W_{e'_2,s}$. In this case we will enumerate $g(n)$ into B_1 at stage $s+1$. The set $W_{e'_2}$ has to respond to this by enumerating some $y \leq g(n)$ into $W_{e'_2}$ after stage s, and, by $[n+1, g(n)] \subseteq W_{e'_2,s}$, this y must be smaller than $n+1$.

We will create the just described setup for a diagonalisation candidate n by enumerating large intervals bit by bit in decreasing order into B_0 and C simultaneously.

The formal construction is a tree construction using the full binary tree $T = \{0,1\}^*$.

In the tree, a node α of length e corresponds to a guess about which of the hypotheses of the join requirements $\mathcal{J}_0, \ldots, \mathcal{J}_{e-1}$ will be true for B_0, B_1 and C. For each $n < e$, if $\alpha(n) = 0$ then α codes the guess that the hypothesis of $\mathcal{J}_n$ is true, otherwise it codes the guess that this hypothesis is false. Depending on these guesses, the node α follows a unique strategy $\mathcal{D}_\alpha$ to satisfy the requirement $\mathcal{D}_e$.

In order to guess whether or not the hypothesis of $\mathcal{J}_e$ is true, we consider the length function

$$l_s(e) = \max(\{x < s : (\forall y < x)(B_{0,s}(x) = \hat{\Phi}_{e_0,s}^{W_{e_2,s}}(x) \text{ and } B_{1,s}(x) = \hat{\Phi}_{e_1,s}^{W_{e_2,s}}(x))\})$$

and we use the observation that – since $\hat{\Phi}_{e_0}$ and $\hat{\Phi}_{e_1}$ are ibT-functionals –

$$\begin{aligned} B_0 = \hat{\Phi}_{e_0}^{W_{e_2}} \text{ and } B_1 = \hat{\Phi}_{e_1}^{W_{e_2}} \quad &\Leftrightarrow \quad \lim\nolimits_{s\to\infty} l_s(e) = \infty \\ &\Leftrightarrow \quad \limsup\nolimits_{s\to\infty} l_s(e) = \infty. \end{aligned} \tag{12}$$

Now, for each node α, we inductively define *α-stages* as follows. Each stage $s \geq 0$ is a λ-stage. If s is an α-stage for some α with $|\alpha| = e$, then s is called *α-expansionary* if $l_s(\mathcal{J}_e) > l_t(\mathcal{J}_e)$ for all α-stages $t < s$, and s is an $\alpha 0$-stage if s is α-expansionary, and s is an $\alpha 1$-stage if s is an α-stage but not α-expansionary.

For each $s \geq 0$, let $\delta_s \in T$ be the unique α of length s such that s is an α-stage. The node δ_s represents the guess about which of the hypotheses of $\mathcal{J}_0, \ldots, \mathcal{J}_{s-1}$ are true which is made at the end of stage s.

Construction. We can now give the construction of B_0, B_1 and C. Let $B_{0,0} = B_{1,0} = C_0 = \emptyset$. For $s \geq 0$, we say that a node α of length e *requires attention* at stage $s+1$ if $\alpha \sqsubseteq \delta_s$ and

(*Case 1*) α has no interval assigned to it at the end of stage s, or

(*Case 2*) α has an interval I assigned to it at the end of stage s such that for all $x \in I$ it holds that $C(x) = \varphi_{e,s}(x, B_{0,s} \upharpoonright g(x), B_{1,s} \upharpoonright g(x))$, $l_s(\mathcal{J}_{e'}) > \max(I)$ for all $e' < e$ with $\alpha(e') = 0$, and $I \cap B_{0,s} = I \cap C_s \subset I$.

If some node requires attention at stage $s+1$, find the least (with respect to $\sqsubseteq$) such α and say that α *is active* at stage $s+1$. We say that α is active due to Case 1 or active due to Case 2, respectively, depending on whether α has an interval assigned or not at the end of stage s. Declare all intervals assigned to nodes $\beta > \alpha$ unassigned (i.e., *initialise* these nodes) and do the following:

If α is active due to Case 1, let $e = |\alpha|$ and assign a new interval $I' = [x, g^{e\cdot(x+1)+1}(x)]$ to α where x is the least number $\geq s+1$ such that x is larger than all numbers from intervals assigned to any node before stage $s+1$. Let $B_{0,s+1} = B_{0,s}$, $B_{1,s+1} = B_{1,s}$ and $C_{s+1} = C_s$.

If α is active due to Case 2, then distinguish the following subcases.

(*Subcase 2.1*) If there exists $x \notin C_s$ such that $g(x) \notin B_{1,s}$, $[x, g(x)] \subseteq I$ and $[x+1, g(x)] \subseteq W_{e'_2,s}$ for every $e' < e$ with $\alpha(e') = 0$ and $e' = \langle e'_0, e'_1, e'_2\rangle$, then let $B_{0,s+1} = B_{0,s}$, $B_{1,s+1} = B_{1,s} \cup \{g(x)\}$ and $C_{s+1} = C_s \cup \{x\}$ for the least such x (we say that α enumerates x into C and $g(x)$ into B_1 at stage $s+1$).

(*Subcase 2.2*) Otherwise, for $y = \max(\{x \in I : x \notin B_{0,s}\})$, let $B_{0,s+1} = B_{0,s} \cup \{y\}$, $B_{1,s+1} = B_{1,s}$ and $C_{s+1} = C_s \cup \{y\}$ (we say that α enumerates n into B_0 and C at stage $s+1$).

If no node requires attention at stage $s+1$, let $B_{0,s+1} = B_{0,s}$, $B_{1,s+1} = B_{1,s}$ and $C_{s+1} = C_s$ and initialise all nodes $\beta > \delta_s$. Proceed to the next stage.

Verification. Obviously, $B_0 \leq_{\mathrm{ibT}} C$ and $B_1 \leq_{\mathrm{ibT}}$ holds by permitting. So it suffices to show that the join and diagonalisation requirements are met. As we will show, this is achieved by the initialisation rules and the strategies for the diagonalisation requirements on the true path.

The *true path* TP of the construction is defined by $\mathrm{TP}(e) = 0$ if the hypothesis of $\mathfrak{J}_e$,

$$B_0 = \hat{\Phi}_{e_0}^{W_{e_2}} \text{ and } B_1 = \hat{\Phi}_{e_1}^{W_{e_2}}, \tag{13}$$

is true (where $e = \langle e_0, e_1, e_2\rangle$), and $\mathrm{TP}(e) = 1$ otherwise.

Claim 1 (True Path Lemma). It holds that $\mathrm{TP} = \liminf_{s\to\infty} \delta_s$*, that is, if* $\alpha \in T$*, then* $\alpha \sqsubset \mathrm{TP}$ *if and only if* $\alpha \sqsubseteq \delta_s$ *for infinitely many* s *and there are only finitely many* s *such that* $\delta_s <_L \alpha$*.*

Proof. The proof is by induction. Let $\alpha = \liminf_{s\to\infty} \delta_s \upharpoonright e = \mathrm{TP} \upharpoonright e$. Then there are infinitely many α-stages. So, by (12), there are infinitely many α-expansionary stages if and only if (13) holds. It follows that $\alpha 0 = \liminf_{s\to\infty} \delta_s \upharpoonright e+1$ if and only if (13) holds.

Claim 2. Let $\alpha \sqsubset \mathrm{TP}$*. Then* α *is initialised, requires attention and is active at only finitely many stages. Moreover, there is an interval* I *permanently assigned to* α *from some stage on.*

Proof. The proof is by induction on $n = |\alpha|$. By $\alpha \sqsubset \mathrm{TP}$ fix s_0 such that $\alpha \leq \delta_s$ for all $s \geq s_0$, and by inductive hypothesis fix $s_1 \geq s_0$ such that no node $\beta \sqsubset \alpha$ requires attention at any stage $s \geq s_1$. Then α will not be initialised after stage s_1 and α will become active at any stage $s+1 > s_1$ at which it requires attention.

Now, by $\alpha \sqsubset \mathrm{TP}$, let s_2 be the least α-stage $\geq s_1$. Then either an interval I is assigned to α at the end of stage s_2 or α will become active at stage s_2+1 and an interval I will be assigned to α at stage s_2+1. Since α is not initialised after stage s_1, this interval I is permanent. It follows that α will act at most $|I|$ times after stage s_2+1, since α can act only via Case 2 after this stage and whenever α acts according to Case 2 after stage s_2 then a new element from I is enumerated into C. Since α acts whenever it requires attention after stage s_1 it follows that α requires attention only finitely often.

Claim 3. Every requirement $\mathfrak{J}_e$, $e \in \mathbb{N}$, is satisfied.

Proof. Let $\alpha = \mathrm{TP} \restriction e$ be the node on the true path of length $e = \langle e_0, e_1, e_2 \rangle$. Assume that (13) holds, that is, $B_0 = \hat{\Phi}_{e_0}^{W_{e_2}}$ and $B_1 = \hat{\Phi}_{e_1}^{W_{e_2}}$ (otherwise $\mathfrak{J}_e$ is trivially satisfied). Then, by the True Path Lemma, $\alpha 0$ is on the true path. So there are infinitely many $\alpha 0$-stages and, by Claim 2, there is an $\alpha 0$-stage s_0 such that no node $\gamma < \alpha 0$ is active at any stage $s \geq s_0$.

Now, in order to compute $C(x)$ with oracle $W_{e_2} \restriction x+1$ for given x, using the oracle compute the least $\alpha 0$-stage $s_1 \geq \max(s_0, x)$ such that

$$W_{e_2,s_1} \restriction x+1 = W_{e_2} \restriction x+1. \tag{14}$$

We claim that $C(x) = C_{s_1}(x)$.

For a contradiction assume that $C(x) \neq C_{s_1}(x)$, i.e., that x enters C at a stage $s+1 > s_1$ and let β_0 be the node which enumerates x into C.

Note that, by choice of s_0 and by $s_0 \leq s_1$, $\alpha 0 \leq \beta_0$. Moreover, since s_1 is an $\alpha 0$-stage, for any node β to the right of $\alpha 0$, β does not become active and is initialised at stage s_1+1. Since a node β will enumerate a number y into C at stage $s+1$ only if there is an interval I assigned to the node at the end of stage s and y is an element of I, and since $\min(I) \geq s'+1$ where $s'+1$ is the stage at which I becomes assigned to β, it follows that a node β to the right of $\alpha 0$ will enumerate only numbers y with $y > s_1+1 > x$ into C after stage s_1.

So $\alpha 0 \sqsubseteq \beta_0$, hence $\beta_0(e) = 0$. Now, when β_0 enumerates x into C at stage $s+1$, then it becomes active via Case 2. It follows, by definition of requiring attention, that x is an element of the interval I assigned to β_0 at the end of stage s,

$$\begin{gathered} B_{0,s} \restriction \max(I)+1 = \hat{\Phi}_{e_0,s}^{W_{e_2,s}} \restriction \max(I)+1 \\ \text{and} \\ B_{1,s} \restriction \max(I)+1 = \hat{\Phi}_{e_1,s}^{W_{e_2,s}} \restriction \max(I)+1 \end{gathered} \tag{15}$$

(since $l_s(\mathfrak{J}_e) > \max(I)$), and

$$C_s(x) = B_{0,s}(x) = 0 \tag{16}$$

(since $I \cap B_{0,s} = I \cap C_s$ and $x \notin C_s$).

Now if Subcase 2.2 applies then x is enumerated not only into C but also into B_0 at stage $s+1$. So, by the first part of (15),

$$B_0(x) = B_{0,s+1}(x) \neq B_{0,s}(x) = \hat{\Phi}_{e_0,s}^{W_{e_2,s}}(x) = \hat{\Phi}_{e_0}^{W_{e_2}}(x),$$

where the final equality follows by (14) since $\hat{\Phi}_{e_0}$ is an ibT-functional. But this contradicts the assumption (13).

So Subcase 2.1 must apply. It follows that $[x, g(x)] \subseteq I$ and $[x+1, g(x)] \subseteq W_{e_2,s}$ and that $g(x)$ is enumerated into B_1 at stage $s+1$. Hence, by the second part of (15),

$$B_1(g(x)) = B_{1,s+1}(g(x)) \neq B_{1,s}(g(x)) = \hat{\Phi}_{e_1,s}^{W_{e_2,s}}(g(x)).$$

On the other hand, by assumption (13), $B_1(g(x)) = \hat{\Phi}_{e_1}^{W_{e_2}}(g(x))$. It follows that a number $y \leq g(x)$ has to enter W_{e_2} after stage $s \geq s_1$. In fact, by $[x+1, g(x)] \subseteq W_{e_0,s}$, this number y has to be $\leq x$. But this contradicts (14).

This completes the proof of Claim 3.

Claim 4. Every requirement $\mathcal{D}_e$, $e \in \mathbb{N}$, is satisfied.

Proof. For a contradiction assume that for all n

$$C(n) = \varphi_e(n, B_0 \upharpoonright g(n), B_1 \upharpoonright g(n)). \tag{17}$$

We start with some notation and observations. Let α be the unique node on the true path for which $|\alpha| = e$. By Claim 2 there is a stage s_α and an interval

$$I_\alpha = [x_\alpha, g^{e\cdot(x_\alpha+1)+1}(x_\alpha)]$$

(for some x_α) such that I_α is assigned to α at stage $s_\alpha + 1$ and I_α is never cancelled. Note that, by permanence of I_α, α is not initialised after stage s_α. So, in particular, $\alpha \leq \delta_s$ for $s > s_\alpha$, no node $\beta < \alpha$ becomes active after stage s_α, and α becomes active whenever it requires attention after stage s_α. Moreover, for a node β with $\alpha < \beta$, β is initialised at stage $s_\alpha + 1$, hence will enumerate only numbers $x > \max(I_\alpha)$ into B_0, B_1 or C after stage s_α. So the numbers $x \leq \max(I_\alpha)$ enumerated into B_0, B_1 or C after stage s_α are just the elements of I_α enumerated by α into B_0, B_1 or C. In particular, $B_i \upharpoonright x_\alpha = B_{i,s_\alpha} \upharpoonright x_\alpha$ for $i \in \{0,1\}$ and $C \upharpoonright x_\alpha = C_{s_\alpha} \upharpoonright x_\alpha$. Also note that

$$I_\alpha \cap B_{0,s_\alpha+1} = I_\alpha \cap B_{1,s_\alpha+1} = I_\alpha \cap C_{s_\alpha+1} = \emptyset \tag{18}$$

holds since when an interval becomes assigned to a node at a stage $s+1$ then no element of this interval has been enumerated into B_0, B_1 or C prior to stage $s+1$ and no number is enumerated into B_0, B_1 or C at stage $s+1$.

Let $E = \{e' : e' < e \;\&\; \alpha(e') = 0\}$ and for $e' \in E$ let $e' = \langle e'_0, e'_1, e'_2 \rangle$. Since α is on the true path there are infinitely many α-stages and, for $e' \in E$, $\lim_{s\to\infty} l_s(\mathfrak{J}_{e'}) = \infty$. So if we call a stage $s > s_\alpha$ *good* if s is an α-stage, $l_s(\mathfrak{J}_{e'}) > \max(I_\alpha)$ for $e' \in E$, and $C_s(n) = \varphi_{e,s}(n, B_{0,s} \restriction g(n), B_{1,s} \restriction g(n))$ for all $n \in I_\alpha$ then by equation (17) there are infinitely many good stages. Note that α requires attention at a stage $s+1 > s_\alpha + 1$ if and only if s is good and $I_\alpha \cap B_{0,s} = I_\alpha \cap C_s \subset I_\alpha$.

In order to get the desired contradiction we will argue that α enumerates all numbers x from I_α into (C and) B_0 according to Case 2.2 whereas Case 2.1 will never apply. We then observe that, for any number $e' \in E$, the enumeration of x into B_0 will force a new number $\leq x$ into $W_{e'_2}$. Moreover, since Case 2.1 does not apply, we may argue that, for any subinterval $[x+1, g(x)]$ of I_α put into B_0 and for some $e' \in E$, $W_{e'_2}$ will not react with the enumeration of $[x+1, g(x)]$ but also some new smaller number(s) will enter $W_{e'_2}$. This will allow us to argue that $|W_{e'_2} \restriction x_\alpha + 1| > x_\alpha + 1$ for some $e' \in E$ which of course is impossible.

We first show that α does not require attention via Case 2.1 after stage $s_\alpha + 1$. For a contradiction assume that α acts via Case 2.1 at stage $s' + 1 > s_\alpha + 1$. Then a number $x \in I_\alpha$ such that

$$C_{s'}(x) = \varphi_{e,s'}(x, B_{0,s'} \restriction g(x), B_{1,s'} \restriction g(x)) = 0$$

is enumerated into C at stage $s'+1$ whereas $B_{0,s'+1} = B_{0,s'}$ and $B_{1,s'+1} = B_{1,s'} \cup \{g(x)\}$. Moreover, α will not act after stage $s'+1$ since $I_\alpha \cap C_s \neq I_\alpha \cap B_{0,s}$ for $s \geq s'+1$. So, by choice of s_α, $B_0 \restriction g(x) = B_{0,s'} \restriction g(x)$ and $B_1 \restriction g(x) = B_{1,s'} \restriction g(x)$, hence $\varphi_e(x, B_0 \restriction g(x), B_1 \restriction g(x)) = \varphi_e(x, B_{0,s'} \restriction g(x), B_{1,s'} \restriction g(x))$. It follows that $C(x) \neq \varphi_e(x, B_0 \restriction g(x), B_1 \restriction g(x))$ contrary to assumption (17).

Next we show that α enumerates all the numbers from I_α into B_0 and C according to Case 2.2 in decreasing order. I.e., for any $x \in I_\alpha$ there is a good stage $s_x > s_\alpha$ such that

$$I_\alpha \cap B_{0,s_x} = I_\alpha \cap C_{s_x} = (x, g^{e\cdot(x_\alpha+1)+1}(x_\alpha)] \;\&\; B_{0,s_x+1} - B_{0,s_x} = C_{s_x+1} - C_{s_x} = \{x\}. \quad (19)$$

The proof is by induction on $g^{e\cdot(x_\alpha+1)+1}(x_\alpha) - x$. Fix x. By (18) (for $x = g^{e\cdot(x_\alpha+1)+1}(x_\alpha)$) or by inductive hypothesis (for $x < g^{e\cdot(x_\alpha+1)+1}(x_\alpha)$) fix a stage $s' > s_\alpha$ such that $I_\alpha \cap B_{0,s'} = I_\alpha \cap C_{s'} = (x, g^{e\cdot(x_\alpha+1)+1}(x_\alpha)]$ and let s'' be the least good stage $\geq s'$. Then α will become active at stage $s''+1$ and – since Case 2.1 does not apply after stage s_α – α will enumerate x into B_0 and C at stage $s''+1$. So s'' is the desired stage s_x.

In the remainder of the proof we will show that

$$(\exists\, e' \in E\,)(|W_{e'_2} \restriction x_\alpha + 1| > x_\alpha + 1) \quad (20)$$

holds (which will give the desired contradiction since there are only $x_\alpha + 1$ numbers less than $x_\alpha + 1$).

For a proof of (20) we first observe that, for $e' \in E$, the enumeration of an element $x > x_\alpha$ of the interval I_α into B_0 at stage $s_x + 1$ forces $W_{e'_2}$ to respond with the enumeration of a new number $\leq x$ before the next smaller element $x - 1$ of I_α is put into B_0:

$$(\forall\, x \in I_\alpha - \{x_\alpha\}\,)(\forall\, e' \in E\,)(W_{e'_2,s_{x-1}} \restriction x + 1 \neq W_{e'_2,s_x} \restriction x + 1). \tag{21}$$

Namely, by goodness of s_x and s_{x-1}, $l_{s_x}(\mathcal{J}_{e'}) > \max(I_\alpha)$ and $l_{s_{x-1}}(\mathcal{J}_{e'}) > \max(I_\alpha)$, hence $B_{0,s_x}(x) = \hat{\Phi}^{W_{e'_2,s_x}}_{e'_0,s_x}(x)$ and $B_{0,s_{x-1}}(x) = \hat{\Phi}^{W_{e'_2,s_{x-1}}}_{e'_0,s_{x-1}}(x)$. Since $B_{0,s_x}(x) = 0 \neq 1 = B_{0,s_x+1}(x) = B_{0,s_{x-1}}(x)$ and $\hat{\Phi}_{e'_0}$ is an ibT-functional, this implies the claim.

Now, recall that $I_\alpha = [x_\alpha, g^{e\cdot(x_\alpha+1)+1}(x_\alpha)]$. Hence we can split $I_\alpha - \{x_\alpha\}$ into $e \cdot (x_\alpha + 1) + 1$ disjoint intervals $(x, g(x)]$ by letting

$$x_n = g^{e\cdot(x_\alpha+1)+1-n}(x_\alpha) \ \ (n \in \{0, \ldots, e \cdot (x_\alpha + 1) + 1\})$$

and

$$I^n_\alpha = (x_{n+1}, x_n] = (x_{n+1}, g(x_{n+1})] \ \ (n \in \{0, \ldots, e \cdot (x_\alpha + 1)\}).$$

We claim that

$$\sum_{e'\in E} |W_{e'_2,s_{x_n}} \restriction x_n + 1| \geq n \text{ (for } n \in \{0, \ldots, e \cdot (x_\alpha + 1) + 1\}). \tag{22}$$

Note that for $n = e \cdot (x_\alpha + 1) + 1$ this amounts to

$$\sum_{e'\in E} |W_{e'_2,s_{x_\alpha}} \restriction x_\alpha + 1| \geq e \cdot (x_\alpha + 1) + 1,$$

since $x_{e\cdot(x_\alpha+1)+1} = x_\alpha$. By $|E| \leq e$ this implies (20).

So it only remains to prove inequality (22). The proof is by induction on n. For $n = 0$ it is trivially true. Assuming that it is true for $n \leq e \cdot (x + 1)$ we show that it is true for $n + 1$. Note that, for the $x_n - x_{n+1}$ numbers y from the interval $I^n_\alpha = (x_{n+1}, x_n]$ it holds that $s_{x_{n+1}} > s_y \geq s_{x_n}$. So, by (21), for any $e' \in E$

$$|(W_{e'_2,s_{x_{n+1}}} \setminus W_{e'_2,s_{x_n}}) \restriction x_n + 1| \geq x_n - x_{n+1}.$$

By inductive hypothesis this implies

$$\sum_{e'\in E} |W_{e'_2,s_{x_{n+1}}} \restriction x_n + 1| \geq n + |E|(x_n - x_{n-1}) = n + |E| \cdot |I^n_\alpha|.$$

Moreover, there is at least one $e' \in E$ such that $I^n_\alpha \not\subseteq W_{e'_2,s_{x_{n+1}}}$, since otherwise by goodness of $s_{x_{n+1}}$ and by (19), α will require attention via Case 2.1 at stage $s_{x_{n+1}}+1$ contrary to our above observation. Since $[0,x_n]=[0,x_{n+1}]\cup(x_{n+1},x_n]=[0,x_{n+1}]\cup I^n_\alpha$ it follows that

$$\begin{array}{rcl}\sum_{e'\in E}|W_{e'_2,s_{x_{n+1}}}\restriction x_{n+1}+1| & = & \sum_{e'\in E}|W_{e'_0,s_{x_{n+1}}}\restriction x_n+1| \\ & & -\sum_{e'\in E}|W_{e'_2,s_{x_{n+1}}}\cap I^n_\alpha| \\ & > & (\sum_{e'\in E}|W_{e'_2,s_{x_{n+1}}}\restriction x_n+1|)-|E|\cdot|I^n_\alpha| \\ & \geq & (n+|E|\cdot|I^n_\alpha|)-(|E|\cdot|I^n_\alpha|). \\ & = & n\end{array}$$

This completes the proof of (22), the proof of Claim 4 and the proof of the theorem. □

3.6 Joins and Meets in Substructures of $\mathcal{R}_{\mathrm{ibT}}$ and $\mathcal{R}_{\mathrm{cl}}$: Simple Degrees

Knowing that, by Corollary 3.15, the answer to the question whether $\mathcal{R}_{\mathrm{ibT}}$ or $\mathcal{R}_{\mathrm{cl}}$ are upper or lower semi-lattices is negative, we may extend this question to substructures of $\mathcal{R}_{\mathrm{ibT}}$ or $\mathcal{R}_{\mathrm{cl}}$. There are two possible directions these questions might lead into. The first is: Given a certain lattice, can we find a substructure of $\mathcal{R}_r$ which is closed under joins and meets and isomorphic to this lattice? This is the question of lattice embeddings, which will be addressed in Chapter 4. The second direction is: Given a certain (naturally definable) substructure of $\mathcal{R}_r$, is it a lattice, or at least an upper or lower semi-lattice?

One substructure of $\mathcal{R}_{\mathrm{ibT}}$ and $\mathcal{R}_{\mathrm{cl}}$, respectively, which was studied recently by Ambos-Spies [Amboa], is the structure of the degrees of simple sets.

Definition 3.21. A c.e. set A is called *simple* if its complement $\mathbb{N}-A$ is infinite but does not contain an infinite c.e. subset, i.e. for every $e\in\mathbb{N}$, if $|W_e|=\infty$, then $A\cap W_e\neq\emptyset$. For $r\in\{\mathrm{ibT},\mathrm{cl},\mathrm{wtt},\mathrm{T}\}$, an r-degree $\mathbf{a}\in\mathcal{R}_r$ is called *simple* if $\mathbf{a}$ contains a simple set and *nonsimple* otherwise. The substructure of $\mathcal{R}_r$ consisting of the simple r-degrees is denoted by $(\mathbf{S}_r,\leq)$.

Simple sets were first defined by Post [Post 44] during his now famous program to find a nontrivial property of c.e. sets which would guarantee Turing incompleteness. Finding such a property which is witnessed by at least one noncomputable set A would immediately have implied the existence of at least three c.e. Turing degrees, namely the degree $\mathbf{0}$ of the computable sets, the degree $\mathbf{0}'$ of the Turing complete sets and the degree $deg_{\mathrm{T}}(A)$. Simplicity was a first candidate for an appropriate property and could be shown to imply many-one incompleteness, but Post also showed that it is not sufficient to ensure Turing incompleteness; in fact, except for $\mathbf{0}$, every c.e. Turing degree contains a simple set. The notion of simplicity has nonetheless been studied in many different contexts during the last decades.

In [Amboa] Ambos-Spies showed that even for *linearly bounded Turing (*lbT*) reducibility*, which is defined like weak truth-table reducibility but with the use function bounded by a linear function (instead of a computable one), it is still true that every lbT-degree except for $\mathbf{0}$ still contains a simple set. Considering ibT- or cl- degrees, however, the distribution of simple degrees turns out to be less trivial. Using a finite-injury argument, Ambos-Spies constructs a noncomputable c.e. set A such that $deg_{\mathrm{ibT}}(A)$ is nonsimple. Remarkably, this implies that $deg_{\mathrm{cl}}(A)$ is nonsimple, too, by the following general theorem.

Theorem 3.22 (Coincidence Theorem)**.** *[Amboa] For any c.e. set A it holds that $deg_{\mathrm{ibT}}(A)$ is simple if and only if $deg_{\mathrm{cl}}(A)$ is simple.*

Concerning the question whether $(\mathbf{S}_r, \leq)$ is a lattice for $r \in \{\mathrm{ibT}, \mathrm{cl}\}$, we get a negative answer again.

Theorem 3.23. *[Amboa] If A is a noncomputable c.e. set, then there exists a simple set B with $A \leq_{\mathrm{ibT}} B$.*

Theorem 3.24. *[Amboa] For $r \in \{\mathrm{ibT}, \mathrm{cl}\}$, there is a maximal pair of simple r-degrees. Hence the partial order $(\mathbf{S}_r, \leq)$ is not an upper semi-lattice.*

Proof. By Theorem 3.5, there exist c.e. sets B_0 and B_1 such that $(deg_{\mathrm{cl}}(B_0), deg_{\mathrm{cl}}(B_1))$ is a cl-maximal pair. By Theorem 3.23, there are simple sets C_0 and C_1 such that $B_0 \leq_{\mathrm{ibT}} C_0$ and $B_1 \leq_{\mathrm{ibT}} C_1$, and a fortiori $B_0 \leq_{\mathrm{cl}} C_0$ and $B_1 \leq_{\mathrm{cl}} C_1$. Then the pair $(deg_{\mathrm{cl}}(C_0), deg_{\mathrm{cl}}(C_1))$ of simple cl-degrees is cl-maximal too.

Since ibT-reducibility implies cl-reducibility, the pair $(deg_{\mathrm{ibT}}(C_0), deg_{\mathrm{ibT}}(C_1))$ is also ibT-maximal. □

Theorem 3.25. *[Amboa] If B is a noncomputable c.e. set, then there exists a simple set A with $A \leq_{\mathrm{ibT}} B$.*

Theorem 3.26. *[Amboa] For $r \in \{\mathrm{ibT}, \mathrm{cl}\}$, there is a minimal pair of simple r-degrees. Hence the partial order $(\mathbf{S}_r, \leq)$ is not a lower semi-lattice.*

Proof. By Theorem 3.9, there exist c.e. sets B_0 and B_1 such that $(deg_r(B_0), deg_r(B_1))$ is an r-minimal pair. By Theorem 3.25 there are simple sets A_0 and A_1 such that $A_0 \leq_{\mathrm{ibT}} B_0$ and $A_1 \leq_{\mathrm{ibT}} B_1$, and a fortiori $A_0 \leq_{\mathrm{cl}} B_0$ and $A_1 \leq_{\mathrm{cl}} B_1$. Then the pair $(deg_r(A_0), deg_r(A_1))$ of simple r-degrees is r-minimal, too.

Since computable sets are not simple, the degree $\mathbf{0}$ is nonsimple. Hence $deg_r(A_0)$ and $deg_r(A_1)$ do not have a meet in $(\mathbf{S}_r, \leq)$. □

Of course, if two simple r-degrees $\mathbf{b_0}$ and $\mathbf{b_1}$ have a meet $\mathbf{a}$ in $\mathcal{R}_r$ and $\mathbf{a}$ is itself simple, then $\mathbf{b_0}$ and $\mathbf{b_1}$ have the same meet in $(\mathbf{S}_r, \leq)$. Similarly, if two simple r-degrees $\mathbf{b_0}$ and $\mathbf{b_1}$ have a join $\mathbf{c}$ in $\mathcal{R}_r$ and $\mathbf{c}$ is itself simple, then $\mathbf{b_0}$ and $\mathbf{b_1}$ have join $\mathbf{c}$ in $(\mathbf{S}_r, \leq)$. Theorem 3.26 shows that the condition that $\mathbf{a}$ be itself simple is necessary, because $\mathbf{S}_r$ is not closed under meets. Note that we cannot analogously conclude from Theorem 3.24 that $\mathbf{S}_r$ is not closed under joins (because r-minimal pairs and r-maximal pairs are not quite dual notions). To complete the picture, we show that the latter is still true.

Theorem 3.27. *For $r \in \{\text{ibT}, \text{cl}\}$, $\mathbf{S}_r$ is not closed under joins, namely there are simple r-degrees $\mathbf{b_0}$ and $\mathbf{b_1}$ such that $\mathbf{b_0}$ and $\mathbf{b_1}$ have a join $\mathbf{c}$ in $\mathcal{R}_r$ and $\mathbf{c}$ is not simple.*

Proof. We will construct c.e. sets B_0, B_1 and C such that B_0 and B_1 are simple but $deg_{\text{ibT}}(C)$ is nonsimple and such that $deg_{\text{ibT}}(B_0) \vee deg_{\text{ibT}}(B_1) = deg_{\text{ibT}}(C)$. By the Coincidence Theorem 3.22 then $deg_{\text{cl}}(C)$ is nonsimple, too, while by the ibT-cl-Join Lemma $deg_{\text{cl}}(B_0) \vee deg_{\text{cl}}(B_1) = deg_{\text{cl}}(C)$. Hence the claim holds for $\mathbf{b_0} = deg_r(B_0)$ and $\mathbf{b_1} = deg_r(B_1)$, where $r \in \{\text{ibT}, \text{cl}\}$.

The sets need to satisfy the following requirements for $i \in \{0, 1\}$ and for all $e = \langle e_0, e_1, e_2 \rangle \in \mathbb{N}$:

- $B_i \leq_{\text{ibT}} C$,
- $\mathcal{J}_e : B_0 = \hat{\Phi}_{e_0}^{W_{e_2}}$ and $B_1 = \hat{\Phi}_{e_1}^{W_{e_2}} \Rightarrow C \leq_{\text{ibT}} W_{e_2}$,
- $\mathcal{S}_e : |W_e| = \infty \Rightarrow W_e \cap B_0 \neq \emptyset$ and $W_e \cap B_1 \neq \emptyset$,
- $\mathcal{N}_e : W_{e_2} = \hat{\Phi}_{e_0}^{C}$ and $C = \hat{\Phi}_{e_1}^{W_{e_2}} \Rightarrow W_{e_2}$ not simple.

If B_0, B_1 and C satisfy all these requirements, then, since $B_0, B_1 \leq_{\text{ibT}} C$, the join requirements $\mathcal{J}_e$ guarantee that $deg_{\text{ibT}}(C)$ is the *least* upper bound of $deg_{\text{ibT}}(B_0)$ and $deg_{\text{ibT}}(B_1)$, that is $deg_{\text{ibT}}(C) = deg_{\text{ibT}}(B_0) \vee deg_{\text{ibT}}(B_1)$. The simplicity requirements $\mathcal{S}_e$ make sure that B_0 and B_1 are simple if $\mathbb{N} - B_0$ and $\mathbb{N} - B_1$ are infinite. By the $\mathcal{N}_e$-requirements C is not ibT-equivalent to any simple set, hence $deg_{\text{ibT}}(C)$ is nonsimple.

The construction will be in stages. $B_{0,s}$, $B_{1,s}$ and C_s denote the finite approximations of B_0, B_1 and C, respectively, after stage s. We define the *length of agreement of requirement* $\mathcal{J}_e$ *($e = \langle e_0, e_1, e_2 \rangle$) at stage* s by

$$l_s(\mathcal{J}_e) = \max(\{x < s : (\forall y < x)(B_{0,s}(y) = \hat{\Phi}_{e_0,s}^{W_{e_2,s}}(y) \text{ and } B_{1,s}(y) = \hat{\Phi}_{e_1,s}^{W_{e_2,s}})\}).$$

Similarly, we define the *length of agreement of requirement* $\mathcal{N}_e$ *($e = \langle e_0, e_1, e_2 \rangle$) at stage* s by

$$l_s(\mathcal{N}_e) = \max(\{x < s : (\forall y < x)(W_{e_2,s}(y) = \hat{\Phi}_{e_0,s}^{C_s}(y) \text{ and } C_s(y) = \hat{\Phi}_{e_1,s}^{W_{e_2,s}})\}).$$

Note that, since $\hat{\Phi}_{e_0}$ and $\hat{\Phi}_{e_1}$ are ibT-functionals, the premise of a join requirement $\mathcal{J}_e$ is true, i.e. $B_0 = \hat{\Phi}_{e_0}^{W_{e_2}}$ and $B_1 = \hat{\Phi}_{e_1}^{W_{e_2}}$, if and only if $\liminf_{s\to\infty} l_s(\mathcal{J}_e) = \limsup_{s\to\infty} l_s(\mathcal{J}_e) = \infty$. Mutatis mutandis, the same holds for nonsimplicity requirements $\mathcal{N}_e$.

Before giving the formal construction we describe the ideas underlying the individual strategies for the requirements and the conflicts between these strategies we have to overcome.

$B_i \leq_{\text{ibT}} C$ will hold by permitting, that is, whenever we enumerate a number x into B_i at stage $s + 1$, then we enumerate a number $y \leq x$ into $C_{s+1} - C_s$.

The simplicity requirements $\mathcal{S}_e$ will be satisfied by waiting until a suitable number x is enumerated into W_e and then enumerating this number into B_0 and B_1. Our definition of "suitable" will be such that almost every number $x \in W_e$ will be suitable at almost every stage. Hence we will eventually find a stage such that $\mathcal{S}_e$ can be satisfied.

The nonsimplicity requirements $\mathcal{N}_{\langle e_0,e_1,e_2\rangle}$ are satisfied as follows. We try to enumerate an infinite c.e. set V of numbers x which we keep out of C. If $V \cap W_{e_2} = \emptyset$, then $V \subseteq \mathbb{N} - W_{e_2}$, witnessing that W_{e_2} is not simple. Otherwise there is some stage t and some $v \in V$ such that $v \in W_{e_2,t}$. Now if the premise of $\mathcal{N}_{\langle e_0,e_1,e_2\rangle}$ is true, then there is a stage $s \geq t$ such that $l_s(\mathcal{N}_{\langle e_0,e_1,e_2\rangle}) > v$, in particular $W_{e_2,s} \upharpoonright v+1 = \hat{\Phi}^{C_s}_{e_0,s} \upharpoonright v+1$ and $C_s(v) = \hat{\Phi}^{W_{e_2,s}}_{e_1,s}(v)$. We enumerate v into C at stage $t+1$. To keep up the second part of the premise, $C = \hat{\Phi}^{W_{e_2}}_{e_1}$, a number $x \leq v$ has to enter W_{e_2} after stage s. In fact, since $v \in W_{e_2,s}$, it must hold that $x < v$. Hence by preserving $C_s \upharpoonright v$ we can destroy the first part of the premise, $W_{e_2} = \hat{\Phi}^C_{e_0}$, and thus satisfy the requirement. We say that v has become a *diagonalisation witness* for $\mathcal{N}_{\langle e_0,e_1,e_2\rangle}$.

To satisfy a join requirement $\mathcal{J}_{\langle e_0,e_1,e_2\rangle}$ the premise of which is true, we enumerate a number x into C at a stage $s+1$ for the sake of a lower priority requirement only if we can enumerate a number $y \geq x$ into $B_{0,s+1} - B_{0,s}$ or $B_{1,s+1} - B_{1,s}$ such that $B_{0,s} \upharpoonright y+1 = \hat{\Phi}^{W_{e_2,s}}_{e_0,s} \upharpoonright y+1$ and $B_{1,s} \upharpoonright y+1 = \hat{\Phi}^{W_{e_2,s}}_{e_1,s} \upharpoonright y+1$ and $[x+1,y] \subseteq W_{e_2,s}$. Then W_{e_2} has to react by enumerating a number $z \leq y$ after stage s. Since $[x+1,y] \subseteq W_{e_2,s}$, indeed $z \leq x$, implying that $C \leq W_{e_2}$ by permitting.

It might be tempting to just let $y = x$, thus making the condition $[x+1,y] \subseteq W_{e_2,s}$ vacuous. In general, however, this is not possible. Namely, consider the c.e. set V enumerated for the sake of some nonsimplicity requirement $\mathcal{N}_{\langle e_0',e_1',e_2'\rangle}$. Moreover, consider some simplicity requirement $\mathcal{S}_{\bar{e}}$. For each enumeration of a number v into V, the number v might later occur in $W_{\bar{e}}$, causing $\mathcal{S}_{\bar{e}}$ to ask for an enumeration of v into B_0 and B_1. Since we do not know whether $W_{\bar{e}} = V$ and V might be infinite, at some stage s we have to allow the enumeration of some such v (otherwise $\mathcal{S}_{\bar{e}}$ might not be satisfied). But now it happens that v enters $W_{e_2'}$ at a stage $t > s$. Following the strategy for the nonsimplicity requirements, we want to enumerate v into C_{t+1} – but v is already in $B_{0,t}$ and $B_{1,t}$, hence we need to choose a number $y > v$ to enumerate it into B_0 or B_1.

This also shows that we need to take care about which numbers we enumerate into V for the sake of some $\mathcal{N}_{\langle e_0',e_1',e_2'\rangle}$. We will enumerate a number v at a stage $s+1$ only if $[v+1,v+n] \subseteq W_{e_2}$ for each join requirement $\mathcal{J}_{\langle e_0,e_1,e_2\rangle}$ of higher priority than $\mathcal{N}_{\langle e_0',e_1',e_2'\rangle}$, where n is sufficiently large. This permits us to enumerate n numbers from $[v,v+n]$ into B_0 and B_1 in order to satisfy up to n simplicity requirements while we still have one number left to enumerate into B_0 or B_1 in order to satisfy the join requirements. By letting the number n grow each time we enumerate any number into any set V, we can thus make sure that every simplicity requirement is eventually allowed to enumerate every number it wishes to enumerate.

This leaves us with the task to create situations where suitable numbers v as above exist, i.e. numbers v with $[v+1,v+n] \subseteq W_{e_2,s}$ for each join requirement $\mathcal{J}_{\langle e_0,e_1,e_2\rangle}$ from a finite set R or requirements with true premise. We say that $[v+1,v+n]$ is *R-safe at stage s* in this case.

To obtain a new suitable v for $\mathcal{N}_{\langle e_0',e_1',e_2'\rangle}$, we assign a finite sufficiently long interval I to

$\mathcal{N}_{\langle e_0',e_1',e_2'\rangle}$. Then we enumerate the elements b of I one by one in decreasing order alternately into B_0 and B_1, always accompanied by an enumeration of the largest possible number $c \leq b$ into C (to satisfy $B_0, B_1 \leq_{\mathrm{ibT}} C$). We only do this at stages $s+1$ such that $l_s(\mathcal{J}_e) > \max(I)$ for each $\mathcal{J}_e \in R$, thus ensuring that for $i = 0$ or $i = 1$

$$\hat{\Phi}_{e_i,s}^{W_{e_2,s}}(b) = B_{i,s}(b) \neq B_i(b) = \hat{\Phi}_{e_i}^{W_{e_2}}(b),$$

and hence $W_{e_2,s} \upharpoonright b+1 \neq W_{e_2} \upharpoonright b+1$, since $\hat{\Phi}_{e_i}$ is an ibT-functional. If $s_0 + 1 < s_1 + 1 < \dots$ is the sequence of stages at which we conduct these enumerations, b_m is the number from I enumerated into B_0 or B_1 at stage $s_m + 1$, and c_m is the number enumerated into C at stage $s_m + 1$ then in fact for each m a new number $z_m \leq b_m$ has to enter W_{e_2} at some stage $s \in [s_m + 1, s_{m+1}]$. If z_m is always the maximum number possible, which is just c_m, then for sufficiently large m we obtain that $|[c_m, b_m]| \geq n-1$ and $[c_m, b_m]$ is R-safe at stage s_m and can set $v = c_m - 1$. On the other hand, if $z_m < c_m$, then at stage $s_{m+1}+1$ we restart the algorithm with the interval $I \cap [0, c_m - 1]$ instead of I. We will see that if we choose I sufficiently large, after finitely many restarts due to lack of space the number z_m will always be maximal.

While we are creating such a suitable v we say that the interval I is *being prepared* by $\mathcal{N}_{\langle e_0',e_1',e_2'\rangle}$.

Once we are finished preparing I, we can allow some simplicity requirements to enumerate a number x from $[v, v+n]$ into B_0 or B_1. Of course, to satisfy $B_0, B_1 \leq_{\mathrm{ibT}} C$, we wish to enumerate a number $y \leq x$ into C at the same time. We will reserve a finite part $[\min(I), \min(I) + n]$ purely for this purpose, and we will use the interval $J = I - [\min(I), \min(I) + n]$ for the actual preparation strategy.

Since we cannot compute the premises of which join requirements are true, the actual construction is a tree construction, using the tree $T = \{0,1\}^*$, where a node of length n corresponds to a guess about which of the first n join requirements are true. The definition of α-stages and α-expansionary stages for a node $\alpha \in T$ and the definition of δ_s are the same as in Theorem 3.20.

3.6.1 The Algorithm

Stage 0: Let $B_{0,0} = B_{1,0} = C_0 = \emptyset$ and $V_{\gamma,0} = \emptyset$ for each node γ. No node has an interval or a diagonalisation witness assigned or is preparing any interval.

Stage $s+1$: We say that a node α of even length $|\alpha| = 2\langle e_0, e_1, e_2\rangle$ *requires attention* at stage $s+1$, if $\alpha \sqsubseteq \delta_s$, α has no diagonalisation witness assigned and there exists some interval I_n assigned to α at stage s and numbers $v, d \in J_n$ such that

(1.1) $v \in V_{\alpha,s} \cap W_{e_2,s}$,

(1.2) $v \notin C_s$,

(1.3) $l_s(\mathcal{N}_{\langle e_0,e_1,e_2\rangle}) > v$, (1.4) v is smaller than $\min(I_m)$ for each interval I_m some node $\beta \sqsubseteq \alpha$ is preparing at the end of stage s,

(1.5) $v \leq d \leq v + n$,

(1.6) $d \notin B_{0,s} \cup B_{1,s}$,

(1.7) $[v+1, d]$ is $\{\mathcal{J}_{e'} : \alpha(e') = 0\}$-safe at stage s

We say that a node α of odd length $|\alpha| = 2e+1$ *requires attention* at stage $s+1$, if $\alpha \sqsubseteq \delta_s$, $B_{i,s} \cap W_{e,s} = \emptyset$ for $i = 0$ or $i = 1$ and there exist numbers x and y such that $y \leq x$ and

(2.1) $x \in W_{e,s}$

(2.2) $x < l_s(\mathcal{J}_{e'})$ for each node β with $\beta 0 \sqsubseteq \alpha$ and $|\beta| = e'$

(2.3) $y > \max(I_n)$ for all intervals I_n assigned to nodes $\alpha' <_L \alpha$ up to stage s and for all intervals I_n such that some $\alpha' \sqsubset \alpha$ has a diagonalisation witness $d \in I_n$ assigned at the end of stage s

(2.4) $y \notin (B_{0,s} \cap B_{1,s}) \cup C_s$

(2.5)
- if $x \in I_n$ for some interval I_n defined up to stage s, then $e < n-1$, and if I_n is assigned to some node $\gamma \sqsubseteq \alpha$ at the end of stage s, then $I_n \cap V_{\gamma,s} \neq \emptyset$
- if $x \in J_n$ for some interval J_n defined up to stage s, then $y \in I_n - J_n$
- if $x \notin J_n$ for any interval J_n defined up to stage s, then $y = x$.

If some node requires attention, let α be the least such node and say that α *is active* at stage $s+1$.

If $|\alpha| = 2\langle e_0, e_1, e_2\rangle$, let I_n be the least α-interval such that v and d satisfying (1.1)-(1.7) exist, let $v \in I_n$ be the least number such that d satisfying (1.5)-(1.7) exists and let $d \in I_n$ be the least number satisfying (1.5)-(1.7). Enumerate v into C_{s+1} and enumerate d into $B_{0,s+1}$ and $B_{1,s+1}$. Assign v as diagonalisation witness to α and say that α is not preparing any interval at stage $s+1$ any more.

If $|\alpha| = 2e+1$, let x be the least number such that there exists y satisfying (2.1)-(2.5); enumerate x into $B_{0,s+1}$ and $B_{1,s+1}$ and enumerate the greatest number $y \leq x$ satisfying (2.4) and (2.5) into $B_{0,s+1}$, $B_{1,s+1}$ and C_{s+1}.

In either case, initialise all nodes $\alpha' > \alpha$, i.e. cancel all assignments of intervals or diagonalisation witnesses to such nodes, say that they are not preparing any intervals any longer, and set $V_{\alpha',s+1} = \emptyset$. If no node requires attention at stage $s+1$, initialise all nodes $\alpha' > \delta_s$.

For all nodes $\alpha' \sqsubset \alpha$ (if α is active at stage $s+1$) or for all nodes $\alpha' \sqsubseteq \delta_s$ (if no node is active at stage $s+1$) of even length do the following in order of priority:

(a) If α' has no diagonalisation witness assigned and is not preparing any interval at the end of stage s, then assign a new interval to α' as follows. Let $k = |\{e' < |\alpha| : \alpha(e') = 0\}|$ and let x be the least number that is greater than k, greater than all numbers enumerated into B_0, B_1 or C up to this point, greater than s and greater than $\max(I_n)$ for any interval I_n already defined. Let m be the least number greater than s such that I_m is not yet defined and set

$$I_m = [x, x + m + (x + m) \cdot (k + 1) \cdot 2m],$$

assign I_m to α' and say that I_m is an α'-interval and that α' is preparing I_m at stage $s+1$. Let

$$J_m = [\min(I_m) + m, \max(I_m)] = [x + m, x + m + (x + m) \cdot (k + 1) \cdot 2m].$$

(b) If α' has no diagonalisation witness assigned and is preparing some interval I_n at the end of stage s and there is a number $c \in J_n$ such that

$$I_n \cap (B_{0,s} \cup B_{1,s}) \subseteq I_n \cap C_s = [c + 1, \max(I_n)], \tag{23}$$

$l_s(\mathfrak{J}_{e'}) > \max(I_n)$ for each e' with $\alpha'(e') = 0$, and for

$$\begin{aligned} b = \max(\{y \in I_n : [c+1, y-1] \cap (B_{0,s} \cup B_{1,s}) = \emptyset,\ y \notin B_{0,s} \cap B_{1,s} \text{ and} \\ [c+1, y] \text{ is } \{\mathfrak{J}_{e'} : \alpha'(e') = 0\}\text{-safe at stage } s\}) \end{aligned} \tag{24}$$

it holds that $b \geq c + n$, then enumerate c into $V_{\alpha',s+1}$ and say that α' is not preparing I_n any more at stage $s + 1$ (note that such b necessarily exists and that $b \geq c$, because $y = c$ satisfies the conditions above).

(c) If the conditions from (b) hold, but $b < c + n$, then enumerate c into C_{s+1}; additionally, if $b \notin B_{0,s}$, enumerate b into $B_{0,s+1}$; otherwise enumerate b into $B_{1,s+1}$.

The assignment of intervals and diagonalisation witnesses and the status of intervals being prepared by nodes, unless mentioned otherwise in the algorithm so far, remains the same at stage $s + 1$ as after stage s.

3.6.2 Verification.

Lemma 3.28. *It holds that $B_0 \leq_{\mathrm{ibT}} C$ and $B_1 \leq_{\mathrm{ibT}} C$.*

Proof. This holds by permitting since whenever some number x is enumerated into B_0 or B_1 at stage $s + 1$, then some $y \leq x$ is enumerated into $C_{s+1} - C_s$. $\square$

The *true path* TP of the construction is defined by $\mathrm{TP}(e) = 0$ if the hypothesis of $\mathfrak{J}_e$,

$$B_0 = \hat{\Phi}_{e_0}^{W_{e_2}} \text{ and } B_1 = \hat{\Phi}_{e_1}^{W_{e_2}}, \tag{25}$$

is true (where $e = \langle e_0, e_1, e_2 \rangle$), and $\mathrm{TP}(e) = 1$ otherwise.

Lemma 3.29 (True Path Lemma)**.** *It holds that* $\mathrm{TP} = \liminf_{s\to\infty} \delta_s$*, that is, if* $\alpha \in T$*, then* $\alpha \sqsubset \mathrm{TP}$ *if and only if* $\alpha \sqsubseteq \delta_s$ *for infinitely many* s *and there are only finitely many* s *such that* $\delta_s <_L \alpha$*.*

Proof. As in Theorem 3.20. □

Lemma 3.30. *Every node* $\alpha \sqsubset \mathrm{TP}$ *is initialised at most finitely often and is active at most finitely often. Moreover, if* α *with* $|\alpha| = 2e$ *is active at some stage* $s > s_0$*, where* s_0 *is the least stage such that* α *is not initialised at any stage* $s > s_0$*, then* $\mathcal{N}_e$ *is satisfied.*

Proof. The proof is by induction. Assume that the claim is true for all $\alpha' \sqsubset \alpha$, and let s_0 be the least stage such that no $\alpha' \sqsubset \alpha$ is active at any stage $s > s_0$ and such that $\delta_s \geq \alpha$ for all $s \geq s_0$. Such a stage exists by the inductive hypothesis and by the True Path Lemma. Then no $\alpha' < \alpha$ is active and s_0 is the least stage such that α is not initialised at any stage $s > s_0$.

If $|\alpha| = 2e + 1$ for some e, then α can be active at most once: If α is active at stage $s + 1$, then some $x \in W_e$ is enumerated into $B_{0,s+1}$ and $B_{1,s+1}$, whence $B_{i,s'} \cap W_{e,s'} \neq \emptyset$ for $i \in \{0, 1\}$ and for all $s' > s$ and α never requires attention after stage $s + 1$.

It remains to consider the case that $|\alpha| = 2\langle e_0, e_1, e_2\rangle$ ($e_0, e_1, e_2 \in \mathbb{N}$). If $W_{e_2} \neq \hat{\Phi}^C_{e_0}$ or $C \neq \hat{\Phi}^{W_{e_2}}_{e_1}$, then $\mathcal{N}_{\langle e_0,e_1,e_2\rangle}$ is trivially satisfied and $\{l_s(\mathcal{N}_{\langle e_0,e_1,e_2\rangle}) : s \geq 0\}$ is bounded. In this case, there are only finitely many v satisfying (1.3) for any s; since each time that α is active some such v is enumerated into C, and hence does not satisfy (1.2) at any later stages, we see that α can be active at most finitely often.

Now assume that

$$W_{e_2} = \hat{\Phi}^C_{e_0} \text{ and } C = \hat{\Phi}^{W_{e_2}}_{e_1}. \tag{26}$$

If α is not active at any stage $s > s_0$, then the claim is proven. For a contradiction assume that α is active at some stage $s + 1 > s_0$. Then a number v from some α-interval I_n satisfying (1.1)-(1.5) is enumerated into C at stage $s + 1$. By equation (26), (1.2) and (1.3) it holds that

$$\hat{\Phi}^{W_{e_2}}_{e_1}(v) = C(v) = C_{s+1}(v) = 1 \neq 0 = C_s(v) = \hat{\Phi}^{W_{e_2,s}}_{e_1,s}(v).$$

Since $\hat{\Phi}_{e_1}$ is an ibT-functional, this implies that there is some $x \leq v$ such that $x \in W_{e_2} - W_{e_2,s}$. Indeed, since $v \in W_{e_2,s}$ by (1.1), $x < v$, hence $W_{e_2} \upharpoonright v \neq W_{e_2,s} \upharpoonright v$. It now suffices to show that $C \upharpoonright v = C_s \upharpoonright v$, because then, by (1.3) and the fact hat $\hat{\Phi}_{e_0}$ is an ibT-functional,

$$\hat{\Phi}^C_{e_0} \upharpoonright v = \hat{\Phi}^{C_s}_{e_0,s} \upharpoonright v = W_{e_2,s} \upharpoonright v \neq W_{e_2} \upharpoonright v,$$

contradicting equation (26).

For a contradiction assume that some number $y < v$ is enumerated into C at a stage $t+1 \geq s+1$. Let α' be the node causing this enumeration. Then $\alpha' \sqsubseteq \delta_t$. Since $\alpha \leq \delta_t$ for $t \geq s_0$, it follows that $\alpha' \not<_L \alpha$.

Moreover, $|\alpha'|$ must be even: Otherwise α' were active at stage $t+1$. If $\alpha' \sqsubset \alpha$, then α were initialised at stage $t+1 \geq s+1 > s_0$, contradicting the choice of s_0. On the other hand, if $\alpha < \alpha'$, then by (2.3) α' enumerates only numbers greater than $\max(I_n) \geq v$ into C at stage $t+1$ (note that α got the diagonalisation witness $v \in I_n$ at stage $s+1$, which is permanent since α is never initialised after stage s_0.)

Since y is enumerated by a node of even length, it follows that y must be an element of some interval $I_{n'}$ assigned to α' at stage t. If $\alpha' > \alpha$, then $I_{n'}$ becomes assigned to α' only at or after stage $s+1$, because otherwise, by initialisation of α', the assignment of $I_{n'}$ to α' were permanently cancelled at stage $s+1$ and $I_{n'}$ would not be assigned to α' at stage t. Hence $y \geq \min(I_{n'}) > \max(I_n) \geq v$, contradicting $y < v$. Consequently, $\alpha' \sqsubseteq \alpha$. Now if $I_{n'}$ becomes assigned to α' only at or after stage $s+1$, we arrive at the same contradiction as above. Hence $I_{n'}$ was assigned to α' before stage $s+1$, and by $\min(I_{n'}) \leq y < v$ and (1.4) α' is not preparing $I_{n'}$ at stage s any more. But this means that $y \in I_{n'}$ can only be enumerated into C at stage $t+1 > s$ by α' being active whence α is initialised at stage $t+1$ – a contradiction to the choice of s_0. ☐

Lemma 3.31. *Let I_n be an α-interval for some $\alpha \sqsubset \mathrm{TP}$. Then α is preparing I_n at only finitely many stages.*

Proof. Let s_0+1 be the stage at which I_n becomes assigned to α. If α is ever initialised after stage s_0, then the claim is certainly true. Hence assume that α is not initialised after stage s_0. For a contradiction assume that α is preparing I_n at infinitely many stages. Then α is preparing I_n at all stages $s \geq s_0+1$ and $I_n \cap V_{\alpha,s} = \emptyset$ for all $s \geq 0$. Let $s_1+1 < \ldots < s_{|J_n|}+1$ be the first $|J_n|$ stages $s+1$ after stage s_0+1 for which s is an α-stage and $l_s(\mathfrak{J}_{e'}) > \max(I_n)$ for all $e' < |\alpha|$ with $\alpha(e') = 0$. These stages exist since $\alpha \sqsubset \mathrm{TP}$ and by the True Path Lemma.

Since $I_n \cap V_{\alpha,s} = \emptyset$ for every s, no node α' enumerates any numbers from I_n into B_0, B_1 or C by being active after stage s_0+1; indeed, for $\alpha' < \alpha$ this holds by the assumption that α is not initialised after stage s_0, for nodes α' of even length it holds by (1.1), for nodes $\alpha' >_L \alpha$ of odd length it holds by (2.3), and for nodes $\alpha' \sqsupseteq \alpha$ of odd length it holds by (2.5). It follows that after stage s_0 numbers from I_n are enumerated into B_0, B_1 or C only by α due to the instruction (c) of the algorithm. In particular, no enumeration of numbers from I_n into B_0, B_1 or C can take place at any stage $s+1$ with $s_m+1 < s+1 \leq s_{m+1}$ for $m \in \{0, \ldots, |J_n|-1\}$, that is,

$$I_n \cap C_{s_m+1} = I_n \cap C_{s_{m+1}} \text{ and } I_n \cap B_{i,s_m+1} = I_n \cap B_{i,s_{m+1}} \text{ for } i \in \{0,1\}. \tag{27}$$

We claim that

$$I_n \cap (B_{0,s_m} \cup B_{1,s_m}) \subseteq I_n \cap C_{s_m} = (\max(I_n) - m + 1, \max(I_n)] \tag{28}$$

and $c_m := \max(I_n) - m + 1$ is enumerated into C at stage $s_m + 1$ for $m \in \{1, \ldots, |J_n|\}$.

To prove this, first notice that $I_n \cap (B_{0,s_0+1} \cup B_{1,s_0+1} \cup C_{s_0+1}) = \emptyset$ by the definition of I_n at stage $s_0 + 1$. By equation (27), this implies $I_n \cap (B_{0,s_1} \cup B_{1,s_1} \cup C_{s_1}) = \emptyset$, so equation (28) is true for $m = 1$.

Let (28) be true for some $m \in \{1, \ldots, |J_n|\}$. Then $c = \max(I_n) - m + 1$ is in J_n (since $\max(I_n) - m + 1 \geq \max(I_n) - |J_n| + 1 = \max(J_n) - |J_n| + 1 = \min(J_n)$) and satisfies (23) for $s = s_m$. It follows that c is enumerated into C_{s_m+1} according to (c) and is the only number from I_n to be enumerated into C at stage $s_m + 1$. If $m < |J_n|$, then by equations (27) and (28), $I_n \cap C_{s_{m+1}} = I_n \cap C_{s_m+1} = (I_n \cap C_{s_m}) \cup \{c\} = (\max(I_n) - (m + 1) + 1, \max(I_n)]$.

Additionally, if some number $b \in I_n$ is enumerated into B_0 or B_1 at stage $s_m + 1$, then $b \geq c$, and hence $b \in [c, \max(I_n)] \subseteq C_{s_m+1}$. Using equations (27) and (28) again, for $m < |J_n|$ we conclude that $I_n \cap (B_{0,s_{m+1}} \cup B_{1,s_{m+1}}) = I_n \cap (B_{0,s_m+1} \cup B_{1,s_m+1}) \subseteq (I_n \cap (B_{0,s_m} \cup B_{1,s_m})) \cup [c, \max(I_n)] = I_n \cap C_{s_{m+1}}$. Hence equation (28) is true for $m + 1$ in place of m.

Let $e' = \langle e'_0, e'_1, e'_2 \rangle < |\alpha|$ such that $\alpha(e') = 0$ and let $1 \leq m \leq |J_n| - 1$. Since c_m is enumerated into C at stage s_m+1 via (c), by the construction there is also some number $b_m \in I_n$ enumerated into $B_{i,s_m+1} - B_{i,s_m}$ for $i = 0$ or $i = 1$. Since $l_{s_{m+1}}(\mathcal{J}_{e'}) > l_{s_m}(\mathcal{J}_{e'}) > \max(I_n) \geq b_m$, it holds that

$$\hat{\Phi}_{e'_i, s_{m+1}}^{W_{e'_2, s_{m+1}}}(b_m) = B_{i,s_{m+1}}(b_m) = 1 \neq 0 = B_{i,s_m}(b_m) = \hat{\Phi}_{e'_i, s_m}^{W_{e'_2, s_m}}(b_m).$$

Since $\hat{\Phi}_{e'_i}$ is an ibT-functional, this means that there must be some number $z_m \leq b_m$ with $z_m \in W_{e'_2, s_{m+1}} - W_{e'_2, s_m}$. But by (24), $[c_m + 1, b_m]$ is already $\{\mathcal{J}_{e'}\}$-safe at stage s_m, that is $[c_m + 1, b_m] \subseteq W_{e'_2, s_m}$. Hence $z_m \leq c_m$ and it follows that (using $c_{m+1} = c_m - 1$)

$$\begin{aligned} |\{x \leq c_{m+1} : x \notin W_{e'_2, s_{m+1}}\}| &\leq |\{x \leq c_m : x \notin W_{e'_2, s_{m+1}}\}| \\ &\leq |\{x \leq c_m : x \notin W_{e'_2, s_m}\}| - 1. \end{aligned} \tag{29}$$

Letting $\{e' < |\alpha| : \alpha(e') = 0\} = \{e_1, \ldots, e_k\}$, set $\pi(e_j) = e'_2$ if $e_j = \langle e'_0, e'_1, e'_2 \rangle$ ($j \in \{1, \ldots, k\}$). Summing up, we conclude that

$$\sum_{j=1}^{k} |\{x \leq c_{m+1} : x \notin W_{\pi(e_j), s_{m+1}}\}| \leq \sum_{j=1}^{k} |\{x \leq c_m : x \notin W_{\pi(e_j), s_m}\}| - k. \tag{30}$$

Furthermore, if $1 \le p < p + 2n \le |J_n|$, then by induction

$$\sum_{j=1}^{k} |\{x \le c_{p+2n} : x \notin W_{\pi(e_j),s_{p+2n}}\}| \le \sum_{j=1}^{k} |\{x \le c_p : x \notin W_{\pi(e_j),s_p}\}| - 2n \cdot k. \tag{31}$$

We will show that this inequality is strict. Indeed, if equality would hold in (31), then for $p \le m < p + 2n$ equality would hold in (30), too, which in turn implies that for all m with $p \le m < p+2n$ and all $j \in \{1, \ldots, k\}$ equality holds in (29) with $\pi(e_j)$ in place of e'_2. The latter is equivalent to $c_m \in W_{\pi(e_j),s_{m+1}}$ for all such m and j, that is, $\{c_m\}$ is $\{\mathcal{J}_{e_j} : 1 \le j \le k\}$-safe at stage s_{m+1}. By induction, since $c_{m+1} = c_m - 1$,

$$[c_m, c_p] \text{ is } \{\mathcal{J}_{e'} : \alpha(e') = 0\}\text{-safe at stage } s_{m+1}. \tag{32}$$

It now follows by another inductive argument that, for all $m \in \{p, \ldots, p + 2n\}$, if $m \in \{p + 2r - 1, p + 2r\}$, then $b_m - c_m \ge r$, and moreover, if $m = p + 2r$, then $c_m + r \notin B_{1,s_m+1}$:

For $m = p$ this holds because $b_m \ge c_m$. If $b_m > c_m$, then still $c_m \notin B_{0,s_m+1} \cup B_{1,s_m+1}$; otherwise, according to the instructions in (c), c_m is enumerated into B_0, but not into B_1 at stage $s_m + 1$.

For the inductive step, let the claim be true for m. First consider the case that $m = p + 2r$. Then $b_m \ge c_m + r$ by the inductive hypothesis. Hence $[\min(I), c_m + r - 1] \cap (B_{0,s_m} \cup B_{1,s_m}) = \emptyset$ by (23) and (24). Since b_m is the only number from I_n enumerated into B_0 or B_1 at any stage s with $s_m + 1 \le s \le s_{m+1}$, it still holds that $[\min(I), c_m + r - 1] \cap (B_{0,s_{m+1}} \cup B_{1,s_{m+1}}) = \emptyset$. Moreover, still by the inductive hypothesis, $c_m + r \notin (I_n \cap B_{1,s_m+1}) = (I_n \cap B_{1,s_{m+1}})$. Since $c_m + r \le c_m + 2r = c_m + (m - p) = c_{m-(m-p)} = c_p$, by equation (32) we know that $[c_{m+1} + 1, c_m + r] = [c_m, c_m + r]$ is $\{\mathcal{J}_{e'} : \alpha(e') = 0\}$-safe at stage s_{m+1}. Hence $y = c_m + r$ satisfies the conditions from equation (24) at stage $s_{m+1} + 1$, whence $b_{m+1} \ge c_m + r$ and $b_{m+1} - c_{m+1} \ge c_m + r - c_{m+1} = c_m + r - (c_m - 1) = r + 1 = r'$, where $m + 1 = p + 2r' - 1$.

Similarly, if $m = p + 2r - 1$, then again $b_m \ge c_m + r$ by the inductive hypothesis and hence $[\min(I), c_m + r - 1] \cap (B_{0,s_m} \cup B_{1,s_m}) = \emptyset$ by (23) and (24). Since b_m is the only number from I_n enumerated into B_0 or B_1 at any stage s with $s_m + 1 \le s \le s_{m+1}$, it still holds that $[\min(I), c_m + r - 1] \cap (B_{0,s_{m+1}} \cup B_{1,s_{m+1}}) = \emptyset$. Since $c_m + r - 1 \le c_m + 2r - 1 = c_m + (m - p) = c_{m-(m-p)} = c_p$, by (32) we know that $[c_{m+1} + 1, c_m + r - 1] = [c_m, c_m + r - 1]$ is $\{\mathcal{J}_{e'} : \alpha(e') = 0\}$-safe at stage s_{m+1}. Hence $y = c_m + r - 1$ satisfies the conditions from (24) at stage $s_{m+1} + 1$, whence $b_{m+1} \ge c_m + r - 1$ and $b_{m+1} - c_{m+1} \ge c_m + r - 1 - c_{m+1} = c_m + r - 1 - (c_m - 1) = r$, where $m + 1 = p + 2r$. Also, since $c_{m+1} + r = c_m + r - 1 \notin B_{0,s_{m+1}} \cup B_{1,s_{m+1}}$, if $b_{m+1} \ne c_{m+1} + r$, then $c_{m+1} + r \notin B_{1,s_{m+1}+1}$; and if $b_{m+1} = c_{m+1} + r$, then b_{m+1} is enumerated only into B_0 at stage $s_{m+1} + 1$, hence again $c_{m+1} + r \notin B_{1,s_{m+1}+1}$.

But now $b_{p+2n} - c_{p+2n} = n$, whence (b) applies to α at stage $s_{p+2n}+1$ and $V_{\alpha,s_{p+2n}+1} \cap I_n \ne \emptyset$, a contradiction. This completes the proof that the inequality in equation (31) is strict, that is

$$\sum_{j=1}^{k} |\{x \leq c_{p+2n} : x \notin W_{\pi(e_j),s_{p+2n}}\}| \leq \sum_{j=1}^{k} |\{x \leq c_p : x \notin W_{\pi(e_j),s_p}\}| - 2n \cdot k - 1.$$

Let $p_q = 1 + q \cdot 2n$ for $0 \leq q < (\min(I_n) + n) \cdot (k+1)$. Then $1 \leq p_q$ and

$$\begin{aligned} p_q + 2n &\leq 1 + (\min(I_n) + n) \cdot (k+1) \cdot 2n \\ &= |[\min(I_n) + n, \min(I_n) + n + (\min(I_n) + n) \cdot (k+1) \cdot 2n]| \\ &= |J_n|. \end{aligned}$$

Hence the above inequality applies to p_q and since $p_{q+1} = p_q + 2n$, by induction on q we get

$$\sum_{j=1}^{k} |\{x \leq c_{p_{q+1}} : x \notin W_{\pi(e_j),s_{p_{q+1}}}\}| \leq \sum_{j=1}^{k} |\{x \leq c_{p_0} : x \notin W_{\pi(e_j),s_{p_0}}\}| - (q+1) \cdot (2n \cdot k) - (q+1).$$

In particular, for $q = (\min(I_n) + n) \cdot (k+1) - 1$ this amounts to

$$\begin{aligned} &\sum_{j=1}^{k} |\{x \leq c_{p_{q+1}} : x \notin W_{\pi(e_j),s_{p_{q+1}}}\}| \\ &\leq \sum_{j=1}^{k} |\{x \leq c_{p_0} : x \notin W_{\pi(e_j),s_{p_0}}\}| - (q+1) \cdot 2n \cdot k - (\min(I_n) + n) \cdot (k+1) \\ &< k \cdot (c_{p_0} + 1) - k \cdot (q+1) \cdot 2n - k \cdot (\min(I_n) + n + 1) \quad [\text{since } k < \min(I_n)] \\ &= k \cdot (c_1 + 1 - (q+1) \cdot 2n - (\min(I_n) + n + 1)). \end{aligned}$$

Consequently, there is some $j \in \{1, \ldots, k\}$ such that

$$\begin{aligned} |\{x \leq c_{p_{q+1}} : x \notin W_{\pi(e_j),s_{p_{q+1}}}\}| &< c_1 + 1 - (q+1) \cdot 2n - (\min(I_n) + n + 1) \\ &= c_1 - (\min(I_n) + n) \cdot (k+1) \cdot 2n - (\min(I_n) + n) \\ &= c_1 - (\min(I_n) + n + (\min(I_n) + n) \cdot (k+1) \cdot 2n) \\ &= \max(I_n) - \max(I_n) \\ &= 0, \end{aligned}$$

a contradiction. This proves the lemma. ☐

Lemma 3.32. *Every simplicity requirement $\mathcal{S}_e$ is satisfied and B_0 and B_1 are simple.*

Proof. Let $\alpha = \mathrm{TP} \upharpoonright 2e + 1$ be the unique node of length $2e + 1$ on the true path. By the True Path Lemma and by Lemma 3.30 there is a stage s_0 such that $\alpha \leq \delta_s$ for all $s \geq s_0$ and no node $\alpha' < \alpha$ is active at any stage $s > s_0$. Then no node $\alpha' <_L \alpha$ gets a new interval assigned at any stage $s+1 > s_0$ (because $\alpha' \not\sqsubseteq \delta_s$); moreover, a node $\alpha' \sqsubset \alpha$ has a diagonalisation witness $v \in I_n$ assigned at stage s for some $s > s_0$ and some n if and only if it has the diagonalisation witness v assigned at stage $s_0 + 1$.

Hence, if we let y_0 be some number that is larger than $\max(I_n)$ for every interval I_n assigned to any node up to stage s_0, then for any $y \geq y_0$ and any $s > s_0$, (2.3) is true. We may assume that $y_0 = \min(I_m)$ for some interval I_m.

If W_e is finite, then $\mathcal{S}_e$ is trivially satisfied. For a contradiction assume that W_e is infinite and $\mathcal{S}_e$ not satisfied. Let $x \in W_e$ be such that $x \geq y_0$ and $x \notin I_n$ for $n \leq e+1$. Further, let $s_1 > s_0$ be a stage such that $x \in W_{e_1,s_1}$. Since $\alpha \sqsubset \mathrm{TP}$, by the definition of the true path and by the True Path Lemma there is an α-stage $s_2 \geq s_1$ such that $l_{s_2}(\mathcal{J}_{e'}) > x$ for all $e' < |\alpha|$ with $\alpha(e') = 0$. Then (2.1) and (2.2) are true for any α-stage $s \geq s_2$.

If $x \in I_n$ for some γ-interval I_n with $\gamma \sqsubseteq \alpha$, then $\gamma \sqsubset \mathrm{TP}$, and by Lemma 3.31, there must be a least α-stage $s_3 \geq s_2$ such that I_n is defined but not in preparation at stage s_3. Then (b) must have applied to I_n at some stage $s \leq s_3$ and $V_{\alpha,s_3} \cap I_n \neq \emptyset$, or γ was initialised since the assignment of I_n and I_n is not assigned to γ at any stage $s \geq s_3$. In both cases, the first item of (2.5) holds for $s = s_3 + 1$. If $x \notin I_n$ for any γ-interval with $\gamma \sqsubseteq \alpha$, let $s_3 = s_2$; then the first item of (2.5) trivially holds at stage $s = s_3$.

We claim that α requires attention due to Case 2 at stage s_3+1, and hence, by the choice of s_0 is active and enumerates x into B_0 and B_1 at stage s_3+1. Then $\mathcal{S}_e$ is satisfied, contradicting the assumption.

First assume that $x \notin J_n$ for any interval J_n defined up to stage s_3. Then (2.5) is true for $y = x$ and $s = s_3$ and it remains to show that (2.4) is also true for these choices of s and y. Since $\mathcal{S}_e$ is not satisfied by our assumption and since $y = x \in W_e$, it follows that $y \notin B_{0,s_3} \cap B_{1,s_3}$. On the other hand, $y = x \notin C_s$ follows from the fact that the only way for a number outside of any interval J_m which is defined until stage s_3 to be enumerated into C until stage s_3 is to be enumerated by some node of odd length being active; but in this case the number is simultaneously enumerated into B_0 and B_1, contrary to what we just showed. Note that $y_0 \leq y \leq x$. Hence (2.1)-(2.5) are all satisified for $y = x$ and $s = s_3$, proving the claim.

Next consider the case that $x \in J_n$ for some interval J_n (with $n > e+1$) defined up to stage s_3. We show that there is a number $y \in I_n - J_n$ with $y \notin C$. Let γ be the node I_n is assigned to. As we just remarked, a number $z \in I_n - J_n$ can be enumerated into C at some stage $s+1$ only by some node α' of odd length $|\alpha'| = 2e'+1$ being active at stage $s+1$. Since for each e' only one node of length $2e'+1$ is active, it is active at most once and enumerates only a single number from I_n into C, there are at most $n-1$ numbers from I_n enumerated into C by nodes α' with $|\alpha'| = 2e'+1$ and $e' < n-1$. On the other hand, no node α' with $|\alpha'| = 2e'+1$ and $e' \geq n-1$ enumerates any $y' \in I_n$ into C by being active at any stage $s+1$: Otherwise I_n would have to be defined until stage s, because for intervals I_m defined at or after stage $s+1$ it would hold that $\min(I_m) > y'$; furthermore, α' would enumerate some x' into B_0 or B_1 at stage $s+1$. Now by the second and third item of (2.5) it would follow that $x' \in J_n \subseteq I_n$ or

$x' = y' \in I_n$, and by the first item of (2.5) it would hold that $e' < n-1$, contradicting the choice of e'.

This shows that at most $n-1$ numbers from $I_n - J_n$ are enumerated into C by nodes of odd length being active. Hence $|\{z \in I_n - J_n : z \notin C\}| \geq |I_n - J_n| - (n-1) = |[\min(I_n), \min(I_n) + n - 1]| - (n-1) = 1$.

Let $y \in I_n - J_n$ be such that $y \notin C$. Note that $y_0 \leq y$, because $y_0 = \min(I_m)$, $y_0 \leq x$ and hence $y_0 \leq \min(I_n) \leq y$; hence (2.1), (2.2) and (2.3) are true for $s = s_3$. Also note that $y \leq x$. As above, $y \notin B_0 \cup B_1$, because if s were minimal with $y \in B_{0,s} \cup B_{1,s}$, then y would be enumerated by some node of odd length being active at stage s, and by (2.5) y would be enumerated into C_s as well. Hence (2.4) is true for y at every stage s and (2.5) is true for $s = s_3$, proving the claim in this case.

To show that B_0 and B_1 are simple it now suffices to show that $\mathbb{N} - B_i$ is infinite for $i = 0$ and $i = 1$. To see this, note that for every m such that I_m gets defined during the construction there exists a number $y_m \in I_m$ such that $y_m \notin B_0 \cup B_1$. The proof is as shown above for $m = n$. Since there are infinitely many intervals defined, the claim follows. $\square$

Lemma 3.33. *Every nonsimplicity requirement $\mathcal{N}_{\langle e_0, e_1, e_2 \rangle}$ ($e_0, e_1, e_2 \in \mathbb{N}$) is satisfied.*

Proof. We only need to consider the case that $W_{e_2} = \hat{\Phi}_{e_0}^C$ and $C = \hat{\Phi}_{e_1}^{W_{e_2}}$, because otherwise $\mathcal{N}_{\langle e_0, e_1, e_2 \rangle}$ is trivially satisfied.

Let $\alpha = \mathrm{TP} \restriction 2e$ be the unique node of length $2e$ on the true path, where $e = \langle e_0, e_1, e_2 \rangle$. By Lemma 3.30 there is a least stage s_0 such that α is never initialised at any stage $s > s_0$; and if α is active at any stage $s > s_0$, then $\mathcal{N}_{\langle e_0, e_1, e_2 \rangle}$ is satisfied.

It remains to consider the case that α is not active at any stage $s > s_0$. Then α has no diagonalisation witness at any stage $s > s_0$. By Lemma 3.30 and by the True Path Lemma, there are infinitely many stages s such that $\alpha \sqsubset \delta_s$ but no $\alpha' \sqsubseteq \alpha$ is active at stage $s+1$. For each such s (a) applies to α at stage $s+1$ unless α is already preparing some interval I_n at the end of stage s. But since by Lemma 3.31 each interval I_n assigned to α is being prepared at only finitely many stages, (a) must apply to α infinitely often and α is preparing infinitely many intervals during the construction. Moreover, after stage s_0, since α is not initialised or active, α only stops preparing some interval I_n if (b) applies to α and I_n, whence $V_\alpha \cap I_n \neq \emptyset$ for $V_\alpha = \bigcup_{s \geq s_0} V_{\alpha,s}$. It follows by Lemma 3.31 again that $|V_\alpha| = \infty$.

Since V_α is effectively enumerated during the construction, V_α is computably enumerable. Hence to show that $\mathcal{N}_e$ is satisfied, i.e. W_{e_2} is not simple, it suffices to show that $W_{e_2} \cap V_\alpha = \emptyset$.

For a contradiction assume that there is some $v \in W_{e_2} \cap V_\alpha$ and let $s_1 > s_0$ be such that $v \in W_{e_2,s_1} \cap V_{\alpha,s_1}$, $l_s(\mathcal{N}_e) > v$ for all $s \geq s_1$ and $v < \min(I_m)$ for any interval I_m some node $\alpha' \sqsubset \alpha$ is preparing at the end of any stage $s \geq s_1$. Such a stage exists by the assumption that $W_{e_2} = \hat{\Phi}_{e_0}^C$ and $C = \hat{\Phi}_{e_1}^{W_{e_2}}$ and by Lemma 3.31. It holds that (1.1), (1.3) and (1.4) are true for every $s \geq s_1$.

Note that $v \in J_n$ for some α-interval I_n. Let $t < s_1$ be the stage such that $v \in V_{\alpha,t+1} - V_{\alpha,t}$. By the definition of V_α it holds that $t+1 \geq s_0$. By (b) I_n is already defined at stage t and $v \notin C_{t+1}$. Then v is not enumerated into C by any node α' of odd length, because such α' does not enumerate numbers from defined intervals J_m into C. Furthermore, v is not enumerated into C by any node $\alpha' \neq \alpha$ of even length, because these nodes do not enumerate numbers from defined α-intervals. Since α is not active after stage s_0, v is not enumerated into C by α being active. Finally, (b) and (c) do not apply to α and I_n after stage $t+1$ (since α stops preparing I_n at stage $t+1$), hence v is not enumerated into C by α due to (b) or (c). Altogether this implies that $v \notin C$, hence (1.2) is true for every stage s.

At stage $t+1$, since $b \geq c+n = v+n$ for b and c defined according to equations (23) and (24) (with t in place of s), it follows that $[v, v+n-1] \cap (B_{0,t} \cup B_{1,t}) = \emptyset$. No number from I_n is enumerated into B_0 or B_1 due to (a), (b) or (c) or due to any node of even length being active after stage t. Since $x \in J_n$ for every $x \in [v, v+n]$, by (2.5) only nodes α' of odd length $|\alpha'| = 2e'+1$ with $e' < n-1$ can enumerate any number from $[v, v+n]$ into B_0 or B_1. Moreover, for every e' there is at most one node of length $2e'+1$ ever active and this node enumerates at most one number from J_n into $B_0 \cup B_1$. Hence for each $e' < n-1$ there is at most one number from $[v, v+n-1]$ enumerated into B_0 or B_1 during the construction. In particular, if $s_2 > s_1$ is an α-stage, then there is some $d \in [v, v+n-1]$ such that $d \notin B_{0,s_2} \cup B_{1,s_2}$. Then (1.5) and (1.6) are satisfied for s_2 in place of s. Since $[v+1, v+n]$ was $\{\mathfrak{J}_{e'} : \alpha(e') = 0\}$-safe at stage t and $s_2 > s_1 > t$, the same is true at stage s_2, so (1.7) is satisfied. Hence α requires attention and is active at stage s_2+1, a contradiction. $\square$

Lemma 3.34. *Every join requirement $\mathfrak{J}_{\langle e_0,e_1,e_2\rangle}$ ($e_0, e_1, e_2 \in \mathbb{N}$) is satisfied.*

Proof. If the premise of $\mathfrak{J}_e$, where $e = \langle e_0, e_1, e_2\rangle$, is false, then $\mathfrak{J}_e$ is trivially true. Hence we may assume that $B_0 = \hat{\Phi}_{e_0}^{W_{e_2}}$ and $B_1 = \hat{\Phi}_{e_1}^{W_{e_2}}$. Let $\beta \sqsubset \mathrm{TP}$ be the unique node of length e on the true path. By the definition of the true path, $\beta 0 \sqsubset \mathrm{TP}$, too.

By Lemma 3.30 there is a stage s_0 such that $\beta 0$ is not initialised at any stage $s \geq s_0$. To compute $C(x)$ with oracle $W_{e_2} \restriction x+1$ for some x, compute a $\beta 0$-stage $s_1 > \max\{s_0, x\}$ such that $l_{s_1}(\mathfrak{J}_e) > x$, $l_{s_1}(\mathfrak{J}_e) > \max(I_n)$ if $x \in I_n$, no $\alpha \sqsubseteq \beta 0$ is preparing any interval I_n with $\max(I_n) < x$ at stage s_1, and $W_{e_2,s_1} \restriction x+1 = W_{e_2} \restriction x+1$. Such a stage exists by the True Path Lemma, because the hypothesis of $\mathfrak{J}_e$ is true, and by Lemma 3.31. We claim that $C_{s_1}(x) = C(x)$.

For a contradiction assume that $x \in C_{s+1} - C_s$ for some $s \geq s_1$.

If x is enumerated into C by some node α being active, then $\alpha \geq \beta 0$, because otherwise $\beta 0$ were initialised at stage $s+1$, contradicting the choice of $s_1 \geq s_0$.

In fact, if $|\alpha|$ is even, then $\alpha \sqsupseteq \beta 0$, because nodes to the right of $\beta 0$ are initialised at stage s_1+1 and only assigned intervals I_n with $\min(I_n) \geq s_1 > x$ after stage s_1, and because a node α of even length only enumerates some number x into C at stage $s+1$ by being active if x is in an interval I_n assigned to α at stage s. Since $\alpha \sqsupseteq \beta 0$, we know that $\beta 0 \sqsubseteq \delta_s$, hence

$l_s(\mathcal{J}_e) > l_{s_1}(\mathcal{J}_e) > \max(I_n)$. Since some $d \in [x, x+n] \subseteq I_n$ is enumerated into $B_{i,s+1} - B_{i,s}$ for $i = 0$ and $i = 1$, it follows that

$$\hat{\Phi}_{e_i}^{W_{e_2}}(d) = B_i(d) = 1 \neq 0 = B_{i,s}(d) = \hat{\Phi}_{e_i,s}^{W_{e_2,s}}(d). \tag{33}$$

Since $\hat{\Phi}$ is an ibT-functional, this implies that $W_{e_2,s} \upharpoonright d+1 \neq W_{e_2} \upharpoonright d+1$. But x was enumerated into V_α at some stage $t+1 \leq s$ according to (b), whence $[x+1, x+n]$ was $\{\mathcal{J}_{e'} : \alpha(e') = 0\}$-safe at stage t. In particular, $[x+1, d] \subseteq [x+1, x+n] \subseteq W_{e_2,t} \subseteq W_{e_2,s}$. Consequently $W_{e_2,s} \upharpoonright x+1 \neq W_{e_2} \upharpoonright x+1$, contradicting the choice of s_1.

If $|\alpha|$ is odd and α enumerates x into C at stage $s+1$, then α enumerates some $y \leq x$ into $B_{i,s+1} - B_{i,s}$ for $i = 0$ or $i = 1$. Then

$$\hat{\Phi}_{e_i}^{W_{e_2}}(y) = B_i(y) = 1 \neq 0 = B_{i,s_1}(y) = \hat{\Phi}_{e_i,s_1}^{W_{e_2,s_1}}(y).$$

As above it follows that $W_{e_2,s_1} \upharpoonright x+1 \neq W_{e_2} \upharpoonright x+1$, contradicting the choice of s_1.

Finally consider the case that x is enumerated into C due to (b). Then $x \in I_n$ for some α-interval I_n. Again, $\alpha \sqsupseteq \beta 0$, because nodes to the right of $\beta 0$ are initialised at stage $s_1 + 1$ and only prepare intervals with numbers greater than x after stage s_1, nodes below $\beta 0$ are not preparing any intervals containing x after stage s_1 and nodes to the left of $\beta 0$ are not on δ_s. Since $l_s(\mathcal{J}_e) > \max(I_n)$ and since some $b \in I_n$ is enumerated into $B_{i,s+1} - B_{i,s}$, where $[x+1, b]$ is $\{\mathcal{J}_{e'} : \alpha(e') = 0\}$-safe at stage s, we conclude that equation (33) holds for b in place of d and that $W_{e_2,s} \upharpoonright b+1 \neq W_{e_2} \upharpoonright b+1$. But since $[x+1, b] \subseteq W_{e_2}$, it must hold that $W_{e_2,s} \upharpoonright x+1 \neq W_{e_2} \upharpoonright x+1$, contradicting the choice of s_1 again. □

□

Chapter 4

Lattice embeddings into $\mathcal{R}_{\mathrm{ibT}}$ and $\mathcal{R}_{\mathrm{cl}}$

While in Chapter 3 we considered existence theorems for joins and meets separately from each other, bringing together both, the present chapter is devoted to the study of lattice embeddings. We give an overview over the known results concerning such embeddings into $\mathcal{R}_{\mathrm{ibT}}$ and $\mathcal{R}_{\mathrm{cl}}$ and prove some new embedding theorems.

4.1 Lattice embeddings

While the common degree structures $\mathcal{R}_r$ are not lattices and usually not even lower semi-lattices, there can be *subsets* of $\mathbf{R}_r$ which are closed under joins and meets and are lattices. The question whether a certain lattice $(P, \leq)$ can be found in $\mathcal{R}_r$ can be formalised by the concept of *lattice embeddings.*

Definition 4.1. Let $\mathcal{P} = (P, \leq_{\mathcal{P}})$ be a partial order. An *embedding* of $\mathcal{P}$ into a degree structure $\mathcal{R}_r$ ($r \in \{\mathrm{ibT}, \mathrm{cl}, \mathrm{wtt}, \mathrm{T}\}$) is a function $h : P \to \mathbf{R}_r$ such that h is one-one and for all $a, b \in P$ it holds that $a \leq_{\mathcal{P}} b$ if and only if $h(a) \leq h(b)$.

An embedding h of $\mathcal{P}$ into $\mathcal{R}_r$ *preserves joins and meets* if for all $a, b, c \in P$ it holds that

$$a \vee b = c \Rightarrow h(a) \vee h(b) = h(c)$$

and

$$a \wedge b = c \Rightarrow h(a) \wedge h(b) = h(c).$$

If $\mathcal{P}$ is a lattice, then an embedding h of $\mathcal{P}$ which preserves joins and meets is called a *lattice embedding.*

If P has a least element a (with respect to $\leq_{\mathcal{P}}$), then an embedding h of $\mathcal{P}$ into $\mathcal{R}_r$ *preserves the least element* if $h(a) = \mathbf{0}$.

$\mathcal{P}$ is called *embeddable into* $\mathcal{R}_r$ (preserving joins and meets or the least element, respectively) if there exist an embedding of $\mathcal{P}$ into $\mathcal{R}_r$ (preserving joins and meets or the least element).

Remark 4.2. *Let $\mathcal{L}'$ be the language consisting of the binary relation symbol $\leq$ and the ternary relation symbols $\vee$ and $\wedge$. If $\mathcal{P}'$ and $\mathcal{R}_r'$ are the expansions of $\mathcal{P}$ and $\mathcal{R}_r$, respectively, to $\mathcal{L}'$-structures, then an embedding of $\mathcal{P}$ into $\mathcal{R}_r$ preserving joins and meets is just an embedding of $\mathcal{P}'$ into $\mathcal{R}_r'$ in the usual sense of mathematical logic.*

Similarly, an embedding preserving the least element is just an embedding in the usual sense if we consider the expansions of the structures to $\mathcal{L}_0$-structures, where $\mathcal{L}_0$ is the language of orders with an additional constant symbol 0.

4.2 Embedding linear orders

The most simple lattices are linear orders $\mathcal{P} = (P, \leq_{\mathcal{P}})$, in which joins and meets become trivial, because $a \vee b = b$ and $a \wedge b = a$ for $a \leq_{\mathcal{P}} b$.

Lemma 4.3. *Every finite linear order is embeddable into $\mathcal{R}_r$ for $r \in \{\mathrm{T}, \mathrm{wtt}, \mathrm{cl}, \mathrm{ibT}\}$ preserving the least element.*

Proof. Let $\mathcal{P} = (\{a_0, \dots, a_n\}, \leq_{\mathcal{P}})$ with $a_0 < \dots < a_n$ be a linear order. Let A_n be a c.e. noncomputable set. By downward induction, using Sack's Splitting Theorem [Sack 63] define c.e. noncomputable sets $A_{n-1}, \dots, A_1$ and $B_{n-1}, \dots, B_1$ such that $A_{k-1} \cap B_{k-1} = \emptyset$, $A_k = A_{k-1} \cup B_{k-1}$ and $A_k \not\leq_{\mathrm{T}} A_{k-1}$ (hence $A_k \not\leq_r A_{k-1}$) for $2 \leq k \leq n-1$. Let $A_0 = \emptyset$. By Lemma 3.4 and since ibT-reducibility implies r-reducibility, it holds that $A_0 <_r \dots <_r A_n$. Hence $h(a_k) = deg_r(A_k)$ defines an embedding of $\mathcal{P}$ into $\mathcal{R}_r$ which preserves the least element. □

It will follow from Theorem 4.8 that not only every finite but indeed every *countable* linear order is embeddable into $\mathcal{R}_r$ for $r \in \{\mathrm{T}, \mathrm{wtt}, \mathrm{cl}, \mathrm{ibT}\}$.

4.3 Embedding distributive lattices

Next we consider the so-called *diamond*.

Definition 4.4. The *diamond (lattice)* is the finite partial order $\mathcal{D} = (\{a, b_0, b_1, c\}, \leq_{\mathcal{D}})$ such that $a <_{\mathcal{D}} b_i <_{\mathcal{D}} c$ for $i \in \{0, 1\}$, $b_0 \wedge b_1 = a$ and $b_0 \vee b_1 = c$.

It is obvious that the diamond is indeed a lattice, that up to isomorphism it is the only lattice with exactly four elements in its domain which is not a linear order, and that every lattice with less than four elements is already a linear order.

Lemma 4.5. *There is a lattice embedding of the diamond into $\mathcal{R}_r$ preserving the least element, for $r \in \{\mathrm{T}, \mathrm{wtt}, \mathrm{cl}, \mathrm{ibT}\}$.*

Proof. By the minimal pair strategy which we described in Theorem 3.9, Lachlan [Lach 66] and independently Yates [Yate 66] constructed c.e. sets $B_0 \subseteq 2\mathbb{N}$ and $B_1 \subseteq 2\mathbb{N}+1$ such that $(deg_{\mathrm{T}}(B_0), deg_{\mathrm{T}}(B_1))$ is a minimal pair in $\mathcal{R}_{\mathrm{T}}$. Since r-reducibility implies Turing reducibility, it follows that the pair $(deg_r(B_0), deg_r(B_1))$ is a minimal pair in $\mathcal{R}_r$. Moreover, by disjointness of B_0 and B_1 and by Lemma 3.4,

$$deg_r(B_0) \vee deg_r(B_1) = deg_r(B_0 \cup B_1) \tag{34}$$

for $r \in \{\mathrm{ibT}, \mathrm{cl}\}$. For $r \in \{\mathrm{wtt}, \mathrm{T}\}$, equation (34) directly follows from the fact that $B_0 \cup B_1 \equiv_r B_0 \oplus B_1$ and $deg_r(B_0 \oplus B_1) = deg_r(B_0) \vee deg_r(B_1)$.

Hence $h(a) = \mathbf{0}$, $h(b_0) = deg_r(B_0)$, $h(b_1) = deg_r(B_1)$ and $h(c) = deg_r(B_0 \cup B_1)$ defines an embedding of $\mathcal{D}$ into $\mathcal{R}_r$ preserving the least element. □

We now turn to a whole class of lattices.

Definition 4.6. Let $\mathcal{P} = (P, \leq_{\mathcal{P}})$ be a lattice.

$\mathcal{P}$ is called *distributive* if for all a, b, $c \in P$ it holds that

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$

and

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c).$$

Otherwise $\mathcal{P}$ is called *nondistributive.*

$\mathcal{P}$ is called a *Boolean algebra* if $\mathcal{P}$ is distributive, P has a least element 0 and greatest element 1 and for every $a \in P$ there exists some $\bar{a} \in P$ such that

$$a \vee \bar{a} = 1 \text{ and } a \wedge \bar{a} = 0.$$

$\mathcal{P}$ is called *atomless* if P has a least element 0 and the set $P - \{0\}$ does not have any minimal elements.

Lemma 4.7 (Folklore)**.** *Up to isomorphism, there exists a unique countably infinite atomless Boolean algebra $\mathcal{B} = (B, \leq_{\mathcal{B}})$. Moreover, for every countable distributive lattice $\mathcal{L}$ there is a lattice embedding of $\mathcal{L}$ into $\mathcal{B}$. If $\mathcal{L}$ has a least element, then the lattice embedding can be chosen to preserve the least element.*

Proof. Let the elements of B be all finite unions of intervals $[a, b) \subseteq [0, 1)$ with $a, b \in \mathbb{Q}$, and let $\leq_{\mathcal{B}}$ be the subset relation restricted to these sets. It is straightforward to verify that this defines a countably infinite atomless Boolean algebra $\mathcal{B}$, where the join of two sets in B is just their union and the meet of two sets in B is their intersection.

The fact that $\mathcal{B}$ is unique with the above conditions is shown by means of a back-and-forth construction. For details and for the lattice embedding of countable distributive lattices into $\mathcal{B}$, see for example [Verm 10]. □

Ambos-Spies observed that the proof of the following theorem, which was given long before the notions of cl- and ibT-reducibility came up, holds for these reducibilities, too.

Theorem 4.8 (Lachlan, Lerman, Thomason [Thom 71])**.** *There is a lattice embedding of the Boolean algebra $\mathcal{B}$ into $\mathcal{R}_r$ preserving the least element, for $r \in \{\mathrm{T}, \mathrm{wtt}, \mathrm{cl}, \mathrm{ibT}\}$.*

Proof (idea). By a computable bijection $\iota : \mathbb{Q} \cap [0, 1) \to \mathbb{N}$ we can define a countably infinite atomess Boolean algebra $\mathcal{B}' = (B', \leq_{\mathcal{B}'})$, where the elements of B' are the sets of the form $V_M = \{\iota(q) : q \in M\}$ with $M \in B$ and $\leq_{\mathcal{B}'}$ is the subset relation. By Lemma 4.7 $\mathcal{B}'$ is isomorphic to $\mathcal{B}$ (the isomorphism is given by $M \mapsto V_M$) and it suffices to embed $\mathcal{B}'$ into $\mathcal{R}_r$.

Given a uniformly computable sequence $(V_n)_{n\in\mathbb{N}}$ of sets such that $\{V_n : n \in \mathbb{N}\} = B'$, Thomason constructs c.e. sets A_i for $i \in \mathbb{N}$ and defines $A_V = \{\langle i, x\rangle : x \in A_i \text{ and } i \in V\}$. Then he sets $h(V) = deg_r(A_V)$ for $V \in B'$.

If $V \subseteq W$, then $\langle i, x\rangle \in A_V$ if and only if $i \in V$ and $\langle i, x\rangle \in A_W$; hence $A_V \leq_{\text{ibT}} A_W$ and a fortiori $A_V \leq_r A_W$, that is $h(V) \leq h(W)$. In particular, $h(V_1), h(V_2) \leq h(V_1 \cup V_2)$ for $V_1, V_2 \in B'$. Moreover, $A_{V_1\cup V_2} = A_{V_1} \cup A_{V_2}$. It follows that $A_{V_1\cup V_2} \leq_{\text{ibT}} W$ whenever $A_{V_1}, A_{V_2} \leq_{\text{ibT}} W$. Hence $deg_{\text{ibT}}(A_{V_1}) \vee deg_{\text{ibT}}(A_{V_2}) = deg_{\text{ibT}}(A_{V_1\cup V_2})$. By Lemmas 3.2 and 3.19 it follows that

$$h(V_1) \vee h(V_2) = deg_r(A_{V_1}) \vee deg_r(A_{V_2}) = deg_r(A_{V_1\cup V_2}) = h(V_1 \cup V_2).$$

It also follows that $h(V_1 \cap V_2) \leq h(V_1), h(V_2)$. To prove that h is a lattice embedding, for $r = \mathrm{T}$ it now suffices to satisfy the requirements

$$\mathcal{D}_{i,j} : \mathbb{N} - A_i \neq W_j$$

and

$$\mathcal{M}_{n_0,n_1,e_0,e_1,e_2} : \Phi_{e_0}^{A_{V_{n_0}}} = \Phi_{e_1}^{A_{V_{n_1}}} = W_{e_2} \Rightarrow W_{e_2} \leq_{\mathrm{T}} A_{V_{n_1}\cap V_{n_2}}.$$

For $r \in \{\mathrm{wtt}, \mathrm{cl}, \mathrm{ibT}\}$, we additonally need to ensure that if the premise of $\mathcal{N}_{n_0,n_1,e_0,e_1,e_2}$ is true, then the use of the oracle computation of $W_{e_2}(x)$ with oracle $A_{V_{n_1}\cap V_{n_2}}$ is bounded by $\max(\{u_{e_0}^{A_{V_{n_0}}}(x), u_{e_1}^{A_{V_{n_1}}}(x)\})$.

The sets A_i satisfying these requirements can be enumerated via a standard tree construction, where a requirement $\mathcal{D}_{i,j}$ is satisfied by chosing a diagonalisation candiate x, waiting until x is enumerated into W_j (if ever) and then enumerating x into A_i. A requirement $\mathcal{M}_{n_0,n_1,e_0,e_1,e_2}$ is satisfied as in the proof of Theorem 3.9. Of course, since we do not construct $A_{V_{n_0}}$ and $A_{V_{n_1}}$ directly, it can now happen that we wish to enumerate a number x into A_i at some stage $s + 1$, where $i \in V_{n_0} \cap V_{n_1}$. In this case $\langle i, x\rangle$ enters both $A_{V_{n_0}}$ and $A_{V_{n_1}}$ (which was not possible in the construction for Theorem 3.9), but also $A_{V_{n_0} \cap V_{n_1}}$. This is why we obtain merely $W_{e_2} \leq_r A_{V_{n_0} \cap V_{n_1}}$ instead of W_{e_2} being computable. $\square$

Corollary 4.9. *For $r \in \{\mathrm{T}, \mathrm{wtt}, \mathrm{cl}, \mathrm{ibT}\}$, every countable distributive lattice is embeddable into $\mathcal{R}_r$ preserving joins and meets and the least element.*

Proof. This is a direct consequence of Lemma 4.7 and Theorem 4.8. $\square$

4.4 Embedding nondistributive lattices

Turning to nondistributive lattices, the situation becomes more complicated. Since every lattice with less than five elements is a linear order or (up to isomorphism) the diamond, which is distributive, the smallest nondistributive lattices can have five elements in their domain. Indeed, there are exactly two such nondistributive lattices, called the $\mathcal{N}_5$ and the $\mathcal{M}_3$[1].

Definition 4.10. The $\mathcal{N}_5$ is the finite partial ordering $\mathcal{N}_5 = (\{a, b_0, b_1, c, d\}, \leq_{\mathcal{N}_5})$ such that

(N1) $a <_{\mathcal{N}_5} b_0 <_{\mathcal{N}_5} b_1 <_{\mathcal{N}_5} d$,

(N2) $a <_{\mathcal{N}_5} c <_{\mathcal{N}_5} d$,

(N3) $b_0 \vee c = b_1 \vee c = d$, and

(N4) $b_0 \wedge c = b_1 \wedge c = a$.

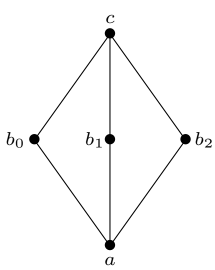
Definition 4.11. Let $n \geq 2$ with $n \in \mathbb{N}$ or $n = \omega$. The $\mathcal{M}_n$ is the partial ordering $\mathcal{M}_n = (\{a, c\} \cup \{b_i : 0 \leq i < n\}, \leq_{\mathcal{M}_n})$ such that

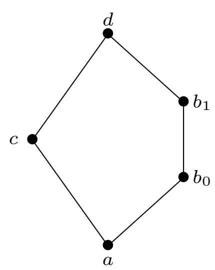
(M1) b_i and b_j are incomparable for $i, j < n$ with $i \neq j$,

(M2) $b_i \vee b_j = c$ for $i, j < n$ with $i \neq j$, and

(M3) $b_i \wedge b_j = a$ for $i, j < n$ with $i \neq j$.

It is easy to see that the $\mathcal{N}_5$ and the $\mathcal{M}_n$ are uniquely defined by these conditions and that they are lattices. Notice that the $\mathcal{M}_2$ is just the diamond lattice. The following diagrams illustrate what the $\mathcal{M}_3$ and the $\mathcal{N}_5$ look like.





[1]The notation found in the literature is not consistent here; elsewhere the $\mathcal{M}_3$ is called the $\mathcal{M}_5$[Birk 79] or the $1-3-1$ [Wein 88].

The $\mathcal{N}_5$ is nondistributive, because

$$b_1 \wedge (b_0 \vee c) = b_1 \wedge d = b_1 \neq b_0 = b_0 \vee a = (b_1 \wedge b_0) \vee (b_1 \wedge c).$$

The $\mathcal{M}_n$ is nondistributive for $n \geq 3$, because

$$b_0 \wedge (b_1 \vee b_2) = b_0 \wedge c = b_0 \neq a = a \vee a = (b_0 \wedge b_1) \vee (b_0 \wedge b_2).$$

The two lattices $\mathcal{N}_5$ and $\mathcal{M}_3$ are of particular interest to the study of lattice embeddings because they are not only the smallest nondistributive lattices with respect to the size of their domains, but also the only minimal elements of the class of all nondistributive lattices (modulo isomorphisms) ordered by lattice embeddability.

Theorem 4.12. *[Dede 00, Birk 79] A lattice $\mathcal{L}$ is nondistributive if and only if there is a lattice embedding of the $\mathcal{N}_5$ or the $\mathcal{M}_3$ into $\mathcal{L}$.*

It was shown by Lachlan [Lach 72] that there are lattice embeddings preserving the least element of both the $\mathcal{N}_5$ and the $\mathcal{M}_3$ into $\mathcal{R}_{\mathrm{T}}$. This led to the question whether *every* finite lattice could be embedded into $\mathcal{R}_{\mathrm{T}}$ by a lattice embedding. Lachlan and Soare [Lach 80] answered this question to the negative by proving that the 8-element-lattice $\mathcal{S}_8$, consisting of a copy of the diamond on top of a copy of the $\mathcal{M}_3$ (see Definition 4.54), cannot be embedded into $\mathcal{R}_{\mathrm{T}}$ preserving joins and meets.

While some sufficient conditions for finite lattices to be or be not lattice embeddable into $\mathcal{R}_{\mathrm{T}}$ have been found (see [Lemp 06],[Ambo 86] or [Ambo 89] for examples), a complete nontrivial characterisation of the embeddable finite lattices is not known up to date.

For $\mathcal{R}_{\mathrm{wtt}}$, the situation looks different.

Theorem 4.13. *[Stob 83] There are no lattice embeddings of the $\mathcal{M}_3$ or the $\mathcal{N}_5$ into $\mathcal{R}_{\mathrm{wtt}}$. Hence by Corollary 4.9 and Theorem 4.12 a countable lattice $\mathcal{L}$ can be embedded into $\mathcal{R}_{\mathrm{wtt}}$ via a lattice embedding if and only if $\mathcal{L}$ is distributive.*

4.5 Embedding the $\mathcal{N}_5$

Now we consider lattice embeddings of nondistributive lattices into $\mathcal{R}_{\mathrm{ibT}}$ and $\mathcal{R}_{\mathrm{cl}}$. For the $\mathcal{N}_5$ we get the same result as in $\mathcal{R}_{\mathrm{T}}$.

Lemma 4.14. *Let $\mathcal{P} = (P, \leq_{\mathcal{P}})$ be a partial order. Then a mapping $h : \{a, b_0, b_1, c, d\} \to P$ is a lattice embedding of the $\mathcal{N}_5$ into $\mathcal{P}$ if and only if*

(N1') $h(a) \leq_{\mathcal{P}} h(b_0) <_{\mathcal{P}} h(b_1) \leq_{\mathcal{P}} h(d)$

(N2') $h(a) \leq_{\mathcal{P}} h(c) \leq_{\mathcal{P}} h(d)$

(N3') $h(b_0) \vee h(c) = h(d)$

(N4') $h(b_1) \wedge h(c) = h(a)$.

Proof. Let h be a lattice embedding of the $\mathcal{N}_5$ into $\mathcal{P}$. Then by the definition of lattice embeddings and by (N1)-(N4) it is immediate that (N1')-(N4') are true.

For the converse, assume that (N1')-(N4') are true. Then since, by (N1') and (N3'), $h(b_0) \leq_{\mathcal{P}} h(b_1) \leq_{\mathcal{P}} h(d) = h(b_0) \vee h(c)$ it follows that $h(b_1) \vee h(c) = h(d)$. By a dual argument, $h(b_0) \wedge h(c) = h(a)$.

Moreover, $h(b_0) \not\leq_{\mathcal{P}} h(c)$, because otherwise $h(b_1) \leq_{\mathcal{P}} h(d) = h(b_0) \vee h(c) = h(c)$ and then $h(b_1) = h(b_1) \wedge h(c) = h(a) \leq_{\mathcal{P}} h(b_0)$, a contradiction to (N1'). It follows that $h(a) <_{\mathcal{P}} h(b_0)$ and $h(c) <_{\mathcal{P}} h(d)$.

By a dual argument, $h(c) \not\leq_{\mathcal{P}} h(b_1)$ and hence $h(b_1) <_{\mathcal{P}} h(d)$ and $h(a) <_{\mathcal{P}} h(c)$.

By $h(b_0) \not\leq_{\mathcal{P}} h(c)$, $h(c) \not\leq_{\mathcal{P}} h(b_1)$ and $h(b_0) \leq_{\mathcal{P}} h(b_1)$ it follows that $h(c)$ is incomparable to both $h(b_0)$ and $h(b_1)$.

This completes the proof that h is one-to-one and that (N1)-(N4) are preserved by h. Hence h is a lattice embedding of the $\mathcal{N}_5$ into $\mathcal{P}$. $\square$

Theorem 4.15. *(Ambos-Spies, Bodewig, Kräling, and Yu [Amboc]) There is a lattice embedding of the $\mathcal{N}_5$ into $\mathcal{R}_{\mathrm{ibT}}$ and $\mathcal{R}_{\mathrm{cl}}$ preserving the least element.*

Proof (idea). To prove the theorem, we need to construct c.e. sets B_0 B_1, C and D such that $B_0 \leq_{\mathrm{ibT}} B_1 \leq_{\mathrm{ibT}} D$ and $C \leq_{\mathrm{ibT}} D$ and the following requirements are satisfied for all $e = \langle e_0, e_1, e_2 \rangle \in \mathbb{N}$

$$\mathcal{D}_e : B_1 \neq \tilde{\Phi}_e^{B_0}$$

$$\mathcal{J}_e : \hat{\Phi}_{e_0}^{W_{e_2}} = B_0 \text{ and } \hat{\Phi}_{e_1}^{W_{e_2}} = C \Rightarrow D \leq_{\mathrm{ibT}} W_{e_2}$$

$$\mathcal{M}_e : \hat{\Phi}_{e_0}^{B_1} = \hat{\Phi}_{e_1}^{C} = W_{e_2} \Rightarrow W_{e_2} \text{ is computable.}$$

We claim that then, for $r \in \{\mathrm{ibT}, \mathrm{cl}\}$, by defining $h(a) = \mathbf{0}$, $h(b_0) = deg_r(B_0)$, $h(b_1) = deg_r(B_1)$, $h(c) = deg_r(C)$ and $h(d) = deg_r(D)$ we obtain a lattice embedding of $\mathcal{N}_5$ into $\mathcal{R}_r$ preserving the least element. To see this, it suffices to show that (N1')-(N4') from Lemma 4.14 are true.

Indeed, if $deg_{\mathrm{ibT}}(B_0) \leq \mathbf{x}$ and $deg_{\mathrm{ibT}}(C) \leq \mathbf{x}$ for some $\mathbf{x} \in \mathbf{R}_{\mathrm{ibT}}$, then choose a c.e. set $W_{e_2} \in \mathbf{x}$ and e_0 and e_1 such that $\hat{\Phi}_{e_0}^{W_{e_2}} = B_0$ and $\hat{\Phi}_{e_1}^{W_{e_2}} = C$. By $\mathcal{J}_{\langle e_0, e_1, e_2 \rangle}$ it follows that $deg_{\mathrm{ibT}}(D) \leq deg_{\mathrm{ibT}}(W_{e_2}) = \mathbf{x}$. Since $B_0, C \leq_{\mathrm{ibT}} D$, this implies that $h(d) = deg_{\mathrm{ibT}}(D) = deg_{\mathrm{ibT}}(B_0) \vee deg_{\mathrm{ibT}}(C) = h(b_0) \vee h(c)$ for $r = \mathrm{ibT}$. Using Lemma 3.2 we obtain the same for $r = \mathrm{cl}$, i.e. (N3') is true.

In an analogous way, satisfaction of the *minimal pair requirements* $\mathcal{M}_e$ implies that $h(a) = \mathbf{0} = deg_r(B_1) \wedge deg_r(C) = h(b_1) \wedge h(c)$, i.e. (N4') holds.

Finally, $B_0 \leq_{\mathrm{ibT}} B_1 \leq_{\mathrm{ibT}} D$ and $C \leq_{\mathrm{ibT}} D$ ensures that $h(a) = \mathbf{0} \leq h(b_0) \leq h(b_1) \leq h(d)$ and $h(a) \leq h(c) \leq h(d)$. Since satisfaction of all *diagonalisation requirements* $\mathcal{D}_e$ ensures that

$B_1 \not\leq_{\mathrm{cl}} B_0$ and hence $B_1 \not\leq_{\mathrm{ibT}} B_0$, it actually holds that $h(b_0) < h(b_1)$, i.e. (N1') and (N2') are true.

Since the formal tree construction of B_0, B_1, C and D can be found in [Amboc] or [Bode 10], we confine ourselves to give an overview of the ideas underlying the construction.

Let $B_{0,s}, B_{1,s}, C_s$ and D_s denote the approximations to the sets B_0, B_1, C and D, respectively, as constructed after stage s. We guarantee $B_0 \leq_{\mathrm{ibT}} B_1 \leq_{\mathrm{ibT}} D$ and $C \leq_{\mathrm{ibT}} D$ by permitting; to be more precise, whenever we enumerate some number x_0 into B_0 at stage s, then we enumerate a number $x_1 \leq x_0$ into B_1 and D at stage s, whenever we enumerate a number y_0 into B_1 at stage s, then we enumerate a number $y_1 \leq y_0$ into D at stage s, and whenever we enumerate a number z_0 into C at stage s, then we enumerate a number $z_1 \leq z_0$ into D at stage s.

To satisfy a diagonalisation requirement $\mathcal{D}_e$ we use the standard diagonalisation strategy, i.e. we wait for a stage $s+1$ and an appropriate number x such that $\tilde{\Phi}_{e,s}^{B_{0,s}}(x)\downarrow= 0$ and $x \notin B_{1,s}$. Then we enumerate x into $B_{1,s+1}$ and restrain all further enumerations of numbers $y \leq x+e$ into B_0, thus ensuring (since $\tilde{\Phi}_e$ is a cl-functional with $\tilde{u}_e^{B_{0,s}}(x) \leq x+e$) that

$$\tilde{\Phi}_e^{B_0}(x) = \tilde{\Phi}_{e,s}^{B_{0,s}}(x) = 0 \neq 1 = B_1(x).$$

To satisfy a minimal pair requirement $\mathcal{M}_e$ with $e = \langle e_0, e_1, e_2\rangle$ the premise of which is true, we use the strategy described in the proof of Theorem 3.9. That is, defining the *length of agreement of* $\mathcal{M}_e$ *at stage* s by

$$l_s(\mathcal{M}_{\langle e_0,e_1,e_2\rangle}) = \max(\{x \leq s : \hat{\Phi}_{e_0,s}^{B_{1,s}} \restriction x = \hat{\Phi}_{e_1,s}^{C_s} \restriction x = W_{e_2,s} \restriction x\}),$$

we take care that, whenever we enumerate some number $x < l_s(\mathcal{M}_e)$ into B_1 or C at stage $s+1$ for the sake of a lower priority requirement, then we enumerate it into *only one* of B_1 and C and do not enumerate any number $y \leq x$ into B_1 or C before the next stage $t > s$ with $l_t(\mathcal{M}_e) > x$. This makes sure that $B_{1,t} \restriction (x+1) = B_{1,s} \restriction x+1$ or $C_t \restriction x+1 = C_s \restriction x+1$, and as in Theorem 3.9 we conclude that W_{e_2} is computable.

To satisfy a join requirement $\mathcal{J}_e$ with $e = \langle e_0, e_1, e_2\rangle$ the premise of which is true, we follow the strategy described in the proof of Theorem 3.27. That is, defining the *length of agreement of* $\mathcal{J}_e$ *at stage* s by

$$l_s(\mathcal{J}_{\langle e_0,e_1,e_2\rangle}) = \max(\{x \leq s : \hat{\Phi}_{e_0,s}^{W_{e_2,s}} \restriction x = B_{0,s} \restriction x \text{ and } \hat{\Phi}_{e_1,s}^{W_{e_2,s}} \restriction x = C_s \restriction x\}),$$

whenever we enumerate a number x into D at stage $s+1$ for the sake of a lower priority requirement, then we simultaneously enumerate some number $y \geq x$ with $l_s(\mathcal{J}_e) > y$ and $[x+1, y] \subseteq W_{e_2,s}$ into $B_{0,s+1} - B_{0,s}$ or $C_{s+1} - C_s$. (We say that the interval $[x+1, y]$ is $\mathcal{J}_e$*-safe* at stage s.) Then

$$\hat{\Phi}_{e_0,s}^{W_{e_2,s}} \upharpoonright y+1 = B_{0,s} \upharpoonright y+1 \neq B_0 \upharpoonright y+1 = \hat{\Phi}_{e_0}^{W_{e_2}} \upharpoonright y+1$$

or

$$\hat{\Phi}_{e_1,s}^{W_{e_2,s}} \upharpoonright y+1 = C_s \upharpoonright y+1 \neq C \upharpoonright y+1 = \hat{\Phi}_{e_1}^{W_{e_2}} \upharpoonright y+1.$$

Since $\hat{\Phi}_{e_0}$ and $\hat{\Phi}_{e_1}$ are ibT-functionals, this implies that $W_{e_2,s+1} \upharpoonright y+1 \neq W_{e_2} \upharpoonright y+1$. Hence by $[x+1, y] \subseteq W_{e_2,s}$ there must be some number $z \leq x$ in $W_{e_2} - W_{e_2,s}$. Then $D \leq_{\text{ibT}} W_{e_2}$ by permitting.

When making these strategies work together, we encounter the following problem. Assume we want to enumerate some number x into B_1 at stage $s+1$ in order to satisfy some requirement $\mathcal{D}_i$. To obtain $B_1 \leq_{\text{ibT}} D$, we need to enumerate some number $x' \leq x$ into D at stage $s+1$. Now assume that there are some join requirement $\mathcal{J}_e$ and some minimal pair requirement $\mathcal{M}_{e'}$ of higher priority than $\mathcal{D}_i$ such that the premises of $\mathcal{J}_e$ and $\mathcal{M}_{e'}$ are true and $x' < l_s(\mathcal{J}_e) < l_s(\mathcal{M}_{e'})$. Following our strategy to satisfy $\mathcal{J}_e$, we need to enumerate some number y with $x' \leq y < l_s(\mathcal{J}_e)$ into B_0 or C at stage $s+1$, where $[x'+1, y]$ is $\mathcal{J}_e$-safe at stage s. By our strategy to satisfy $\mathcal{M}_{e'}$, since we already enumerate x into B_1 at stage $s+1$, we may not enumerate y into C at stage $s+1$. Hence we must enumerate y into B_0. But by our strategy to satisfy $\mathcal{D}_i$ we are only allowed to do so if $y > x+i$.

The upshot of the previous discussion is that for any $n \geq 0$ we need to ensure that there is a stage s and an interval J of length n such that J is $\mathcal{J}_e$-safe at stage s and such that $x = x' = \min(J) - 1 \notin B_{1,s} \cup D_s$ and $y = \max(J) \notin B_{0,s}$. Only then can we safely use x as diagonalisation witness for $\mathcal{D}_i$ with $i < n$.

To create such an interval J, starting at some stage $s_0 + 1$ we assign a long interval I with $I \cap (B_{0,s_0} \cup B_{1,s_0} \cup C_{s_0} \cup D_{s_0}) = \emptyset$ to $\mathcal{D}_i$. We now enumerate the numbers from I one by one in decreasing order into C and D. The k-th such enumeration is performed at a stage $s_k + 1$ $(k \geq 1)$, where $l_{s_k}(\mathcal{J}_e) > \max(I)$ and $l_{s_k}(\mathcal{M}_{e'}) > \max(I)$. Note that this is compatible with our strategies to satisfy the join and meet requirements and with our strategy to make $C \leq_{\text{ibT}} D$. Let $x_k = \max(I) - k + 1$ be the number enumerated into C and D at stage $s_k + 1$. It holds that

$$\hat{\Phi}_{e_1,s_k}^{W_{e_2,s_k}}(x_k) = C_{s_k}(x_k) \neq C_{s_{k+1}}(x_k) = \hat{\Phi}_{e_1,s_{k+1}}^{W_{e_2,s_{k+1}}}(x_k).$$

Since $\hat{\Phi}_{e_1}$ is an ibT-functional, this implies that there must be some number $y_k \leq x_k$ in $W_{e_2,s_{k+1}} - W_{e_2,s_k}$. In particular,

$$|W_{e_2,s_{k+1}} \cap [0, \max(I)]| \geq k. \tag{35}$$

If there is any k such that $[x_{k+n-1}, x_k] \subseteq W_{e_2,s_{k+n}}$, then $J = [x_{k+n-1}, x_k]$ is an interval of length n which is $\mathcal{J}_e$-safe at stage s_{k+n} and satisfies the conditions that $\max(J) \notin B_{0,s_{k+n}}$ and, provided that $\min(J) - 1 \in I$, $\min(J) - 1 \notin B_{1,s_{k+n}} \cup D_{s_{k+n}}$.

On the other hand, if there is no such k, then for each k there must be some number $z_k \in [x_{k+n-1}, x_k]$ such that $z_k \neq y_{k'}$ for $k' \leq k+n-1$. Since for $k' \geq k+n$ we have that $y_{k'} \leq x_{k'} \leq x_{k+n} < x_{k+n-1}$, it follows that $z_k \neq y_{k'}$ for every k. Hence we can refine the inequality given above to

$$|W_{e_2,s_{k+1}} \cap ([0, \max(I)] - \{z_{k'} : k' \geq 1\})| \geq k.$$

Since we enumerate the complete interval I into C and D in this case, we get

$$|W_{e_2,s_{|I|}} \cap ([0, \max(I)] - \{z_{k'} : k' \geq 1\})| \geq |I| - 1. \tag{36}$$

By choosing the interval I sufficiently large (without changing $\min(I)$), we can ensure that $|\{z_{k'} : k' \geq 1\}| > \min(I) + 1$. Then the left-hand side of equation (36) becomes smaller than $\max(I) + 1 - (\min(I) + 1) = \max(I) - \min(I) = |I| - 1$, which contradicts equation (36).

In the actual construction, where we have to deal with several join requirements $\mathcal{J}_{e_1}, \ldots, \mathcal{J}_{e_m}$ at once, the combinatorics become a bit more difficult, but the strategy to obtain intervals of a desired length which are simulatenously $\mathcal{J}_e$-safe for $e \in \{e_1, \ldots, e_m\}$ basically remains the same. Since we cannot compute whether the premise of some join or minimal pair requirement is true, additionally we have to use a tree argument, and the strategies we described so far only work along the true path of this tree. For details see the references given above. □

4.6 Embedding the $\mathcal{S}_7$

Before we turn to lattice embeddings of the $\mathcal{M}_3$, we consider a nondistributive lattice which looks somewhat similar to the $\mathcal{M}_3$ but imposes less restrictions on an embedding.

Definition 4.16. The $\mathcal{S}_7$ is the finite partial ordering $(\{a, a_1, a_2, b_0, b_1, b_2, c\}, \leq_{\mathcal{S}_7})$ such that

(S1) $a <_{\mathcal{S}_7} a_i <_{\mathcal{S}_7} b_0, b_i <_{\mathcal{S}_7} c$ for $i \in \{1, 2\}$,

(S2) b_0, b_1 and b_2 are pairwise incomparable,

(S3) a_1 and a_2 are incomparable,

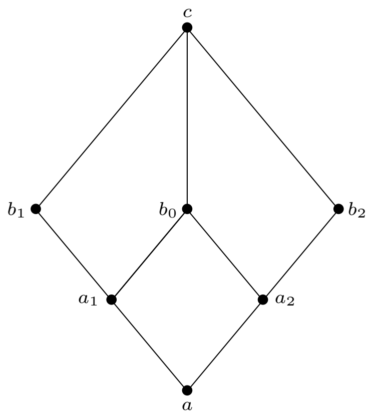
(S4) $a_1 \vee b_2 = a_2 \vee b_1 = c$,

(S5) $a_1 \vee a_2 = b_0$,

(S6) $b_0 \wedge b_i = a_i$ for $i \in \{1, 2\}$

(S7) $b_1 \wedge b_2 = a$

It is not hard to see that these conditions indeed define a unique partial ordering and that the $\mathcal{S}_7$ is a lattice with least element a and greatest element c as illustrated below.



We prove the following equivalent characterisation.

Lemma 4.17. *Let $\mathcal{P} = (P, \leq_{\mathcal{P}})$ be a partial order. Then a mapping $h : \{a, a_1, a_2, b_0, b_1, b_2, c\} \to P$ is a lattice embedding of the $\mathcal{S}_7$ into $\mathcal{P}$ if and only if*

(S1') $h(a) \leq_{\mathcal{P}} h(a_i) \leq_{\mathcal{P}} h(b_0), h(b_i) \leq_{\mathcal{P}} h(c)$ for $i \in \{1, 2\}$,

(S2') $h(b_i) \not\leq_{\mathcal{P}} h(a_{3-i})$ for $i \in \{1, 2\}$,

(S3') $h(c) \not\leq_{\mathcal{P}} h(b_0)$

(S4') $h(a_1) \vee h(b_2) = h(a_2) \vee h(b_1) = h(c)$,

(S5') $h(a_1) \vee h(a_2) = h(b_0)$,

(S6') $h(b_0) \wedge h(b_i) = h(a_i)$ for $i \in \{1, 2\}$

(S7') $h(b_1) \wedge h(b_2) = h(a)$.

Proof. Let h be a lattice embedding of the $\mathcal{S}_7$ into $\mathcal{P}$. Then by (S1)-(S7) and the definition of lattice embeddings it follows that (S1')-(S7') must hold (for (S2') note that $a_{3-i} \leq_{\mathcal{S}_7} b_0$ and $b_i \not\leq_{\mathcal{S}_7} b_0$ imply $b_i \not\leq_{\mathcal{S}_7} a_{3-i}$).

For the converse, assume that (S1')-(S7') are true. It suffices to prove that h is an embedding of the $\mathcal{S}_7$ into $\mathcal{P}$. Then the fact that h preserves joins and meets (i.e. h is a lattice embedding) follows easily by (S1') and (S4')-(S7').

First we show that $h(b_i) \neq h(c)$ for $i \in \{1, 2\}$. Otherwise $h(b_{3-i}) \leq_{\mathcal{P}} h(c) = h(b_i)$ by (S1') and $h(b_{3-i}) = h(b_i) \wedge h(b_{3-i}) = h(a_i) \leq_{\mathcal{P}} h(a_3)$ by (S7'), contradicting (S2').

It follows that $h(a_{3-i}) \not\leq_{\mathcal{P}} h(b_i)$ for $i \in \{1, 2\}$, because otherwise $h(c) = h(a_{3-i}) \vee h(b_i) = h(b_i)$ by (S4'), contradicting the above. By (S2') we conclude that $h(b_i)$ and $h(a_{3-i})$ are incomparable for $i \in \{1, 2\}$. Then $h(a_1)$ and $h(a_2)$ must be incomparable too, because otherwise $h(a_{3-i}) \leq_{\mathcal{P}} h(a_i) \leq_{\mathcal{P}} h(b_i)$ for $i = 1$ or $i = 2$.

Next, $h(a) \neq h(a_1), h(a_2)$. Indeed, if $h(a_i) = h(a)$ for $i = 1$ or $i = 2$, then $h(a_i) \leq_{\mathcal{P}} h(b_{3-i})$, contradicting the incomparability of $h(a_i)$ and $h(b_{3-i})$.

Further, it holds that $h(a_1), h(a_2) \neq h(b_0)$, because by (S1') otherwise $h(a_i) \leq_{\mathcal{P}} h(b_0) = h(a_{3-i})$ for $i = 1$ or $i = 2$, contradicting the incomparability of $h(a_1)$ and $h(a_2)$.

Moreover, $h(b_0) \neq h(c)$ by (S3'). It also follows from this fact that $h(a_i) \neq h(b_i)$ for $i \in \{1, 2\}$, because $h(a_i) \vee h(a_{3-i}) = h(b_0) \neq h(c) = h(b_i) \vee h(a_{3-i})$ by (S5') and (S4').

So we have shown that all the inequalities occuring in (S1') are strict. It remains to show that $h(b_0)$, $h(b_1)$ and $h(b_2)$ are pairwise incomparable. By (S6') and since $h(a_i) \neq h(b_0), h(b_i)$ it holds that $h(b_0)$ and $h(b_i)$ are incomparable for $i \in \{1, 2\}$. By (S7') and since $h(a) <_{\mathcal{P}} h(a_i) \leq_{\mathcal{P}} h(b_i)$ for $i \in \{1, 2\}$ it holds that $h(b_1)$ and $h(b_2)$ are incomparable. □

Now we can use this characterisation to embed the $\mathcal{S}_7$ into $\mathcal{R}_{\mathrm{ibT}}$ and $\mathcal{R}_{\mathrm{cl}}$.

Theorem 4.18. *There is a lattice embedding of the $\mathcal{S}_7$ into $\mathcal{R}_{\mathrm{ibT}}$ and $\mathcal{R}_{\mathrm{cl}}$ preserving the least element.*

Proof. We will describe a stage-by-stage construction of c.e. sets A_1, A_2, B_0, B_1, B_2 and C such that the desired embedding is given by $h(a) = \mathbf{0}$, $h(a_i) = deg_r(A_i)$, $h(b_j) = deg_r(B_j)$ and $h(c) = deg_r(C)$ for $i \in \{1, 2\}$, $j \in \{0, 1, 2\}$.

During the construction we will meet the following requirements for all $e = \langle e_0, e_1, e_2 \rangle \in \mathbb{N}$ and all $i \in \{1, 2\}$.

- $A_i \leq_{\mathrm{ibT}} B_0, B_i \leq_{\mathrm{ibT}} C$,
- $\mathcal{D}^i_e : B_i \neq \tilde{\Phi}^{A_{3-i}}_e$,
- $\mathcal{D}^0_e : C \neq \tilde{\Phi}^{B_0}_e$,
- $\mathcal{J}^i_e : (B_i = \hat{\Phi}^{W_{e_0}}_{e_1}$ and $A_{3-i} = \hat{\Phi}^{W_{e_0}}_{e_2}) \Rightarrow C \leq_{\mathrm{ibT}} W_{e_0}$,
- $\mathcal{J}^0 : B_0 = A_1 \cup A_2$ and $A_1 \cap A_2 = \emptyset$,
- $\mathcal{M}^i_e : (W_{e_0} = \hat{\Phi}^{B_0}_{e_1} = \hat{\Phi}^{B_i}_{e_2}) \Rightarrow W_{e_0} \leq_{\mathrm{ibT}} A_i$,
- $\mathcal{M}^0_e : (W_{e_0} = \hat{\Phi}^{B_1}_{e_1} = \hat{\Phi}^{B_2}_{e_2}) \Rightarrow W_{e_0}$ is computable.

Using the fact that ibT-reducibility implies cl-reducibility and Lemmas 3.2 and 3.3 as well as Lemma 3.4 we see that satisfaction of these requirements ensures (S1')-(S7') and hence that h is a lattice embedding of the $\mathcal{S}_7$ into $\mathcal{R}_r$ by Lemma 4.17.

Let $A_{i,s}, B_{j,s}, C_s$ denote the finite approximation to the sets A_i, B_j and C, respectively, as given after stage s of the construction ($i \in \{1, 2\}$, $j \in \{0, 1, 2\}$).

Again, we will obtain $A_i \leq_{\mathrm{ibT}} B_0, B_i \leq_{\mathrm{ibT}} C$ by permitting. The definition of the lengths of agreement $l_s(\mathcal{J}^i_e)$ ($i \in \{1, 2\}$) and $l_s(\mathcal{M}^i_e)$ ($i \in \{0, 1, 2\}$), and the basic strategies to satisfy the *diagonalisation requirements* $\mathcal{D}^i_e$ ($i \in \{0, 1, 2\}$), the *join requirements* $\mathcal{J}^i_e$ ($i \in \{1, 2\}$) and the *minimal pair requirements* $\mathcal{M}^0_e$ are – with the obvious changes imposed by the sets and functions occuring in the requirements having different names – the same as in the proof of Theorem 4.15. Remember that we say that an interval $[x, y]$ is $\mathcal{J}_e^i$*-safe at stage* s if $[x, y] \subseteq W_{e_0,s}$, where $e = \langle e_0, e_1, e_2\rangle$; and we say that $[x, y]$ is R*-safe at stage* s for a set R of requirements if it is $\mathcal{J}_e^i$-safe at stage s for every join requirement $\mathcal{J}_e^i \in R$.

The basic strategy to satisfy some *meet requirement* $\mathcal{M}_e^i$ ($i \in \{1, 2\}, e = \langle e_0, e_1, e_2\rangle$) the premise of which is true is a slight relaxation of the minimal pair strategy. Let s be a stage such that $l_s(\mathcal{M}_e^i) > x$, assume that we enumerate x into B_j ($j \in \{0, i\}$) at stage $s + 1$, and let $t > s$ be the next stage such that $l_t(\mathcal{M}_e^i) > x$. While the minimal pair strategy did not permit us to enumerate any number $y \leq x$ into B_{i-j} at any stage s' with $s + 1 \leq s' \leq t$, the modified strategy allows for such an enumeration, provided that we also enumerate a number $z \leq x$ into A_i at or after stage s'. In particular, if we know a stage s_0 such that $l_{s_0}(\mathcal{M}_e^i) > x$ and $A_{i,s_0+1} \upharpoonright x + 1 = A_i \upharpoonright x + 1$, then for any s, t as above with $s, t \geq s_0$ with $l_s(\mathcal{M}_e^i) > x$ and $l_t(\mathcal{M}_e^i) > x$ we know that $B_{0,s} \upharpoonright x + 1 = B_{0,t} \upharpoonright x + 1$ or $B_{i,s} \upharpoonright x + 1 = B_{i,t} \upharpoonright x + 1$ and hence

$$W_{e_0,s} \upharpoonright x + 1 = \hat{\Phi}_{e_1,s}^{B_{0,s}} \upharpoonright x + 1 = \hat{\Phi}_{e_1,t}^{B_{0,t}} \upharpoonright x + 1 = W_{e_0,t} \upharpoonright x + 1$$

or

$$W_{e_0,s} \upharpoonright x + 1 = \hat{\Phi}_{e_2,s}^{B_{i,s}} \upharpoonright x + 1 = \hat{\Phi}_{e_2,t}^{B_{i,t}} \upharpoonright x + 1 = W_{e_0,t} \upharpoonright x + 1.$$

By induction it follows that $W_{e_0,s} \upharpoonright x + 1 = W_{e_0,s_0} \upharpoonright x + 1$ for all $s \geq s_0$ and hence $W_{e_0} \upharpoonright x + 1 = W_{e_0,s_0} \upharpoonright x + 1$. This shows that we can compute $W_{e_0} \upharpoonright x + 1$ given a stage s_0 as above; but such a stage is computable from $A_i \upharpoonright x + 1$, implying $W_{e_0} \leq_{\mathrm{ibT}} A_i$ as desired.

4.6.1 Conflicts between the requirements

We now take a look at dynamics and the problems which occur when we make these basic strategies work together. For a start we use the simplified assumption that we know which join, meet and minimal pair requirements have a true premise.

The diagonalisation requirements $\mathcal{D}_e^i$, $i \in \{1, 2\}$, do not impose big problems. We can just reserve some number x for the satisfaction of these requirements and wait for a stage s with $\tilde{\Phi}_{e,s}^{A_{3-i,s}}(x) \downarrow= 0 = B_{i,s}(x)$ and $l_s(\mathcal{M}_{e'}^j) > x$, $l_s(\mathcal{J}_{e'}^j) > x$ for all requirements $\mathcal{M}_{e'}^j$, $\mathcal{J}_{e'}^i$ of higher priority than $\mathcal{D}_e^i$ the premise of which is true. Then, following the basic diagonalisation strategy, we enumerate x into B_i at stage $s + 1$ and preserve $A_{3-i,s} \upharpoonright x + e + 1$. To satisfy $B_i \leq_{\mathrm{ibT}} C$ we also enumerate x into C. Note that this is compatible with the strategy to satisfy join requirements $\mathcal{J}_{e'}^i$ of higher priority than $\mathcal{D}_e^i$ (because we enumerated x into B_i and the interval $[x + 1, x] = \emptyset$ is trivially $\mathcal{J}_{e'}^i$-safe at every stage). However, for the sake of join requirements $\mathcal{J}_{e'}^{3-i}$ of higher priority than $\mathcal{D}_e^i$ we also have to enumerate some number $y \leq x$ into B_{3-i} or A_i. Since the strategy for the minimal pair requirements does not allow for an enumeration of x into B_i and y into B_{3-i} at the same stage, we let $y = x$ and enumerate x into A_i. To satisfy $A_i \leq_{\mathrm{ibT}} B_0$ we also enumerate x into B_0. (The enumeration of x into B_0 and B_i at stage $s + 1$

is compatible with the strategy for meet requirements $\mathcal{M}^i_{e'}$, because we enumerated x into A_i, too.) Since we only enumerated x into A_i, B_0, B_i and C, but not into A_{3-i} and B_{3-i}, we did not hurt the requirement $\mathcal{J}^0$.

Satisfying a diagonalisation requirement $\mathcal{D}^0_e$ requires for a more advanced technique. Indeed, suppose that there are join requirements $\mathcal{J}^1_{e_1}$ and $\mathcal{J}^2_{e_2}$ and a minimal pair requirement $\mathcal{M}^0_{e'}$, all of higher priority than $\mathcal{D}^0_e$ and with true premises. If we want to enumerate some number x into C at a stage $s+1$ such that $\tilde{\Phi}^{B_{0,s}}_{e,s}(x)\downarrow= 0 = C_s(x)$, then by the strategy to satisfy $\mathcal{J}^1_{e_1}$ we need to enumerate some $y \geq x$ such that $[x+1, y]$ is $\mathcal{J}^1_{e_1}$-safe at stage s into B_1 or A_2; by the same reasoning for $\mathcal{J}^2_{e_2}$ we need to enumerate some $y' \geq x$ such that $[x+1, y']$ is $\mathcal{J}^2_{e_2}$-safe at stage s into B_2 or A_1. Since $l_s(\mathcal{M}^0_{e'})$ may be much larger than x, by the minimal pair strategy we are not allowed to enumerate y into B_1 and y' into B_2 at the same time. Hence say we enumerate y' into A_1. Now to satisfy $A_1 \leq_{\mathrm{ibT}} B_0$, we need to enumerate some $z \leq y'$ into B_0 as well. Since the diagonalisation strategy for $\mathcal{D}^0_e$ requires us to restrain all enumerations of numbers less than $x+e+1$ into B_0 after stage s, this is only possible if $y' \geq x+e$. In this case we can enumerate y' into A_1 and B_0 and, to satisfy $A_1 \leq_{\mathrm{ibT}} B_1$, enumerate $y := x$ into B_1 (thus following the strategy for $\mathcal{J}^1_{e_1}$, because $[x+1, x] = \emptyset$ is trivially $\mathcal{J}^1_{e_1}$-safe at every stage). Note that the simultaneous enumeration of x into B_1 and y' into B_0 is compatible with the meet requirements since we enumerate y' into A_1, too. Also note that requirement $\mathcal{J}^0$ is not hurt. Hence these enumerations are compatible with the basic strategies for all join, meet and minimal pair requirements.

So, as in the proof of Theorem 4.15, we are left with the task of creating a $\mathcal{J}^2_{e_2}$-safe interval $J = [x+1, y']$ of length e such that $y' \notin A_{1,s} \cup B_{0,s}$. Again, to do this we assign a long interval I to $\mathcal{D}^0_e$ at some stage s_0+1, where $I \cap (A_{1,s_0} \cup A_{2,s_0} \cup B_{0,s_0} \cup B_{1,s_0} \cup B_{2,s_0} \cup C_{s_0}) = \emptyset$. The goal is to obtain $J \subseteq I$.

If we try to proceed by the naive approach and – in analogy to the proof of Theorem 4.15 – let $\mathcal{D}^i_e$ enumerate I one by one in decreasing order into B_2, then we run into a new problem, which is caused by the fact that now we have to respect *two* kinds of join requirements (not counting the requirement $\mathcal{J}^0$). Indeed, each enumeration w into B_2 has to be accompagnied by an enumeration $v \leq w$ into C to satisfy $B_2 \leq_{\mathrm{ibT}} C$, and a fortiori by an enumeration of some $v' \geq v$ into B_1 or A_2, where $[v+1, v']$ is $\mathcal{J}^1_{e_1}$-safe at the stage before the enumeration. But the minimal pair requirements do not allow for an enumeration into B_1 and B_2 at the same time; hence we must enumerate v' into A_2. Then, since we enumerate v' into A_2 and v into C, we need to enumerate a number v'' with $v \leq v'' \leq v'$ into B_0. So if we choose $v = v'$ in each such situation, then we must enumerate v into B_0 and will finally obtain that $J \subseteq B_{0,s}$, contradicting $y' \notin A_{1,s} \cup B_{0,s}$.

This shows that before we can create a $\mathcal{J}^2_{e_2}$-safe interval, we need a $\mathcal{J}^1_{e_1}$-safe interval, the construction of which in turn relies on the existence of a $\mathcal{J}^2_{e_2}$-safe interval and so on. Hence the above approach fails.

Instead, we will give the responsibility for the creation of safe intervals to the join requirements themselves. For the next section assume that $\mathcal{J}^1_{e_1}$ has higher priority than $\mathcal{J}^2_{e_2}$.

4.6.2 Building safe intervals for two requirements under a maximal response hypothesis

Assume that t is a stage such that $l_t(\mathcal{J}^i_{e_i}) > x$ ($i \in \{1,2\}$) and we enumerate x into $B_{i,t+1} - B_{i,t}$ or $A_{3-i,t+1} - A_{3-i,t}$, and let t' be the first stage after stage t such that $l_{t'}(\mathcal{J}^i_{e_i}) > x$. Then we have that, for $e_i = \langle e_{i0}, e_{i1}, e_{i2}\rangle$,

$$\hat{\Phi}^{W_{e_{i0},t}}_{e_{i1},t}(x) = B_{i,t}(x) \neq B_{i,t'}(x) = \hat{\Phi}^{W_{e_{i0},t'}}_{e_{i1},t'}(x)$$

or

$$\hat{\Phi}^{W_{e_{i0},t}}_{e_{i2},t}(x) = A_{3-i,t}(x) \neq A_{3-i,t'}(x) = \hat{\Phi}^{W_{e_{i0},t'}}_{e_{i2},t'}(x).$$

Since $\hat{\Phi}_{e_{i1}}$ and $\hat{\Phi}_{e_{i2}}$ are ibT-functionals, this implies that there is some $y \leq x$ in $W_{e_{i0},t'} - W_{e_{i0},t}$. To simplify the construction, for a moment we will assume the following hypothesis to be true.

Hypothesis 4.19 (Maximal response hypothesis)**.** *In each situation as described above it holds that* $\max(\{z \leq x : z \notin W_{e_{i0},t}\}) \in W_{e_{i0},t'}$. *In particular,* $\{x\}$ *is* $\mathcal{J}^i_{e_i}$*-safe at stage* t'*.*

The interval I assigned to $\mathcal{D}^0_e$ at stage $s_0 + 1$ is partitioned into the singleton $\{\min(I)\}$ and $\iota(I)+1$ subintervals $[\min(I)+1, y_1) = [y_0, y_1), [y_1+1, y_2), \ldots, [y_{\iota(I)}, y_{\iota(I)+1})$, where it will hold that $|[y_k, y_{k+1}-1)| > y_k$ for $0 \leq k \leq \iota(I)$.

We first use the rightmost of these subintervals, $[y_{\iota(I)}, y_{\iota(I)+1})$, to obtain an $\emptyset$-safe subinterval $J_0 \subseteq [y_{\iota(I)}, y_{\iota(I)+1})$ of length $y_{\iota(I)}$. Of course, since every interval is $\emptyset$-safe at every stage, this is a trivial task, because by our assumption that $|[y_{\iota(I)}, y_{\iota(I)+1}-1)| > y_{\iota(I)}$ we can just take the subinterval $[y_{\iota(I)}, y_{\iota(I)+1})$ itself. Set $c^0 = y_{\iota(I)}$.

Next, we use the two rightmost intervals, $[y_{\iota(I)-1}, y_{\iota(I)+1})$, to create a $\mathcal{J}^1_{e_1}$-safe subinterval $J_1 \subseteq [y_{\iota(I)-1}, y_{\iota(I)}+1)$ of length $y_{\iota(I)-1}$. This task is less trivial, but still easy. Let $s_0 < s^1_1 < s^1_2 < \ldots$ be a computable sequence of stages with $l_{s^1_m}(\mathcal{J}^1_{e_1}) > \max(I)$ for $m \geq 1$. Then at the stages $s^1_1+1, s^1_2+1, \ldots$ we enumerate the elements from $[y_{\iota(I)-1}, y_{\iota(I)}-1)$ one by one in decreasing order into B_1 and C. By the maximal response hypothesis, if we enumerate $c_{s^1_m}$ into B_1 at stage s^1_m+1, then $\{c_{s^1_m}\}$ is $\mathcal{J}^1_{e_1}$-safe at stage s^1_{m+1}. Hence once we have enumerated the complete subinterval $[y_{\iota(I)-1}, y_{\iota(I)}-1)$ into B_1 at some stage s^1_m+1, then $[y_{\iota(I)-1}, c^0-1) = [y_{\iota(I)-1}, y_{\iota(I)}-1)$ is $\mathcal{J}^1_{e_1}$-safe at stage s^1_{m+1}, and by assumption, $|[y_{\iota(I)-1}, c^0-1)| > y_{\iota(I)-1}$. Let m be minimal such that $|[c_{s^1_m}, c^0-1)| > y_{\iota(I)-1}$. The interval $J_1 = [c_{s^1_m}, c^0-1)$ is $\mathcal{J}^1_{e_1}$-safe at stage s^1_{m+1}. Set $s^1 = s^1_{m+1}$ and set $c^1 = c_{s^1_m}$. We stop the enumerations into B_1 and C at stage s^1.

Note that what we did so far was compatible with the basic strategies for all requirements, except for $\mathcal{J}^2_{e_2}$. When we enumerated $c_{s^1_m}$ into C at stage s^1_m+1, then if $l_{s^1_m}(\mathcal{J}^2_{e_2}) > c_{s^1_m}$, the basic strategy for requirement $\mathcal{J}^2_{e_2}$ asked for the enumeration of some number into A_1 or B_2, which we did not yet perform. Instead, we satisfy this strategy with some delay and enumerate the number $c^1 - 1 \leq c_{s^1_m}$ into A_1, B_0, B_1 and C at stage $s^1 + 1$. Note that this is again compatible with the basic strategies for all requirements. Also note that by the maximal response hypothesis $\{c^1 - 1\}$ is $\mathcal{J}^1_{e_1}$ safe at the next stage s after stage s^1 with $l_s(\mathcal{J}^1_{e_1}) > \max(I)$.

Finally, we use the three rightmost intervals, $[y_{\iota(I)-2}, y_{\iota(I)+1})$, to create a $\{\mathcal{J}^1_{e_1}, \mathcal{J}^2_{e_2}\}$-safe subinterval $J_2 \subseteq [y_{\iota(I)-2}, y_{\iota(I)} + 1)$ of length $y_{\iota(I)-2}$. Let $s^1 < s^2_1 < \ldots$ be a computable sequence of stages with $l_{s^2_m}(\mathcal{J}^1_{e_1}) > \max(I)$ and $l_{s^2_m}(\mathcal{J}^2_{e_2}) > \max(I)$ for $m \geq 1$. Then at the stages $s^1_2+1, s^1_2+1, \ldots$ we enumerate the elements from $[y_{\iota(I)-2}, c^1-1)$ one by one in decreasing order into B_2 and C. By the maximal response hypothesis, if we enumerate $c_{s^2_m}$ into B_2 at stage s^2_m+1, then $\{c_{s^2_m}\}$ is $\mathcal{J}^2_{e_2}$-safe at stage s^2_{m+1}. At the same time, we enumerate the elements from J_1 one by one in decreasing order into A_2 and B_0. By the maximal response hypothesis, if we enumerate $a_{s^2_m}$ into A_2 at stage $s^2_m + 1$, then $\max(\{z \leq a_{s^2_m} : z \notin W_{e10,s^2_m}\}) \in W_{e10,s^2_{m+1}}$. Since J_1 and $\{c^1 - 1\}$ are already $\mathcal{J}^1_{e_1}$-safe at stage s^2_m, it follows by induction that $c_{s^2_m} \in W_{e10,s^2_{m+1}}$, i.e. $\{c_{s^2_m}\}$ is $\mathcal{J}^1_{e_1}$-safe at stage s^2_{m+1}. Hence once we have enumerated the complete subinterval $[y_{\iota(I)-2}, c^1 - 1)$ into B_2 at some stage $s^2_m + 1$, then $[y_{\iota(I)-2}, c^1 - 1)$ is $\{\mathcal{J}^1_{e_1}, \mathcal{J}^2_{e_2}\}$-safe at stage s^2_{m+1}, and by assumption $|[y_{\iota(I)-2}, c^1-1)| \geq |[y_{\iota(I)-2}, y_{\iota(I)-1}-1)| > y_{\iota(I)-2}$. Let m be minimal such that $|[c_{s^2_m}, c^1 - 1)| > y_{\iota(I)-1}$. The interval $J_2 = [c_{s^2_m}, c^1 - 1)$ is $\{\mathcal{J}^1_{e_1}, \mathcal{J}^2_{e_2}\}$-safe at stage s^2_{m+1}. Set $s^2 = s^2_{m+1}$ and set $c^2 = c_{s^2_m}$. We stop the enumerations into B_2 and C at stage s^2.

To end the construction, we enumerate c^2-1 into A_2, B_0, B_2 and C at stage s^2+1 (although in the case of only two requirements, this would not be necessary). By the maximal response hypothesis, the interval $[c^2 - 1, c^1 - 1)$ is still $\{\mathcal{J}^1_{e_1}, \mathcal{J}^2_{e_2}\}$-safe at the next stage s after stage s_2 such that $l_s(\mathcal{J}^1_{e_1}) > \max(I)$ and $l_s(\mathcal{J}^2_{e_2}) > \max(I)$.

Since we may assume that $y_{\iota(I)-2} \geq y_0 \geq e$, the interval $J = [c^2-1, c^1-1)$ is now as desired and can be used for the diagonalisation strategy of $\mathcal{D}^0_e$ (note that indeed $\max(J) = c^1 - 2$ has not been enumerated into A_1 or B_0 so far).

4.6.3 Building safe intervals for n requirements under a maximal response hypothesis

It is straightforward to generalise the strategy described above to the case that $\mathcal{D}^0_e$ has to respect n join requirements $\mathcal{J}^{i_1}_{e_1}, \ldots, \mathcal{J}^{i_n}_{e_n}$ $(i_1, \ldots, i_n \in \{1, 2\})$, all with true hypothesis and such that $\mathcal{J}^{i_k}_{e_k}$ has higher priority than $\mathcal{J}^{i_{k+1}}_{e_{k+1}}$. We still assume that a maximal response hypothesis as above holds for all these join requirements. Let $E_k = \{\mathcal{J}^{i_{k'}}_{e_{k'}} : k' \leq k\}$ for $0 \leq k \leq n$.

By induction $0 \leq k \leq n$ we define stages s^k and t^k and intervals $J_k = [c^k, c^{k-1} - 1) \subseteq I$ such that

(i) $s^k < t^k$

(ii) J_k is E_k-safe at stage s^k

(iii) $\{c^k - 1\}$ is E_k-safe at stage t^k

(iv) $[\min(I), c^k - 1) \cap (A_{0,t_k} \cup A_{1,t_k} \cup B_{0,t_k} \cup B_{1,t_k} \cup B_{2,t_k} \cup C_{t_k}) = \emptyset$

(v) $J_k \cap (A_{0,t_k} \cup A_{1,t_k} \cup B_{0,t_k}) = \emptyset$

(vi) $c^k \geq y_{\iota(I)-k}$

(vii) $|J_k| > y_{\iota(I)-k}$.

Indeed, for $k = 0$ we can set $s^0 = s_0$, $t^0 = s_0 + 1$ and $J_0 = [y_{\iota(I)}, y_{\iota(I)+1})$. Assume that J_k, s^k and t^k have already been defined. Then $\mathfrak{J}^{i_{k+1}}_{e_{k+1}}$ is responsible for building J_{k+1}. Let $t^k = s^{k+1}_1 < s^{k+1}_2 < \ldots$ be a computable sequence of stages such that $l_{s^{k+1}_m}(\mathfrak{J}^{i_0}_{e_0}) > \max(I), \ldots, l_{s^{k+1}_m}(\mathfrak{J}^{i_{k+1}}_{e_{k+1}}) > \max(I)$ for $m \geq 1$. Then at the stages $s^{k+1}_1 + 1, s^{k+1}_2 + 1, \ldots$ we enumerate the interval $[y_{\iota(I)-(k+1)}, c^k - 1)$ one by one and in decreasing order into $B_{i_{k+1}}$ and C. Let $c_{s^{k+1}_m}$ be the number thus enumerated at stage $s^{k+1}_m + 1$. At the same time, we enumerate the numbers from J_k one by one and in decreasing order into B_0 and $A_{i_{k+1}}$. Every join requirement $\mathfrak{J}^{i_{k'}}_{e_{k'}}$, $k' \leq k$, has to respond to the enumeration into B_0, and $\mathfrak{J}^{i_{k+1}}_{e_{k+1}}$ has to respond to the enumeration into $B_{i_{k+1}}$. By the fact that J_k is already $\{\mathfrak{J}^{i_{k'}}_{e_{k'}} : k' \leq k\}$-safe at stage s^{k+1}_m and by the maximal response hypothesis it follows (by induction on m) that $[c^{k+1}_m, c^k - 1)$ is $\{\mathfrak{J}^{i_{k'}}_{e_{k'}} : k' \leq k+1\}$-safe at stage s^{k+1}_{m+1}. Let m be minimal such that $|[c_{s^{k+1}_m}, c^k - 1)| > y_{\iota(I)-(k+1)}$ and set $c^{k+1} = c_{s^{k+1}_m}$, set $s^{k+1} = s^{k+1}_{m+1}$ and set $t^{k+1} = s^{k+1}_{m+2}$. By the inductive hypothesis that $c^k \geq y_{\iota(I)-k}$ and by the assumption that $|[y_{\iota(I)-(k+1)}, y_{\iota(I)-k} - 1)| > y_{\iota(I)-(k+1)}$ it holds that $c^{k+1} \geq y_{\iota(I)-(k+1)}$.

At stage $s^{k+1} + 1$ we enumerate $c^{k+1} - 1$ into $A_{i_{k+1}}$, B_0, $B_{i_{k+1}}$ and C. Hence by the minimal response hypothesis, $\{c^{k+1} - 1\}$ is $\{\mathfrak{J}^{i_{k'}}_{e_{k'}} : k' \leq k+1\}$-safe at stage t^{k+1}. At this stage we stop the construction for J_{k+1}. We see that s^{k+1}, t^{k+1} and J_{k+1} are as desired.

4.6.4 Building safe intervals without a maximal response hypothesis

Note that in the previous discussion, to build a safe interval for n requirements we just needed $n + 1$ subintervals $[y_k, y_{k+1}]$ of I, that is, we could have chosen $\iota(I) = n$.

Now we drop the minimal response hypothesis. To have the construction still work, we will have to "restart" certain parts of it, which will require the use of exponentially more than $n + 1$ subintervals.

We start the construction of $\{\mathfrak{J}^{i_1}_{e_1}, \ldots, \mathfrak{J}^{i_n}_{e_n}\}$-safe intervals as before. However, now it can happen that, why we try to create an interval J_k which is $E_{k'}$-safe (not necessarily $k = k'$ anymore), we find that $\{c_{s^k_m}\}$ is not $E_{k'}$-safe at stage s^k_{m+1} for some minimal m. In this case we define $c^k = c_{s^k_m}$, $s^k = s^k_{m+1}$, $t^k = s^k_{m+2}$ and we enumerate $c^k - 1$ into $A_{i_{k'}}$, B_0, $B_{i_{k'}}$ and C. Of course, we cannot guarantee any longer that $\{c^k - 1\}$ is $E_{k'}$-safe at stage t^k. Instead, we define $k'' \leq n$ to be maximal such that $\{c^k - 1\}$ is $E_{k''-1}$-safe at stage t^k and we can define some $E_{k''-1}$-safe interval $J_k = [c^k, x]$ at stage t^k with

$$|J_k - (A_{1,t^k} \cup A_{2,t^k} \cup B_{0,t^k})| > y_{\iota(I)-k}. \tag{37}$$

(Before, this was implied by (v) and (vii).)

Then (i), (iv), (vi) and (vii) still hold, and (ii) and (iii) hold with $E_{k''-1}$ in place of E_k. While (v) does not hold anymore, it can be replaced by the weaker condition (37). As described above, we can then proceed by trying to build an $E_{k''}$-safe interval J_{k+1}.

4.6.5 Eliminating requirements

For the construction to work, there are still two obstacles. We have to tackle the following questions:

(1) How does (37) ensure that whenever we need to enumerate a new number from J_k into B_0, there still exists such a number which is not already enumerated?

(2) If we need too many "restarts" as described above, then finally we may have enumerated all numbers in I into C and cannot go on. How can we put a computable upper bound to the number of "restarts" needed (in order to choose I large enough in advance)?

The important new feature that helps us overcome these questions is that if some join requirement $\mathfrak{J}^{i_k}_{e_k}$ ($e_k = \langle e_{k0}, e_{k1}, e_{k2}\rangle$) gives too many non-maximal responses to our enumerations of numbers from I, then we will be able to actively diagonalise against $\mathfrak{J}^{i_k}_{e_k}$.

To wit, assume that some requirements which have lower priority than $\mathfrak{J}^{i_k}_{e_k}$ enumerate the number c_{s_m} into C at stage $s_m + 1$ and the next smaller number $c_{s_{m+1}} = c_{s_m} - 1$ into C at stage $s_{m+1} + 1$. Then by the construction, as described above, we enforce that there must be some $y \leq c_{s_m}$ in $W_{e_{k0}, s_{m+1}} - W_{e_{k0}, s_m}$. Hence

$$|\{x \leq c_{s_{m+1}} : x \notin W_{e_{k0}, s_{m+1}}\}| \leq |\{x \leq c_{s_m} : x \notin W_{e_{k0}, s_m}\}| - 1.$$

Moreover, if $\{c_{s_m}\}$ is not $\mathfrak{J}^{i_k}_{e_k}$-safe at stage s_{m+1}, then the above inequality is strict.

Let $[s, t]$ be an interval of stages during which only requirements of lower priority than $\mathfrak{J}^{i_k}_{e_k}$ are active, and let $s_1 + 1 < s_2 + 1 < \ldots < s_{r+1} + 1$ be the stages in $[s, t]$ at which some number from I is enumerated into C. Let c_{s_m} be the number enumerated at stage $s_m + 1$. Furthermore let

$$p = |\{m \in \{1, \ldots, r\} : \{c_{s_m}\} \text{ is not } \mathfrak{J}^{i_k}_{e_k}\text{-safe at stage } s_{m+1}\}|.$$

Then

$$|\{x \leq c_{s_{r+1}} : x \notin W_{e_{k0}, s_{r+1}}\}| \leq |\{x \leq c_{s_1} : x \notin W_{e_{k0}, s_1}\}| - r - p \leq c_{s_1} + 1 - r - p = c_{s_{r+1}} + 1 - p.$$

Here, the final equality follows from the fact that we enumerate the numbers from I one by one and in decreasing order into C.

If $p > \min(I) + 1$, then we derive that

$$|\{x \leq c_{s_{r+1}} : x \notin W_{e_{k0}, s_{r+1}}\}| < c_{s_{r+1}} + 1 - \min(I) - 1 = |(\min(I), c_{s_{r+1}}]|. \tag{38}$$

But by the construction we will ensure that

$$(\min(I), c_{s_{r+1}}] \cap (A_{1,s_{r+1}} \cup A_{2,s_{r+1}} \cup B_{0,s_{r+1}} \cup B_{1,s_{r+1}} \cup B_{2,s_{r+1}} \cup C_{s_{r+1}}) = \emptyset;$$

hence we can change our strategy at stage $s_{r+1}+1$ and start to enumerate the numbers from $(\min(I), c_{s_{r+1}}]$ one by one and in decreasing order into A_0, B_0, B_1 and C at stages t with $l_t(\mathcal{J}^{i_k}_{e_k}) > \max(I)$. By (38) and the fact that $\mathcal{J}^{i_k}_{e_k}$ has to respond to each of these enumerations by enumerating a new number $y \leq c_{s_{r+1}}$ into $W_{e_k 0}$ we arrive at a contradiction. We say that $\mathcal{J}^{i_k}_{e_k}$ is *ready for elimination.*

Note that if $\mathcal{J}^{i_k}_{e_k}$ gives sufficiently many non-maximal responses while we try to construct safe-intervals, then there will always be at least $\min(I)+1$ such responses during an interval $[s,t]$ of stages at which no requirement of higher priority is active, *unless* some requirement of higher priority gives a non-maximal response. Hence by a combinatorial argument we can determine an upper bound to the number of non-maximal responses given by *any* requirement to enumerations of numbers from I. Since this upper bound only depends on $\min(I)$, we can choose $\iota(I)$ greater than this upper bound. This solves problem (2).

By a refined argument we can make sure that once we had at least k non-maximal responses to enumerations of numbers from I, then we need to enumerate at most $y_{\iota(I)-k}$ further numbers from I into C. Hence it suffices for the interval J_k defined at stage s^k+1 to contain at least $y_{\iota(I)-k}$ many numbers not yet enumerated into A_1, A_2 or B_0. This solves problem (1).

4.6.6 Bringing the strategies together on a tree

At last, we drop the assumption that we know the premises of which join and meet requirements are true. Instead, we have to use a tree argument for the construction. We use the tree $T = \{0,1\}^*$.

Let $\rho : \mathbb{N} \to \{\mathcal{J}^j_e : e \in \mathbb{N}, j \in \{1,2\}\} \cup \{\mathcal{M}^i_e : e \in \mathbb{N}, i \in \{0,1,2\}\}$ be a computable one-to-one enumeration of all join and meet requirements (except for $\mathcal{J}^0$). A node $\alpha \in T$ of length n corresponds to a guess about which of the first n join and meet requirements $\rho(0),\dots,\rho(n-1)$ have true premises. Moreover, nodes of length $3e+i$ will be responsible for the strategy to satisfy the diagonalisation requirement $\mathcal{D}^i_e$ ($i \in \{0,1,2\}$).

The definition of α-stages and α-expansionary stages, and the definition of δ_s and the true path TP are analogous to the definitions given in the proof of Theorem 3.20.

Instead of assigning an interval to a certain diagonalisation requirement $\mathcal{D}^i_e$ ($i \in \{0,1,2\}$), each node α of length $3e+i$ will now get assigned its own interval. When we refer to an interval I as being an *α-interval*, we mean that I was assigned to α at some point during the construction. As long as α gets not initialised, I will be a candidate to construct a long $\{\rho(|\beta|) : \beta 0 \sqsubseteq \alpha\}$-safe interval within. Whenever α is responsible for enumerating numbers from I at stage $s+1$, this will be denoted by $cand_s(\alpha) = I$. In particular, we will have $cand_s(\alpha) = I$ if I becomes freshly assigned to α at stage s, if s corresponds to one of the stages t^k from the informal description of the construction above, or if some node β with $\beta 0 \sqsubseteq \alpha$ corresponds to a join requirement which is ready for elimination at stage s (we say that β is ready for elimination at stage s).

In this status, α tries to give responsibility for the enumeration of numbers from I to some node β with $\beta 0 \sqsubseteq \alpha$ corresponding to a join requirement to create a $\{\rho(|\beta'|) : \beta' 0 \sqsubseteq \beta 0\}$-safe interval J_{k+1} if there already exists a $\{\rho(|\beta'|) : \beta' 0 \sqsubset \beta 0\}$-safe interval J_k satisfying the desired conditions for such an interval. When β becomes responsible for enumerations of numbers from I, we say that I *is demoted to* β; henceforth we write $job_s(\beta) = I$ at all stages s at which β is responsible for these enumerations. The node β will return the responsibility for enumerations of numbers from I to α as soon as it has finished the construction of J_{k+1} or some join requirement corresponding to a node β' with $\beta' 0 \sqsubseteq \beta 0$ gives a nonmaximal response.

Finally, the node α can be assigned a diagonalisation witness from I. In this construction, by the term "diagonalisation witness" we mean a pair (x, y) of numbers, where x is the number to be enumerated into C and B_i and y the corresponding number to be enumerated into A_i and B_0 according to the basic strategy to satisfy a diagonalisation requirement $\mathcal{D}_e^i$ ($i \in \{1, 2\}$). To avoid case distinctions, diagonalisation requirements $\mathcal{D}_e^0$ will also formally be appointed pairs (x, y) as diagonalisation witnesses, but in this case only x really matters.

We have now explained all parts of the construction. Putting these together, the algorithm is as follows.

4.6.7 The construction

Stage 0: Let $A_{1,0} = A_{2,0} = B_{0,0} = B_{1,0} = B_{2,0} = C_0 = \emptyset$ and $job_0(\beta)\uparrow$ and $cand_0(\beta)\uparrow$ for each node β. No node has an interval or a diagonalisation witness assigned.

Stage $s + 1$:

We say that a node α *requires attention at stage* $s + 1$ if $\alpha \sqsubseteq \delta_s$ and one of the following holds.

(Case 1.1) α has no interval assigned to it at the end of stage s.

(Case 1.2) α has an interval I assigned to it at the end of stage s, $cand_s(\alpha) = I$, for every β with $\beta 0 \sqsubseteq \alpha$ it holds that $l_s(\rho(|\beta|)) > \max(I)$, there is a number $c_s \in I$ such that

$$I \cap (A_{1,s} \cup A_{2,s} \cup B_{0,s} \cup B_{1,s} \cup B_{2,s}) \subseteq I \cap C_s = [c_s + 1, \max(I)] \tag{39}$$

and there is a join requirement $\mathcal{J}_e^i = \rho(|\beta|)$ ($i \in \{1, 2\}, e = \langle e_0, e_1, e_2\rangle$) for some β with $\beta 0 \sqsubseteq \alpha$ such that

$$|\{x \leq c_s : x \notin W_{e_0,s}\}| < |\{x \leq c_s : x \in I\}|. \tag{40}$$

(Case 1.3) α has an interval I assigned to it at the end of stage s, $cand_s(\alpha) = I$, for every β with $\beta 0 \sqsubseteq \alpha$ and every α'-interval $I' = job_s(\beta)$ it holds that $\alpha \leq \alpha'$ and $l_s(\rho(|\beta|)) > \max(I)$, and there are numbers a_s and $c_s \in I$ and a node β such that

- equation (39) holds for c_s

- β is the longest node such that either $\beta = \alpha$, or $\rho(|\beta|)$ is a join requirement and $\beta 0 \sqsubseteq \alpha$, and such that there is some $y > c_s$, $y \in I$, for which

$$[c_s + 1, y] \text{ is } E'\text{-safe at stage } s, \text{ where } E' = \{\rho(|\beta'|) : \beta' \sqsubset \beta \text{ and } \beta' 0 \sqsubseteq \alpha\} \tag{41}$$

  and if I is relevant above $y_k(I)$ at stage s, then

$$|\{z \in [c_s + 1, y] \cap I : z \notin A_{1,s} \cup A_{2,s} \cup B_{0,s}\}| \geq y_k(I) \tag{42}$$

- $a_s = y$ for the least such y (note that $a_s \notin A_{1,s} \cup A_{2,s} \cup B_{0,s}$ by minimality of a_s and by $y_k(I) > 0$).

(Case 1.4) α has an interval I and a diagonalisation witness (x, y), $x, y \in I$, assigned at the end of stage s, for every β with $\beta 0 \sqsubseteq \alpha$ and every α'-interval $I' = job_s(\beta)$ it holds that $\alpha \leq \alpha'$ and $l_s(\rho(|\beta|)) > y$ and

(a) if $|\alpha| = 3e$, then $x + e + 1 \leq y$, $\tilde{\Phi}_e^{B_{0,s}}(x) \downarrow = B_{1,s}(x) = C_s(x) = 0$ and $y \notin A_{1,s} \cup A_{2,s} \cup B_{0,s}$;

(b) if $|\alpha| = 3e + i$, $i \in \{1, 2\}$, then $\tilde{\Phi}_e^{A_{3-i,s}}(x) \downarrow = 0$ and $x \notin A_{1,s} \cup A_{2,s} \cup B_{0,s} \cup B_{1,s} \cup B_{2,s} \cup C_s$.

We also say that a node β *requires attention at stage $s+1$ and is α-linked* if

(*Case 2*) $\beta 0 \sqsubseteq \delta_s$ and $\alpha \leq \delta_s$, $job_s(\beta) = I$ for some α-interval I, for every β' with $\beta' 0 \sqsubseteq \beta 0$ it holds that $l_s(\rho(|\beta'|)) > \max(I)$ and there exists a number $c_s \in I$ satisfying equation (39).

Let η be the least node that requires attention at stage $s + 1$ (such a node exists, because δ_s always requires attention due to Case 1 at stage $s + 1$). Let $X \in \{1.1, 1.2, 1.3, 1.4, 2\}$ be minimal such that η requires attention due to Case X. We say that η *receives attention and is active* due to Case X.

(*Case 1.1*) If $\eta = \alpha$ is active due to Case 1.1, then assign a new interval I to α in the following way. Let $|\alpha| = 3e + i$, $i \in \{0, 1, 2\}$. Let x be the least number that is larger than $\max(|\alpha|, s + 1)$ and larger than all numbers from intervals assigned to any node before stage $s + 1$. Let $E = \{\beta : \beta 0 \sqsubseteq \alpha\}$. Define

$$\iota(I) = (x + 3)^{2|E|} - 1$$

and

$$I = [x, 2^{\iota(I)+1} \cdot ((x + 1) + (\iota(I) + 1)),$$

i.e. $I = \{x\} \cup \bigcup_{k=0}^{\iota(I)} [y_k, y_{k+1})$, where $y_k = y_k(I) = 2^k(x + 1 + k)$ for $0 \leq k \leq \iota(I) + 1$. For $k < 0$, define $y_k(I) = x$. Say that I is an *α-interval.*

Let $c_s = y_{\iota(I)}$. Set $\hat{A}_{1,s+1} = A_{1,s} \cup \{y_{\iota(I)}\}$, $\hat{A}_{2,s+1} = A_{2,s}$, $\hat{B}_{0,s+1} = B_{0,s} \cup \{y_{\iota(I)}\}$, $\hat{B}_{1,s+1} = B_{1,s} \cup \{y_{\iota(I)}\}$, $\hat{B}_{2,s+1} = B_{2,s}$ and $\hat{C}_{s+1} = C_s \cup [y_{\iota(I)}, y_{\iota(I)+1})$. If $E = \emptyset$ or $i \neq 0$, assign

$(x, x+e+2)$ as diagonalisation witness to α. Otherwise set $cand_{s+1}(\alpha) = I$. Say that I is relevant above $y_{\iota(I)}$ at stage $s+1$.

(*Case 1.2*) If $\eta = \alpha$ is active due to Case 1.2, set $\hat{A}_{1,s+1} = A_{1,s} \cup \{c_s\}$, $\hat{A}_{2,s+1} = A_{2,s}$, $\hat{B}_{0,s+1} = B_{0,s} \cup \{c_s\}$, $\hat{B}_{1,s+1} = B_{1,s} \cup \{c_s\}$, $\hat{B}_{2,s+1} = B_{2,s}$ and $\hat{C}_{s+1} = C_s \cup \{c_s\}$. We say that α *is ready for elimination at stage* $s+1$.

(*Case 1.3*) If $\eta = \alpha$ is active due to Case 1.3, let $cand_s(\alpha) = I$, let a_s, c_s and β be as in the hypothesis of Case 1.2. Say that $cand_{s+1}(\alpha)$ is undefined.

If $\beta = \alpha$, assign (c_s, a_s) as diagonalisation witness to α and set $\hat{A}_{i,s+1} = A_{i,s}$ for $i \in \{1,2\}$, $\hat{B}_{j,s+1} = B_{j,s}$ for $j \in \{0,1,2\}$ and $\hat{C}_{s+1} = C_s$.

If $\beta \neq \alpha$ and $\rho(|\beta|) = \mathfrak{J}^i_e$, $i \in \{1,2\}$, then set $job_{s+1}(\beta) = I$, $\hat{A}_{i,s+1} = A_{i,s} \cup \{a_s\}$, $\hat{A}_{3-i,s+1} = A_{3-i,s}$, $\hat{B}_{0,s+1} = B_{0,s} \cup \{a_s\}$, $\hat{B}_{i,s+1} = B_{i,s} \cup \{c_s\}$, $\hat{B}_{3-i,s+1} = B_{3-i,s}$ and $\hat{C}_{s+1} = C_s \cup \{c_s\}$. Say that I is *demoted to* β.

If I was relevant above $y_k(I)$ at stage s, say that I is relevant above $y_{k-1}(I)$ at stage $s+1$.

(*Case 1.4*) If $\eta = \alpha$ is active due to Case 1.4 and has the diagonalisation witness (x, y) assigned, then let $c_s = x$ and $a_s = y$ and

(a) if $|\alpha| = 3e$, set $\hat{A}_{1,s+1} = A_{1,s} \cup \{a_s\}$, $\hat{A}_{2,s+1} = A_{2,s}$, $\hat{B}_{0,s+1} = B_{0,s} \cup \{a_s\}$, $\hat{B}_{1,s+1} = B_{1,s} \cup \{c_s\}$, $\hat{B}_{2,s+1} = B_{2,s}$ and $\hat{C}_{s+1} = C_s \cup \{c_s\}$.

(b) if $|\alpha| = 3e + i$ with $i \in \{1,2\}$, set $\hat{A}_{i,s+1} = A_{i,s} \cup \{c_s\}$, $\hat{A}_{3-i,s+1} = A_{3-i,s}$, $\hat{B}_{0,s+1} = B_{0,s} \cup \{c_s\}$, $\hat{B}_{i,s+1} = B_{i,s} \cup \{c_s\}$, $\hat{B}_{3-i,s+1} = B_{3-i,s}$ and $\hat{C}_{s+1} = C_s \cup \{c_s\}$.

(*Case 2*) If $\eta = \beta$ is active due to Case 2 and is α-linked at stage $s+1$, let $I = job_s(\beta)$ and let k be such that I is relevant above $y_k(I)$ at stage s. Let $\rho(|\beta|) = \mathfrak{J}^i_e$.

Let a_s be the greatest number $y \in I$ such that

$$y \geq c_s, \tag{43}$$

$$y \notin (A_{1,s} \cup A_{2,s} \cup B_{0,s}), \text{ and} \tag{44}$$

$$[c_s + 1, y] \text{ is } E\text{-safe at stage } s \text{ for } E = \{\rho(|\beta'|) : \beta' 0 \sqsubset \beta 0\}. \tag{45}$$

(Note that such a number exists, because by (39) $y = c_s$ satisfies all three conditions.)

(a) If $[c_s{+}1, c_s{+}y_k(I)]$ is $\{\rho(|\beta'|) : \beta'0 \sqsubseteq \beta 0\}$-safe and $[c_s{+}1, c_s{+}y_k(I)] \cap (A_{1,s} \cup A_{2,s} \cup B_{0,s}) = \emptyset$, or if $\{c_s+1\}$ is not $\{\rho(|\beta'|) : \beta'0 \sqsubseteq \beta 0\}$-safe at stage s, set $\hat{A}_{i,s+1} = A_{i,s} \cup \{c_s\}$, $\hat{A}_{3-i,s+1} = A_{3-i,s}$, $\hat{B}_{0,s+1} = B_{0,s} \cup \{c_s\}$, $\hat{B}_{i,s+1} = B_{i,s} \cup \{c_s\}$, $\hat{B}_{3-i,s+1} = B_{3-i,s}$ and $\hat{C}_{s+1} = C_s \cup \{c_s\}$. Set $job_{s+1}(\beta) \uparrow$ and $cand_{s+1}(\alpha) = I$.

(b) If $\{c_s{+}1\}$ is $\{\rho(|\beta'|) : \beta'0 \sqsubseteq \beta 0\}$-safe at stage s, but $[c_s{+}1, c_s{+}y_k(I)]$ is not or has nonempty intersection with $A_{1,s} \cup A_{2,s} \cup B_{0,s}$, then set $\hat{A}_{i,s+1} = A_{i,s} \cup \{a_s\}$, $\hat{A}_{3-i,s+1} = A_{3-i,s}$, $\hat{B}_{0,s+1} = B_{0,s} \cup \{a_s\}$, $\hat{B}_{i,s+1} = B_{i,s} \cup \{c_s\}$, $\hat{B}_{3-i,s+1} = B_{3-i,s}$ and $\hat{C}_{s+1} = C_s \cup \{c_s\}$.

In all cases, initialise all nodes $\alpha' > \delta_s$, i.e. declare all intervals I' assigned to these nodes unassigned and not relevant above any number, set $cand_{s+1}(\alpha') \uparrow$ and set $job_{s+1}(\tilde{\beta}) \uparrow$ for all $\tilde{\beta}$ with $job_s(\tilde{\beta}) = I'$. Also initialise every node $\alpha' > \alpha$.

Let

$$Z = \{\min(I') : (\exists \tilde{\beta} 0 \sqsubseteq \delta_s)(job_s(\tilde{\beta})) = I' \text{ and } I' \text{ is an } \alpha'\text{-interval for some } \alpha' \text{ initialised at stage } s+1\}.$$

If $\hat{A}_{1,s+1} \neq A_{1,s}$, set $A_{1,s+1} = \hat{A}_{1,s+1} \cup Z$, $A_{2,s+1} = \hat{A}_{2,s+1}$, $B_{0,s+1} = \hat{B}_{0,s+1} \cup Z$, $B_{1,s+1} = \hat{B}_{1,s+1} \cup Z$, $B_{2,s+1} = \hat{B}_{2,s+1}$ and $C_{s+1} = \hat{C}_{s+1} \cup Z$.

Otherwise set $A_{1,s+1} = \hat{A}_{1,s+1}$, $A_{2,s+1} = \hat{A}_{2,s+1} \cup Z$, $B_{0,s+1} = \hat{B}_{0,s+1} \cup Z$, $B_{1,s+1} = \hat{B}_{1,s+1}$, $B_{2,s+1} = \hat{B}_{2,s+1} \cup Z$ and $C_{s+1} = \hat{C}_{s+1} \cup Z$.

For all nodes α', unless stated otherwise before, leave the assignment of intervals and diagonalisation witnesses, the values of $cand(\alpha)$ and $job(\alpha)$ and the relevant parts of α-intervals at stage $s+1$ as they were at stage s.

Quit the stage.

4.6.8 Verification

Lemma 4.20 (True Path Lemma). *It holds that* TP $= \liminf_{s \to \infty} \delta_s$, *i.e. if* $\alpha \in T$, *then* $\alpha \sqsubset$ TP *if and only if* $\alpha \sqsubseteq \delta_s$ *for infinitely many* s *and there are only finitely many* s *such that* $\delta_s <_L \alpha$.

Proof. Analogous to the proof of Theorem 3.20. □

Lemma 4.21. *For every* α*-interval* I *and every stage* s *exactly one of the following holds:*

- I *is not assigned to* α *at stage* s
- $cand_s(\alpha) = I$
- $job_s(\beta) = I$ *for some* $\beta \sqsubseteq \alpha$
- α *has a diagonalisation witness* (x, y) *with* $x, y \in I$ *assigned at stage* s

Proof. Immediate by induction on s. □

Lemma 4.22. *Let* I *be an* α*-interval and let* β *be a node such that* $\beta 0 \sqsubseteq \alpha$. *Assume that* I *is demoted to* β *and is relevant above* $y_k := y_k(I)$ *at stage* $t_0 + 1$ *and that for some* $r \leq y_k$ *there are minimal stages* $t_0 < t_1 < \ldots < t_r$ *such that* $I \cap C_{t_n+1} \neq I \cap C_{t_n}$ *for* $n \in \{0, \ldots, r\}$ *and* α *is not initialised at any stage* $s \in [t_0 + 1, t_r + 1]$. *Furthermore assume that* $\{c_{t_n}\}$ *is* $\{\rho(|\beta'|) : \beta' 0 \sqsubseteq \beta 0\}$*-safe at stage* t_{n+1} *for* $0 \leq n < r$.

Then for $1 \leq n \leq r$ *the node* β *is active due to Case 2(b) at stage* $t_n + 1$ *if* $n < y_k$ *and due to Case 2(a) at stage* $t_n + 1$ *if* $n = y_k$, *and for* $0 \leq n \leq r$ *it holds that*

$$[c_{t_n} + 1, c_{t_n} + n] \cap (A_{1,t_n} \cup A_{2,t_n} \cup B_{0,t_n}) = \emptyset \text{ and } c_{t_n} + n + 1 \in B_{0,t_n}. \tag{46}$$

Proof. The proof is by induction on n. If $n = 0$, then the first part of (46) is trivially true because $[c_{t_n} + 1, c_{t_n} + n] = \emptyset$. For the second part, note that $cand_{t_0}(\alpha) = I$. Let $s < t_0$ be maximal such that $I \cap C_s \neq I \cap C_{s+1}$. Then $cand_{s+1}(\alpha) = I$. Hence by Lemma 4.21 the enumeration into $I \cap C$ at stage $s+1$ must be caused by some node being active due to Case 1.1, Case 1.2 or Case 2(a). In each case, $c_s \in B_{0,s+1} \subseteq B_{0,t_0}$. But by (39) it holds that $c_{t_0} + 1 = c_s$, so the second part of (46) is true, too.

Now fix $n < r$ and assume that the inductive hypothesis is true for all numbers up to n.

By the inductive hypothesis, since $n < r \leq y_k$, at stage $t_n + 1$ either I was demoted to β (if $n = 0$) or β was active due to Case 2(b). Hence $job_{t_n+1}(\beta) = I$. Then still $job_{t_{n+1}}(\beta) = I$. Since $I \cap C_{t_{n+1}+1} \neq I \cap C_{t_{n+1}}$, but I is not initialised at stage $t_{n+1} + 1$, β must be active due to Case 2 at stage $t_{n+1} + 1$.

By (39) we know that $c_{t_{n+1}} = c_{t_n} - 1$ and hence $c_{t_{n+1}} + (n+1) + 1 = c_{t_n} + n + 1 \in B_{0,t_n} \subseteq B_{0,t_{n+1}}$ by the inductive hypothesis. Hence the second part of (46) is true for $n+1$. For the first part, since $[c_{t_{n+1}} + 1, c_{t_{n+1}} + (n+1)] = [c_{t_n}, c_{t_n} + n]$ and $[c_{t_n}, c_{t_n} + n] \cap (A_{1,t_n} \cup A_{2,t_n} \cup B_{0,t_n}) = \emptyset$ by the inductive hypothesis and by (39), it suffices to show that no number from $[c_{t_n}, c_{t_n} + n]$ is enumerated into A_1, A_2 or B_0 at any stage $t \in [t_n + 1, t_{n+1}]$. In fact, since each such enumeration is accompanied by an enumeration of a new number from I into C at the same stage and since $I \cap C_{t_n+1} = I \cap C_{t_{n+1}}$, it suffices to show that no number from $[c_{t_n}, c_{t_n} + n]$ is enumerated into A_1, A_2 or B_0 at stage $t_n + 1$. This means to show that $a_{t_n} > c_{t_n} + n = c_{t_0}$.

If $n = 0$, this is clear by definition of a_{t_0} in Case 1.3. If $n > 0$, by (42), there are at least $y_{k+1}(I)$ numbers $z \in [c_{t_0} + 1, a_{t_0}]$ which are not in $A_{1,t_0} \cup A_{2,t_0} \cup B_{0,t_0}$. Since at each of the stages $t_0 + 1, t_1 + 1, \ldots, t_{n-1} + 1$ only one such number z is enumerated into $A_1 \cup A_2 \cup B_0$, while at stages $t \in [t_0 + 1, t_n] - \{t_0 + 1, t_1 + 1, \ldots, t_{n-1} + 1\}$ there are no such enumerations, there are still $y_{k+1}(I) - n \geq y_k - n > 0$ numbers $z \in [c_{t_0} + 1, a_{t_0}]$ which are not in $A_{1,t_n} \cup A_{2,t_n} \cup B_{0,t_n}$. Let y be the greatest such number. Then $[c_{t_0} + 1, y]$ was $\{\rho(|\beta'|) : \beta'0 \sqsubset \beta0\}$-safe at stage t_0 by (41). On the other hand each number $x \in [c_{t_n} + 1, c_{t_0}]$ has the form $x = c_{t_m}$ for some $m \in [0, n-1]$. Since by the hypothesis $\{c_{t_m}\}$ is $\{\rho(|\gamma|) : \gamma 0 \sqsubseteq \beta 0\}$-safe at stage t_{m+1}, hence at stage t_n, it follows that $[c_{t_n} + 1, y]$ is $\{\rho(|\beta'|) : \beta'0 \sqsubset \beta0\}$-safe at stage t_n. Hence the number y satisfies equations (43) to (45) with t_n instead of s. It follows that $a_{t_n} \geq y > c_{t_0}$.

This completes the proof of (46).

Now if $1 \leq n + 1 < y_k$, then it follows from the second part of (46) (for $n + 1$) that β must be active due to Case 2(b) at stage $t_{n+1} + 1$.

If $n + 1 = y_k$, then $[c_{t_{n+1}} + 2, c_{t_0}] = [c_{t_n} + 1, c_{t_{n+1}} + y_k]$ is $\{\rho(|\beta'|) : \beta'0 \sqsubseteq \beta0\}$-safe at stage t_n as we argued above, and $\{c_{t_{n+1}} + 1\} = \{c_{t_n}\}$ is $\{\rho(|\beta'|) : \beta'0 \sqsubseteq \beta0\}$-safe at stage t_{n+1} by the hypothesis. Together with the first part of (46) this implies that β must be active due to Case 2(a) at stage $t_{n+1} + 1$. $\square$

Lemma 4.23. *Let I be an α-interval. Assume that for some $k \in \{0, \ldots, \iota(I) - 1\}$ there is a sequence of minimal stages $t_0 < t_1 < \ldots < t_{y_k(I)+1}$ such that $C_{t_n+1} \cap I \neq C_{t_n} \cap I$ and I is relevant above $y_k := y_k(I)$ at stage $t_n + 1$ for $n \in \{0, \ldots, y_k + 1\}$. Then there must be a stage*

t_n such that α is ready for elimination at stage $t_n + 1$.

Proof. For $n \in \{0, \dots, y_k + 1\}$ let β_{t_n} be the node that is active at stage $t_n + 1$. For a contradiction assume that α is not ready for elimination at any stage $t_n + 1$.

If I would already become relevant above y_k at a stage $s+1$ before stage $t_0 + 1$, then since $C_s \cap I = C_{s+1} \cap I$ (by minimality of t_0), α would get the interval I and a diagonalisation witness assigned at stage $s+1$. In this case α could only be active due to Case 1.4 at stage $t_0 + 1$ and never enumerate anything into $C \cap I$ after stage $t_0 + 1$, contradicting the fact that $C_{t_{y_k+1}} \cap I \neq C_{t_{y_k+1}+1} \cap I$. Hence I becomes relevant above y_k at stage $t_0 + 1$, i.e. $\beta_{t_0} = \alpha$ and I is demoted to β_{t_1} at stage $t_0 + 1$.

Let r be the greatest number in $\{0, \dots, y_k\}$ such that $\{c_{t_n}\}$ is $\{\rho(|\beta'|) : \beta' 0 \sqsubseteq \beta_{t_1} 0\}$-safe at stage t_{n+1} for all $n < r$. If $r = y_k$, then by Lemma 4.22 β_{t_1} is active due to Case 2(a) at stage $t_{y_k} + 1$. If $r < y_k$, then by Lemma 4.22 β_{t_1} is active due to Case 2(b) at all stages $t_1 + 1, \dots, t_r + 1$. Then β_{t_1} must be active due to Case 2 at stage $t_{r+1} + 1$, too (because α is not initialised but a new number from I is enumerated into C at stage $t_{r+1} + 1$). But since $\{c_{t_{r+1}} + 1\} = \{c_{t_r}\}$ is not $\{\rho(|\beta'|) : \beta' 0 \sqsubseteq \beta_{t_1} 0\}$-safe at stage t_{r+1}, β_{t_1} is active due to Case 2(a) at stage $t_{r+1} + 1$.

This shows that there is a least stage t_p with $p \leq y_k$ such that β_{t_1} is active due to Case 2(a) at stage $t_p + 1$. Then $cand_{t_{p+1}}(\alpha) = I$. Since α is not initialised or active due to Case 1.3 at stage $t_{p+1} + 1$ (otherwise I would not be relevant above y_k any more), it must be active due to Case 1.2, proving the claim. $\square$

Lemma 4.24. *Let I be an α-interval and let*

$$E = \{\beta : \beta 0 \sqsubseteq \alpha \text{ and } \rho(|\beta|) \text{ is a join requirement}\} = \{\beta'_0, \dots, \beta'_{|E|-1}\}$$

with $\beta'_0 \sqsubset \beta'_1 \sqsubset \ldots \sqsubset \beta'_{|E|-1}$. Let $y_0 = y_0(I)$. Then for $\bar{e} \in \{0, \dots, |E|-1\}$ there is no sequence of minimal stages $s_0 < s_1 < \ldots < s_{|E| \cdot y_0}$ such that I is demoted to $\beta'_{\bar{e}}$ at stage $s_n + 1$ for $n \in \{0, \dots, |E| \cdot y_0\}$ and I is not demoted to any β'_e with $e < \bar{e}$ at any stage $s \in [s_0 + 1, s_{|E| \cdot y_0} + 1]$.

Proof. For a contradiction assume that there is such a sequence $s_0 < s_1 < \ldots < s_{|E| \cdot y_0}$ for some $\bar{e}$. We show by induction on m that, for $m \in \{0, \dots, |E| \cdot y_0\}$,

$$\sum_{e=0}^{\bar{e}} |\{w \leq c_{s_m} : w \notin W_{e_0, s_m}\}| \leq (\bar{e} + 1) \cdot (c_{s_m} + 1) - m, \tag{47}$$

where $\rho(|\beta'_e|) = \mathfrak{J}^{i(e)}_{\langle e_0, e_1, e_2 \rangle}$ for $e < |E|$.

For $m = 0$ (47) is trivially true, because $|[0, c_{s_0}]| = c_{s_0} + 1 - 0$.

Fix $m < |E| \cdot y_0$ and assume that (47) is true for m. Define a sequence of stages $s_m = t_0 < t_1 < \ldots < t_{p+1} = s_{m+1}$ such that $\{t_q : 0 \leq q \leq p+1\} = \{t \in [s_m, s_{m+1}] : I \cap C_t \neq I \cap C_{t+1}\}$. Let β_{t_q} be the node that is active at stage $t_q + 1$.

We will show that, for all $q \in \{0, \ldots, p\}$ and all $e \leq \bar{e}$,

$$|\{w \leq c_{t_{q+1}} : w \notin W_{e_0,t_{q+1}}\}| \leq |\{w \leq c_{t_q} : w \notin W_{e_0,t_q}\}| - 1 \tag{48}$$

and that there is some $q \in \{0, \ldots, p\}$ and some $e \leq \bar{e}$ such that

$$|\{w \leq c_{t_{q+1}} : w \notin W_{e_0,t_{q+1}}\}| \leq |\{w \leq c_{t_q} : w \notin W_{e_0,t_q}\}| - 2. \tag{49}$$

Then, using the inductive hypothesis (47) and $t_0 = s_m$ we know that

$$\sum_{e=0}^{\bar{e}} |\{w \leq c_{t_0} : w \notin W_{e_0,t_0}\}| \leq (\bar{e}+1) \cdot (c_{t_0}+1) - m,$$

and by (48) and (49) it follows that, for $q \in \{0, \ldots, p\}$,

$$\sum_{e=0}^{\bar{e}} |\{w \leq c_{t_{q+1}} : w \notin W_{e_0,t_{q+1}}\}| \leq (\bar{e}+1) \cdot (c_{t_0}+1) - m - (q+1) \cdot (\bar{e}+1)$$

and

$$\sum_{e=0}^{\bar{e}} |\{w \leq c_{t_{p+1}} : w \notin W_{e_0,t_{p+1}}\}| \leq (\bar{e}+1) \cdot (c_{t_0}+1) - m - (p+1) \cdot (\bar{e}+1) - 1. \tag{50}$$

Since $c_{t_{q+1}} = c_{t_q} - 1$ must hold for $q \in \{0, \ldots, p\}$, we see that $c_{s_{m+1}} = c_{t_{p+1}} = c_{t_0} - (p+1)$. Using (50) this implies

$$\begin{aligned}
\sum_{e=0}^{\bar{e}} |\{w \leq c_{s_{m+1}} : w \notin W_{e_0,s_{m+1}}\}| &\leq (\bar{e}+1) \cdot (c_{t_0}+1) - (m+1) - (p+1) \cdot (\bar{e}+1) \\
&= (\bar{e}+1) \cdot (c_{t_0} - (p+1) + 1) - (m+1) \\
&= (\bar{e}+1) \cdot (c_{s_{m+1}}+1) - (m+1),
\end{aligned}$$

i.e. (47) is true for $m+1$. So to prove (47), it suffices to prove (48) and (49).

To prove (48), fix $e \leq \bar{e}$ and $q \in \{0, \ldots, p\}$. Let $\rho(|\beta'_e|) = \mathfrak{J}^i_{\langle e_0,e_1,e_2\rangle}$. Since I is demoted to $\beta'_{\bar{e}}$ at stage $s_m + 1 = t_0 + 1$ and not demoted to any node below $\beta'_{\bar{e}}$ at any stage t with $t_0 + 1 \leq t \leq t_{p+1}$, it holds that $\beta'_e \sqsubseteq \beta'_{\bar{e}} \sqsubseteq \beta_{t_q} \sqsubseteq \delta_{t_q}$ and $\delta_{t_q}(|\beta'_e|) = 0$, as well as $\beta'_e \sqsubseteq \beta'_{\bar{e}} \sqsubseteq \beta_{t_{q+1}} \sqsubseteq \delta_{t_{q+1}}$ and $\delta_{t_{q+1}}(|\beta'_e|) = 0$. Since β_{t_q} is active at stage $t_q + 1$ and $\beta_{t_{q+1}}$ is active at stage $t_{q+1} + 1$ (due to Case 1.2, Case 1.3 or Case 2), it follows that

$$l_{t_q}(\mathfrak{J}^i_{\langle e_0,e_1,e_2\rangle}) = l_{t_q}(\rho(|\beta'_e|) > \max(I), \tag{51}$$

$$l_{t_{q+1}}(\mathfrak{J}^i_{\langle e_0,e_1,e_2\rangle}) = l_{t_{q+1}}(\rho(|\beta'_e|) > \max(I). \tag{52}$$

If $\beta'_e = \beta_{t_q}$, then $c = c_{t_q}$ is enumerated into $A_{3-i,t_q+1} - A_{3-i,t_q}$ or $B_{i,t_q+1} - B_{i,t_q}$ at stage $t_q + 1$ (note that $c_{t_q} \notin A_{3-i,t_q} \cup B_{i,t_q}$ by (39)). By (51) and (52) it follows that

$$\hat{\Phi}_{e_1,t_{q+1}}^{W_{e_0,t_{q+1}}} \upharpoonright (c+1) = B_{i,t_{q+1}} \upharpoonright (c+1) \neq B_{i,t_q} \upharpoonright (c+1) = \hat{\Phi}_{e_1,t_q}^{W_{e_0,t_q}} \upharpoonright (c+1) \tag{53}$$

or

$$\hat{\Phi}_{e_2,t_{q+1}}^{W_{e_0,t_{q+1}}} \upharpoonright (c+1) = A_{3-i,t_{q+1}} \upharpoonright (c+1) \neq A_{3-i,t_q} \upharpoonright (c+1) = \hat{\Phi}_{e_2,t_q}^{W_{e_0,t_q}} \upharpoonright (c+1). \tag{54}$$

Since $\hat{\Phi}_{e_1}$ and $\hat{\Phi}_{e_2}$ are ibT-functionals, this implies

$$W_{e_0,t_{q+1}} \upharpoonright (c+1) \neq W_{e_0,t_q} \upharpoonright (c+1), \tag{55}$$

in particular there is some minimal $z_q^e \leq c_{t_q}$ in $W_{e_0,t_{q+1}} - W_{e_0,t_q}$.

On the other hand, if $\beta'_e \sqsubset \beta_{t_q}$, then some $c \in \{a_{t_q}, c_{t_q}\}$ is enumerated into $A_{3-i,t_q+1} - A_{3-i,t_q}$ or $B_{i,t_q+1} - B_{i,t_q}$ at stage t_q+1. As above we can deduce (55) and there is some minimal $z_q^e \leq c$ in $W_{e_0,t_{q+1}} - W_{e_0,t_q}$. In the case that $c = c_{t_q}$, trivially $z_q^e \leq c_{t_q}$. In the case that $c = a_{t_q}$, by the conditions on a_{t_q} in Case 1.3 or Case 2 $[c_{t_q}+1, a_{t_q}]$ is $\{\rho(|\beta'|) : \beta'0 \sqsubset \beta_{t_q}\}$-safe at stage t_q, hence in particular $[c_{t_q}+1, a_{t_q}] \subseteq W_{e_0,t_q}$. It follows that $z_q^e \leq c_{t_q}$ again.

Consequently,

$$\{w \leq c_{t_{q+1}} : w \notin W_{e_0,t_{q+1}}\} \subseteq \{w \leq c_{t_q} : w \notin W_{e_0,t_q}\} - \{z_q^e\},$$

and (48) follows.

To prove (49), it suffices to show that for some $q \in \{0, \ldots, p\}$ and some $e \leq \bar{e}$, $c_{t_q} \notin W_{e_0,t_{q+1}}$, whence $z_q^e \neq c_{t_q}$ and

$$\{w \leq c_{t_{q+1}} : w \notin W_{e_0,t_{q+1}}\} \subseteq \{w \leq c_{t_q} : w \notin W_{e_0,t_q}\} - \{z_q^e, c_{t_q}\},$$

proving (49).

For a contradiction assume that for all $q \in \{0, \ldots, p\}$ and all $e \leq \bar{e}$,

$$c_{t_q} \in W_{e_0,t_{q+1}}. \tag{56}$$

Let I be relevant above $y_k(I)$ at stage t_0+1 and let $r \in \{1, \ldots, p\}$ be maximal such that $\beta_{t_r} = \beta'_{\bar{e}}$ (since I is demoted to $\beta'_{\bar{e}}$ at stage t_0+1, it must hold that $\beta_{t_1} = \beta'_{\bar{e}}$, hence such r exists). Then from Lemma 4.22 we can conclude that $r = y_k(I)$ and that $\beta'_{\bar{e}}$ is active due to Case 2.2(a) at stage t_r+1. Since $c_{t_r}+1 = c_{t_{r-1}} \in W_{e_0,t_r}$ by assumption, it must hold that $[c_{t_r}+1, c_{t_r}+y_k(I)]$ is $\{\rho(|\beta'|) : \beta'0 \sqsubseteq \beta'_{\bar{e}}\}$-safe at stage t_r and $[c_{t_r}+1, c_{t_r}+y_k(I)] \cap (A_{1,t_r} \cup A_{2,t_r} \cup B_{0,t_r}) = \emptyset$. Indeed, then by (39), $[\min(I), c_{t_r}+y_k(I)] \cap (A_{1,t_r} \cup A_{2,t_r} \cup B_{0,t_r}) = \emptyset$. At each stage $t \in \{t_r+1, \ldots, t_p+1\}$ at most one number from I is enumerated into $A_1 \cup A_2 \cup B_0$ and at stages $t \in [t_r+1, t_{p+1}] - \{t_r+1, \ldots, t_p+1\}$ no such number is enumerated into

$A_1 \cup A_2 \cup B_0$. Hence $|[c_{t_{p+1}}+1, c_{t_r}+y_k(I)] \cap (A_{1,t_{p+1}} \cup A_{2,t_{p+1}} \cup B_{0,t_{p+1}})| \leq p+1-r$. Since $|[c_{t_{p+1}}+1, c_{t_r}+y_k(I)]| = c_{t_r}+y_k(I)-c_{t_{p+1}} = p+1-r+y_k(I)$, it follows that

$$|\{z \in [c_{t_{p+1}}+1, c_{t_r}+y_k(I)] : z \notin A_{1,t_{p+1}} \cup A_{2,t_{p+1}} \cup B_{0,t_{p+1}}\}| \geq y_k(I) \geq y_{k'}(I),$$

where I is relevant above $y_{k'}(I)$ at stage t_{p+1}. Moreover, since each $c \in [c_{t_{p+1}}+1, c_{t_r}+y_k(I)]$ is of the form $c = c_{t_q}$ for some $q \in [0,\ldots,p]$, by (56) we know that $[c_{t_{p+1}}+1, c_{t_r}+y_k(I)]$ is $\{\rho(|\beta'|) : \beta' \sqsubset \beta \text{ and } \beta'0 \sqsubseteq \alpha\}$-safe at stage t_{p+1}, where $\beta = \beta'_{\bar{e}+1}$ if $\bar{e} < |E|-1$ and $\beta = \alpha$ otherwise. But then by (41) and (42) at stage $t_{p+1}+1$ the interval I is demoted to β or α gets a diagonalisation witness from I assigned, contradicting the fact that $t_{p+1} = s_{m+1}$ and the hypothesis that I is demoted to $\beta'_{\bar{e}}$ at stage $s_{m+1}+1$.

This completes the proof of (49).

Now substituting $m = |E| \cdot y_0$ in (47), we get

$$\begin{aligned}\sum_{e=0}^{\bar{e}} |\{w \leq c_{s_{|E|\cdot y_0}} : w \notin W_{e0,s_{|E|\cdot y_0}}\}| &\leq (\bar{e}+1)\cdot(c_{s_{|E|\cdot y_0}}+1) - |E|\cdot y_0 \\ &\leq (\bar{e}+1)\cdot(c_{s_{|E|\cdot y_0}}+1-y_0).\end{aligned}$$

Hence there must be some $e \leq \bar{e}$ with

$$\begin{aligned}&|\{w \leq c_{s_{|E|\cdot y_0}} : w \notin W_{e0,s_{|E|\cdot y_0}}\}| \\ \leq\ & c_{s_{|E|\cdot y_0}}+1-y_0 \\ <\ & |[\min(I), c_{s_{|E|\cdot y_0}}]| \\ =\ & |\{w \leq c_{s_{|E|\cdot y_0}} : w \in I \text{ and } w \notin A_{1,s_{|E|\cdot y_0}} \cup B_{0,s_{|E|\cdot y_0}} \cup B_{1,s_{|E|\cdot y_0}} \cup C_{s_{|E|\cdot y_0}}\}|.\end{aligned}$$

Then α is ready for elimination at stage $s_{|E|\cdot y_0}+1$ and I is not demoted to $\beta'_{\bar{e}}$, contradicting the assumption.

□

Lemma 4.25. *Let I be an α-interval such that $I \subseteq C$ and such that there is a node β with $\beta 0 \sqsubset \alpha$. Let $I_0 = [\min(I), y_{\iota(I)}(I)]$, let $t_0 < \ldots < t_{|I_0|-1}$ be such that for $n \in \{0,\ldots,|I_0|-1\}$, at stage t_n+1 the number $c_{t_n} = y_{\iota(I)}(I) - n$ is enumerated into C and let β_{t_n} be the node that is active at stage t_n+1. Then either α is initialised and the assignment of I to α is cancelled at stage $t_{|I_0|-1}+1$, or $cand_{t_{|I_0|-1}+1}(\alpha) = I$ and $\alpha \not\sqsubseteq \delta_s$ for any $s > t_{|I_0|-1}$.*

Proof. We claim that there is some $m \in \{0,\ldots,|I_0|-2\}$ such that α is ready for elimination at stage t_m+1. Let $E = \{\beta' : \beta'0 \sqsubseteq \alpha\} = \{\beta'_0,\ldots,\beta'_{|E|-1}\}$ with $\beta'_0 \sqsubset \beta'_1 \ldots \sqsubset \beta'_{|E|-1}$. Note that by the hypothesis $E \neq \emptyset$. By Lemma 4.23, if for some $k \in \{0,\ldots,\iota(I)-1\}$ there is a sequence of stages $t_n+1 < t_{n+1}+1 < \ldots < t_{n+y_k+1}+1 \leq t_{|I_0|-2}+1$ at which I is relevant above $y_k := y_k(I)$, then the claim is true.

For a contradiction assume that there is no such sequence and no m as above. Since $cand_{t_0+1}(\alpha) = I$ at stage t_0+1 (when I becomes assigned to α), α must be active due to Case 1.3 at stage t_1+1 and I becomes relevant above $y_{\iota(I)-1}$. Hence I is relevant above $y_{\iota(I)}(I)$ at stage t_n+1 if and only if $n=0$. Using the assumption that I is relevant above y_k at at most y_k+1 many stages t_n+1 for $k \in \{0,\ldots,\iota(I)-1\}$, we see that there are at most

$$\begin{aligned}
1+\sum_{k=0}^{\iota(I)-1}(y_k+1) &= 1+\sum_{k=0}^{\iota(I)-1}(2^k(y_0+k)+1) \\
&= 1+\sum_{k=0}^{\iota(I)-1}(2\cdot 2^k(y_0+k)-2^k(y_0+k)+1) \\
&< 1+\sum_{k=0}^{\iota(I)-1}(2^{k+1}(y_0+(k+1))-2^k(y_0+k)) \\
&= 1+\sum_{k=0}^{\iota(I)-1}(y_{k+1}-y_k) \\
&= 1+y_{\iota(I)}(I)-y_0(I) \\
&= |I_0|-1
\end{aligned}$$

many stages t_n+1, $n \in \{0,\ldots,|I_0|-1\}$ at which I is relevant above some $y_k(I)$ with $k \geq 0$. In particular, at stage $t_{|I_0|-2}+1$ it is relevant above some $y_k(I)$ with $k<0$. Since I is relevant above $y_{\iota(I)}(I)$ at stage t_0+1, there must be at least $\iota(I)+1$ many stages $t_{i_0}+1,\ldots,t_{i_{\iota(I)}}+1$ (with $0 < i_0 < \ldots < i_{\iota(I)} \leq |I_0|-2$ chosen to be minimal) at which α is active due to Case 1.3 and the number y_k that I is relevant above is decreased.

For $e \in \{0,\ldots,|E|-1\}$, let

$$D_e = \{t \in \{t_{i_0},\ldots,t_{i_{\iota(I)}}\} : job_{t+1}(\beta'_e) = I\}.$$

Then each t_{i_k}, $0 \leq k \leq \iota(I)$ is in D_e for exactly one e. We show that there must be some e such that

$$|D_e| \geq (|E|\cdot y_0+1)\cdot(\sum_{e'<e}|D_{e'}|+1). \tag{57}$$

If this were not true, then for all $e<|E|$,

$$|D_e| < (|E|\cdot y_0+1)\cdot(y_0+2)^{2e}, \tag{58}$$

as we can see by induction on e: In fact, if (57) fails, then $|D_0| < (|E|\cdot y_0+1) = (|E|\cdot y_0+1)\cdot$

$(y_0+2)^{2\cdot 0}$, and once we have shown that $|D_{e'}| < (|E|\cdot y_0+1)\cdot(y_0+2)^{2e'}$ for $e' \leq e$, we see that

$$\begin{aligned}
|D_{e+1}| &< (|E|\cdot y_0+1)\cdot(\sum_{e'=0}^{e}|D_{e'}|+1) \quad \text{[by failure of (57)]}\\
&< (|E|\cdot y_0+1)\cdot(\sum_{e'=0}^{e}(|E|\cdot y_0+1)\cdot(y_0+2)^{2e'}+1) \quad \text{[by inductive hypothesis]}\\
&\leq (|E|\cdot y_0+1)\cdot((|E|\cdot y_0+1)\frac{(y_0+2)^{2e+2}-1}{(y_0+2)^2-1}+1)\\
&\leq (|E|\cdot y_0+1)\cdot(((y_0+2)^2-1)\frac{(y_0+2)^{2e+2}-1}{(y_0+2)^2-1}+1) \quad \text{[since } 0<|E|\leq|\alpha|<y_0]\\
&= (|E|\cdot y_0+1)\cdot(y_0+2)^{2(e+1)},
\end{aligned}$$

proving (58) for $e+1$.

But then it also follows that

$$\begin{aligned}
(y_0+2)^{2|E|} &= \iota(I)+1 \quad \text{[by definition of } \iota(I) \text{ and } y_0]\\
&= \sum_{e=0}^{|E|-1}|D_e| \quad \text{[since } (D_0,\dots D_{|E|-1}) \text{ is a partition of } \{t_{i_0},\dots,t_{i_{\iota(I)}}\}]\\
&< \sum_{e=0}^{|E|-1}(|E|\cdot y_0+1)\cdot(y_0+2)^{2e} \quad \text{[by(58)]}\\
&= (|E|\cdot y_0+1)\cdot\sum_{e=0}^{|E|-1}(y_0+2)^{2e}\\
&\leq ((y_0+2)^2-1)\cdot\frac{(y_0+2)^{2|E|}-1}{(y_0+2)^2-1}\\
&= (y_0+2)^{2|E|}-1,
\end{aligned}$$

which is not possible. This shows that (57) must be true for some e.

Let $\bar{e}$ be the least e such that (57) holds. Assume that $D_{\bar{e}} = \{t_{j_0},\dots,t_{j_{|D_{\bar{e}}|-1}}\}$ with $t_{j_0} < \dots < t_{j_{|D_{\bar{e}}|-1}}$. Since there are at least $\sum_{e<\bar{e}}|D_e|+1$ pairwise disjoint sequences of $|E|\cdot y_0+1$ stages $t_{j_n},\dots,t_{j_{n+|E|\cdot y_0}}$ in $D_{\bar{e}}$, there must be at least one such sequence with

$$[t_{j_n}, t_{j_{n+|E|\cdot y_0}}] \cap D_e = \emptyset \text{ for all } e < \bar{e}. \tag{59}$$

But the existence of such a sequence is a contradiction to Lemma 4.24. We have thus proven that α is ready for eliminiation at stage t_m+1 for some $m \in \{0,\dots,|I_0|-2\}$.

Now assume that α is not initialised at stage $t_{|I_0|-1}+1$. Let $m < |I_0|-1$ be such that α is ready for elimination at stage t_m+1, that is

$$|\{x \leq c_{t_m} : x \notin W_{e_0,t_m}\}| < |\{x \leq c_{t_m} : x \in I\}|, \tag{60}$$

for some $\beta'0 \sqsubseteq \alpha$ with $\rho(|\beta'|) = \mathcal{J}^{i'}_{e'}$, $e' = \langle e'_0, e'_1, e'_2 \rangle$. We claim that α is ready for elimination at stage $t_{m+1}+1$, too.

In fact, α requires attention at stage $t_m + 1$ and – since $cand_{t_{m+1}}(\alpha) = cand_{t_m+1}(\alpha) = I$ – at stage $t_{m+1}+1$. Hence

$$l_{t_m}(\mathcal{J}^{i'}_{e'}) > \max(I)$$

and

$$l_{t_{m+1}}(\mathcal{J}^{i'}_{e'}) > \max(I).$$

Since (using (39)) c_{t_m} is enumerated into $B_{i',t_m+1} - B_{i',t_m}$ or $A_{3-i',t_m+1} - A_{3-i',t_m}$ at stage $t_m + 1$, it follows that

$$\hat{\Phi}^{W_{e'_0,t_{m+1}}}_{e'_1,t_{m+1}} \restriction (c_{t_m}+1) = B_{i',t_{m+1}} \restriction (c_{t_m}+1) \neq B_{i',t_m} \restriction (c_{t_m}+1) = \hat{\Phi}^{W_{e'_0,t_m}}_{e'_1,t_m} \restriction (c_{t_m}+1)$$

or

$$\hat{\Phi}^{W_{e'_0,t_{m+1}}}_{e'_2,t_{m+1}} \restriction (c_{t_m}+1) = A_{3-i',t_{m+1}} \restriction (c_{t_m}+1) \neq A_{3-i',t_m} \restriction (c_{t_m}+1) = \hat{\Phi}^{W_{e'_0,t_m}}_{e'_2,t_m} \restriction (c_{t_m}+1)$$

Consequently, since $\hat{\Phi}_{e'_1}$ and $\hat{\Phi}_{e'_2}$ are ibT-functionals, there is some $z \leq c_{t_m}$ in $W_{e'_0,t_{m+1}} - W_{e'_0,t_m}$. Hence

$$|\{x \leq c_{t_{m+1}} : x \notin W_{e'_0,t_{m+1}}| < |\{x \leq c_{t_m} : x \notin W_{e'_0,t_m}\}|. \tag{61}$$

Also,

$$|\{x \leq c_{t_{m+1}} : x \in I\}| = |\{x \leq c_{t_m} : x \in I\}| - 1, \tag{62}$$

since $c_{t_m} = c_{t_{m+1}} + 1$.

Using (60), (61) and (62) it follows that

$$|\{x \leq c_{t_{m+1}} : x \notin W_{e'_0,t_{m+1}}\}| < |\{x \leq c_{t_{m+1}} : x \in I\}|. \tag{63}$$

Hence α is ready for elimination at stage $t_{m+1}+1$.

By induction, this shows that α is ready for elimination at every stage $t_{m'}+1$, $m' \in \{m, \ldots, |I_0|-1\}$. In particular, $cand_{t_{|I_0|-1}+1}(\alpha) = I$ and $|\{x \leq \min(I) : x \notin W_{e_0,t_{|I_0|-1}}\}| = 0$ and it follows that for $s > t_{|I_0|-1}$

$$\hat{\Phi}^{W_{e'_0,s}}_{e'_1,s}(\min(I)) = \hat{\Phi}^{W_{e'_0,t_{|I_0|-1}}}_{e'_1,t_{|I_0|-1}}(\min(I)) = B_{i',t_{|I_0|-1}}(\min(I)) = 0 \neq 1 = B_{i',s}(\min(I))$$

if $i' = 1$ and

$$\hat{\Phi}^{W_{e'_0,s}}_{e'_2,s}(\min(I)) = \hat{\Phi}^{W_{e'_0,t_{|I_0|-1}}}_{e'_2,t_{|I_0|-1}}(\min(I)) = A_{3-i',t_{|I_0|-1}}(\min(I)) = 0 \neq 1 = B_{3-i',s}(\min(I))$$

if $i' = 2$.

Then $l_s(\mathcal{J}^{i'}_{e'}) \leq \min(I)$ and $\alpha \not\sqsubseteq \delta_s$ for all $s > t_{|I_0|-1}$, proving the lemma.

$\square$

Lemma 4.26. *(i) Let $\beta 0 \sqsubseteq \delta_s$ and let $job_s(\beta)) = I$ be an α-interval. Assume that $\alpha \leq \delta_s$ and for every β' with $\beta' 0 \sqsubseteq \beta 0$ it holds that $l_s(\rho(|\beta'|)) > \max(I)$. Then β requires attention and is α-linked at stage $s+1$.*

(ii) Let $\alpha \sqsubseteq \delta_s$ and let $cand_s(\alpha) = I$. Assume that for every β' with $\beta' 0 \sqsubseteq \alpha$ it holds that $l_s(\rho(|\beta'|)) > \max(I)$ and if $job_s(\beta') = I'$ is an α'-interval, then $\alpha \leq \alpha'$. Then α requires attention at stage $s+1$.

Proof. (i) From Lemma 4.25 it is clear that $I \not\subseteq C_s$, because otherwise, if the last enumeration into $I \cap C$ were at some stage $t \leq s$, then $cand_{t'}(\alpha) = I$ for all $t' \geq t$ until α becomes initialised and the assignment of I to α is cancelled, contradicting the fact that $job_s(\beta) = I$. Note that by (39) the numbers from I are enumerated into C in decreasing order (unless the assignment of I to α is cancelled, which does not happen until stage s). Moreover, for $i \in \{1,2\}$ and $j \in \{0,1,2\}$, a number $b \in I$ is enumerated into A_i or B_j at some stage only if a number $c \in I$ with $c \leq b$ is enumerated into C at the same stage. From these facts it follows that there is $c_s \in I$ satisfying (39) at stage s and hence that β requires attention and is α-linked at stage $s+1$.

(ii) Again, from Lemma 4.25 it follows that $I \not\subseteq C_s$ (there is a node β with $\beta 0 \sqsubseteq \alpha$, because otherwise $cand_s(\alpha)$ were undefined at every stage). As in (i) we can conclude that c_s satisfying (39) exists.

Let β be the least node such that $\beta 0 \sqsubseteq \alpha$ and $\rho(|\beta|)$ is a join requirement, if such a node exists, and $\beta = \alpha$ otherwise. Then (41) is trivially satisfied for every $y > c_s$.

At the stage s_0+1 when I is assigned to α, for $c_{s_0} = y_{\iota(I)}(I)$ there are $y_{\iota(I)+1}(I) - y_{\iota(I)}(I) - 1$ numbers $z > c_{s_0}$ with $z \in I - (A_{1,s_0} \cup A_{2,s_0} \cup B_{0,s_0})$. Since a new number from I is enumerated into $A_1 \cup A_2 \cup B_0$ only if a new number from I is enumerated into C, and since the numbers from I are enumerated into C in decreasing order, it still holds that there are $y_{\iota(I)+1}(I) - y_{\iota(I)}(I) - 1$ numbers $z \geq c_s$ with $z \in I - (A_{1,s} \cup A_{2,s} \cup B_{0,s})$. But

$$\begin{aligned}
y_{\iota(I)+1}(I) - y_{\iota(I)}(I) - 1 &= 2^{\iota(I)+1}(y_0(I) + (\iota(I)+1)) - 2^{\iota(I)}(y_0(I) + \iota(I)) - 1 \\
&\geq 2^{\iota(I)}(y_0(I) + \iota(I)) \\
&= y_{\iota(I)}(I) \\
&\geq y_k(I).
\end{aligned}$$

for $k \leq \iota(I)$. Hence for the greatest number $y \in I$ with $y \notin (A_{1,s} \cup A_{2,s} \cup B_{0,s})$, (42) is true. It follows that a_s exists and α requires attention due to Case 1.3 at stage $s+1$. $\square$

Lemma 4.27. *(i) Let $\alpha \sqsubset \mathrm{TP}$. Then α is initialised only finitely many times.*

(ii) Let $\beta 0 \sqsubset \mathrm{TP}$. Then for each interval I, there are only finitely many stages s such that $job_s(\beta) = I$.

Proof. (i) The proof is by induction on $|\alpha|$. Let the claim be true for all $\alpha' \sqsubset \alpha$ and let s_0 be a stage such that no $\alpha' \sqsubset \alpha$ is initialised at any stage $s \geq s_0$ and such that $\alpha \leq \delta_s$ for all $s \geq s_0$. Such a stage exists by the True Path Lemma. Then whenever an interval is assigned to some $\alpha' \sqsubset \alpha$ after stage s_0, this assignment is permanent. Hence there are only finitely many α'-intervals for $\alpha' \sqsubset \alpha$ defined during the construction. Whenever α is initialised at some stage $s \geq s_0$, this is because some $\alpha' \sqsubset \alpha$ is active at stage s. Then either α' gets a diagonalisation witness assigned at stage s – which is possible only once for each α', because these assignments are permanent, too –, or a new number from some α'-interval is enumerated into C at stage s, which by the above observation can happen only finitely often. The claim follows.

(ii) For a contradiction assume that there is some least node $\beta 0 \sqsubset \mathrm{TP}$ such that there exists an α-interval I with $job_s(\beta) = I$ for infinitely many s. Note that $job_s(\beta) = I$ for *almost every* s. This holds because whenever $job_s(\beta) = I$ and $job_{s+1}(\beta) \neq I$, then either α is initialised at stage $s+1$ and $job_{s'}(\beta) \neq I$ for all $s' \geq s+1$, or β is active due to Case 2.2 and a new number c_s is enumerated into $I \cap C_{s+1} - I \cap C_s$. Since I is finite, the latter can happen only finitely often.

Let s_0 be the least stage such that $job_s(\beta) = I$ for all $s \geq s_0$. In particular, for $s+1 \geq s_0$, α is not initialised at stage $s+1$; hence $\alpha \leq \delta_s$. By the fact that $\beta 0 \sqsubset \mathrm{TP}$, it holds that $\lim_{s\to\infty} l_s(\rho(|\beta'|)) = \infty$ for every β' with $\beta' 0 \sqsubseteq \beta 0$. Hence at almost every $\beta 0$-stage $s \geq s_0$ it holds that $\beta 0 \sqsubseteq \delta_s$, $\alpha \leq \delta_s$ and $l_s(\rho(|\beta'|)) > \max(I)$ for $\beta' 0 \sqsubseteq \beta 0$. By Lemma 4.26 β requires attention and is α-linked at stage $s+1$ for each such s.

Note that by the True Path Lemma there are infinitely many $\beta 0$-stages. We can now conclude that β must be active at infinitely many stages $s \geq s_0$. Let s_1 be the least $\beta 0$-stage after stage s_0 such that β requires attention and is α-linked at stage $s+1$ whenever $s \geq s_1$ is a $\beta 0$-stage. Then no node $\alpha' \sqsupseteq \beta 0$ is active at stages $s > s_1$. This implies that there are only finitely many α'-intervals with $\alpha' \sqsupseteq \beta 0$ defined during the construction. By the True Path Lemma, the same holds for α'-intervals with $\alpha' <_L \beta 0$. Finally, since $\beta 0$ is initialised only finitely often by (i), there are only finitely many α'-intervals with $\alpha' \sqsubseteq \beta$ defined during the construction. Hence by minimality of β there is a stage $s_2 \geq s_1$ such that for all $\beta' \sqsubset \beta$, all α'-intervals I' with $\alpha' \leq \beta 0$ or $\alpha' \sqsupseteq \beta 0$ and all $s \geq s_2$, $job_s(\beta') \neq I'$. In other words, whenever $I' = job_s(\beta')$ for such β', I' and s, it holds that $\beta 0 <_L \alpha'$.

Hence no $\beta' \sqsubset \beta$ requires attention due to Case 2 at any stage $s+1$ where $s \geq s_3$ is a $\beta 0$-stage. Moreover, no $\alpha' \sqsubseteq \beta$ requires attention due to Case 1.1, Case 1.2, Case 1.3 or Case 1.4 at any stage $s \geq s_3$, because otherwise $\beta 0$ and α were initialised and $I \neq job_{s+1}(\beta)$. It now follows that β receives attention and is active at every stage $s+1$ where $s \geq s_3$ is a $\beta 0$-stage. Hence β is active due to Case 2 infinitely often.

But each time that happens, a new number from I is enumerated into C, which is impossible. This is a contradiction, so the claim must be true. $\square$

We are now ready to show that all requirements are satisfied.

Lemma 4.28. *It holds that $A_1 \leq_{\mathrm{ibT}} B_0, B_1 \leq_{\mathrm{ibT}} C$, $A_2 \leq_{\mathrm{ibT}} B_0, B_2 \leq_{\mathrm{ibT}} C$, $B_0 = A_0 \cup A_1$ and $A_0 \cap A_1 = \emptyset$.*

Proof. The desired reductions hold by permitting. This and the fact that $\mathcal{J}^0$ is true can directly be verified by considering the conditions and the respective actions in the various cases of the construction. $\square$

Lemma 4.29. *Every meet requirement $\mathcal{M}_e^0$ ($e = \langle e_0, e_1, e_2 \rangle \in \mathbb{N}$) is satisfied.*

Proof. Let $n = \rho^{-1}(\mathcal{M}_e^0)$ and let $\gamma \sqsubset \mathrm{TP}$ be the unique node of length n on the true path. Assume that the premise of $\mathcal{M}_e^0$ is true, that is $W_{e_0} = \hat{\Phi}_{e_1}^{B_1} = \hat{\Phi}_{e_2}^{B_2}$ (otherwise $\mathcal{M}_e^0$ is trivially satisfied). Since $\gamma 0 \sqsubset \mathrm{TP}$ by the definition of the true path, due to the True Path Lemma there are infinitely many $\gamma 0$-stages. By Lemma 4.27 (i), there is a $\gamma 0$-stage s_0 such that $\gamma 0$ is never initialised at any stage $s \geq s_0$. Then for no node $\alpha \leq \gamma$ any numbers from an α-interval are enumerated into B_1 or B_2 at any stage $s \geq s_0$.

Now, in order to compute $W_{e_0}(x)$ for some given x, compute the least $\gamma 0$-stage $s_1 \geq \max(\{s_0, x\})$ such that $l_{s_1}(\mathcal{M}_e^0) > x$, such that $l_{s_1}(\rho(|\beta|)) > \max(I)$ for every node $\beta 0 \sqsubseteq \gamma 0$ and every interval I assigned to any node up to stage x, and such that $x < \min(I')$ for every interval I' such that $job_{s_1}(\beta) = I'$ for some $\beta 0 \sqsubseteq \gamma 0$. Such a stage exists because $\lim_{s\to\infty} l_s(\rho(|\beta|)) = \infty$ for all $\beta 0 \sqsubseteq \gamma 0$ and by Lemma 4.27 (ii). We claim that $x \in W_{e_0}$ if and only if $x \in W_{e_0,s_1}$.

Let $s_1 \leq s_2 \leq \ldots$ be the sequence of $\gamma 0$-stages, starting with s_1. An inductive proof shows that for every $n \geq 1$, there is at most one interval I with $\min(I) \leq x$ and $job_s(\beta) = I$ for any $s \in (s_n, s_{n+1}]$ and any $\beta 0 \sqsubseteq \gamma 0$. Indeed, any such interval I must be an α-interval for some $\alpha \sqsupseteq \gamma 0$. This is true because if I is an α-interval with $\alpha < \gamma 0$, then $I \neq job_{s_1}(\beta)$ for any $\beta 0 \sqsubseteq \gamma 0$ and I is never demoted at stages $t \geq s_0$ (otherwise $\gamma 0$ would be initialised at stage t); if I is an α-interval with $\gamma 0 <_L \alpha$, then the assigment of I to α is cancelled at stage $s_1 + 1$, or I only becomes assigned to α after stage $s_1 + 1$, in which case $\min(I') > s_1 \geq x$.

So let above claim be true for $n - 1$. By the inductive hypothesis there is at most one interval I with $\min(I) \leq x$ and $job_{s_n}(\beta')$ for any $\beta' 0 \sqsubseteq \gamma 0$. If there is no demotion of such an interval at stage $s_n + 1$, then the same holds with $s_n + 1$ in place of s_n. On the other hand, if any α-interval I with $\min(I) \leq x$ is demoted to any β with $\beta 0 \sqsubseteq \gamma 0$ at stage $s_n + 1$, then for all $\beta' 0 \sqsubseteq \alpha$ (hence by the argument from the previous paragraph, for all $\beta' 0 \sqsubseteq \gamma 0$) and all α'-intervals I' with $job_{s_n}(\beta') = I'$ it holds that $\alpha \leq \alpha'$ and in fact $\alpha < \alpha'$, because α can only have one interval assigned at any time; but then all these α' are initialised at stage $s_n + 1$ and $job_{s_n+1}(\beta')$ is set to undefined for all such $\beta' \neq \beta$. Hence I is the only interval I' with $\min(I') \leq x$ and $job_{s_n+1}(\beta') = I'$ for any $\beta' 0 \sqsubseteq \gamma 0$.

Furthermore, at stages $t+1 \in (s_n+1, s_{n+1}]$ there are no demotions of α-intervals I with $\min(I) \leq x$, because otherwise $\gamma 0 \sqsubseteq \alpha \sqsubseteq \delta_t$, as we argued, contradicting the choice of s_n and s_{n+1}. This shows that the claim above holds for n.

We claim that, for $n \geq 1$,

$$B_{1,s_{n+1}} \restriction (x+1) = B_{1,s_n} \restriction (x+1) \text{ or } B_{2,s_{n+1}} \restriction (x+1) = B_{2,s_n} \restriction (x+1). \tag{64}$$

Indeed, the only way that numbers $y \leq x$ can be enumerated into B_1 or B_2 at stages $s+1$ with $s_n + 1 < s+1 \leq s_{n+1}$ is because some node β with $\beta 0 \sqsubset \gamma 0$ is active due to Case 2 and y is enumerated into an α-interval $I = job_s(\beta)$, or because y does belong to an α'-interval, where α' is initialised at stage $s+1$. As we have argued, the first reason holds for only one interval I. Since $\alpha \not\sqsubseteq \delta_s$ for any $s \in (s_n, s_{n+1})$, the interval I is not demoted at any stage $s+1 \in (s_n+1, s_{n+1}]$. But between two demotions of I numbers from I are enumerated into at most one of B_1 or B_2 (because each node β performs enumerations into at most one of these sets when it is active due to Case 2). On the other hand, if y belongs to an α'-interval, where α' is initialised at stage $s+1$, then $\alpha' \sqsupset \gamma 0$, since nodes $\alpha' \leq \gamma 0$ are not initialised and nodes $\alpha' >_L \gamma 0$ have intervals I' with $\min(I') > x$ assigned. But then α' is only initialised because some $\beta 0$ with $\beta 0 \sqsubset \gamma 0$ is active due to Case 2 at stage $s+1$, and y is enumerated into B_1 if β enumerates some number into B_1 and into B_2 otherwise. This shows that we may neglect enumerations into B_1 or B_2 which are due to initialisation at stages s with $s_n + 1 < s \leq s_{n+1}$.

Hence

$$B_{1,s_{n+1}} \restriction (x+1) = B_{1,s_n+1} \restriction (x+1) \text{ or } B_{2,s_{n+1}} \restriction (x+1) = B_{2,s_n+1} \restriction (x+1). \tag{65}$$

Assume that $B_{1,s_{n+1}} \restriction (x+1) \neq B_{1,s_n+1} \restriction (x+1)$. Let I be an α-interval such that some $y \in I$ with $y \leq x$ is enumerated into B_1 at some stage $s \in (s_n+1, s_{n+1}]$. Since I is not demoted at any such stage s, there is some β with $\beta 0 \sqsubseteq \gamma 0$ such that either $job_{s_n}(\beta) = I$ or I is demoted to β at stage s_n+1. If I is demoted at stage s_n+1, then $\rho(|\beta|) = \mathcal{J}^1_{e'}$ for some e', because the enumeration of y into B_1 must be caused by β being active due to Case 2. Then no number is enumerated into $I \cap B_2$ at stage s_n+1, that is $B_{2,s_n} \restriction (x+1) = B_{2,s_n+1} \restriction (x+1) = B_{2,s_{n+1}} \restriction (x+1)$ by (65). If I is not demoted at stage s_n+1, then $\beta 0 \sqsubseteq \gamma 0 \sqsubseteq \delta_{s_n}$ and $\alpha \leq \delta_{s_n}$ (otherwise α were cancelled at stage s_n+1 and y would not be enumerated later). Also, $l_{s_n}(\rho(|\beta'|)) \geq l_{s_1}(\rho(|\beta'|)) > \max(I)$ for all β' with $\beta' 0 \sqsubseteq \beta 0$ (note that I must become assigned to α up to stage x, because after stage x only intervals I' with $\min(I') > x$ are assigned to nodes). By Lemma 4.26, β requires attention at stage s_n+1. Since there are no nodes β' with $\beta' 0 \sqsubset \beta 0$ and $job_{s_n}(\beta') = I'$ for any I' with $\min(I') \leq x$ and since no node $\alpha' \sqsubseteq \beta$ can be active due to Case 1.1, Case 1.2, Case 1.3 or Case 1.4 at stage s_n+1, no node below β enumerates any number $z \leq x$ into B_2 at stage s_n+1, while by β requiring attention nodes above β cannot be active at stage s_n+1. If β is active at stage s_n+1, then there is neither an enumeration into B_2 at stage s_n+1, because otherwise this enumeration would be a number from I – but then again there were no enumeration of numbers from I into B_1 until the next demotion of I, that is until stage s_{n+1}. Hence in all cases $B_{2,s_n} \restriction (x+1) = B_{2,s_n+1} \restriction (x+1) = B_{2,s_{n+1}} \restriction (x+1)$. This completes the proof of (64).

From (64) it follows that, if $x \notin W_{e_0,s_1}$, inductively

$$\hat{\Phi}^{B_{1,s_n}}_{e_1,s_n}(x) = \hat{\Phi}^{B_{2,s_n}}_{e_2,s_n}(x) = W_{e_0,s_n}(x) = 0.$$

This is true for s_1, because $l_{s_1}(\mathcal{M}^0_e) > x$. If it is true for s_n, then $B_{1,s_{n+1}} \restriction (x+1) = B_{1,s_n} \restriction (x+1)$ or $B_{2,s_{n+1}} \restriction (x+1) = B_{2,s_n} \restriction (x+1)$. If, say $B_{1,s_{n+1}} \restriction (x+1) = B_{1,s_n} \restriction (x+1)$, then

$$0 = \hat{\Phi}^{B_{1,s_n}}_{e_1,s_n}(x) = \hat{\Phi}^{B_{1,s_{n+1}}}_{e_1,s_{n+1}}(x) = \hat{\Phi}^{B_{2,s_{n+1}}}_{e_2,s_{n+1}}(x) = W_{e_0,s_{n+1}}(x),$$

proving the equality for s_{n+1}.

Since $W_{e_0}(x) = \lim_{n\to\infty} W_{e_0,s_n}(x)$, this implies that $W_{e_0}(x) = 0$. □

Lemma 4.30. *Every meet requirement $\mathcal{M}^i_e$ ($i \in \{1,2\}$, $e = \langle e_0, e_1, e_2\rangle \in \mathbb{N}$) is satisfied.*

Proof. Let $n = \rho^{-1}(\mathcal{M}^i_e)$ and let $\gamma \sqsubset \mathrm{TP}$ be the unique node of length n on the true path. Assume that the hypothesis of $\mathcal{M}^i_e$ is true, that is $W_{e_0} = \hat{\Phi}^{B_0}_{e_1} = \hat{\Phi}^{B_i}_{e_2}$ (otherwise $\mathcal{M}^i_e$ is trivially satisfied). Since $\gamma 0 \sqsubset \mathrm{TP}$ by the definition of the true path, due to the True Path Lemma there are infinitely many $\gamma 0$-stages. By Lemma 4.27(i), there is a $\gamma 0$-stage s_0 such that $\gamma 0$ is never initialised at any stage $s \geq s_0$. Then for no node $\alpha \leq \gamma$ any numbers from an α-interval are enumerated into B_0 or B_i at any stage $s \geq s_0$.

Now, in order to compute $W_{e_0}(x)$ for some given x with oracle $A_i \restriction (x+1)$, compute the least $\gamma 0$-stage $s_1 \geq \max(\{s_0, x\})$ such that $A_{i,s_1} \restriction (x+1) = A_i \restriction (x+1)$, such that $l_{s_1}(\mathcal{M}^i_e) > x$, such that $l_{s_1}(\rho(|\beta|)) > \max(I)$ for every node $\beta 0 \sqsubseteq \gamma 0$ and such that $x < \min(I')$ for every interval I' such that $job_{s_1}(\beta) = I'$ for some $\beta 0 \sqsubseteq \gamma 0$. Such a stage exists by Lemma 4.27 (ii).

Let $s_1 \leq s_2 \leq \ldots$ be the sequence of $\gamma 0$-stages, starting with s_1. We claim that, for $n \geq 2$,

$$B_{0,s_{n+1}} \restriction (x+1) = B_{0,s_n} \restriction (x+1) \text{ or } B_{i,s_{n+1}} \restriction (x+1) = B_{i,s_n} \restriction (x+1). \tag{66}$$

For a contradiction, assume that there are minimal stages $t_0, t_i \in [s_n, s_{n+1})$ and numbers $y_0, y_i \leq x$ such that $y_0 \in B_{0,t_0+1} - B_{0,t_0}$ and $y_i \in B_{i,t_i+1} - B_{i,t_i}$. There must be some α_0-interval I_0 such that $y_0 \in I_0$ and some α_i-interval I_i such that $y_i \in I_i$.

Note that $\gamma 0 \sqsubset \alpha_0$ and $\gamma 0 \sqsubset \alpha_i$: By the choice of s_0 there are no enumerations into α-intervals for $\alpha \leq \gamma$ after stage s_0, and nodes $\alpha >_L \gamma 0$ are initialised at stage $s_1 + 1$ and only assigned intervals I with $\min(I) > x$ after stage $s_1 + 1$.

We consider the following cases.

- Assume that $\alpha_0 \neq \alpha_i$ and neither α_0 is initialised at stage $t_0 + 1$ nor α_i is initialised at stage $t_i + 1$. Since $\alpha_0 \neq \alpha_i$, it follows that $I_0 \neq I_i$. Then also $t_0 \neq t_i$. Hence there is $j \in \{0, i\}$ such that $t_j > s_n$. Let β_0 and β_i be the nodes which are active at stage

t_0+1 and t_i+1, respectively. Necessarily $\beta_j \sqsubseteq \alpha_j$; in fact $\beta_j \sqsubseteq \gamma$, because $\gamma 0 \not\sqsubseteq \delta_{t_j}$ and $\beta_j \sqsubseteq \delta_{t_j}$. It follows that β_j is active due to Case 2 at stage t_j+1. Hence $job_{t_j}(\beta_j) = I$, and in fact $job_{s_n+1}(\beta_j) = I_j$, because α_j is not active and cannot demote I_j at stages $s \in (s_n+1, s_{n+1}]$. There are two cases: Either $job_{s_n}(\beta_j) = I_j$, or I_j is demoted at stage s_n+1.

If the second case holds, i.e. I_j is demoted at stage s_n+1, then α_j is active and α_{i-j} is not active. By the conditions on demotion of I, if $\alpha_{i-j} < \alpha_j$, then there is no node β' with $\beta' 0 \sqsubseteq \gamma 0$ and $job_{s_n}(\beta') = I_{i-j}$; then, since I_{i-j} is not demoted at stage s_n+1 and cannot be demoted at any stage $s \in (s_n+1, s_{n+1}]$, there is no node β' with $\beta' 0 \sqsubseteq \gamma 0$ and $job_{t_{i-j}}(\beta') = I_{i-j}$; but then $\gamma 0 \sqsubset \beta_{i-j}$ and β_{i-j} cannot become active at stage $t_{i-j}+1$, a contradiction. On the other hand, if $\alpha_j < \alpha_{i-j}$, then α_{i-j} is initialised at stage s_n+1, which implies that $t_{i-j} = s_n$ and contradicts the assumption on α_0 and α_i.

If the first case holds, i.e. $job_{s_n}(\beta_j) = I_j$, then there is no node β' with $\beta' 0 \sqsubseteq \gamma 0$ and $job_{s_n}(\beta') = I_{i-j}$. Indeed, otherwise there were a stage $s < s_n$ such that $job_s(\beta') = I_{i-j}$ and I_j were demoted at stage $s+1$, or the other way round. Again, this would lead to the contradiction that one of α_0 and α_i would be initialised at stage $s+1$. It holds that $\alpha_j \leq \delta_{s_n}$ (otherwise α_j were initialised at stage s_n+1, a contradiction again), $\beta_j 0 \sqsubseteq \gamma 0 \sqsubseteq \delta_{s_n}$ and for every $\beta' 0 \sqsubseteq \beta_j 0$ we know that $l_{s_n}(\rho(|\beta'|)) > l_{s_1}(\rho(|\beta'|)) > \max(I)$ by the choice of s_1. Hence by Lemma 4.26 (a) β_j requires attention at stage s_n+1. A fortiori, α_{i-j} is not active and I_{i-j} is not demoted at stage s_n+1. Again, since I_{i-j} cannot be demoted at any stage $s \in (s_n+1, s_{n+1}]$, there is no node β' with $\beta' 0 \sqsubseteq \gamma 0$ and $job_{t_{i-j}}(\beta') = I_{i-j}$; then β_{i-j} cannot be active at stage $t_{i-j}+1$, a contradiction.

- Assume that α_i is initialised at stage t_i+1. Then y_i is enumerated not only into B_i, but also into $A_{i,t_i+1} - A_{i,t_i}$ (note that $y_i = \min(I_i) \notin A_{i,t_i}$, because whenever a number from I_i is enumerated into A_i, then the same or a smaller number from I_i is enumerated into B_i; hence if $\min(I)$ were in A_{i,t_i}, then it would also be in B_{i,t_i}). This contradicts the choice of s_1.

- Assume that α_0 is initialised at stage t_0+1 and that $t_0 > s_n$. Then there is a node $\beta \leq \alpha_0$ which is active at stage t_0+1. By the choice of s_0, $\beta \not<_L \gamma 0$, and since $\gamma 0 \not\sqsubseteq \delta_{t_0}$, neither $\beta \sqsupseteq \gamma 0$. Hence $\beta \sqsubseteq \gamma$. Again by the choice of s_0, β can only be active due to Case 2 and is α-linked for some $\alpha \geq \gamma 0$ at stage t_0; in particular, $job_{t_0}(\beta) = I$ for some α-interval I. By the fact that α_0 is initialised at stage t_0+1 it follows that $\gamma 0 \sqsubseteq \alpha \sqsubset \alpha'$. Then I was assigned to α before I_0 was assigned to α_0, and hence $\max(I) < \min(I_0) < x$. At stage t_0+1 some number $z_0 \in I$ is enumerated into B_0. Hence we can substitute y_0 by z_0 and α_0 by α and are in the case that α_0 is not initialised at stage t_0+1.

- Assume that α_0 is initialised at stage $t_0+1 = s_n+1$, but α_i is not initialised at stage t_i+1. If $t_i = s_n$, then y_i is enumerated into B_i at stage s_n+1 and some number is enumerated into $A_{i,s_n+1} - A_{i,s_n}$. But then y_0 is also enumerated into $A_{i,s_n+1} - A_{i,s_n}$, contradicting the choice of s_1.

Hence $t_i > s_n$. Let β_i be the node which is active at stage $t_i + 1$. As in the first case considered above we conclude that either α_i is active at stage $s_n + 1$ and I_i is demoted to β_i, or $job_{s_n}(\beta_i) = job_{s_n+1}(\beta_i) = I_i$. Since β_i enumerates y_i into B_i by being active due to Case 2, it must hold that $\rho(|\beta_i|) = \mathcal{J}^i_{e'}$ ($e' \in \mathbb{N}$). Hence if I_i is demoted to β_i at stage $s_n + 1$, then a_{s_n} is enumerated into $A_{i,s_n+1} - A_{i,s_n}$; then by the instructions for initialisation y_0 is enumerated into $A_{i,s_n+1} - A_{i,s_n}$, too, contradicting the choice of s_1. On the other hand, if $job_{s_n}(\beta_i) = job_{s_n+1}(\beta_i) = I_i$, then as in the first case above we argue that β_i requires attention at stage $s_n + 1$, and similarly to the argument above we see that for all nodes $\alpha' \sqsupseteq \gamma 0$ there is no α'-interval I' with $job_{s_n}(\beta') = I'$ for any β' with $\beta' 0 \sqsubseteq \gamma 0$. The latter holds for nodes $\alpha' <_L \gamma 0$, too, because if $job_{s_n}(\beta') = I'$ for an α'-interval I' with $\alpha' <_L \gamma$, then by Lemma 4.27 there is a least stage $s > s_n$ with $job_{s_n}(\beta') \neq I$; then α' is initialised at stage s or β' is active and α'-linked at stage s – in either case, $\gamma 0$ is initialised at stage s, contradicting the choice of s_0.

It follows that no node $\beta' \sqsubset \beta_i$ requires attention due to Case 2 at stage $s_n + 1$. Hence β_i receives attention and is active due to Case 2 at stage s_n+1. Then β_i enumerates a number into $A_{i,s_n+1} - A_{i,s_n}$, and hence by the instructions for initialisation y_0 is enumerated into $A_{i,s_n+1} - A_{i,s_n}$, too, contradicting the choice of s_1.

- Finally, assume that none of the above holds, i.e. $\alpha_0 = \alpha_i$ and neither α_0 is initialised at stage $t_0 + 1$ nor α_i is initialised at stage $t_i + 1$. Then for $j \in \{0, i\}$ the interval I_j must already be assigned to α_0 up to stage $s_n + 1$, because α_i is not active at any stage $s \in (s_n + 1, s_{n+1}]$. In fact I_j must be assigned to α_0 at stage s_n or $s_n + 1$. It now follows that $I_0 = I_i$, because α has only one interval assigned at any time, and if α were initialised at stage $s_n + 1$, then it would not get a new interval assigned at stage $s_n + 1$.

  If $t_0 = t_i = s_n$, then y_0 is enumerated into $A_{i,t_0+1} - A_{i,t_0}$, contradicting the choice of s_1. If $s_n < t_i$, let β_i be the node that is active at stage t_i+1. Then again $job_{t_i}(\beta_i) = I_0$. Since y_i is enumerated into B_i at stage $t_i + 1$, it holds that $\rho(|\beta_i|) = \mathcal{J}^i_{e'}$ ($e' \in \mathbb{N}$). Moreover, as above, β_i is active at stage $t_0 + 1$, or I_0 is demoted to β_i at stage $t_0 + 1 = s_n + 1$. In either case, y_0 is enumerated into $A_{i,t_0+1} - A_{i,t_0}$, contradicting the choice of s_1. The case that $s_n = t_i < t_0$ leads to a contradiction in a similar way.

Now from (66) it follows that, if $x \notin W_{e_0,s_2}$, then for $n \geq 2$,

$$\hat{\Phi}^{B_{0,s_n}}_{e_1,s_n}(x) = \hat{\Phi}^{B_{i,s_n}}_{e_2,s_n}(x) = W_{e_0,s_n}(x) = 0.$$

This is true for $n = 1$, because $l_{s_2}(\mathcal{M}^i_e) > l_{s_1}(\mathcal{M}^i_e) > x$. If it is true for s_n, then $B_{0,s_{n+1}} \upharpoonright (x+1) = B_{0,s_n} \upharpoonright (x+1)$ or $B_{i,s_{n+1}} \upharpoonright (x+1) = B_{i,s_n} \upharpoonright (x+1)$. If, say $B_{0,s_{n+1}} \upharpoonright (x+1) = B_{0,s_n} \upharpoonright (x+1)$, then

$$0 = \hat{\Phi}^{B_{0,s_n}}_{e_1,s_n}(x) = \hat{\Phi}^{B_{0,s_{n+1}}}_{e_1,s_{n+1}}(x) = \hat{\Phi}^{B_{i,s_{n+1}}}_{e_2,s_{n+1}}(x) = W_{e_0,s_{n+1}}(x),$$

proving the equality for s_{n+1}. The case that $B_{i,s_{n+1}} \upharpoonright (x+1) = B_{i,s_n} \upharpoonright (x+1)$ is analogue.

Since $W_{e_0}(x) = \lim_{n\to\infty} W_{e_0,s_n}(x)$, this implies that $W_{e_0}(x) = 0$.

Hence $W_{e_0}(x) = W_{e_0,s_2}(x)$, and $W_{e_0} \leq_{\mathrm{ibT}} A_i$. □

Lemma 4.31. *Every join requirement $\mathfrak{J}^i_e$ ($e \in \mathbb{N}$, $i \in \{1,2\}$) is satisfied.*

Proof. Let $n = \rho^{-1}(\mathfrak{J}^i_e)$ and let $\beta \sqsubset \mathrm{TP}$ be the unique node of length n on the true path. For $e = \langle e_0, e_1, e_2\rangle$, assume that the premise of $\mathfrak{J}^i_e$ is true, that is $B_i = \hat{\Phi}^{W_{e_0}}_{e_1}$ and $A_{3-i} = \hat{\Phi}^{W_{e_0}}_{e_2}$ (otherwise $\mathfrak{J}^i_e$ is trivially satisfied). Since $\beta 0 \sqsubset \mathrm{TP}$ by the definition of the true path, due to the True Path Lemma there are infinitely many $\beta 0$-stages. By Lemma 4.27(i), there is a $\beta 0$-stage s_0 such that $\beta 0$ is never initialised and no node $\alpha \leq \beta 0$ is active due to Case 1 at any stage $s \geq s_0$.

Now, in order to compute $C(x)$ with oracle $W_{e_0} \upharpoonright (x+1)$ for some given x, using the oracle compute the least $\beta 0$-stage $s_1 \geq \max(\{s_0, x\})$ such that

$$W_{e_0,s_1} \upharpoonright (x+1) = W_{e_0} \upharpoonright (x+1) \tag{67}$$

and $l_{s_1}(\mathfrak{J}^i_e) > x$.

We claim that $x \in C$ if and only if $x \in C_{s_1+1}$. If $x \notin I$ for any interval I assigned to any node during the construction, this is clearly true. So assume that $x \in I$, where I is an α-interval.

Since at stages $s > s_1$ only intervals I' with $\min(I') > s_1 \geq x$ become assigned to any node, I must be assigned to α at some stage $s \leq s_1$. If $\alpha < \beta 0$, then x is not enumerated into C at any stage $s \geq s_1$, because otherwise $\beta 0$ were initialised at stage s, contradicting the choice of s_0. Furthermore, if $\beta 0 <_L \alpha$, then $\delta_{s_1} <_L \alpha$, whence α is initialised at stage $s_1 + 1$ and I is not assigned to α at any stage $s > s_1$; then there are no enumerations of numbers from I into C at stages $s > s_1 + 1$.

Hence it suffices to consider the case that $\beta 0 \sqsubseteq \alpha$. For a contradiction assume that $C(x) \neq C_{s_1+1}(x)$, i.e. that x enters C at some stage $s+1 > s_1+1$. We now consider the possible cases why x enters C at stage $s+1$. Let β_s be the node which is active at stage $s+1$.

- If α is initialised at any stage $t+1$ with $s_1 + 1 \leq t+1 \leq s+1$, then $y = \min(I) \leq x$ is enumerated into $B_{i,t+1} - B_{i,t}$ or $A_{3-i,t+1} - A_{3-i,t}$. Then

$$\hat{\Phi}^{W_{e_0}}_{e_1}(y) = B_i(y) = 1 \neq 0 = B_{i,s_1}(y) = \hat{\Phi}^{W_{e_0,s_1}}_{e_1,s_1}(y) \tag{68}$$

or

$$\hat{\Phi}^{W_{e_0}}_{e_2}(y) = A_{3-i}(y) = 1 \neq 0 = A_{3-i,s_1}(y) = \hat{\Phi}^{W_{e_0,s_1}}_{e_2,s_1}(y) \tag{69}$$

So there must be some $z \leq y \leq x$ in $W_{e_0} - W_{e_0,s_1}$, contradicting the assumption. Hence for the remaining cases we may assume that α is not initialised up to stage $s+1$.

- If β_s is active due to Case 1.2, Case 1.4 (b) or Case 2 (a) at stage $s+1$, or if β_s is active due to Case 1.4 (a) and $i = 1$, then $x = c_s$ is enumerated into $B_{i,s+1} - B_{i,s}$ or $A_{3-i,s+1} - A_{3-i,s}$ at stage $s+1$ and it follows that (68) or (69) is true with x in place of y. Again it follows that there must be some $z \leq x$ in $W_{e_0} - W_{e_0,s_1}$.

- If $\beta_s = \alpha$ is active due to Case 1.4 (a) at stage $s+1$ and $i = 2$, then there is some stage $t+1 < s+1$ such that $x = c_t$ and (c_t, a_t) has been assigned as diagonalisation witness to α via Case 1.3 at stage $t+1$. By (41) $[c_t+1, a_t]$ was $\{\mathcal{J}^i_e\}$-safe at stage t. By the hypothesis of Case 1.4, $l_s(\rho(|\beta|)) > a_t$ and $y = a_t$ is enumerated into $A_{3-i,s+1} - A_{3-i,s}$. Then
$$\hat{\Phi}^{W_{e_0}}_{e_2}(y) = A_{3-i}(y) = 1 \neq 0 = A_{3-i,s}(y) = \hat{\Phi}^{W_{e_0,s}}_{e_2,s}(y). \tag{70}$$
Since $\hat{\Phi}_{e_2}$ is an ibT-functional, there must be some $z \leq y$ in $W_{e_0} - W_{e_0,s}$. But since $[x+1, y] = [c_t+1, a_t] \subseteq W_{e_0,t} \subseteq W_{e_0,s}$, it follows that $z \leq x$, contradicting the choice of s_1.

- If $\beta_s = \alpha$ is active due to Case 1.3 at stage $s+1$ and I is demoted to some β' with $\beta \sqsubseteq \beta'$, or if $\beta' = \beta_s$ is active due to Case 2 (b) and $\beta 0 \sqsubseteq \beta_s 0$, then $x = c_s$ and x is enumerated into $B_{i,s+1} - B_{i,s}$ or $y = a_s$ is enumerated into $A_{3-i,s+1} - A_{3-i,s}$. Then
$$\hat{\Phi}^{W_{e_0}}_{e_1}(x) = B_i(x) = 1 \neq 0 = B_{i,s}(x) = \hat{\Phi}^{W_{e_0,s}}_{e_1,s}(x) \tag{71}$$
or
$$\hat{\Phi}^{W_{e_0}}_{e_2}(y) = A_{3-i}(y) = 1 \neq 0 = A_{3-i,s}(y) = \hat{\Phi}^{W_{e_0,s}}_{e_2,s}(y). \tag{72}$$
If the first inequality holds (in particular, this is the case if $\beta = \beta'$), there must be some $z \leq c_s = x$ in $W_{e_0} - W_{e_0,s}$. If the second inequality holds, there must be some $z \leq a_s$ in $W_{e_0} - W_{e_0,s}$. Since $\beta 0 \sqsubset \beta' 0$ in this case, by(41) or (45), respectively, $[c_s+1, a_s] \subseteq W_{e_0,s}$. Hence $z \leq c_s = x$ again.

- If $\beta_s = \alpha$ is active due to Case 1.3 at stage $s+1$ and I is demoted to some β' with $\beta' \sqsubset \beta$, or if $\beta' = \beta_s$ is active due to Case 2 (b) and $\beta_s 0 \sqsubset \beta 0$, then by Lemma 4.27 there is a least stage $s' > s$ such that $job_{s'}(\beta') = I \neq job_{s'+1}(\beta')$. If α is initialised and the assignment of I to α cancelled at stage $s'+1$, then $I \not\subseteq C_{s'}$ by Lemma 4.25, and in particular $\min(I) \notin C_{s'}$, hence $\min(I) \notin A_{1,s'} \cup A_{2,s'} \cup B_{0,s'} \cup B_{1,s'} \cup B_{2,s'}$. Then for $y = \min(I) \leq x$ the same analysis as in the first case above shows that there is some $z \leq x$ in $W_{e_0} - W_{e_0,s_1}$.

  If I is not cancelled at stage $s'+1$, then β' must be active due to Case 2.2 (a) at stage $s'+1 > s+1$. As we have seen above, then some $z \leq c_{s'} \leq c_s = x$ enters $W_{e_0} - W_{e,s_1}$.

In all cases we have arrived at a contradiction to (67). Hence $C(x) = C_{s_1+1}(x)$, completing the proof of the lemma. $\square$

Lemma 4.32. *Every diagonalisation requirement $\mathcal{D}_e^j$ ($e \in \mathbb{N}$, $j \in \{0, 1, 2\}$) is satisfied.*

Proof. Let $\alpha \sqsubset \mathrm{TP}$ be the unique node of length $3e + j$ on the true path. By Lemma 4.27 and the True Path Lemma, there is some stage s_0 such that for any stage $s \geq s_0$, α is not initialised at stage s and for all $\alpha' \leq \alpha$, all $\beta 0 \sqsubseteq \alpha$ and every α'-interval I, $job_s(\beta) \neq I$.

For a contradiction assume that α has no interval assigned at any stage $s \geq s_0$. Then no $\alpha' \sqsupseteq \alpha$ is active after stage s_0, because α requires attention at every α-stage $s \geq s_0$. For any α-stage $s \geq s_0$ some node $\beta \sqsubset \alpha$ must be active at stage $s + 1$ and enumerate some number c in $C_{s+1} - C_s$, where c is in an α'-interval for some $\alpha' \leq \delta_s$. Since no $\alpha' \sqsupseteq \alpha$ is active after stage s_0, almost all of these enumerations are into α'-intervals with $\alpha' < \alpha$. But by the choice of stage s_0 no such enumerations are possible.

Hence there must be a least α-stage $s_1 \geq s_0$ such that α has some interval I assigned at stage $s_1 + 1$. Since α is never initialised after stage s_1, I is assigned to α at all stages $s \geq s_1 + 1$. Since $s_1 \geq s_0$ and by the choice of s_0, for all $s > s_1$ it holds that either α has a diagonalisation witness (x, y) with $x, y \in I$ assigned at stage s or $cand_s(\alpha) = I$.

Since $\alpha \sqsubseteq \mathrm{TP}$, there is an α-stage $s_2 \geq s_1 + 1$ such that $l_{s_2}(\rho(|\beta|)) > \max(I)$ for all $\beta 0 \sqsubseteq \alpha$.

For a contradiction assume that $cand_s(\alpha) = I$ for all $s \geq s_2$. By Lemma 4.26 (ii) α requires attention at every stage $s + 1$ for which $s \geq s_2$ and s is an α-stage. Hence no node above α is active after stage s_2 and there are only finitely many intervals assigned to such nodes. Let $s_3 \geq s_2$ be an α-stage such that for all β with $\beta 0 \sqsubseteq \alpha$ and all $\alpha \sqsubseteq \alpha'$, there is no α'-interval I' with $job_s(\beta) = I'$. Such a stage exists by Lemma 4.27. Nodes $\alpha' \sqsubseteq \alpha$ do not require attention due to Case 1.1, Case 1.2, Case 1.3 or Case 1.4 at any stage $s + 1$, where $s \geq s_3$ is an α-stage, because otherwise α were initialised, and they do not require attention due to Case 2 at such a stage $s + 1$, because if $job_s(\beta) = I'$ for some β with $\beta 0 \sqsubseteq \alpha$, then I' is an α'-interval for some $\alpha' >_L \alpha$, hence $\alpha' >_L \delta_s$. It follows that α is active at every such stage $s + 1$. In fact, since α has no diagonalisation witness assigned at stage s and since $cand_{s+1}(\alpha) = I$ by assumption, α must be active due to Case 1.2 and enumerates the number $c_s \in I$ into $C_{s+1} - C_s$. But since there are infinitely many α-stages, this implies that infinitely many numbers from I are enumerated into C, which is impossible.

Hence there is a stage $s \geq s_2$ such that $cand_s(\alpha) \neq I$ and α has a diagonalisation witness (x, y) with $x, y \in I$ assigned at stage s.

Let (x, y) become assigned as diagonalisation witness to α at stage $\bar{s}+1$. Then $x \notin A_{1,\bar{s}+1} \cup A_{2,\bar{s}+1} \cup B_{0,\bar{s}+1} \cup B_{1,\bar{s}+1} \cup B_{2,\bar{s}+1} \cup C_{\bar{s}+1}$.

Case A: $j \in \{1, 2\}$

Then α is active due to Case 1.1 at stage $\bar{s}+1$ and $y = x+e+2$. Now if x is never enumerated into B_j, then $B_j(x) = 0 \neq \tilde{\Phi}_e^{A_{3-j}}(x)$: For a contradiction assume that $\tilde{\Phi}_e^{A_{3-j}}(x) = 0$. Then there is an α-stage $s_4 \geq \max(\{\bar{s}, s_2\})$ such that $l_s(\rho(|\beta|)) > x + e + 2$ for all β with $\beta 0 \sqsubseteq \alpha$,

$A_{3-j,s_4} \upharpoonright (x+e+1) = A_{3-j} \upharpoonright (x+e+1)$ and $\tilde{\Phi}_{e,s_4}^{A_{3-j,s_4}}(x) = 0$. By the choice of $s_4 \geq s_2$, α requires attention due to Case 1.4 at every stage $s+1$, where $s \geq s_4$ is an α-stage. Similar as above we can argue that α must be active at some such stage $s+1$, whence x is enumerated into B_j, contradicting the hypothesis.

On the other hand, if x is enumerated into B_j at some stage $t+1 > \bar{s}$, then, since α is not initialised after stage $\bar{s}$, α is active due to Case 1.3 at stage $t+1$ and

$$B_j(x) = 1 \neq 0 = B_{j,t}(x) = \tilde{\Phi}_{e,t}^{A_{3-j,t}}(x).$$

Since $\tilde{u}_e^{A_{3-j,t}}(x) \leq x+e$, it suffices to show that $A_{3-j,t} \upharpoonright (x+e+1) = A_{3-j} \upharpoonright (x+e+1)$, because then $\tilde{\Phi}_{e,t}^{A_{3-j,t}}(x) = \tilde{\Phi}_e^{A_{3-j}}(x)$ and $\mathcal{D}_e^j$ is satisfied.

But this is true because there are no enumerations into A_{3-j} from any α'-intervals with $\alpha' \leq \alpha$ after stage t (otherwise α would be initialised and I cancelled), while nodes α' with $\alpha < \alpha'$ are initialised at stage $t+1$ and only get intervals I' with $\min(I') > \max(I)$ assigned at later stages.

Case B: $j = 0$

If there is no $\beta 0 \sqsubseteq \alpha$, then α is active due to Case 1.1 at stage $\bar{s}+1$ and $\iota(I) = 0$, $I = [x, 2\cdot(x+2))$, and since $e \leq |\alpha| < x$ it holds that $y_0(I) = x+1 < x+e+2 \leq \max(I)$. Then $y = x+e+2 \notin A_{1,\bar{s}+1} \cup A_{2,\bar{s}+1} \cup B_{0,\bar{s}+1}$.

If there is some $\beta 0 \sqsubseteq \alpha$, then α is active due to Case 1.3 at stage $\bar{s}+1$, and by (42) $|[x+1,y]| = |[c_{\bar{s}}+1,y]| \geq y_k(I) \geq x \geq \min(I) > |\alpha| \geq e$, where I is relevant above $y_k(I)$ at stage $\bar{s}$, hence $y \geq x+e+1$. Moreover, $y \notin A_{1,\bar{s}+1} \cup A_{2,\bar{s}+1} \cup B_{0,\bar{s}+1}$.

Now if x is never enumerated into C, then $C(x) = 0 \neq \tilde{\Phi}_e^{B_0}(x)$: For a contradiction assume that $\tilde{\Phi}_e^{B_0}(x) = 0$. Then there is a stage $s_4 \geq \max(\{s_2, \bar{s}\})$ such that $B_{0,s_4} \upharpoonright (x+e+1) = B_0 \upharpoonright (x+e+1)$ and $\tilde{\Phi}_{e,s_4}^{B_{0,s_4}}(x) = 0$. By the choice of $s_4 \geq s_2$, α requires attention at every stage $s+1$, where $s \geq s_4$ is an α-stage. Similar as above we can argue that α must be active at some such stage $s+1$, whence x is enumerated into C, contradicting the hypothesis.

On the other hand, if x is enumerated into C at some stage $t+1 > \bar{s}$, then

$$C(x) = 1 \neq 0 = C_t(x) = \tilde{\Phi}_{e,t}^{B_{0,t}}(x).$$

It suffices to show that $B_{0,t} \upharpoonright (x+e+1) = B_0 \upharpoonright (x+e+1)$, because then $\tilde{\Phi}_{e,t}^{B_{0,t}}(x) = \tilde{\Phi}_e^{B_0}(x)$ and $\mathcal{D}_e^j$ is satisfied.

But this is true because there are no enumerations into B_0 from any α'-intervals with $\alpha' \leq \alpha$ after stage t (otherwise α would be initialised and I cancelled), while α'-intervals with $\alpha < \alpha'$ are cancelled at stage $t+1$ and nodes $\alpha' > \alpha$ are assigned intervals I' with $\min(I') > \max(I)$ at later stages. □

This completes the verification and the proof of Theorem 4.18.

□

In the search for a characterisation of the finite lattices for which a lattice embedding into $\mathcal{R}_{\mathrm{T}}$ exists, the notion of a so-called *critical triple* turned out to be of interest.

Definition 4.33. Let $\mathcal{P} = (P, \leq_{\mathcal{P}})$ be a partial order. A *critical triple* in $\mathcal{P}$ is a triple $(b_0, b_1, b_2) \in P^3$, such that b_0, b_1 and b_2 are pairwise incomparable, $b_0 \vee b_1 = b_0 \vee b_2$ and $b_1 \wedge b_2 \leq b_0$ (in particular, the two joins and the meet exist).

For example, Downey [Down 90] showed that if a lattice $\mathcal{L}$ has a critical triple, then there is a c.e. Turing degree $\mathbf{a} \neq \mathbf{0}$ such that there exists no lattice embedding of $\mathcal{L}$ into $(\mathbf{R}_{\mathrm{T}}(\leq \mathbf{a}), \leq)$, where $\mathbf{R}_{\mathrm{T}}(\leq \mathbf{a}) = \{\mathbf{b} \in \mathbf{R}_{\mathrm{T}} : \mathbf{b} \leq \mathbf{a}\}$. He conjectured that the converse were true as well, that is, if $\mathcal{L}$ has no critical triple, then there is a lattice embedding of $\mathcal{L}$ into every initial segment $\mathbf{R}_{\mathrm{T}}(\leq \mathbf{a})$ with $\mathbf{a} \neq \mathbf{0}$. This conjecture was later refuted by Lempp and Lerman [Lemp 97], but only by giving a rather complicated lattice with 20 elements in its domain as a counterexample.

Since the $\mathcal{S}_7$ is lattice embeddable into $\mathcal{R}_r$ for $r \in \{\mathrm{ibT}, \mathrm{cl}\}$, we get the following result as a corollary.

Corollary 4.34. *For $r \in \{\mathrm{ibT}, \mathrm{cl}\}$ there exists a critical triple $(\mathbf{b_0}, \mathbf{b_1}, \mathbf{b_2})$ in $\mathcal{R}_r$ such that $\mathbf{b_1} \wedge \mathbf{b_2} = \mathbf{0}$.*

Proof. For a lattice embedding h of the $\mathcal{S}_7$ into $\mathcal{R}_r$ preserving the least element, such a triple is given by $(h(b_0), h(b_1), h(b_2))$. □

4.7 Embedding the $\mathcal{M}_3$

So far all our results on lattice embeddings were the same for $\mathcal{R}_{\mathrm{ibT}}$ and $\mathcal{R}_{\mathrm{cl}}$ as for $\mathcal{R}_{\mathrm{T}}$ and $\mathcal{R}_{\mathrm{wtt}}$. Now, investigating lattice embeddings of the $\mathcal{M}_3$ into $\mathcal{R}_{\mathrm{ibT}}$ and $\mathcal{R}_{\mathrm{cl}}$ preserving the least element, we will see that here the situation looks different.

Note that a critical triple $(\mathbf{b_0}, \mathbf{b_1}, \mathbf{b_2})$ in $\mathcal{R}_r$ with $\mathbf{b_1} \wedge \mathbf{b_2} = \mathbf{0}$ would give rise to a lattice embedding h of the $\mathcal{M}_3$ preserving the least element into $\mathcal{R}_r$ if $\mathbf{b_1} \vee \mathbf{b_2} = \mathbf{b_0} \vee \mathbf{b_1}$ and $(\mathbf{b_0}, \mathbf{b_1})$ and $(\mathbf{b_0}, \mathbf{b_2})$ both were minimal pairs. We could just set $h(a) = \mathbf{0}$, $h(b_i) = \mathbf{b_i}$ for $i \in \{0, 1, 2\}$ and $h(c) = \mathbf{b_0} \vee \mathbf{b_1}$. However, a critical triple with these additional requirements does not exist in $\mathcal{R}_r$ for $r \in \{\mathrm{ibT}, \mathrm{cl}\}$.

To prove this, we need the following lemma, which is the dual of Lemma 3.19.

Lemma 4.35 (cl-wtt-Meet Lemma [Ambo 13b])**.** *Let $B_0, \ldots, B_n, C$ be c.e. sets such that*

$$deg_{\mathrm{cl}}(B_0) \wedge \ldots \wedge deg_{\mathrm{cl}}(B_n) = \deg_{\mathrm{cl}}(C).$$

Then

$$deg_{\mathrm{wtt}}(B_0) \wedge \ldots \wedge deg_{\mathrm{wtt}}(B_n) = \deg_{\mathrm{wtt}}(C).$$

Proof. Let W be a c.e. set such that $W \leq_{\mathrm{wtt}} B_0, \ldots, B_n$. For $k \in \{0, \ldots, n\}$, let $W = \Phi_{e_k}^{B_k}$ and let f_k be a computable function such that $u_{e_k}^{B_k}(x) \leq f_k(x)$ for every x. Let f be a computable, strictly increasing function such that $f_k(x) \leq f(x)$ for every x and every $k \in \{0, \ldots, n\}$. Then $W_f = \{f(x) : x \in W\} \leq_{\mathrm{cl}} B_0, \ldots, B_n$: To compute $W_f(y)$ with oracle B_k, first check whether $y = f(x)$ for any x; if not, then $W_f(y) = 0$. Otherwise $W_f(y) = \Phi_{e_k}^{B_k}(x)$, hence we can compute $W_f(y)$ with a use bound of $f_k(x) \leq f(x) = y$.

Since $deg_{\mathrm{cl}}(B_0) \wedge \ldots \wedge deg_{\mathrm{cl}}(B_n) = \deg_{\mathrm{cl}}(C)$, it follows that $W_f \leq_{\mathrm{cl}} C$, in particular $W_f \leq_{\mathrm{wtt}} C$. But since $W_f \equiv_{\mathrm{wtt}} W$, it holds that $W \leq_{\mathrm{wtt}} C$, proving the lemma. □

Lemma 4.36. *(Ambos-Spies, Bodewig, Kräling, and Yu [Amboc]) Let* $r \in \{\mathrm{ibT}, \mathrm{cl}\}$ *and let* $h : \{a, b_0, b_1, b_2, c\} \to \mathbf{R}_r$ *be a lattice embedding of the* $\mathcal{M}_3$ *into* $\mathcal{R}_r$*. Then* $h(a), h(b_0), h(b_1), h(b_2)$ *and* $h(c)$ *are all contained in the same* wtt*-degree.*

Proof. Let $A \in h(a)$, $B_0 \in h(b_0)$, $B_1 \in h(b_1)$, $B_2 \in h(b_2)$ and $C \in h(c)$ be c.e. sets. We need to show that $deg_{\mathrm{wtt}}(A) = deg_{\mathrm{wtt}}(B_0) = deg_{\mathrm{wtt}}(B_1) = deg_{\mathrm{wtt}}(B_2) = deg_{\mathrm{wtt}}(C)$.

For a contradiction, first assume that $deg_{\mathrm{wtt}}(A) \neq deg_{\mathrm{wtt}}(C)$. Since h is an embedding of the $\mathcal{M}_3$ into $\mathcal{R}_r$, it holds that $deg_r(A) \leq deg_r(B_i) \leq deg_r(C)$ for $i \in \{0, 1, 2\}$. Since r-reducibility implies wtt-reducibility, then $deg_{\mathrm{wtt}}(A) \leq deg_{\mathrm{wtt}}(B_i) \leq deg_{\mathrm{wtt}}(C)$ for $i \in \{0, 1, 2\}$. Moreover, $deg_r(B_i) \wedge deg_r(B_j) = deg_r(A)$ and $deg_r(B_i) \vee deg_r(B_j) = deg_r(C)$ for $i \neq j$, $i, j \in \{0, 1, 2\}$, and by Lemmas 3.2, 3.3, 3.19 and 4.35 it follows that $deg_{\mathrm{wtt}}(B_i) \wedge deg_{\mathrm{wtt}}(B_j) = deg_{\mathrm{wtt}}(A)$ and $deg_{\mathrm{wtt}}(B_i) \vee deg_{\mathrm{wtt}}(B_j) = deg_{\mathrm{wtt}}(C)$.

Now if $deg_{\mathrm{wtt}}(B_0) \leq deg_{\mathrm{wtt}}(B_1)$, then $deg_{\mathrm{wtt}}(B_0) = deg_{\mathrm{wtt}}(B_0) \wedge deg_{\mathrm{wtt}}(B_1) = deg_{\mathrm{wtt}}(A)$ and $deg_{\mathrm{wtt}}(B_1) = deg_{\mathrm{wtt}}(B_0) \vee deg_{\mathrm{wtt}}(B_1) = deg_{\mathrm{wtt}}(C)$; then further $deg_{\mathrm{wtt}}(B_0) \leq deg_{\mathrm{wtt}}(B_2) \leq deg_{\mathrm{wtt}}(B_1)$ and hence

$$deg_{\mathrm{wtt}}(C) = deg_{\mathrm{wtt}}(B_0) \vee deg_{\mathrm{wtt}}(B_2) = deg_{\mathrm{wtt}}(B_2) = deg_{\mathrm{wtt}}(B_1) \wedge \deg_{\mathrm{wtt}}(B_2) = deg_{\mathrm{wtt}}(A),$$

contradicting the assumption that $deg_{\mathrm{wtt}}(A) \neq deg_{\mathrm{wtt}}(C)$. Hence $deg_{\mathrm{wtt}}(B_0) \not\leq deg_{\mathrm{wtt}}(B_1)$.

For symmetric reasons, $deg_{\mathrm{wtt}}(B_0)$, $deg_{\mathrm{wtt}}(B_1)$ and $deg_{\mathrm{wtt}}(B_2)$ are pairwise incomparable. Since $deg_{\mathrm{wtt}}(A) \leq deg_{\mathrm{wtt}}(B_i) \leq deg_{\mathrm{wtt}}(C)$ for $i \in \{0, 1, 2\}$, then $deg_{\mathrm{wtt}}(A) \neq deg_{\mathrm{wtt}}(B_i)$ and $deg_{\mathrm{wtt}}(B_i) \neq deg_{\mathrm{wtt}}(C)$.

This shows that $h' : \{a, b_0, b_1, b_2, c\}$ with $h(a) = deg_{\mathrm{wtt}}(A)$, $h(b_i) = deg_{\mathrm{wtt}}(B_i)$ for $i \in \{0, 1, 2\}$ and $h(c) = deg_{\mathrm{wtt}}(C)$ is a lattice embedding of the $\mathcal{M}_3$ into $\mathcal{R}_{\mathrm{wtt}}$. But the existence of such an embedding contradicts Theorem 4.13.

We conclude that $deg_{\mathrm{wtt}}(A) = deg_{\mathrm{wtt}}(C)$. Again, since $deg_r(A) \leq deg_r(B_i) \leq deg_r(C)$ and r-reducibility implies wtt-reducibility, it follows that $deg_{\mathrm{wtt}}(A) = deg_{\mathrm{wtt}}(B_i) = deg_{\mathrm{wtt}}(C)$ for $i \in \{0, 1, 2\}$, proving the lemma. □

Theorem 4.37. *(Ambos-Spies, Bodewig, Kräling, and Yu [Amboc]) For* $r \in \{\mathrm{ibT}, \mathrm{cl}\}$*, there is no lattice embedding of the* $\mathcal{M}_3$ *into* $\mathcal{R}_r$ *that preserves the least element.*

Proof. If h were such a lattice embedding, then by Lemma 4.36 it would hold that $h(c)$ and $h(a) = \mathbf{0} = deg_r(\emptyset)$ were contained in the same wtt-degree. But since $h(c) \neq h(a)$, i.e. every set $C \in h(c)$ is noncomputable, it holds that $C \neq_{\mathrm{wtt}} \emptyset$. This is a contradiction. □

We will now show that if we drop the requirement of preserving the least element, then the $\mathcal{M}_3$ and, more generally, the $\mathcal{M}_n$ can be embedded by a lattice embedding into $\mathcal{R}_r$. This shows that not every lattice which can be embedded by a lattice embedding can also be embedded by a lattice embedding preserving the least element (an unpublished result of Ambos-Spies and Wang exhibits the $\mathcal{S}_7^*$, the dual lattice of the $\mathcal{S}_7$, as another example of such a lattice).

For the desired embedding, first we simplify the requirement derived from (M1) of Definition 4.11 that $h(b_0), \ldots, h(b_n)$ are pairwise incomparable.

Lemma 4.38. *Let $\mathcal{P} = (P, \leq_{\mathcal{P}})$ be a partial order. Let $n \geq 3$ (possibly $n = \omega$). Then a mapping $h : \{a, c\} \cup \{b_i : i < n\} \to P$ is a lattice embedding of the $\mathcal{M}_n$ into $\mathcal{P}$ if and only if*

(M1') $h(a) \leq_{\mathcal{P}} h(b_i) \leq_{\mathcal{P}} h(c)$ for $i < n$

(M2') $h(c) \not\leq_{\mathcal{P}} h(a)$

(M3') $h(b_i) \vee h(b_j) = h(c)$ for $i, j < n$ with $i \neq j$

(M4') $h(b_i) \wedge h(b_j) = h(a)$ for $i, j < n$ with $i \neq j$

Proof. Let $h : \{a, c\} \cup \{b_i : i < n\} \to P$ be a lattice embedding of the $\mathcal{M}_n$ into $\mathcal{P}$. Then (M3') and (M4') clearly hold by (M2) and (M3) and the definition of lattice embeddings. By (M2) and (M3), $a \leq_{\mathcal{M}_n} b_i \leq_{\mathcal{M}_n} c$ for $i < n$, hence (M1') holds. And since $a \neq c$ (otherwise $b_0 = b_1$, contradicting the incomparability of b_0 and b_1) (M2') follows, too.

On the other hand, assume that the mapping $h : \{a, c\} \cup \{b_i : i < n\} \to P$ satisfies (M1')-(M4'). For a contradiction assume that $h(b_i) \leq h(b_j)$ for some $i, j < n$ with $i \neq j$. Then $h(b_i) = h(b_i) \wedge h(b_j) = h(a)$ by (M4') and $h(b_j) = h(b_i) \vee h(b_j) = h(c)$ by (M3'). Since $n \geq 3$, there is some $k < n$ with $k \neq i$ and $k \neq j$. Then further $h(b_i) \leq_{\mathcal{P}} h(b_k) \leq_{\mathcal{P}} h(b_j)$ and hence

$$h(c) = h(b_i) \vee h(b_k) = h(b_k) = h(b_j) \wedge h(b_k) = h(a),$$

contradicting (M2').

This proves that $h(b_i)$ and $h(b_j)$ must be incomparable for all $i, j < n$ with $i \neq j$. By (M3') and (M4') it follows that $h(a) \neq h(b_i)$ and $h(b_i) \neq h(c)$ for all $i < n$. Hence h is an embedding; since by (M3') and (M4') h preserves joins and meets, it is a lattice embedding. $\square$

Now we have all the ingredients to prove the next theorem, which for the case $n = 3$ was obtained in joint work with Ambos-Spies, Bodewig, and Wang. The details of the proof are worked out here for the first time.

Theorem 4.39. *(Ambos-Spies, Bodewig, Kräling, and Wang) Let $n \geq 3$ (possibly $n = \omega$) and let $r \in \{\mathrm{ibT}, \mathrm{cl}\}$. Then there is a lattice embedding of the $\mathcal{M}_n$ into $\mathcal{R}_r$.*

Proof. Since the identity function on $\{a, b_0, \ldots, b_n\}$ is a lattice embedding of the $\mathcal{M}_n$ into the $\mathcal{M}_\omega$, it suffices to show that there is a lattice embedding h of the $\mathcal{M}_\omega$ into $\mathcal{R}_r$. We describe a stage-by-stage construction of c.e. sets A, B_i ($i \in \mathbb{N}$) and C such that the desired embedding is defined by $h(a) = deg_r(A)$, $h(b_i) = deg_r(B_i)$ for $i \in \mathbb{N}$ and $h(c) = deg_r(C)$.

We will satisfy the following requirements for all $e = \langle e_0, e_1, e_2\rangle$ and all $i, j \in \mathbb{N}$ with $i \neq j$.

- $A \leq_{\mathrm{ibT}} B_i \leq_{\mathrm{ibT}} C$,
- $\mathcal{D}_e : C \neq \tilde{\Phi}_e^A$,
- $\mathcal{J}_e^{i,j} : (B_i = \hat{\Phi}_{e_1}^{W_{e_0}}$ and $B_j = \hat{\Phi}_{e_2}^{W_{e_0}}) \Rightarrow C \leq_{\mathrm{ibT}} W_{e_0}$,
- $\mathcal{M}_e^{i,j} : (W_{e_0} = \hat{\Phi}_{e_1}^{B_i} = \hat{\Phi}_{e_2}^{B_j}) \Rightarrow W_{e_0} \leq_{\mathrm{ibT}} A$.

The *diagonalisation requirements* $\mathcal{D}_e$ ensure that $deg_{\mathrm{cl}}(C) \not\leq deg_{\mathrm{cl}}(A)$ and, since ibT-reducibility is stronger than cl-reducibility, $deg_{\mathrm{ibT}}(C) \not\leq deg_{\mathrm{ibT}}(A)$. The *join requirements* $\mathcal{J}_e^{i,j}$ together with $B_i, B_j \leq_{\mathrm{ibT}} C$ ensure that $deg_{\mathrm{ibT}}(B_i) \vee deg_{\mathrm{ibT}}(B_j) = deg_{\mathrm{ibT}}(C)$ for $i \neq j$ ($i, j \in \mathbb{N}\}$). By Lemma 3.2, this also implies that $deg_{\mathrm{cl}}(B_i) \vee deg_{\mathrm{cl}}(B_j) = deg_{\mathrm{cl}}(C)$. The *meet requirements* $\mathcal{M}_e^{i,j}$ together with $A \leq_{\mathrm{ibT}} B_i, B_j$ ensure that $deg_{\mathrm{ibT}}(B_i) \wedge deg_{\mathrm{ibT}}(B_j) = deg_{\mathrm{ibT}}(A)$ for $i \neq j$ ($i, j \in \mathbb{N}$). By Lemma 3.3, this also implies that $deg_{\mathrm{cl}}(B_i) \wedge deg_{\mathrm{cl}}(B_j) = deg_{\mathrm{cl}}(A)$. Hence, by Lemma 4.38, h is an embedding into $\mathcal{R}_r$ for both $r = \mathrm{ibT}$ or $r = \mathrm{cl}$.

The construction is quite similar to the one described in the proof of Theorem 4.18.

Let A_s, $B_{i,s}$ and C_s denote the finite approximation to the sets A, B_i and C, respectively, as given after stage s of the construction.

Again, we will obtain $B_i \leq_{\mathrm{ibT}} C$ by permitting. Since we want the construction to be effective, i.e. we may only perform finitely many actions at each stage, however, we will not enumerate any numbers into B_i before stage $i+1$. Hence enumerations into A before stage $i+1$ will not necessarily be permitted by B_i. But if we let $\hat{A}_i = A - A_i$, then (since A_i is finite) $\hat{A}_i \equiv_{\mathrm{ibT}} A$ and $\hat{A}_i \leq_{\mathrm{ibT}} B_i$ will hold by permitting. Hence $A \leq_{\mathrm{ibT}} B_i$.

The definition of the lengths of agreement $l_s(\mathcal{J}_e^{i,j})$ and $l_s(\mathcal{M}_e^{i,j})$ ($i, j \in \mathbb{N}$, $i \neq j$), and the basic strategies to satisfy the diagonalisation requirements $\mathcal{D}_e$, the join requirements $\mathcal{J}_e^{i,j}$ and the meet requirements $\mathcal{M}_e^{i,j}$ are – with the obvious changes imposed by the sets and functions occuring in the requirements having different names – the same as in the proof of Theorem 4.18. Remember that we say that an interval $[x, y]$ is $\mathcal{J}_e^{i,j}$*-safe at stage* s if $[x, y] \subseteq W_{e_0,s}$, where $e = \langle e_0, e_1, e_2\rangle$; and we say that $[x, y]$ *is* R*-safe at stage* s for a set R of requirements if it is $\mathcal{J}_e^{i,j}$-safe at stage s for every join requirement $\mathcal{J}_e^{i,j} \in R$.

Suppose that for some pairwise different numbers $i, j, k \in \mathbb{N}$ there are a diagonalisation requirement $\mathcal{D}_e$, three join requirements $\mathcal{J}_{e_0}^{i,j}$, $\mathcal{J}_{e_1}^{i,k}$ and $\mathcal{J}_{e_2}^{j,k}$ and three meet requirements $\mathcal{M}_{e_0'}^{i,j}$, $\mathcal{M}_{e_1'}^{i,k}$ and $\mathcal{M}_{e_2'}^{j,k}$ the premises all of which are true and such that all meet requirements have higher priority than each of the join requirements and the join requirements all have higher priority than the diagonalisation requirement.

Then in order to satisfy $\mathcal{D}_e$ we want to enumerate a number x into C at some stage $s+1$ such that $\tilde{\Phi}_{e,s}^{A_s}(x) \downarrow= 0$ and we wish to restrain the enumerations of numbers $z \leq x + e$ into A after stage s. However, by the basic strategy to satisfy $\mathcal{J}_{e_0}^{i,j}$ we need to enumerate a number

$y^{i,j} \geq x$ such that $[x+1, y^{i,j}]$ is $\mathcal{J}^{i,j}_{e_0}$-safe at stage s into $B_i - B_{i,s}$ or $B_j - B_{j,s}$, and analogous for $\mathcal{J}^{i,k}_{e_1}$ and $\mathcal{J}^{j,k}_{e_2}$. Say, for example, we enumerate $y^{i,j}$ into B_i and $y^{j,k}$ into B_j. Then due to the basic strategy for $\mathcal{M}^{i,j}_{e'_0}$ we need to enumerate a number $z \leq \max(\{y^{i,j}, y^{j,k}\})$ into $A - A_s$. Hence in order to make this compatible with the basic strategy for $\mathcal{D}_e$ we need to require that at least one of $y^{i,j}$ and $y^{j,k}$ is greater than $x + e$, i.e. we need to construct an interval J of length $e+1$ which is $\mathcal{J}^{i,j}_{e_0}$- or $\mathcal{J}^{j,k}_{e_2}$-safe at stage s.

The construction of R-safe intervals of a desired length is very much as in the proof of Theorem 4.18. If, in the situation described above, $\mathcal{J}^{i,j}_{e_0}$ had higher priority than $\mathcal{J}^{i,k}_{e_1}$ and $\mathcal{J}^{i,k}_{e_1}$ had higher priority than $\mathcal{J}^{j,k}_{e_2}$, then first we would create a very long $\mathcal{J}^{i,j}_{e_0}$-safe interval J_0, use this interval to obtain a long $\{\mathcal{J}^{i,j}_{e_0}, \mathcal{J}^{i,k}_{e_1}\}$-safe interval J_1, and finally use that interval to create the desired $\{\mathcal{J}^{i,j}_{e_0}, \mathcal{J}^{i,k}_{e_1}, \mathcal{J}^{j,k}_{e_2}\}$-safe interval J of length at least $e+1$.

There is one difference to the strategy in the proof of Theorem 4.18. There, every join requirement $\mathcal{J}^i_e$ $(i \in \{1,2\})$ had to react to an enumeration of a number x into $A_j \cup B_j$, $j \in \{1,2\}$ (provided that $l_s(\mathcal{J}^i_e) > x$), irrespective of the values of i and j. For this reason, in (37) it was sufficient to require that there were many numbers in the interval J_k which were not in $B_0 = A_1 \cup A_2$ at stage t_k.

Now, in general we will need to enumerate a number a from J_k into many sets B_i in order to make all join requirements which we have to respect give a response. Since we may also need to respect some meet requirements, this forces us enumerate a into A as well. But then, if the enumeration of x takes place at stage $s+1$, in order to obtain $A \leq B_i$ we have to enumerate a number a_i less than or equal to a into *every* set B_i with $i \leq s$.

Therefore, in order to make the construction as uniform as possible with respect to the different join requirements we change condition (37) to

$$|J_k - (A_{t^k} \cup \bigcup_{i \geq 0} B_{i,t_k})| > y_{\iota(I)-k}. \tag{73}$$

In order to obtain a $\mathcal{J}^{i,j}_e$-safe interval J containing many numbers that are not yet enumerated into *any* set $B_{i'}$ when J is defined, we change the enumerations performed by $\mathcal{J}^{i,j}_e$ as follows. Starting with an interval I_0 at a stage s such that $I_0 \cap (A_s \cup \bigcup_{i \geq 0} B_{i,s} \cup C_s) = \emptyset$, we enumerate the numbers from I_0 into C one by one in decreasing order and at the same time we enumerate the numbers from I_0 alternately into B_i and B_j one by one and in decreasing order (unlike in the proof of Theorem 4.18, where we always enumerated the same number into C and one of B_1 and B_2). That way after $2m$ stages we end up with an interval of length $2m$ which has completely been enumerated into C but the lower half of which, an interval of length m, has not yet been enumerated into B_i or B_j. We can now only take the lower half of I_0 into account for the desired interval J, but since we may choose I_0 sufficiently large, this does not provide an obstacle to the construction.

Once again, the construction takes place on the tree $T = \{0,1\}^*$. Let $\rho : \mathbb{N} \to \{\mathcal{J}^{i,j}_e : e, i, j \in \mathbb{N}, i \neq j\} \cup \{\mathcal{M}^{i,j}_e : e, i, j \in \mathbb{N}, i \neq j\}$ be a computable one-to-one enumeration of all join and meet requirements, where we assume that if $\rho(n) = \mathcal{J}_e^{i,j}$, then $i, j \le n$. A node $\alpha \in T$ of length n corresponds to a guess about which of the premises of the first n join and meet requirements $\rho(0), \ldots, \rho(n-1)$ are true. Nodes of length e will also be responsible for the strategy to satisfy the diagonalisation requirement $\mathcal{D}_e$. The definitions of α-stages and α-expansionary stages, of δ_s and of the true path TP are analogous to the ones from the proof of Theorem 4.18. Note that if $\beta \sqsubseteq \delta_s$, then $|\beta| \le s$; hence if $\rho(|\beta|) = \mathcal{J}_e^{i,j}$, then $i, j \le s$ by the choice of ρ.

4.7.1 The construction

Stage 0: Let $A_0 = B_{i,0} = C_0 = \emptyset$ for $i \in \mathbb{N}$ and $job_0(\beta) = \uparrow$ and $cand_0(\beta) \uparrow$ for each node β. No node has an interval or a diagonalisation witness assigned.

Stage $s+1$:

We say that a node α *requires attention at stage* $s+1$ if $\alpha \sqsubseteq \delta_s$ and one of the following holds.

(Case 1.1) α has no interval assigned to it at the end of stage s.

(Case 1.2) α has an interval I assigned to it at the end of stage s, $cand_s(\alpha) = I$, for every β with $\beta 0 \sqsubseteq \alpha$ it holds that $l_s(\rho(|\beta|)) > \max(I)$, there is a number $c_s \in I$ such that

$$I \cap (A_s \cup B_{0,s} \cup B_{1,s} \cup \ldots \cup B_{s,s}) \subseteq I \cap C_s = [c_s + 1, \max(I)] \tag{74}$$

and there is a join requirement $\mathcal{J}_e^{i,j} = \rho(|\beta|)$ $(i, j \in \mathbb{N}, e = \langle e_0, e_1, e_2 \rangle)$ for some β with $\beta 0 \sqsubseteq \alpha$ such that

$$|\{x \le c_s : x \notin W_{e_0,s}\}| < |\{x \le c_s : x \in I\}|. \tag{75}$$

(Case 1.3) α has an interval I assigned to it at the end of stage s, $cand_s(\alpha) = I$, for every β with $\beta 0 \sqsubseteq \alpha$ and every α'-interval $I' = job_s(\beta)$ it holds that $\alpha \le \alpha'$ and $l_s(\rho(|\beta|)) > \max(I)$, and there are numbers a_s and $c_s \in I$ and a node β such that

- eqation (74) holds for c_s
- β is the longest node such that either $\beta = \alpha$, or $\rho(|\beta|)$ is a join requirement and $\beta 0 \sqsubseteq \alpha$, and such that there is some $y > c_s$, $y \in I$, for which

  $$[c_s + 1, y] \text{ is } E'\text{-safe at stage } s \text{ for } E' = \{\rho(|\beta'|) : \beta' \sqsubset \beta \text{ and } \beta' 0 \sqsubseteq \alpha\} \tag{76}$$

  and if I is relevant above $y_k(I)$ at stage s, then

  $$|\{z \in [c_s + 1, y] \cap I : z \notin A_s \cup B_{0,s} \cup B_{1,s} \cup \ldots \cup B_{s,s}\}| \ge y_k(I) \tag{77}$$

- $a_s = y$ for the least such y (note that $a_s \notin A_s \cup B_{0,s} \cup B_{1,s} \cup \ldots \cup B_{s,s}$ by minimality of a_s and by $y_k(I) > 0$).

(Case 1.4) α has an interval I and a diagonalisation witness (x, y), $x, y \in I$ assigned at the end of stage s, for every β with $\beta 0 \sqsubseteq \alpha$ and every α'-interval $I' = job_s(\beta)$ it holds that $\alpha \leq \alpha'$ and $l_s(\rho(|\beta|)) > y$ and if $|\alpha| = e$, then $x + e + 1 \leq y$, $\tilde{\Phi}_e^{A_s}(x) \downarrow = C_s(x) = 0$ and $y \notin A_s \cup B_{0,s} \cup B_{1,s} \cup \ldots \cup B_{s,s}$.

We also say that a node β *requires attention at stage $s+1$ and is α-linked* if

(*Case 2*) $\beta 0 \sqsubseteq \delta_s$ and $\alpha \leq \delta_s$, $job_s(\beta) = I$ for some α-interval I, for every β' with $\beta' 0 \sqsubseteq \beta 0$ it holds that $l_s(\rho(|\beta'|)) > \max(I)$ and there exists a number $c_s \in I$ satisfying (74)

Let η be the least node that requires attention at stage $s+1$ (such a node exists because δ_s always requires attention due to Case 1 at stage $s+1$). Let $X \in \{1.1, 1.2, 1.3, 1.4, 2\}$ be minimal such that η requires attention due to Case X. We say that η *receives attention and is active* due to Case X.

(*Case 1.1*) If $\eta = \alpha$ is active due to Case 1.1, then assign a new interval I to α in the following way. Let $|\alpha| = e$. Let x be the least number that is larger than $\max(|\alpha|, s+1)$ and larger than all numbers from intervals assigned to any node before stage $s+1$. Let $E = \{\beta : \beta 0 \sqsubseteq \alpha\}$. Define

$$\iota(I) = (x+3)^{2|E|} - 1$$

and

$$I = [x, 3^{\iota(I)+1} \cdot ((x+1) + (\iota(I)+1)),$$

i.e. $I = \{x\} \cup \bigcup_{k=0}^{\iota(I)} [y_k, y_{k+1})$, where $y_k = y_k(I) = 3^k(x+1+k)$ for $0 \leq k \leq \iota(I)+1$. For $k < 0$, define $y_k(I) = x$. Say that I is an *α-interval.*

Let $c_s = y_{\iota(I)}$. Set $\hat{A}_{s+1} = A_s \cup \{y_{\iota(I)}\}$, $\hat{B}_{i,s+1} = B_{i,s} \cup \{y_{\iota(I)}\}$ for $i \leq s$ and $\hat{C}_{s+1} = C_s \cup [y_{\iota(I)}, y_{\iota(I)+1})$. If $E = \emptyset$, assign $(x, x+e+2)$ as diagonalisation witness to α. Otherwise set $cand_{s+1}(\alpha) = I$. Say that I is relevant above $y_{\iota(I)}$ at stage $s+1$.

(*Case 1.2*) If $\eta = \alpha$ is active due to Case 1.2, define $b_s = c_s$ and set $\hat{A}_{s+1} = A_s \cup \{c_s\}$, $\hat{B}_{i,s+1} = B_{i,s} \cup \{c_s\}$ for $i \leq s$ and $\hat{C}_{s+1} = C_s \cup \{c_s\}$. We say that α *is ready for elimination.*

(*Case 1.3*) If $\eta = \alpha$ is active due to Case 1.3, let $cand_s(\alpha) = I$, let a_s, c_s and β be as in the hypothesis of Case 1.2 and let $b_s = c_s$. Say that $cand_{s+1}(\alpha)$ is undefined.

If $\beta = \alpha$, assign (c_s, a_s) as diagonalisation witness to α and let $\hat{A}_{s+1} = A_s$, $\hat{B}_{i,s+1} = B_{i,s}$ for $i \leq s$ and $C_{s+1} = C_s$.

If $\beta \neq \alpha$ and $\rho(|\beta|) = \mathfrak{J}_e^{i,j}$, then set $job_{s+1}(\beta) = I$, $\hat{A}_{s+1} = A_s \cup \{a_s\}$, $\hat{B}_{i',s+1} = B_{i',s} \cup \{a_s\}$ for $i' \leq s$ and $i' \neq i$, $\hat{B}_{i,s+1} = B_{i,s} \cup \{c_s\}$ and $\hat{C}_{s+1} = C_s \cup \{c_s\}$. Say that I is *demoted to β.*

If I was relevant above $y_k(I)$ at stage s, say that I is relevant above $y_{k-1}(I)$ at stage $s+1$.

(*Case 1.4*) If $\eta = \alpha$ is active due to Case 1.4 and α has the diagonalisation witness (x, y) assigned, then let $c_s = x$ and $a_s = y$ and set $\hat{A}_{s+1} = A_s \cup \{a_s\}$, $\hat{B}_{i,s+1} = B_{i,s} \cup \{a_s\}$ for $i \leq s$ and $\hat{C}_{s+1} = C_s \cup \{c_s\}$.

(*Case 2*) If $\eta = \beta$ is active due to Case 2 and is α-linked at stage $s+1$, let $I = job_s(\beta)$ and let k be such that I is relevant above $y_k(I)$ at stage s. Let $\rho(|\beta|) = \mathfrak{J}_e^{i,j}$.

Let a_s be the greatest number $y \in I$ such that

$$y \geq c_s \tag{78}$$
$$y \notin (A_s \cup B_{0,s} \cup B_{1,s} \cup \ldots \cup B_{s,s}) \text{ and} \tag{79}$$
$$[c_s+1, y] \text{ is } E\text{-safe for } E = \{\rho(|\beta'|) : \beta' 0 \sqsubset \beta 0\}. \tag{80}$$

(Note that such a number exists, because by (74) $y = c_s$ satisfies all three conditions.)

Let b_s be the greatest number $x \in I$ such that

$$c_s \leq x \leq a_s \tag{81}$$
$$[\min(I), x-1] \cap (B_{i,s} \cup B_{j,s}) = \emptyset \tag{82}$$
$$x \notin B_{i,s} \cap B_{j,s} \text{ and} \tag{83}$$
$$[c_s+1, x] \text{ is } E\text{-safe for } E = \{\rho(|\beta'|) : \beta' 0 \sqsubseteq \beta 0\}. \tag{84}$$

(Note that such a number exists, because again by (74) $y = c_s$ satisfies all three conditions.)

(a) If $[c_s+1, c_s+y_k]$ is $\{\rho(|\beta'|) : \beta' 0 \sqsubseteq \beta 0\}$-safe and $[c_s+1, c_s+y_k(I)] \cap (A_s \cup B_{0,s} \cup B_{1,s} \cup \ldots \cup B_{s,s}) = \emptyset$, or if $\{c_s+1\}$ is not $\{\rho(|\beta'|) : \beta' 0 \sqsubseteq \beta 0\}$-safe at stage s, set $\hat{A}_{s+1} = A_s \cup \{c_s\}$, $\hat{B}_{j',s+1} = B_{j',s} \cup \{c_s\}$ for $j' \leq s$ and $\hat{C}_{s+1} = C_s \cup \{c_s\}$. Set $job_{s+1}(\beta) \uparrow$ and $cand_{s+1}(\alpha) = I$.

(b) If $\{c_s+1\}$ is $\{\rho(|\beta'|) : \beta' 0 \sqsubseteq \beta 0\}$-safe at stage s, but $[c_s+1, c_s+y_k(I)]$ is not or has nonempty intersection with $A_s \cup B_{0,s} \cup B_{1,s} \cup \ldots \cup B_{s,s}$, then let $i' = i$ if $b_s \notin B_{i,s}$ and $i' = j$ otherwise and set $\hat{A}_{s+1} = A_s \cup \{a_s\}$, $\hat{B}_{j',s+1} = B_{j',s} \cup \{a_s\}$ for $j' \leq s$ with $j' \neq i'$, $\hat{B}_{i',s+1} = B_{i',s} \cup \{b_s\}$ and $\hat{C}_{s+1} = C_s \cup \{c_s\}$.

In all cases, initialise all nodes $\alpha' >_L \delta_s$, i.e. declare all intervals I' assigned to these nodes unassigned and not relevant above any number, set $cand_{s+1}(\alpha') =\uparrow$ and set $job_{s+1}(\beta') \uparrow$ for all β' with $job_s(\beta') = I'$. Also initialise every node $\alpha' > \alpha$.

Let

$$Z = \{\min(I') : (\exists \tilde{\beta} 0 \sqsubseteq \delta_s)(job_s(\tilde{\beta}) = I' \text{ and } I' \text{ is an } \alpha'\text{-interval for some } \alpha' \text{ initialised at stage } s{+}1\}.$$

Set $A_{s+1} = \hat{A}_{s+1} \cup Z$, $B_{i,s+1} = \hat{B}_{i,s+1} \cup Z$ for $i \leq s$ and $C_{s+1} = \hat{C}_{s+1} \cup Z$.

Set $B_{i,s+1} = B_{i,s}$ for all $i > s$. For all nodes α, unless stated otherwise before, leave the assignment of intervals and diagonalisation witnesses, the values of $cand(\alpha)$ and $job(\alpha)$ and the relevant parts of α intervals at stage $s+1$ as they were at stage s.

Quit the stage.

4.7.2 Verification

Lemma 4.40 (True Path Lemma). *It holds that* $\mathrm{TP} = \liminf_{s\to\infty} \delta_s$*, i.e. if* $\alpha \in T$*, then* $\alpha \sqsubset \mathrm{TP}$ *if and only if* $\alpha \sqsubseteq \delta_s$ *for infinitely many* s *and there are only finitely many* s *such that* $\delta_s <_L \alpha$*.*

Proof. Analogous to the proof of Theorem 3.20. □

Lemma 4.41. *For every* α*-interval* I *and every stage* s *exactly one of the following holds:*

- I *is not assigned to* α *at stage* s
- $cand_s(\alpha) = I$
- $job_s(\beta) = I$ *for some* $\beta \sqsubseteq \alpha$
- α *has a diagonalisation witness* (x, y) *with* $x, y \in I$ *assigned at stage* s

Proof. Immediate by induction on s. □

Lemma 4.42. *For* $i \geq s$ *it holds that* $B_{i,s} = \emptyset$*.*

Proof. Immediate by induction on s. □

Lemma 4.43. *Let* I *be an* α*-interval and let* β *be a node such that* $\beta 0 \sqsubseteq \alpha$ *and* $\rho(|\beta|) = \mathfrak{J}_e^{i,j}$*. Assume that* I *is demoted to* β *and becomes relevant above* $y_k := y_k(I)$ *at stage* $t_0 + 1$ *and that for some* $r \leq 2y_k$ *there are minimal stages* $t_0 + 1 < t_1 + 1 < \ldots < t_r + 1$ *such that* $I \cap C_{t_n+1} \neq I \cap C_{t_n}$ *for* $n \in \{0, \ldots, r\}$ *and* α *is not initialised at any stage* $s \in [t_0 + 1, t_r + 1]$*. Furthermore assume that* $\{c_{t_n}\}$ *is* $\{\rho(|\beta'|) : \beta' 0 \sqsubseteq \beta 0\}$*-safe at stage* t_{n+1} *for* $0 \leq n < r$*.*

Then for $1 \leq 2n+q \leq r$ $(q \in \{0,1\})$ *the node* β *is active due to Case 2(b) at stage* $t_{2n+q}+1$ *if* $2n + q < 2y_k$ *and due to Case 2(a) at stage* $t_{2n+q} + 1$ *if* $2n + q = 2y_k$ *and for* $0 \leq 2n + q \leq r$ *it holds that*

$$[c_{t_{2n+q}} + 1, c_{t_{2n+q}} + n] \cap (A_{t_{2n+q}} \cup \bigcup_{i' \in \mathbb{N}} B_{i',t_{2n+q}}) = \emptyset \tag{85}$$

and

$$c_{t_{2n+q}} + n + 1 \in \begin{cases} B_{i,t_{2n+q}} \cap B_{j,t_{2n+q}} & \textit{if } q = 0 \\ B_{i,t_{2n+q}} - B_{j,t_{2n+q}} & \textit{if } q = 1. \end{cases} \tag{86}$$

Proof. The proof is by induction on $2n + q$. If $2n + q = 0$, then (85) is trivially true because $[c_{t_0} + 1, c_{t_0} + 0] = \emptyset$. For (86), note that $cand_{t_0}(\alpha) = I$. Let $s < t_0$ be maximal such that $I \cap C_s \neq I \cap C_{s+1}$. Then $cand_{s+1}(\alpha) = I$. Hence the enumeration into $I \cap C$ at stage $s + 1$ must be caused by some node being active due to Case 1.1, Case 1.2 or Case 2(a). In each case,

$c_s \in B_{i,s+1} \cap B_{j,s+1} \subseteq B_{i,t_0} \cap B_{j,t_0}$ (remember that $i, j \leq |\beta| \leq s$). But by (74) it holds that $c_{t_0} + 1 = c_s$, so (86) is true, too.

Now fix $2n + q < r$ and assume that the inductive hypothesis is true for all numbers up to $2n + q$.

By the inductive hypothesis, since $2n + q < r \leq y_k$, at stage $t_{2n+q} + 1$ either I was demoted to β (if $n = q = 0$) or β was active due to Case 2(b). Hence $job_{t_{2n+q}+1}(\beta) = I$. Then still $I \in jobs_{t_{2n+q+1}}(\beta)$. Since $I \cap C_{t_{2n+q+1}+1} \neq I \cap C_{t_{2n+q+1}}$, but α is not initialised at stage $t_{2n+q+1} + 1$, β must be active due to Case 2 at stage $t_{2n+q+1} + 1$.

For the proof of equations (85) and (86) we first show that $a_{t_{2n+q}} > c_{t_{2n+q}} + n$. If $2n+q = 0$, this is immediate by the definition of a_{t_0} via Case 1.4. If $2n + q > 0$, since I is demoted at stage $t_0 + 1$, by (77), there are at least $y_{k+1}(I)$ numbers $z \in [c_{t_0} + 1, a_{t_0}]$ which are not in $A_{t_0} \cup B_{0,t_0} \cup \ldots \cup B_{t_0,t_0}$. By Lemma 4.42, these numbers are not in $\bigcup_{i' \in \mathbb{N}} B_{i',t_0}$. Since at each of the stages $t_0 + 1, t_1 + 1, \ldots, t_{2n+q-1} + 1$ only one such number z is enumerated into $A \cup \bigcup_{i' \in \mathbb{N}} B_{i'}$, while at stages $t \in [t_0 + 1, t_{2n+q-1}] - \{t_0 + 1, t_1 + 1, \ldots, t_{2n+q-1} + 1\}$ there are no such enumerations, there are still $y_{k+1}(I) - (2n + q) \geq y_k - (2n + q) > 0$ numbers $z \in [c_{t_0} + 1, a_{t_0}]$ which are not in $A_{t_{2n+q}} \cup \bigcup_{i' \in \mathbb{N}} B_{i',t_{2n+q}}$. Let y be the greatest such number. Then $[c_{t_0} + 1, y]$ was $\{\rho(|\beta'|) : \beta'0 \sqsubset \beta0\}$-safe at stage t_0. On the other hand each number $x \in [c_{t_{2n+q}} + 1, c_{t_0}]$ has the form $x = c_{t_m}$ for some $m \in [0, 2n + q - 1]$. Since by the hypothesis $\{c_{t_m}\}$ is $\{\rho(|\beta'|) : \beta'0 \sqsubseteq \beta0\}$-safe at stage t_{m+1}, hence at stage t_{2n+q}, it follows that $[c_{t_{2n+q}}+1, y]$ is $\{\rho(|\beta'|) : \beta'0 \sqsubset \beta0\}$-safe at stage t_{2n+q}. Hence the number y satisfies equations (78) to (80) with t_{2n+q} in place of s. It follows that $a_{t_{2n+q}} \geq y > c_{t_0}$. But $c_{t_{m+1}} + 1 = c_{t_m}$ for all $m < r$, and hence by induction $c_{t_{2n+q}} + n = c_{t_{2n+q-n}} = c_{t_{n+q}} \leq c_{t_0} < a_{t_{2n+q}}$.

Now first consider the case $q = 0$. We claim that $b_{t_{2n+q}} = c_{t_{2n+q}} + n$. This is clear by definition if $n = q = 0$. For $n > 0$, indeed $x = c_{t_{2n+q}}+n$ satisfies equation (81) with t_{2n+q} instead of s, as we have just shown. Moreover, x satisfies equations (82) and (83) by equation (74) and equation (85). Finally, x satisfies equation (84) because for every number $x' \in [c_{t_{2n+q}}+1, c_{t_0}]$ the set $\{x'\}$ is $\{\rho(|\beta'|) : \beta'0 \sqsubseteq \beta0\}$-safe at stage t_{2n+q}, as already mentioned, and because $x \leq c_{t_0}$. Altogether, this shows that $b_{t_{2n+q}} \geq x$. On the other hand, by equation (86), x is the greatest number in I satisfying both equations (82) and (83) (for $x' = x + 1$ equation (83) fails and for $x' > x + 1$ equation (82) fails). Hence $b_{t_{2n+q}} = c_{t_{2n+q}} + n$. Since by the inductive hypothesis $b_{t_{2n+q}} = c_{t_{2n+q}} + n = c_{t_{2n+q+1}} + n + 1$ is enumerated into B_i at stage $t_{2n+q} + 1$ due to Case 2 (b), but is not in $B_{j,t_{2n+q}+1}$ (this follows from equation (85)), since no other number from I is enumerated into A or any set $B_{i'}$ at stage $t_{2n+q} + 1$, no number from I is enumerated into A or any set $B_{i'}$ at any stage $s \in (t_{2n+q}+1, t_{2n+q+1}]$, and since $c_{t_{2n+q+1}} \notin A_{t_{2n+q+1}} \cup \bigcup_{i' \in \mathbb{N}} B_{i',t_{2n+q+1}}$ by Lemma 4.42 and equation (74), we can conclude that equations (85) and (86) are true for $2n + q + 1$.

Now we turn to the case $q = 1$ and claim that $b_{t_{2n+q}} = c_{t_{2n+q}}+n+1$. Similar to the case $q = 0$, $x = c_{t_{2n+q}} + n + 1$ satisfies equation (81) with t_{2n+q} instead of s by what we have shown above, x satisfies equations (82) and (83) by equations (74), (85) and (86), and x satisfies equation (84) because $x = c_{t_{n+q}} + 1 \leq c_{t_0}$ and because $\{x'\}$ is $\{\rho(|\beta'|) : \beta' 0 \sqsubseteq \beta 0\}$-safe at stage t_{2n+q} for every $x' \in [c_{t_{2n+q}} + 1, c_{t_0}]$. This shows that $b_{t_{2n+q}} \geq x$. Since $x \in B_{t_{2n+q}}$ by (86), x is the greatest number in I satisfying (82); hence $b_{t_{2n+q}} = x$, as claimed. Since by the inductive hypothesis $b_{t_{2n+q}}$ is in $B_{i,t_{2n+q}}$ and is enumerated into B_j at stage $t_{2n+q}+1$ due to Case 2 (b), it holds that $c_{t_{2n+q+1}}+(n+1)+1 = c_{t_{2n+q}}+n+1 = b_{t_{2n+q}} \in B_{i,t_{2n+q+1}} \cap B_{j,t_{2n+q+1}}$. Since $c_{t_{2n+q+1}} = c_{t_{2(n+1)+0}}$, this proves equation (86) for $2n+q+1$. Equation (85) for $2n+q+1$ follows from the fact that no number from I less than $b_{t_{2n+q+1}}$ is in $A_{t_{2n+q}} \cup \bigcup_{i' \in \mathbb{N}} B_{i',t_{2n+q}}$ (by Lemma 4.42, (74) and (85)) or is enumerated into A or any set $B_{i'}$ at any stage $s \in [t_{2n+q}+1, t_{2n+q+1}]$.

This completes the inductive proof of equations (85) and (86).

Now if $1 \leq 2n+q+1 < 2y_k$, that is, $n+1 \leq y_k$, then it follows from equation (86) that β must be active due to Case 2(b) at stage $t_{2n+q+1}+1$.

If $2n+q+1 = 2y_k$, that is, $n+1 = y_k$ and $q = 1$, then $[c_{t_{2n+q+1}} + 1, c_{t_{2n+q+1}} + y_k] \subseteq [c_{t_{2n+q+1}} + 1, c_0]$ is $\{\rho(|\beta'|) : \beta' 0 \sqsubseteq \beta 0\}$-safe at stage t_{2n+q+1} as we argued above. Together with equation (85) this implies that β must be active due to Case 2(a) at stage $t_{2n+q+1}+1$. □

Lemma 4.44. *Let I be an α-interval. Assume that for some $k \in \{0, \ldots, \iota(I)-1\}$ there is a sequence of minimal stages $t_0 < t_1 < \ldots < t_{2y_k(I)+1}$ such that $C_{t_n+1} \cap I \neq C_{t_n} \cap I$ and I is relevant above $y_k := y_k(I)$ at stage t_n+1 for $n \in \{0, \ldots, 2y_k+1\}$. Then there must be a stage t_n such that α is ready for elimination at stage t_n+1.*

Proof. For $n \in \{0, \ldots, 2y_k+1\}$ let β_{t_n} be the node that is active at stage t_n+1. For a contradiction assume that α is not ready for elimination at any stage t_n+1.

If I would already become relevant above y_k at a stage $s+1$ before stage t_0+1, then since $C_s \cap I = C_{s+1} \cap I$ by minimality of t_0, α would get the interval I and a diagonalisation witness assigned at stage $s+1$. In this case α could only be active due to Case 1.4 at stage t_0+1 and never enumerate anything into $C \cap I$ after stage t_0+1, contradicting the fact that $C_{t_{2y_k+1}} \cap I \neq C_{t_{2y_k+1}} \cap I$. Hence I becomes relevant above y_k at stage t_0+1, i.e. $\beta_{t_0} = \alpha$ and I is demoted to β_{t_1} at stage t_0+1.

Let r be the greatest number in $\{0, \ldots, 2y_k\}$ such that $\{c_{t_n}\}$ is $\{\rho(|\beta'|) : \beta' 0 \sqsubseteq \beta_{t_1} 0\}$-safe at stage t_{n+1} for all $n < r$. If $r = 2y_k$, then by Lemma 4.43 β_{t_1} is active due to Case 2 (a) at stage $t_{2y_k}+1$. If $r < 2y_k$, then by Lemma 4.43 β_{t_1} is active due to Case 2 (b) at all stages $t_1+1, \ldots, t_r+1$. Then β_{t_1} must be active due to Case 2 at stage $t_{r+1}+1$, too (since α is not initialised but a new number is enumerated into $I \cap C$ at that stage). But since $\{c_{t_{r+1}} + 1\} = \{c_{t_r}\}$ is not $\{\rho(|\beta'|) : \beta' 0 \sqsubseteq \beta_{t_1} 0\}$-safe at stage t_{r+1}, β_{t_1} is active due to Case 2(a) at stage $t_{r+1}+1$.

This shows that there is a least stage t_p with $p \leq 2y_k$ such that β_{t_1} is active due to Case 2 (a) at stage t_p+1. Then $cand_{t_{p+1}}(\alpha) = I$. Since α is not initialised and not active due to Case 1.3 at stage $t_{p+1}+1$ (otherwise I would not be relevant above y_k any more), it must be active due to Case 1.2, proving the claim. □

Lemma 4.45. *Let I be an α-interval and let $s < s'$ be stages such that $cand_s(\alpha) = I$ and α is ready for elimination at stage $s+1$, α is not initialised at stage $s'+1$ and $C_{s'} \cap I \neq C_{s'+1} \cap I$. Then α is ready for elimination at stage $s'+1$, too.*

Proof. It suffices to show this for the case that s' is the *least* stage $t > s$ such that $C_t \cap I \neq C_{t+1} \cap I$; then the general claim follows by induction on s'.

By the hypothesis it holds that

$$|\{x \leq c_s : x \notin W_{e_0,s}\}| < |\{x \leq c_s : x \in I\}| \tag{87}$$

for some $\beta 0 \sqsubseteq \alpha$ with $\rho(|\beta|) = \mathcal{J}^{i,j}_{\langle e_0,e_1,e_2\rangle}$.

By the conditions of α requiring attention at stage $s+1$,

$$l_s(\mathcal{J}^{i,j}_e) > \max(I).$$

Since $cand_{s'}(\alpha) = cand_{s+1}(\alpha) = I$, α must require attention and be active at stage $s'+1$, too, hence

$$l_{s'}(\mathcal{J}^{i,j}_e) > \max(I).$$

Since c_s is enumerated into $B_{i,s+1} - B_{i,s}$ at stage $s+1$, this means that

$$\hat{\Phi}^{W_{e_0,s'}}_{e_1,s'} \restriction (c_s+1) = B_{i,s'} \restriction (c_s+1) \neq B_{i,s} \restriction (c_s+1) = \hat{\Phi}^{W_{e_0,s}}_{e_1,s} \restriction (c_s+1).$$

Consequently, since $\hat{\Phi}_{e_1}$ is an ibT-functional, there is some $z \leq c_s$ in $W_{e_0,s'} - W_{e_0,s}$. Since $c_{s'} = c_s - 1$ by (74) and the fact that no number from I is enumerated into C between stages $s+1$ and $s'+1$, it follows that

$$|\{x \leq c_{s'} : x \notin W_{e_0,s'}| < |\{x \leq c_s : x \notin W_{e_0,s}| \tag{88}$$

and

$$|\{x \leq c_{s'} : x \in I\}| = |\{x \leq c_s : x \in I\}| - 1. \tag{89}$$

Using (87), (88) and (89) it follows that

$$|\{x \leq c_{s'} : x \notin W_{e_0,s'}\}| < |\{x \leq c_{s'} : x \in I\}|. \tag{90}$$

Hence α is ready for elimination at stage $s'+1$.

□

Lemma 4.46. *Let I be an α-interval and let*

$$E = \{\beta : \beta 0 \sqsubseteq \alpha \textit{ and } \rho(|\beta|) \textit{ is a join requirement}\} = \{\beta'_0, \ldots, \beta'_{|E|-1}\}$$

with $\beta'_0 \sqsubset \beta'_1 \sqsubset \ldots \sqsubset \beta'_{|E|-1}$. Let $y_0 = y_0(I)$. Then for $\bar{e} \in \{0, \ldots, |E|-1\}$ there is no sequence of minimal stages $s_0 < s_1 < \ldots < s_{|E|\cdot y_0}$ such that, for $n \in \{0, \ldots, |E| \cdot y_0\}$, I is relevant above some $y_k(I)$ with $k \geq 0$ at stage s_n, I is demoted to $\beta'_{\bar{e}}$ at stage $s_n + 1$, and I is not demoted to any β'_e with $e < \bar{e}$ at any stage $s \in [s_0 + 1, s_{|E|\cdot y_0} + 1]$.

Proof. For a contradiction assume that there is such a sequence $s_0 < s_1 < \ldots < s_{|E|\cdot y_0}$ for some $\bar{e}$. We show by induction on m that, for $m \in \{0, \ldots, |E| \cdot y_0\}$,

$$\sum_{e=0}^{\bar{e}} |\{w \leq c_{s_m} : w \notin W_{\pi_0(e),s_m}\}| \leq (\bar{e}+1) \cdot (c_{s_m} + 1) - m, \tag{91}$$

where $\rho(|\beta'_e|) = \eth^{i_e,j_e}_{\langle \pi_0(e),\pi_1(e),\pi_2(e)\rangle}$ for $e \in E$.

For $m = 0$ (91) is trivially true, because

$$\sum_{e=0}^{\bar{e}} |\{w \leq c_{s_0} : w \notin W_{\pi_0(e),s_0}\}| \leq (\bar{e}+1) \cdot |[0, c_{s_0}]| = (\bar{e}+1) \cdot (c_{s_0} + 1) - 0.$$

Assume that (91) is true for m. Define a sequence of stages $s_m = t_0 < t_1 < \ldots < t_{p+1} = s_{m+1}$ such that $\{t_q : 0 \leq q \leq p+1\} = \{t \in [s_m, s_{m+1}] : I \cap C_t \neq I \cap C_{t+1}\}$. Let β_{t_q} be the node that is active at stage $t_q + 1$.

We will show that, for all $q \in \{0, \ldots, p\}$ and all $e \leq \bar{e}$,

$$|\{w \leq c_{t_{q+1}} : w \notin W_{\pi_0(e),t_{q+1}}\}| \leq |\{w \leq c_{t_q} : w \notin W_{\pi_0(e),t_q}\}| - 1 \tag{92}$$

and that there is some $q \in \{0, \ldots, p\}$ and some $e \leq \bar{e}$ such that

$$|\{w \leq c_{t_{q+1}} : w \notin W_{\pi_0(e),t_{q+1}}\}| \leq |\{w \leq c_{t_q} : w \notin W_{\pi_0(e),t_q}\}| - 2. \tag{93}$$

Then, using the inductive hypothesis (91) and $t_0 = s_m$ we know that

$$\sum_{e=0}^{\bar{e}} |\{w \leq c_{t_0} : w \notin W_{\pi_0(e),t_0}\}| \leq (\bar{e}+1) \cdot (c_{t_0} + 1) - m,$$

and by (92), (93) and $c_{t_{q+1}} = c_{t_q} - 1$ (by (74) it follows that, for $q \in \{0, \ldots, p\}$,

$$\sum_{e=0}^{\bar{e}} |\{w \leq c_{t_{q+1}} : w \notin W_{\pi_0(e),t_{q+1}}\}| \leq (\bar{e}+1) \cdot (c_{t_0} + 1) - m - (q+1) \cdot (\bar{e}+1)$$

and

$$\sum_{e=0}^{\bar{e}} |\{w \leq c_{t_{p+1}} : w \notin W_{\pi_0(e),t_{p+1}}\}| \leq (\bar{e}+1) \cdot (c_{t_0} + 1) - m - (p+1) \cdot (\bar{e}+1) - 1. \tag{94}$$

Since $c_{t_{q+1}} = c_{t_q} - 1$ for $q \in \{0, \dots, p\}$, we see that $c_{s_{m+1}} = c_{t_{p+1}} = c_{t_0} - (p+1)$. Using (94) this implies

$$\begin{aligned}\sum_{e=0}^{\bar{e}} |\{w \leq c_{s_{m+1}} : w \notin W_{\pi_0(e), s_{m+1}}\}| &\leq (\bar{e}+1) \cdot (c_{t_0} + 1) - (m+1) - (p+1) \cdot (\bar{e}+1) \\ &= (\bar{e}+1) \cdot (c_{t_0} - (p+1) + 1) - (m+1) \\ &= (\bar{e}+1) \cdot (c_{s_{m+1}} + 1) - (m+1),\end{aligned}$$

i.e. (91) is true for $m+1$. So to prove (91), it suffices to prove (92) and (93).

To prove (92), fix $e \leq \bar{e}$ and $q \in \{0, \dots, p\}$. Let $\rho(|\beta'_e|) = \mathfrak{J}^{i,j}_{\langle e_0, e_1, e_2 \rangle}$ (in particular, $\pi_0(e) = e_0$). Since I is demoted to $\beta'_{\bar{e}}$ at stage $s_m + 1 = t_0 + 1$ and not demoted to any node below $\beta'_{\bar{e}}$ at any stage t with $t_0 + 1 \leq t \leq t_{p+1}$, it holds that $\beta'_e \sqsubseteq \beta'_{\bar{e}} \sqsubseteq \beta_{t_q} \sqsubseteq \delta_{t_q}$ and $\delta_{t_q}(|\beta'_e|) = 0$, as well as $\beta'_e \sqsubseteq \beta'_{\bar{e}} \sqsubseteq \beta_{t_{q+1}} \sqsubseteq \delta_{t_{q+1}}$ and $\delta_{t_{q+1}}(|\beta'_e|) = 0$. Since β_{t_q} is active at stage $t_q + 1$ and $\beta_{t_{q+1}}$ is active at stage $t_{q+1} + 1$ (due to Case 1.2, Case 1.3 or Case 2), it follows that

$$l_{t_q}(\mathfrak{J}^{i,j}_{\langle e_0, e_1, e_2 \rangle}) = l_{t_q}(\rho(|\beta'_e|)) > \max(I) \tag{95}$$

$$l_{t_{q+1}}(\mathfrak{J}^{i,j}_{\langle e_0, e_1, e_2 \rangle}) = l_{t_{q+1}}(\rho(|\beta'_e|)) > \max(I). \tag{96}$$

If $\beta'_e = \beta_{t_q}$, then β_{t_q} is active due to Case 2 at stage $t_{q+1} + 1$ and some $b \in \{c_{t_q}, b_{t_q}\}$ is enumerated into $B_{i, t_q+1} - B_{i, t_q}$ or $B_{j, t_q+1} - B_{j, t_q}$ at stage $t_q + 1$ (note that $c_{t_q} \notin B_{i, t_q} \cup B_{j, t_q}$ by equation (74), and if $b_{t_q} \neq c_{t_q}$, then $b_{t_q} \notin B_{i, t_q} \cap B_{j, t_q}$ by (83)). By (95) and (96) it follows that

$$\hat{\Phi}^{W_{e_0, t_{q+1}}}_{e_1, t_{q+1}} \restriction (b+1) = B_{i, t_{q+1}} \restriction (b+1) \neq B_{i, t_q} \restriction (b+1) = \hat{\Phi}^{W_{e_0, t_q}}_{e_1, t_q} \restriction (b+1) \tag{97}$$

or

$$\hat{\Phi}^{W_{e_0, t_{q+1}}}_{e_2, t_{q+1}} \restriction (b+1) = B_{j, t_{q+1}} \restriction (b+1) \neq B_{j, t_q} \restriction (b+1) = \hat{\Phi}^{W_{e_0, t_q}}_{e_2, t_q} \restriction (b+1). \tag{98}$$

Since $\hat{\Phi}_{e_1}$ and $\hat{\Phi}_{e_2}$ are ibT-functionals, this implies

$$W_{e_0, t_{q+1}} \restriction (b+1) \neq W_{e_0, t_q} \restriction (b+1), \tag{99}$$

in particular there is some minimal $z^e_q \leq b_{t_q}$ in $W_{e_0, t_{q+1}} - W_{e_0, t_q}$. Since $[c_{t_q} + 1, b] \subseteq [c_{t_q} + 1, b_{t_q}]$ is $\{\beta_{\bar{e}}\}$-safe at stage t_q by equation (84), necessarily $z^e_q \leq c_{t_q}$.

On the other hand, if $\beta'_e \sqsubset \beta_{t_q}$, then some $b \in \{a_{t_q}, c_{t_q}\}$ is enumerated into $B_{i, t_q+1} - B_{i, t_q}$ at stage $t_q + 1$ (note that $i \leq |\delta_{t_q}| \leq t_q$ by the choice of ρ, $c_{t_q} \notin B_{i, t_q}$ by equation (74), and $a_{t_q} \notin B_{i, t_q}$ by equation (77) or equation (79), respectively). As above we can deduce (99) and there is some minimal $z^e_q \leq b$ in $W_{e_0, t_{q+1}} - W_{e_0, t_q}$. In the case that $b = c_{t_q}$, trivially $z^e_q \leq c_{t_q}$. In the case that $c = a_{t_q}$, by the conditions on a_{t_q} in Case 1.3 or Case 2 $[c_{t_q} + 1, a_{t_q}]$ is $\{\rho(|\gamma|) : \gamma 0 \sqsubset \beta_{t_q}\}$-safe, hence in particular $[c_{t_q} + 1, a_{t_q}] \subseteq W_{e_0, t_q}$. It follows that $z^e_q \leq c_{t_q}$ again.

Consequently,

$$\{w \leq c_{t_{q+1}} : w \notin W_{e_0,t_{q+1}}\} \subseteq \{w \leq c_{t_q} : w \notin W_{e_0,t_q}\} - \{z_q^e\},$$

and (92) follows.

To prove (93), it suffices to show that for some $q \in \{0, \ldots, p\}$ and some $e \leq \bar{e}$ with $\pi_0(e) = e_0$, $c_{t_q} \notin W_{e_0,t_{q+1}}$, whence $z_q^e \neq c_{t_q}$ and

$$\{w \leq c_{t_{q+1}} : w \notin W_{e_0,t_{q+1}}\} \subseteq \{w \leq c_{t_q} : w \notin W_{e_0,t_q}\} - \{z_q^e, c_{t_q}\},$$

proving (93).

For a contradiction assume that for all $q \in \{0, \ldots, p\}$ and all $e \leq \bar{e}$,

$$c_{t_q} \in W_{e_0,t_{q+1}}. \tag{100}$$

Let I become relevant above $y_k(I)$ at stage $t_0 + 1$ and let $r \in \{1, \ldots, p\}$ be maximal such that $\beta_{t_r} = \beta'_{\bar{e}}$. Then using equation (100) from Lemma 4.43 we can conclude that $r = 2y_k(I)$ and that $\beta'_{\bar{e}}$ is active due to Case 2.2 (a) at stage t_r+1. Since $c_{t_r}+1 = c_{t_{r-1}} \in W_{e_0,t_r}$ by assumption, it must hold that $[c_{t_r} + 1, c_{t_r} + y_k(I)]$ is $\{\rho(|\beta'|) : \beta' 0 \sqsubseteq \beta_{\bar{e}}\}$-safe and

$$[c_{t_r} + 1, c_{t_r} + y_k(I)] \cap (A_{t_r} \cup B_{0,t_r} \cup B_{1,t_r} \cup \ldots \cup B_{t_r,t_r}) = \emptyset. \tag{101}$$

Let k' be such that I is relevant above $y_{k'}(I)$ at stage t_{p+1}. By the hypothesis $0 \leq k' \leq k$. By Lemma 4.45, α is not ready for elimination at any stage $s \in [t_r + 1, t_{p+1}]$, because $I \cap C_{t_{p+1}+1} \neq I \cap C_{t_{p+1}}$ but α is not ready for elimination at stage $t_{p+1} + 1$. By Lemma 4.44, then for each $\tilde{k}$ with $k' \leq \tilde{k} < k$ there are at most $2y_{\tilde{k}}(I)+1$ stages $s \in [t_r+1, t_{p+1}]$ such that I is relevant above $y_{\tilde{k}}(I)$ at stage s and some new number from I is enumerated into C at stage s. For $\tilde{k} = k$ stage $s = t_r + 1$ is the only such stage (because when the next number from I is enumerated into C after stage $t_r + 1$, then I is demoted), while for $\tilde{k} \leq k'$ there are no stages

s as above. Hence there are at most

$$\begin{aligned} 1+\sum_{\tilde{k}=k'}^{k-1}(2y_{\tilde{k}}+1) &= 1+\sum_{\tilde{k}=k'}^{k-1}(2\cdot 3^{\tilde{k}}(y_0+\tilde{k})+1) \\ &= 1+\sum_{\tilde{k}=k'}^{k-1}(3\cdot 3^{\tilde{k}}(y_0+\tilde{k})-3^{\tilde{k}}(y_0+\tilde{k})+1) \\ &< 1+\sum_{\tilde{k}=k'}^{k-1}(3^{\tilde{k}+1}(y_0+(\tilde{k}+1))-3^{\tilde{k}}(y_0+\tilde{k})) \\ &= 1+\sum_{\tilde{k}=k'}^{k-1}(y_{\tilde{k}+1}(I)-y_{\tilde{k}}(I)) \\ &= 1+y_k(I)-y_{k'}(I) \end{aligned} \tag{102}$$

enumerations of numbers from I into C at stages $s+1 \in [t_r+1, t_{p+1}]$. Each such enumeration is accompanied by the enumeration of at most one number from $[c_{t_r}+1, c_{t_r}+y_k(I)]$ into $A\cup\bigcup_{i\in\mathbb{N}} B_i$. Indeed, since c_{t_r} is enumerated into $B_0,\dots,B_{t_r}$ at stage t_r+1, by equation (83) and (74) it follows that if $b_s \in I$ at stages $s+1\in[t_r+1,t_{p+1}]$, then $b_s \le c_{t_r}$; hence a_s is the only number from $[c_{t_r}+1, c_{t_r}+y_k(I)]$ that can possibly be enumerated into $A\cup\bigcup_{i\in\mathbb{N}} B_i$ at such a stage $s+1$. Since there are no enumerations into $A\cup\bigcup_{i\in\mathbb{N}} B_i$ at stages $s+1$ with $I\cap C_s = I\cap C_{s+1}$, by (101) and Lemma 4.42 we can conclude that

$$|[c_{t_r}+1, c_{t_r}+y_k(I)]\cap(A_{t_p}\cup\bigcup_{i\in\mathbb{N}} B_{i,t_p})| \le y_k(I)-y_{k'}(I).$$

If y is the largest number in $[c_{t_r}+1, c_{t_r}+y_k(I)]$ which is not in $A_{t_p}\cup\bigcup B_{i,t_p}$, then this means that

$$|\{z\in[c_{t_{p+1}}+1, y]\cap I : z\notin A_{t_p}\cup\bigcup B_{i,t_p}\}| \ge y_{k'},$$

that is, equation (77) (with t_{p+1} in place of s) is true.

Moreover, since each $c\in[c_{t_{p+1}}+1, c_{t_0}]$ is of the form $c=c_{t_q}$ for some $q\in\{0,\dots,p\}$, by (100) we know that $[c_{t_{p+1}}+1, c_{t_r}+y_k(I)]$ is $\{\rho(|\beta'|) : \beta'\sqsubset\beta \text{ and } \beta'0\sqsubseteq\alpha\}$-safe at stage t_{p+1}, where $\beta=\beta'_{\bar{e}+1}$ if $\bar{e}<|E|-1$ and $\beta=\alpha$ otherwise. But then by (76) and (77) at stage $t_{p+1}+1$ either the interval I is demoted to β (and not to $\beta_{\bar{e}}$) or α gets a diagonalisation witness assigned, contradicting the fact that $t_{p+1}=s_{m+1}$ and the assumption on s_{m+1}.

This completes the proof of (93).

Now substituting $m = |E| \cdot y_0$ in (91), we get

$$\begin{aligned}\sum_{e=0}^{\bar{e}} |\{w \leq c_{s_{|E|\cdot y_0}} : w \notin W_{\pi_0(e), s_{|E|\cdot y_0}}\}| &\leq (\bar{e}+1)\cdot(c_{s_{|E|\cdot y_0}}+1) - |E|\cdot y_0 \\ &\leq (\bar{e}+1)\cdot(c_{s_{|E|\cdot y_0}}+1-y_0).\end{aligned}$$

Hence there must be some $e \leq \bar{e}$ with

$$\begin{aligned} & |\{w \leq c_{s_{|E|\cdot y_0}} : w \notin W_{\pi_0(e), s_{|E|\cdot y_0}}\}| \\ \leq \; & c_{s_{|E|\cdot y_0}} + 1 - y_0 \\ < \; & |[\min(I), c_{s_{|E|\cdot y_0}}]| \\ = \; & |\{w \leq c_{s_{|E|\cdot y_0}} : w \in I\}|.\end{aligned}$$

Then α is active due to Case 2 at stage $s_{|E|\cdot y_0}+1$ and I is not demoted to $\beta_{\bar{e}}$, contradicting the assumption.

□

Lemma 4.47. *Let I be an α-interval such that $I \subseteq C$ and such that there is a node β with $\beta 0 \sqsubset \alpha$. Let $I_0 = [\min(I), y_{\iota(I)}(I)]$, let $t_0 < \ldots < t_{|I_0|-1}$ be such that for $n \in \{0, \ldots, |I_0|-1\}$, at stage t_n+1 the number $c_{t_n} = y_{\iota(I)}(I) - n$ is enumerated into C and let β_{t_n} be the node that is active at stage t_n+1. Then either α is initialised and the assignment of I to α is cancelled at stage $t_{|I_0|-1}+1$, or $cand_{t_{|I_0|-1}+1}(\alpha) = I$ and $\alpha \not\sqsubseteq \delta_s$ for any $s > t_{|I_0|-1}$.*

Proof. It suffices to show that α is ready for elimination at some stage t_m+1, $m \in \{0, \ldots, |I_0|-2\}$. Indeed, by Lemma 4.45, in this case, if α is not initialised at stage $t_{|I_0|-1}+1$, then α is ready for elimination at stage $t_{|I_0|-1}+1$, whence $cand_{t_{|I_0|-1}+1}(\alpha) = I$ and there is some β with $\beta 0 \sqsubseteq \alpha$ such that, for $\rho(|\beta|) = \mathfrak{J}^{i,j}_{\langle e_0, e_1, e_2\rangle}$,

$$|\{x \leq \min(I) : x \notin W_{e_0, t_{|I_0|-1}}\}| < |\{x \leq \min(I) : x \in I\} \leq 1,$$

that is $W_{e_0} \restriction \min(I)+1 = W_{e_0, t_{|I_0|-1}} \restriction \min(I)+1 = [0, \min(I)]$. Moreover,

$$l_{t_{|I_0|-1}}(\mathfrak{J}^{i,j}_{\langle e_0, e_1, e_2\rangle}) > \max(I)$$

and $\min(I)$ is enumerated into B_i at stage $t_{|I_0|-1}+1$ (remember that $i \leq |\delta_{t_{|I_0|-1}}| = t_{|I_0|-1}$ by our assumptions on ρ).

Since $\hat{\Phi}_{e_1}$ is an ibT-functional, it follows that for $s > t_{|I_0|-1}$

$$\hat{\Phi}^{W_{e_0,s}}_{e_1,s}(\min(I)) = \hat{\Phi}^{W_{e_0, t_{|I_0|-1}}}_{e_1, t_{|I_0|-1}}(\min(I)) = B_{i, t_{|I_0|-1}}(\min(I)) = 0 \neq 1 = B_{i,s}(\min(I)).$$

Then $l_s(\mathfrak{J}^{i,j}_{\langle e_0, e_1, e_2\rangle}) \leq \min(I)$ for all $s > t_{|I_0|-1}$ and $\beta 0 \not\sqsubseteq \delta_s$, hence $\alpha \not\sqsubseteq \delta_s$, proving the lemma.

Let $E = \{\beta' : \beta'0 \sqsubseteq \alpha\} = \{\beta'_0, \ldots, \beta'_{|E|-1}\}$ with $\beta'_0 \sqsubset \beta'_1 \ldots \sqsubset \beta'_{|E|-1}$. Note that by the hypothesis $E \neq \emptyset$. By Lemma 4.44, if for some $k \in \{0, \ldots, \iota(I) - 1\}$ there is a sequence of stages $t_n + 1 < t_{n+1} + 1 < \ldots < t_{n+2y_k+1} + 1 \leq t_{|I_0|-2} + 1$ at which I is relevant above $y_k(I)$, then α is ready for elimination at some such stage $t_m + 1$, as we claimed.

For a contradiction assume that there is no such sequence. Then for each $k \in \{0, \ldots, \iota(I)-1\}$ there are at most $2y_k(I)+1$ stages t_n, $n \in \{0, |I_0|-2\}$, at which I is relevant above $y_k := y_k(I)$. Moreover, since $cand_{t_0+1}(\alpha) = I$ at stage $t_0 + 1$ (when I becomes assigned to α), α must be active due to Case 1.3 at stage $t_1 + 1$ and I becomes relevant above $y_{\iota(I)-1}(I)$. Hence I is relevant above $y_{\iota(I)}(I)$ at stage t_n if and only if $n = 0$. As in equation (102) we see that there are at most

$$1 + \sum_{k=0}^{\iota(I)-1} (2y_k(I) + 1) < 1 + y_{\iota(I)}(I) - y_0(I) = |I_0| - 1$$

many stages $t_n + 1$, $n \in \{0, \ldots, |I_0| - 1\}$ at which I is relevant above some $y_k(I)$ with $k \geq 0$. In particular, at stage $t_{|I_0|-2} + 1$ it is relevant above some $y_k(I)$ with $k < 0$. Since I is relevant above $y_{\iota(I)}(I)$ at stage $t_0 + 1$, there must be at least $\iota(I) + 1$ many stages $t_{i_0} + 1, \ldots, t_{i_{\iota(I)}} + 1$ (with $0 < i_0 < \ldots < i_{\iota(I)} \leq |I_0| - 2$ chosen to be minimal) at which I is demoted and the number $y_k(I)$ that I is relevant above is decreased.

For $e \in \{0, \ldots, |E| - 1\}$, let

$$D_e = \{t \in \{t_{i_0}, \ldots, t_{i_{\iota(I)}}\} : job_{t+1}(\beta'_e) = I\}.$$

Then each t_{i_k}, $0 \leq k \leq \iota(I)$ is in D_e for exactly one e. We show that there must be some e such that

$$|D_e| \geq (|E| \cdot y_0 + 1) \cdot (\sum_{e' < e} |D_{e'}| + 1). \tag{103}$$

If this were not true, then for all $e < |E|$,

$$|D_e| < (|E| \cdot y_0 + 1) \cdot (y_0 + 2)^{2e} \tag{104}$$

, as we can see by induction on e: In fact, if (103) fails, then $|D_0| < (|E| \cdot y_0 + 1) = (|E| \cdot y_0 + 1) \cdot (y_0 + 2)^{2 \cdot 0}$, and once we have shown that $|D_{e'}| < (|E| \cdot y_0 + 1) \cdot (y_0 + 2)^{2e'}$ for $e' \leq e$, we see that

$$
\begin{aligned}
|D_{e+1}| &< (|E| \cdot y_0 + 1) \cdot (\sum_{e'=0}^{e} |D_{e'}| + 1) \quad \text{[by failure of (103)]} \\
&< (|E| \cdot y_0 + 1) \cdot (\sum_{e'=0}^{e} (|E| \cdot y_0 + 1) \cdot (y_0 + 2)^{2e'} + 1) \quad \text{[by inductive hypothesis]} \\
&\leq (|E| \cdot y_0 + 1) \cdot ((|E| \cdot y_0 + 1) \frac{(y_0 + 2)^{2e+2} - 1}{(y_0 + 2)^2 - 1} + 1) \\
&\leq (|E| \cdot y_0 + 1) \cdot (((y_0 + 2)^2 - 1) \frac{(y_0 + 2)^{2e+2} - 1}{(y_0 + 2)^2 - 1} + 1) \quad \text{[since } 0 < |E| \leq |\alpha| < y_0\text{]} \\
&= (|E| \cdot y_0 + 1) \cdot (y_0 + 2)^{2(e+1)}.
\end{aligned}
$$

But then it also follows that

$$
\begin{aligned}
(y_0 + 2)^{2|E|} &= \iota(I) + 1 \\
&= \sum_{e=0}^{|E|-1} |D_e| \\
&< \sum_{e=0}^{|E|-1} (|E| \cdot y_0 + 1) \cdot (y_0 + 2)^{2e} \\
&= (|E| \cdot y_0 + 1) \cdot \sum_{e=0}^{|E|-1} (y_0 + 2)^{2e} \\
&\leq ((y_0 + 2)^2 - 1) \cdot \frac{(y_0 + 2)^{2|E|} - 1}{(y_0 + 2)^2 - 1} \\
&= (y_0 + 2)^{2|E|} - 1,
\end{aligned}
$$

which is not possible. This shows that (104) must be true for some $e < |E|$.

Let $\bar{e}$ be the least e such that (103) holds. Assume that $D_{\bar{e}} = \{t_{j_0} < \ldots < t_{j_{|D_{\bar{e}}|-1}}\}$. Since there are at least $\sum_{e<\bar{e}} |D_e| + 1$ pairwise disjoint sequences of $|E| \cdot y_0 + 1$ stages $t_{j_n}, \ldots, t_{j_{n+|E| \cdot y_0}}$ in $D_{\bar{e}}$, there must be at least one such sequence with

$$
[t_{j_n}, t_{j_{n+|E| \cdot y_0}}] \cap D_e = \emptyset \text{ for all } e < \bar{e}. \tag{105}
$$

But the existence of such a sequence is a contradiction to Lemma 4.46.

□

Lemma 4.48. *(i) Let $\beta 0 \sqsubseteq \delta_s$ and let $I = job_s(\beta)$ be an α-interval. Assume that $\alpha \leq \delta_s$ and for every β' with $\beta' 0 \sqsubseteq \beta 0$ it holds that $l_s(\rho(|\beta'|)) > \max(I)$. Then β requires attention and is α-linked stage $s + 1$.*

(ii) Let $\alpha \sqsubseteq \delta_s$ and let $cand_s(\alpha) = I$. Assume that for every β' with $\beta' 0 \sqsubseteq \alpha$ it holds that

$l_s(\rho(|\beta'|)) > \max(I)$ and if $I' \in jobs_s(\beta')$ is an α'-interval, then $\alpha \leq \alpha'$. Then α requires attention at stage $s+1$.

Proof. (i) From Lemma 4.47 it is clear that $I \not\subseteq C_s$, because otherwise, if the last enumeration into $I \cap C$ were at some stage $t \leq s$, then $cand_{t'}(\alpha) = I$ for all $t' \geq t$ until α becomes initialised and the assignment of I to α is cancelled, contradicting the fact that $I \in jobs_s(\beta)$. Note that by (74) the numbers from I are enumerated into C in decreasing order and without gaps (unless α is initialised, which does not happen until stage s). Moreover, for $i \in \mathbb{N}$, a number $b \in I$ is enumerated into A or B_i at some stage only if a number $c \in I$ with $c \leq b$ is enumerated into C at the same stage. From these facts it follows that there is $c_s \in I$ satisfying (74) at stage s and hence that β requires attention and is α-linked at stage $s+1$.

(ii) Again, from Lemma 4.47 it follows that $I \not\subseteq C_s$ (there is a node β with $\beta 0 \sqsubseteq \alpha$, because otherwise $cand_s(\alpha)$ were undefined at every stage). As in (i) we can conclude that c_s satisfying (74) exists.

Let β be the least node such that $\beta 0 \sqsubseteq \alpha$ and $\rho(|\beta|)$ is a join requirement, if such a node exists, and $\beta = \alpha$ otherwise. Then (76) is trivially satisfied for every $y > c_s$.

At the stage s_0+1 when I is assigned to α, for $c_{s_0} = y_{\iota(I)}(I)$ there are $y_{\iota(I)+1}(I) - y_{\iota(I)}(I) - 1$ numbers $z > c_{s_0}$ with $z \in I - (A_{s_0} \cup \bigcup_{i\in\mathbb{N}} B_{i,s_0})$. A new number from I is enumerated into $A \cup \bigcup_{i\in\mathbb{N}} B_i$ only when a new number from I is enumerated into C; and at every stage $t+1$, at most two numbers from I are enumerated into $A \cup \bigcup_{i\in\mathbb{N}} B_i$ (namely, a_t and b_t). Since the numbers from I are enumerated into C in decreasing order and there at most $y_{\iota(I)}+1-\min(I) \leq y_{\iota(I)}$ many stages at which such an enumeration takes place, it follows that there are still $y_{\iota(I)+1}(I) - 2 \cdot y_{\iota(I)}(I) - 1$ numbers $z \geq c_s$ with $z \in I - (A_s \cup \bigcup_{i\in\mathbb{N}} B_{i,s})$. But

$$\begin{aligned} y_{\iota(I)+1} - 2y_{\iota(I)} - 1 &= 3^{\iota(I)+1}(y_0(I) + (\iota(I)+1)) - 2 \cdot 3^{\iota(I)}(y_0(I) + \iota(I)) - 1 \\ &\geq 3^{\iota(I)}(y_0(I) + \iota(I)) \\ &= y_{\iota(I)}(I) \\ &\geq y_k(I) \end{aligned}$$

for $k \leq \iota(I)$. Hence for the greatest number $y \in I$ with $y \notin (A_s \cup \bigcup_{i\in\mathbb{N}} B_{i,s})$, (77) is true. It follows that a_s exists and α requires attention due to Case 1.3 at stage $s+1$. $\square$

Lemma 4.49. *(i) Let $\alpha \sqsubset \mathrm{TP}$. Then α is initialised only finitely many times.*

(ii) Let $\beta 0 \sqsubset \mathrm{TP}$. Then for each interval I, there are only finitely many stages s such that $job_s(\beta) = I$.

Proof. Literally repeat the proof of Lemma 4.27 (using Case 2 where we used Case 2.2 before, and Lemma 4.48 where we used Lemma 4.26 before). $\square$

Lemma 4.50. *It holds that $A \leq_{\mathrm{ibT}} B_i \leq_{\mathrm{ibT}} C$ for $i \in \mathbb{N}$.*

Proof. It can directly be verified by looking at the construction that $B_i \leq_{\mathrm{ibT}} C$ by permitting. Since ibT-reducibility is invariant under finite variants, for $A \leq_{\mathrm{ibT}} B_i$ it suffices to prove that $A - A_i \leq_{\mathrm{ibT}} B_i$. Again, this can immediately be verified by the construction. $\square$

Lemma 4.51. *Every meet requirement $\mathcal{M}_e^{i,j}$ ($i, j \in \mathbb{N}$, $i \neq j$, $e = \langle e_0, e_1, e_2 \rangle \in \mathbb{N}$) is satisfied.*

Proof. Let $n = \rho^{-1}(\mathcal{M}_e^{i,j})$ and let $\gamma \sqsubset \mathrm{TP}$ be the unique node of length n on the true path. Assume that the premise of $\mathcal{M}_e^{i,j}$ is true, that is $W_{e_0} = \hat{\Phi}_{e_1}^{B_i} = \hat{\Phi}_{e_2}^{B_j}$ (otherwise $\mathcal{M}_e^{i,j}$ is trivially satisfied). Since $\gamma 0 \sqsubset \mathrm{TP}$ by the definition of the true path, due to the True Path Lemma there are infinitely many $\gamma 0$-stages. By Lemma 4.49(i), there is a $\gamma 0$-stage s_0 such that $\gamma 0$ is never initialised at any stage $s \geq s_0$. Then for no node $\alpha \leq \gamma$ any numbers from an α-interval are enumerated into B_i or B_j at any stage $s \geq s_0$.

Now, in order to compute $W_{e_0}(x)$ for some given x with oracle $A \restriction (x+1)$, compute the least $\gamma 0$-stage $s_1 \geq \max(\{s_0, x\})$ such that $A_{s_1} \restriction (x+1) = A \restriction (x+1)$ and $l_{s_1}(\mathcal{M}_e^{i,j}) > x$. We claim that $x \in W_{e_0}$ if and only if $x \in W_{e_0, s_1}$.

Let $s_1 \leq s_2 \leq \ldots$ be the sequence of $\gamma 0$-stages, starting with s_1. We claim that, for $n \geq 1$,

$$B_{i,s_{n+1}} \restriction (x+1) = B_{i,s_n} \restriction (x+1) \text{ or } B_{j,s_{n+1}} \restriction (x+1) = B_{j,s_n} \restriction (x+1). \tag{106}$$

The proof is by induction. Let equation (106) be true for n. For a contradiction, assume that there are stages $t_i, t_j \in [s_n, s_{n+1})$ and numbers $y_i, y_j \leq x$ such that $y_i \in B_{i,t_i+1} - B_{i,t_i}$ and $y_j \in B_{j,t_j+1} - B_{j,t_j}$. The enumeration of y_i into B_i at stage $t_i + 1$ cannot be caused by the initialisation of any node or by any node being active due to Case 1.1, Case 1.2 or Case 2(a), because otherwise y_i were enumerated into $A_{t_i+1} - A_{t_i}$ at stage $t_i + 1$, contradicting $t_i \geq s_1$ (if $y_i = c_{t_i}$, then $y_i \notin A_{t_i}$ by (74); if $y = \min(I)$ for some α-interval I, where α is initialised at stage $t_i + 1$, then $y_i \notin A_{t_i}$, because each enumeration of a number $w \in I$ into A is accompanied by an enumeration of a number $v \in I$ with $v \leq w$ into C, and $y_i \notin C_{t_i}$). If it caused by some node α being active due to Case 1.3 or Case 1.4, then $\alpha \not< \gamma 0$ (because otherwise $\gamma 0$ were initialised at stage $t_i + 1$, contradicting $t_i + 1 \geq s_0$) and $\alpha \not>_L \gamma 0$ (because otherwise α is initialised at stage $s_n + 1$ and only enumerates numbers into intervals I' with $\min(I') > s_n \geq s_1 \geq x$ at later stages). Hence in this case $\alpha \sqsupseteq \gamma 0$; since t_i must be an α-stage in this case, it follows that $t_i = s_n$.

The same analysis holds with j in place of i. There is only one node which can be active at stage s_n+1 and if such a node enumerates both y_i into $B_{i,s_n+1} - B_{i,s_n}$ and y_j into $B_{j,s_n+1} - B_{j,s_n}$, then it enumerates y_i or y_j into $A_{s_n+1} - A_{s_n}$, contradicting $y_i, y_j \leq x$ and $s_n \geq s_1$.

Hence $t_i \neq t_j$ and at least one of the enumerations of y_i and y_j must be caused by some node β being active due to Case 2 (b) at some stage $t \in (s_n + 1, s_{n+1})$. Without loss of generality assume that this is the case for y_i. Let $y_i \in I$, where I is an α-interval. Since $\beta 0 \sqsubseteq \delta_{t_i}$ but $\gamma 0 \not\sqsubseteq \delta_{t_i}$, we know that $\gamma 0 \not\sqsubseteq \beta 0$; since $\gamma 0 \leq \delta_{t_i}$, we also know that $\beta 0 \not<_L \gamma 0$; finally, $\gamma 0 \not<_L \beta 0$, because otherwise $\gamma 0 <_L \alpha$, α were initialised at stage $s_n + 1$ and there were only enumerations into α-intervals I' with $\min(I') > s_n \geq s_1 \geq x$ at later stages. Hence $\beta 0 \sqsubseteq \gamma 0 \sqsubset \mathrm{TP}$.

But then, by Lemma 4.49 (b), there is a least stage $t' \geq t_i$ such that $job_{t'+1}(\beta) \neq I$. Then either β is active due to Case 2 (a) at stage $t'+1$ or α is initialised at stage $t'+1$. In either case, some number $y \leq y_i \leq x$ is enumerated into $A_{t'+1} - A_{t'}$, contradicting $t' \geq s_1$. This completes the inductive step of the proof of equation (106).

Now from (106) it follows that, if $x \notin W_{e_0,s_1}$, for $n \geq 1$,

$$\hat{\Phi}^{B_{i,s_n}}_{e_1,s_n}(x) = \hat{\Phi}^{B_{j,s_n}}_{e_2,s_n}(x) = W_{e_0,s_n}(x) = 0.$$

This is true for $n = 1$, because $l_{s_1}(\mathcal{M}^{i,j}_e) > x$. If it is true for s_n, then $B_{i,s_{n+1}} \upharpoonright (x+1) = B_{i,s_n} \upharpoonright (x+1)$ or $B_{j,s_{n+1}} \upharpoonright (x+1) = B_{j,s_n} \upharpoonright (x+1)$. If, say $B_{i,s_{n+1}} \upharpoonright (x+1) = B_{i,s_n} \upharpoonright (x+1)$, then

$$0 = \hat{\Phi}^{B_{i,s_n}}_{e_1,s_n}(x) = \hat{\Phi}^{B_{i,s_{n+1}}}_{e_1,s_{n+1}}(x) = \hat{\Phi}^{B_{j,s_{n+1}}}_{e_2,s_{n+1}}(x) = W_{e_0,s_{n+1}}(x),$$

proving the equality for s_{n+1}.

Since $W_{e_0}(x) = \lim_{n\to\infty} W_{e_0,s_n}(x)$, this implies that $W_{e_0}(x) = 0$. □

Lemma 4.52. *Every join requirement $\mathcal{J}^{i,j}_e$ ($i, j \in \mathbb{N}$, $i \neq j$, $e = \langle e_0, e_1, e_2\rangle \in \mathbb{N}$) is satisfied.*

Proof. Let $n = \rho^{-1}(\mathcal{J}^{i,j}_e)$ and let $\beta \sqsubset \mathrm{TP}$ be the unique node of length n on the true path. Assume that the hypothesis of $\mathcal{J}^{i,j}_e$ is true, that is $B_i = \hat{\Phi}^{W_{e_0}}_{e_1}$ and $B_j = \hat{\Phi}^{W_{e_0}}_{e_2}$ (otherwise $\mathcal{J}^{i,j}_e$ is trivially satisfied). Since $\beta 0 \sqsubset \mathrm{TP}$ by the definition of the true path, due to the True Path Lemma there are infinitely many $\beta 0$-stages. By Lemma 4.49, there is a $\beta 0$-stage s_0 such that $\beta 0$ is never initialised and no node $\gamma \leq \beta 0$ is active due to Case 1 at any stage $s \geq s_0$.

Now, in order to compute $C(x)$ with oracle $W_{e_0} \upharpoonright (x+1)$ for some given x, using the oracle compute the least $\beta 0$-stage $s_1 \geq \max(\{s_0, x\})$ such that

$$W_{e_0,s_1} \upharpoonright (x+1) = W_{e_0} \upharpoonright (x+1) \tag{107}$$

and $l_{s_1}(\mathcal{J}^{i,j}_e) > x$.

We claim that $x \in C$ if and only if $x \in C_{s_1+1}$. If $x \notin I$ for any interval I assigned to any node during the construction, then $C(x) = 0$, and the claim is true. Hence assume that $x \in I$, where I is an α-interval. We also assume that I is assigned to α at stage s_1, because at stages $s > s_1$ only intervals I' with $\min(I') > s_1 \geq x$ become assigned to any node, and if the assignment of I to α has already been cancelled at or before stage s_1, then no numbers from I are enumerated into C after stage s_1.

If $\alpha < \beta 0$, then x is not enumerated into C at any stage $s \geq s_1$, because otherwise $\beta 0$ were initialised at stage s, contradicting the choice of s_0. Furthermore, if $\beta 0 <_L \alpha$, then $\delta_{s_1} <_L \alpha$, whence α is initialised at stage $s_1 + 1$ and I is cancelled if it was not cancelled before; then there are no enumerations into I at stages $s \geq s_1 + 1$.

Hence it suffices to consider the case that $\beta 0 \sqsubseteq \alpha$. For a contradiction assume that $C(x) \neq C_{s_1+1}(x)$, i.e. that x enters C at some stage $s + 1 > s_1 + 1$. We now consider the possible cases why x enters C at stage $s + 1$. Let β_s be the node which is active at stage $s + 1$.

- If α is initialised at stage $s+1$, then $x = \min(I)$ is enumerated into $B_{i,s+1} - B_{i,s}$. Then

$$\hat{\Phi}_{e_1}^{W_{e_0}}(x) = B_i(x) = 1 \neq 0 = B_{i,s_1}(x) = \hat{\Phi}_{e_1,s_1}^{W_{e_0,s_1}}(x). \tag{108}$$

Then there must be some $z \leq x$ in $W_{e_0} - W_{e_0,s_1}$, contradicting the choice of s_1. So for the remaining cases we may assume that α is not initialised at any stage t with $s_1 \leq t \leq s+1$ (because I is assigned to α at stage s_1 and still assigned to α at stage $s+1$).

- If β_s is active due to Case 1.2 or Case 2(a) at stage $s+1$, then $x = c_s$ is enumerated into $B_{i,s+1} - B_{i,s}$ and as above we conclude that there must be some $z \leq x$ in $W_{e_0} - W_{e_0,s_1}$, contradicting the choice of s_1.

- If $\beta_s = \alpha$ is active due to Case 1.4 at stage $s+1$, then there is some stage $t+1 < s+1$ such that $x = c_t$ and (c_t, a_t) has been assigned as diagonalisation witness to α via Case 1.3 at stage $t+1$. By (76) $[c_t+1, a_t]$ was $\{\mathfrak{J}_e^{i,j}\}$-safe at stage t. By the hypothesis of Case 1.4, $l_s(\rho(|\beta|)) > a_t$ and $y = a_t$ is enumerated into $B_{i,s+1} - B_{i,s}$. Then (108) holds with y in place of x.

  Since $\hat{\Phi}_{e_1}$ is an ibT-functional, there must be some $z \leq y$ in $W_{e_0} - W_{e_0,s}$. But since $[x+1, y] \subseteq W_{e_0,s}$, it follows that $z \leq x$, contradicting the choice of s_1.

- If $\beta_s = \alpha$ is active due to Case 1.3 and I is demoted to some β' with $\beta \sqsubset \beta'$, or if $\beta_s = \beta'$ is active due to Case 2 (b) and $\beta 0 \sqsubset \beta_s 0$ at stage $s+1$, then $x = c_s$, and a_s is enumerated into $B_{i,s+1} - B_{i,s}$ or $B_{j,s+1} - B_{j,s}$. If a_s is enumerated into B_i, then equation (108) holds with a_s in place of x and s in place of s_1; if a_s is enumerated into B_j, then

$$\hat{\Phi}_{e_2}^{W_{e_0}}(a_s) = B_j(x) = 1 \neq 0 = B_{j,s}(a_s) = \hat{\Phi}_{e_2,s}^{W_{e_0,s}}(a_s). \tag{109}$$

  In either case, there must be some $z \leq a_s$ in $W_{e_0} - W_{e_0,s}$. Since $\beta 0 \sqsubset \beta' 0$ in this case, by (76) or (79), respectively, $[c_s+1, a_s] \subseteq W_{e_0,s}$. Hence $z \leq c_s = x$, contradicting the choice of s_1 again.

- If $\beta_s = \alpha$ is active due to Case 1.3 and I is demoted to some β' with $\beta' \sqsubseteq \beta$, or if $\beta_s = \beta'$ is active due to Case 2 (b) and $\beta_s 0 \sqsubseteq \beta 0$ at stage $s+1$, then $job_{s+1}(\beta') = I$, while by Lemma 4.49 there is a least stage $s' > s$ such that $job_{s'+1}(\beta_s) \neq I$. If α is initialised at stage $s'+1$, then $I \not\subseteq C_{s'}$ by Lemma 4.47, and in particular $\min(I) \notin C_{s'}$, hence $\min(I) \notin B_{i,s'}$. Then the same analysis as in the first case above shows that there is some $z \leq \min(I) \leq x$ in $W_{e_0} - W_{e_0,s_1}$, contradicting the choice of s_1.

  If I is not cancelled at stage $s'+1$, then β' must be active due to Case 2.2 (a) at stage $s'+1 > s+1$. As we have seen above, then some $z \leq c_{s'} \leq c_s = x$ enters $W_{e_0} - W_{e,s_1}$, contradicting the choice of s_1 again.

In all cases we have arrived at a contradiction to (107). Hence $C(x) = C_{s_1+1}(x)$, completing the proof of the lemma. $\square$

Lemma 4.53. *Every diagonalisation requirement $\mathcal{D}_e$ ($e \in \mathbb{N}$) is satisfied.*

Proof. Let $\alpha \sqsubset \mathrm{TP}$ be the unique node of length e on the true path. By Lemma 4.49, there is some stage s_0 such that for any stage $s \geq s_0$, α is not initialised at stage s and for all $\alpha' \leq \alpha$, all $\beta 0 \sqsubseteq \alpha$ and every α'-interval I, $job_s(\beta) \neq I$.

For a contradiction assume that α has no interval assigned at any stage $s \geq s_0$. Then no $\alpha' \sqsupseteq \alpha$ is active after stage s_0, because α requires attention at stage $s+1$ for every α-stage $s \geq s_0$. For any α-stage $s \geq s_0$ some node $\beta \sqsubset \alpha$ must be active due to Case 2 at stage $s+1$ and enumerate some number c in $C_{s+1} - C_s$, where c is in an α'-interval for some $\alpha' \leq \delta_s$. Since no $\alpha' \sqsupseteq \alpha$ is active after stage s_0, almost all of these enumerations are into α'-intervals with $\alpha' < \alpha$. But by the choice of stage s_0 no such enumerations are possible.

Hence there must be a least α-stage $s_1 \geq s_0$ such that α has some interval I assigned at stage s_1+1. Since α is never initialised after stage s_1, I is assigned to α at all stages $s \geq s_1+1$. Since $s_1 \geq s_0$ and by the choice of s_0, for all $s > s_1$ it holds that either α has a diagonalisation witness (x, y) with $x, y \in I$ assigned at stage s, or $cand_s(\alpha) = I$.

Since $\alpha \sqsubset \mathrm{TP}$, there is an α-stage $s_2 \geq s_1+1$ such that $l_{s_2}(\rho(|\beta|)) > \max(I)$ for all $\beta 0 \sqsubseteq \alpha$.

For a contradiction assume that $cand_s(\alpha) = I$ for all $s \geq s_2$. By Lemma 4.48 (ii) and the choice of s_0, α requires attention due to Case 1.3 at stage $s+1$ for every α-stage $s \geq s_2$. Hence no node above α is active after stage s_2 and there are only finitely many intervals assigned to such nodes. Let $s_3 \geq s_2$ be an α-stage such that for all β with $\beta 0 \sqsubseteq \alpha$ and all $\alpha' \sqsupseteq \alpha$, there is no α'-interval I' with $job_s(\beta) = I'$. Such a stage exists by Lemma 4.49. Nodes $\alpha' \sqsubseteq \alpha$ do not require attention due to Case 1.1, Case 1.2, Case 1.3 or Case 1.4 at stage $s+1$ for any α-stage $s \geq s_3$, because otherwise α were initialised, and they do not require attention due to Case 2 at such a stage $s+1$, because if $job_s(\beta) = I'$ for some β with $\beta 0 \sqsubseteq \alpha$, then I' is an α'-interval for some $\alpha' >_L \alpha$, hence $\alpha' >_L \delta_s$. It follows that α is active at every such stage $s+1$. In fact, since α has no diagonalisation witness assigned and since $cand_{s+1}(\alpha) = I$ by assumption, α must be active due to Case 1.2 and enumerates the number $c_s \in I$ into $C_{s+1} - C_s$. But since there are infinitely many α-stages, this implies that infinitely many numbers from I are enumerated into C, which is impossible.

Hence there is a stage $s \geq s_2$ such that $cand_s(\alpha) \neq I$ and α has a diagonalisation witness (x, y) with $x, y \in I$ assigned at stage s.

Let (x, y) become assigned as diagonalisation witness to α at stage $\bar{s}+1$. Then $x \notin C_{\bar{s}+1}$.

If there is no $\beta 0 \sqsubseteq \alpha$, then α is active due to Case 1.1 at stage $\bar{s}+1$ and $\iota(I) = 0$, $I = [x, 3 \cdot (x+2))$, and since $e = |\alpha| < x$ it holds that $y_0(I) = x+1 < x+e+2 \leq \max(I)$. Then $y = x+e+2 \notin A_{\bar{s}+1} \cup \bigcup_{i \in \mathbb{N}} B_{i,\bar{s}+1}$.

If there is some $\beta 0 \sqsubseteq \alpha$, then α is active due to Case 1.3 at stage $\bar{s}+1$, and by (77) $|[c_{\bar{s}}+1, y]| \geq y_k(I) \geq x \geq \min(I) > |\alpha| = e$, where I is relevant above $y_k(I)$ at stage $\bar{s}$, hence

$y \geq c_s + e + 1 = x + e + 1$. Moreover, $y \notin \bigcup_{i \in \mathbb{N}} A_{\bar{s}+1} \cup B_{i,\bar{s}+1}$.

Now if x is never enumerated into C, then $C(x) = 0 \neq \tilde{\Phi}_e^A(x)$: For a contradiction assume that $\tilde{\Phi}_e^A(x) = 0$. Then there is a stage $s_4 \geq \bar{s}$ such that $A_{s_4} \upharpoonright (x+e+1) = A \upharpoonright (x+e+1)$ and $\tilde{\Phi}_{e,s_4}^{A,s_4}(x) = 0$. By the choice of $s_4 \geq s_2$, α requires attention due to Case 1.4 at stage $s+1$ for every α-stage $s \geq s_4$. Similar as above we can argue that α must be active at some such stage $s+1$, whence x is enumerated into C, contradicting the hypothesis.

On the other hand, if x is enumerated into C at some stage $t+1 > \bar{s}$, then by the premises of Case 1.4,

$$C(x) = 1 \neq 0 = C_t(x) = \tilde{\Phi}_{e,t}^{A_t}(x).$$

It suffices to show that $A_t \upharpoonright (x+e+1) = A \upharpoonright (x+e+1)$, because then $\tilde{\Phi}_{e,t}^{A_t}(x) = \tilde{\Phi}_e^A(x)$ and $\mathcal{D}_e$ is satisfied.

But this is true because there are no enumerations into A from any α'-intervals with $\alpha' \leq \alpha$ after stage t (otherwise α would be initialised), while nodes α' with $\alpha < \alpha'$ are initialised at stage $t+1$ and only get intervals I' with $\min(I') > \max(I)$ assigned at later stages.

□

□

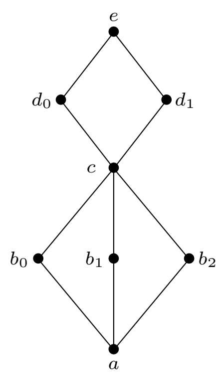
At the end of this chapter, we state that there is a finite lattice for which a lattice embedding into $\mathcal{R}_{\mathrm{ibT}}$ and $\mathcal{R}_{\mathrm{cl}}$ exists but no lattice embedding into $\mathcal{R}_{\mathrm{T}}$.

Definition 4.54. The $\mathcal{S}_8$ is the partial ordering $\mathcal{S}_8 = (\{a, b_0, b_1, b_2, c, d_0, d_1, e\}, \leq_{\mathcal{S}_8})$ such that

1. $h(a) = a, h(b_i) = b_i$ for $i \in \{0,1,2\}$ and $h(c) = c$ defines a lattice embedding of the $\mathcal{M}_3$ into $\mathcal{S}_8$,

2. $h'(a) = c$, $h'(b_0) = d_0$, $h'(b_1) = d_1$ and $h'(c) = e$ defines a lattice embedding of the diamond into $\mathcal{S}_8$.

The following diagram illustrates what the $\mathcal{S}_8$ looks like.



Theorem 4.55. *[Lach 80] There is no lattice embedding of the $\mathcal{S}_8$ into $\mathcal{R}_{\mathrm{T}}$.*

To show that the $\mathcal{S}_8$ is lattice embeddable into $\mathcal{R}_r$ for $r \in \{\mathrm{ibT}, \mathrm{cl}\}$, we cite a result which is interesting in its own right.

Theorem 4.56. *(Ambos-Spies, Bodewig, Kräling, and Yu [Amboc]) For $r \in \{\mathrm{ibT}, \mathrm{cl}\}$, $\mathcal{R}_r$ is branching, that is for every c.e. r-degree $\mathbf{c}$ there are incomparable c.e. r-degrees $\mathbf{d_0}, \mathbf{d_1} > \mathbf{c}$ such that $\mathbf{d_0} \wedge \mathbf{d_1} = \mathbf{c}$. Moreover, we can choose $\mathbf{d_0}$ and $\mathbf{d_1}$ in such a way that $\mathbf{d_0}$ and $\mathbf{d_1}$ have a join $\mathbf{e}$.*

Corollary 4.57. *For $r \in \{\mathrm{ibT}, \mathrm{cl}\}$ there is a lattice embedding of the $\mathcal{S}_8$ into $\mathcal{R}_r$.*

Proof. By Theorem 4.39 there is a lattice embedding h of the $\mathcal{M}_3$ into $\mathcal{R}_r$. For $h(c) = \mathbf{c}$, let $\mathbf{d_0}$, $\mathbf{d_1}$ and $\mathbf{e}$ be as in Theorem 4.56. Then we can extend h to an embedding h' of the $\mathcal{S}_8$ into $\mathcal{R}_r$ by defining $h'(d_0) = \mathbf{d_0}$, $h'(d_1) = \mathbf{d_1}$ and $h'(e) = \mathbf{e}$. $\square$

Chapter 5

Cuppable degrees and the theories of $\mathcal{R}_{\mathrm{ibT}}$ and $\mathcal{R}_{\mathrm{cl}}$

5.1 Elementary equivalence of degree structures

Almost all of the results stated about some degree structure $\mathcal{R}_r$ in the previous chapters could be formalised as "$\mathcal{R}_r$ satisfies some theorem of first-order predicate logic in the language containing $\leq$ as the only non-logical symbol". For example, the statement of Lemma 4.5 that the diamond can be lattice embedded into $\mathcal{R}_r$ preserving the least element (for $r \in \{\mathrm{T}, \mathrm{wtt}, \mathrm{cl}, \mathrm{ibT}\}$) could be formalised as

$$\mathcal{R}_r \models (\exists a)(\exists b_0)(\exists b_1)(\exists c)(\forall x)((a \leq x) \wedge (b_0 \not\leq b_1) \wedge (b_1 \not\leq b_0) \wedge (b_0 \leq c) \wedge (b_1 \leq c) \\ \wedge (x \leq b_0 \wedge x \leq b_1 \rightarrow x \leq a) \wedge (b_0 \leq x \wedge b_1 \leq x \rightarrow c \leq x)).$$

This observation leads to the question whether two degree structures $\mathcal{R}_r$ and $\mathcal{R}_{r'}$ satisfy the same first-order theorems.

Definition 5.1. Two partial orders are *elementarily equivalent* if they satisfy the same theorems of first-order predicate logic in the language containing $\leq$ as the only non-logical symbol.

For the most common degree structures studied in the literature, $\mathcal{R}_1$ (the structure of the c.e. degrees with respect to one-one-reducibility), $\mathcal{R}_{\mathrm{m}}$, $\mathcal{R}_{\mathrm{tt}}$ (the structure of the c.e. degrees with respect to truth-table reducibility), $\mathcal{R}_{\mathrm{wtt}}$ and $\mathcal{R}_{\mathrm{T}}$, it is well known that they are pairwise not elementarily equivalent. This follows from the facts that $\mathcal{R}_1$ does not constitute an upper semi-lattice (in contrast to $\mathcal{R}_r$ for $r \in \{\mathrm{m}, \mathrm{tt}, \mathrm{wtt}, \mathrm{T}\}$), that the c.e. incomplete m-degrees are closed under joins (in contrast to the c.e. incomplete r-degrees for $r \in \{\mathrm{tt}, \mathrm{wtt}, \mathrm{T}\}$), that there exist minimal c.e. tt-degrees (but no minimal c.e. wtt- or T-degrees) and that the upper semi-lattice of c.e. wtt-degrees is distributive but the upper semi-lattice of c.e. Turing-degrees is not (see [Odif 99] for details).

Since, for $r \in \{\mathrm{ibT}, \mathrm{cl}\}$, $\mathcal{R}_r$ does not have a greatest element (since there are r-maximal pairs by Theorem 3.5), while $\mathcal{R}_1$, $\mathcal{R}_{\mathrm{m}}$, $\mathcal{R}_{\mathrm{tt}}$, $\mathcal{R}_{\mathrm{wtt}}$ and $\mathcal{R}_{\mathrm{T}}$ all have a greatest element (the respective degree of the halting problem), it follows that $\mathcal{R}_{\mathrm{ibT}}$ and $\mathcal{R}_{\mathrm{cl}}$ are not elementarily equivalent to any of $\mathcal{R}_1$, $\mathcal{R}_{\mathrm{m}}$, $\mathcal{R}_{\mathrm{tt}}$, $\mathcal{R}_{\mathrm{wtt}}$ and $\mathcal{R}_{\mathrm{T}}$.

On the other hand, all theorems we have proven for $\mathcal{R}_{\mathrm{ibT}}$ so far were also true for $\mathcal{R}_{\mathrm{cl}}$. In some cases (the existence of maximal pairs, for example), the result for $\mathcal{R}_{\mathrm{ibT}}$ directly carries over to $\mathcal{R}_{\mathrm{cl}}$ by general observations like the ibT-cl-Join and -Meet Lemma, in other cases the proofs are very similar and just slightly more involved for $\mathcal{R}_{\mathrm{cl}}$ (for example, the embedding of the $\mathcal{M}_3$ into $\mathcal{R}_{\mathrm{ibT}}$ and $\mathcal{R}_{\mathrm{cl}}$). In other cases, however, the known proofs for $\mathcal{R}_{\mathrm{cl}}$ are considerably more complex than the respective proofs for $\mathcal{R}_{\mathrm{ibT}}$. Examples are the proof of Theorem 4.56 or of the fact that neither $\mathcal{R}_{\mathrm{ibT}}$ nor $\mathcal{R}_{\mathrm{cl}}$ is dense (shown by Barmpalias and Lewis for $\mathcal{R}_{\mathrm{ibT}}$ [Barm 06] and by Day for $\mathcal{R}_{\mathrm{cl}}$ [Day 10]).

Thus it would simplify matters if $\mathcal{R}_{\mathrm{ibT}}$ and $\mathcal{R}_{\mathrm{cl}}$ were elementarily equivalent, whence properties of the c.e. ibT-degrees would always carry over to the c.e. cl-degrees. In this chapter we will establish a property witnessing that this is not the case. Most results in this chapter are from joint work with Klaus Ambos-Spies, Philipp Bodewig and Yun Fan, and were published in [Ambo 13a].

5.2 Cuppability in $\mathcal{R}_{\mathrm{ibT}}$

The property we consider is defined in terms of *cuppability.*

Definition 5.2. Let $\mathcal{P} = (\mathrm{P}, \leq_{\mathcal{P}})$ be a partial order and let $a, b \in \mathrm{P}$ such that $b \leq_{\mathcal{P}} a$. Then b is *a-cuppable* if there is an element c of P such that $c <_{\mathcal{P}} a$ and $a = b \vee c$; and b is *a-noncuppable* otherwise, i.e. if, for all $c <_{\mathcal{P}} a$, a is not the join of b and c (i.e. $b \vee c$ does not exist or $b \vee c < a$).

Let $r \in \{\mathrm{ibT}, \mathrm{cl}\}$. For any c.e. r-degree $\mathbf{a}$ let $\mathbf{NCu}_r(\mathbf{a})$ denote the class of the $\mathbf{a}$-noncuppable c.e. r-degrees. Ambos-Spies [Ambob] has shown that in case of $r = \mathrm{ibT}$, for each r-degree $\mathbf{a} > \mathbf{0}$ the class $\mathbf{NCu}_r(\mathbf{a})$ is bounded by some r-degree $\mathbf{c} < \mathbf{a}$.

Theorem 5.3 (Ambos-Spies [Ambob])**.** *For any c.e.* ibT*-degree* $\mathbf{a} > \mathbf{0}$*,*

$$\mathbf{NCu}_{\mathrm{ibT}}(\mathbf{a}) \subseteq \{\mathbf{b} \in \mathbf{R}_{\mathrm{ibT}} : \mathbf{b} \leq \mathbf{a} + 1\}.$$

For the sake of completeness we give the short proof of this result, which uses the following easy lemma.

Lemma 5.4 (Disjoint Sets Lemma; [Ambob])**.** *Let D and E be disjoint noncomputable c.e. sets such that $D \leq_{\mathrm{ibT}} E$. Then $D \leq_{\mathrm{ibT}} E + 1$.*

Proof. By Lemma 2.7 we may assume that there are enumeration functions d for D and e for E such that for all n, $e(n) \leq d(n)$. Since D and E are disjoint, in fact $e(n) < d(n)$. Then

$h(n) := e(n) + 1 \leq d(n)$, and h is an enumeration function for $E + 1$; hence $D \leq_{\mathrm{ibT}} E + 1$ by permitting. $\square$

Proof of Lemma 5.4. Given a noncomputable c.e. set A and a c.e. set $B \leq_{\mathrm{ibT}} A$ such that $B \not\leq_{\mathrm{ibT}} A + 1$, it suffices to find a c.e. set $C <_{\mathrm{ibT}} A$ such that

$$deg_{\mathrm{ibT}}(A) = deg_{\mathrm{ibT}}(B) \vee deg_{\mathrm{ibT}}(C). \tag{110}$$

By Lemma 2.7, without loss of generality we may assume that there are computable one-to-one enumerations $\{a(n)\}_{n\geq 0}$ and $\{b(n)\}_{n\geq 0}$ of A and B, respectively, such that $a(n) \leq b(n)$ for all $n \geq 0$. Split A and B into c.e. sets

$$A_0 = \{a(n) : a(n) = b(n)\} \text{ and } A_1 = \{a(n) : a(n) < b(n)\}$$

and

$$B_0 = \{b(n) : a(n) = b(n)\} \text{ and } B_1 = \{b(n) : a(n) < b(n)\},$$

respectively. Note that $A_0 = B_0$. Hence, by the Splitting Lemma (Lemma 3.4),

$$deg_{\mathrm{ibT}}(A) = deg_{\mathrm{ibT}}(A_0) \vee deg_{\mathrm{ibT}}(A_1) = deg_{\mathrm{ibT}}(B_0) \vee deg_{\mathrm{ibT}}(A_1).$$

So, since (again by the Splitting Lemma) $B_0 \leq_{\mathrm{ibT}} B$, it suffices to show that $A_1 <_{\mathrm{ibT}} A$. (Then (110) will hold for $C = A_1$.)

For a contradiction assume that $A_1 \equiv_{\mathrm{ibT}} A$. Then $A_0 \leq_{\mathrm{ibT}} A_1$. Hence, by the Disjoint Sets Lemma, $A_0 \leq_{\mathrm{ibT}} (A_1) + 1$. Since, by definition of A_1 and B_1, $B_1 \leq_{\mathrm{ibT}} (A_1) + 1$ by permitting, it follows, by $B_0 = A_0$ and by the Splitting Lemma, that $B \leq_{\mathrm{ibT}} (A_1) + 1$. Since $(A_1) + 1 \leq_{\mathrm{ibT}} A + 1$, this contradicts the assumption that $B \not\leq_{\mathrm{ibT}} A + 1$. $\square$

We can also look at the converse of Theorem 5.3 and ask whether

$$\{\mathbf{b} \in \mathbf{R}_{\mathrm{ibT}} : \mathbf{b} \leq \mathbf{a} + 1\} \subseteq \mathbf{NCu}_{\mathrm{ibT}}(\mathbf{a})$$

holds for all c.e. ibT-degrees $\mathbf{a} > \mathbf{0}$. Since $\mathbf{NCu}_{\mathrm{ibT}}(\mathbf{a})$ is closed downwards, this is equivalent to the question whether $\mathbf{a} + 1 \in \mathbf{NCu}_{\mathrm{ibT}}(\mathbf{a})$. Ambos-Spies [Ambob] has shown that the latter is indeed true if there exists some sufficiently scattered c.e. set $A \in \mathbf{a}$, for example a set A containing only even numbers.

Lemma 5.5 (Ambos-Spies [Ambob]). *Let A be a noncomputable c.e. set such that $A \subseteq 2\mathbb{N} = \{2n : n \in \mathbb{N}\}$. Then $deg_{\mathrm{ibT}}(A + 1)$ is $deg_{\mathrm{ibT}}(A)$-noncuppable.*

Another partial positive result is that, for any noncomputable c.e. set A, the ibT-degree of $A + 1$ does not cup to the ibT-degree of A by the ibT-degree of any c.e. set $B <_{\mathrm{ibT}} A$ such that A and B are cl-equivalent. This is a consequence of the following lemma since, for any c.e. sets A and B such that $A \equiv_{\mathrm{cl}} B$, $A + k \leq_{\mathrm{ibT}} B$ for some $k \geq 1$.

Lemma 5.6. *Let* $\mathbf{a}, \mathbf{b}$ *be c.e.* ibT*-degrees such that* $\mathbf{a} + 1 \vee \mathbf{b} = \mathbf{a}$. *Then* $\mathbf{a} + k \vee \mathbf{b} = \mathbf{a}$ *for all* $k \geq 1$.

Proof. We give the proof for $k = 2$. The general claim follows by induction. Since the bounded shifts induce automorphisms of the partial ordering of the c.e. ibT-degrees (see [Ambob]), it follows from $\mathbf{a} + 1 \vee \mathbf{b} = \mathbf{a}$ that $\mathbf{a} + 2 \vee \mathbf{b} + 1 = \mathbf{a} + 1$ (note that $\mathbf{a} + 2 = (\mathbf{a} + 1) + 1$). So $\mathbf{a} + 2 \vee \mathbf{b} + 1 \vee \mathbf{b} = \mathbf{a}$. Since $\mathbf{b} + 1 \leq \mathbf{b}$ this implies $\mathbf{a} + 2 \vee \mathbf{b} = \mathbf{a}$. $\square$

Despite the above observations, however, in general $\mathbf{a} + 1$ is *not* $\mathbf{a}$-noncuppable. In fact, as we will show now, for any computable shift f there is a nonzero c.e. ibT-degree $\mathbf{c} = deg_{\mathrm{ibT}}(C)$ such that the f-shift $deg_{\mathrm{ibT}}(C_f)$ of $\mathbf{c}$ cups to $\mathbf{c}$.

Theorem 5.7. *Let f be a computable shift. Then there are c.e. sets B and C such that $B <_{\mathrm{ibT}} C$ and $deg_{\mathrm{ibT}}(C_f) \vee deg_{\mathrm{ibT}}(B) = deg_{\mathrm{ibT}}(C)$.*

Proof. W.l.o.g. we may assume that $f(x) > x$ for all numbers x (if this is not the case we may replace f by $f + 1$). It suffices to effectively enumerate sets B and C satisfying the following requirements for all $e = \langle e_0, e_1, e_2 \rangle \in \mathbb{N}$.

$$B \leq_{\mathrm{ibT}} C$$

$$\mathcal{D}_e : C \neq \hat{\Phi}_e^B$$

$$\mathcal{J}_e : (C_f = \hat{\Phi}_{e_1}^{W_{e_0}} \text{ and } B = \hat{\Phi}_{e_2}^{W_{e_0}}) \Rightarrow C \leq_{\mathrm{ibT}} W_{e_0}.$$

Satisfaction of the *diagonalisation requirements* $\mathcal{D}_e$ ensures that $C \not\leq_{\mathrm{ibT}} B$. Satisfaction of the *join requirements* $\mathcal{J}_e$ ensures that for every c.e. ibT-degree $\mathbf{d}$, if $\mathbf{d} \geq deg_{\mathrm{ibT}}(C_f)$ and $\mathbf{d} \geq deg_{\mathrm{ibT}}(B)$, then $\mathbf{d} \geq deg_{\mathrm{ibT}}(C)$. Namely, if e_0, e_1 and e_2 are chosen in such a way that W_{e_0} is some c.e. set with $\mathbf{d} = deg_{\mathrm{ibT}}(W_{e_0})$, $C_f = \hat{\Phi}_{e_1}^{W_{e_0}}$ and $B = \hat{\Phi}_{e_2}^{W_{e_0}}$, then with $e = \langle e_0, e_1, e_2 \rangle$ the satisfaction of $\mathcal{J}_e$ implies that $deg_{\mathrm{ibT}}(C) \leq deg_{\mathrm{ibT}}(W_{e_0}) = \mathbf{d}$.

$B \leq_{\mathrm{ibT}} C$ will be satisfied by direct coding. That is, if a new number x is enumerated into B at stage $s + 1$ then $x \notin C_s$ and x is simultaneously enumerated into C at stage $s + 1$.

Note the similarity between the requirements given above and the requirements in the proof of Theorem 3.20. If we substitute the sets B_0 and B_1 from that proof by C_f and B, then the join requirements are exactly the same, and the order requirements are also the same (note that $C_f \leq_{\mathrm{cl}} C$ automatically holds by the Computable Shift Lemma 2.6). The diagonalisation requirements are different, of course; however, the strategies to satisfy them are very much alike. In the proof of Theorem 3.20 they basically consisted in putting some number x into C and restraining all further enumerations of numbers $y \leq g(x)$ into B_0 or B_1. In the current proof, to satisfy the requirement $\mathcal{D}_e$ by the usual diagonalisation strategy, we wait for a stage s such that $\hat{\Phi}_{e,s}^{B_s}(x) = 0$ for some appropriate witness $x \notin C_s$ and then put x into C_{s+1} and restrain all enumerations of numbers $y \leq x$ into B. Since we satisfy $B \leq_{\mathrm{ibT}} C$ by direct coding and since $x \leq f(x)$, a stronger condition would be to restrain all enumerations of numbers $y \leq f(x)$ into

B and all enumerations of numbers $y \leq x$ into C, i.e. all enumerations of numbers $y \leq f(x)$ into C_f. But this is – substituting B_0 by B and B_1 by C_f again, and substituting g by f – just the strategy in the proof of Theorem 3.20.

To clarify matters, we state the modified construction. The notation is as in the proof of Theorem 3.20.

Construction. Let $B_0 = C_0 = \emptyset$. For $s \geq 0$, we say that a node α of length e *requires attention* at stage $s+1$ if $\alpha \sqsubseteq \delta_s$ and

(*Case 1*) α has no interval assigned to it at the end of stage s, or

(*Case 2*) α has an interval I assigned to it at the end of stage s such that $C_s(x) = \hat{\Phi}_{e,s}^{B_s}(x)$ for all $x \in I$, $l_s(\mathfrak{I}_{e'}) > \max(I)$ for all $e' < e$ with $\alpha(e') = 0$ and $I \cap B_s = I \cap C_s \subset I$.

If some node requires attention at stage $s+1$, find the least (with respect to $\sqsubseteq$) such α and say that α *is active* at stage $s+1$. We say that α is active due to Case 1 or active due to Case 2, respectively, depending on whether α has an interval assigned or not at the end of stage s. Declare all intervals assigned to nodes $\beta > \alpha$ unassigned (i.e., *initialise* these nodes) and do the following:

If α is active due to Case 1, let $e = |\alpha|$ and assign a new interval $I' = [x, f^{e\cdot(x+1)+1}(x)]$ to α where x is the least number $\geq s+1$ such that x is larger than all numbers from intervals assigned to any node before stage $s+1$. Let $C_{s+1} = C_s$ and $B_{s+1} = B_s$.

If α is active due to Case 2, then distinguish the following subcases.

(*Subcase 2.1*) If there exists $x \notin C_s$ such that $[x, f(x)] \subseteq I$ and $[x+1, f(x)] \subseteq W_{e'_0,s}$ for every $e' < e$ with $\alpha(e') = 0$ and $e' = \langle e'_0, e'_1, e'_2 \rangle$, then let $C_{s+1} = C_s \cup \{x\}$ for the least such x and $B_{s+1} = B_s$ (we say that α enumerates x into C at stage $s+1$).

(*Subcase 2.2*) Otherwise, for $y = \max(\{x \in I : x \notin B_s\})$, let $C_{s+1} = C_s \cup \{y\}$ and $B_{s+1} = B_s \cup \{y\}$ (we say that α enumerates y into C and B at stage $s+1$).

If no node requires attention at stage $s+1$, let $C_{s+1} = C_s$, $B_{s+1} = B_s$ and initialise all nodes $\beta > \delta_s$. Proceed to the next stage.

The verification is completely analogous to the proof of Theorem 3.20 and is left to the reader. It can also be found in [Ambo 13a]. □

5.3 Cuppability in $\mathcal{R}_{\mathrm{cl}}$

Turning to the c.e. cl-degrees, we first notice that the analogue of Theorem 5.7 is true as well.

Theorem 5.8. *Let f be an unbounded computable shift. Then there are c.e. sets B and C such that $B <_{\mathrm{cl}} C$ and $deg_{\mathrm{cl}}(C_f) \vee deg_{\mathrm{cl}}(B) = deg_{\mathrm{cl}}(C)$.*

Proof. By Theorem 5.7 there are c.e. sets B and C such that $B <_{\mathrm{ibT}} C$ and $deg_{\mathrm{ibT}}(C_f) \vee deg_{\mathrm{ibT}}(B) = deg_{\mathrm{ibT}}(C)$. We will show that B and C have the required properties. By the ibT-Join Lemma, $deg_{\mathrm{cl}}(C_f) \vee deg_{\mathrm{cl}}(B) = deg_{\mathrm{cl}}(C)$. So, since ibT-reducibility is stronger than cl-reducibility, it suffices to show that $B \not\equiv_{\mathrm{cl}} C$.

For a contradiction assume that $B \equiv_{\mathrm{cl}} C$. Then $C + k \leq_{\mathrm{ibT}} B$ for some $k \geq 0$. Since f is an unbounded computable shift, it follows that $f(x) \geq x + k$ for almost every x, hence $C_f \leq_{\mathrm{ibT}} C + k \leq_{\mathrm{ibT}} B$. So

$$deg_{\mathrm{ibT}}(C_f) \vee deg_{\mathrm{ibT}}(B) = deg_{\mathrm{ibT}}(B) < deg_{\mathrm{ibT}}(C).$$

But this contradicts the choice of B and C. □

This shows that for some degrees $\mathbf{a} \in \mathcal{R}_{\mathrm{cl}}$ there exist cl-degrees $\mathbf{d} < \mathbf{a}$ which are "much smaller" than $\mathbf{a}$ and still $\mathbf{a}$-cuppable. On the other hand, as we will see, there are degrees $\mathbf{a} \in \mathcal{R}_{\mathrm{cl}}$ for which we can find $\mathbf{a}$-*non*cuppable degrees which are arbitrarily close to $\mathbf{a}$ in the sense that they can avoid any given lower cone $\{\mathbf{d} : \mathbf{d} \leq \mathbf{b}\}$, where $\mathbf{b} < \mathbf{a}$. Indeed, this will hold for *2-scattered cl*-degrees $\mathbf{a}$.

Definition 5.9. Let $R_2 = \{2^m : m \geq 0\}$. Call a set A 2-*scattered* if $A \subseteq R_2$, and call a c.e. degree 2-*scattered* if it contains a 2-scattered c.e. set.

Note that, for any c.e. set A, $\hat{A} = \{2^n : n \in A\}$ is 2-scattered, c.e. and wtt-equivalent to A. So any c.e. wtt-degree contains a c.e. 2-scattered set.

Theorem 5.10. *Let* $\mathbf{a}$ *and* $\mathbf{b}$ *be c.e.* cl-*degrees such that* $\mathbf{a}$ *is* 2-*scattered and* $\mathbf{b} < \mathbf{a}$. *There is an* $\mathbf{a}$-*noncuppable c.e.* cl-*degree* $\mathbf{c} \leq \mathbf{a}$ *such that* $\mathbf{c} \not\leq \mathbf{b}$.

For the proof of Theorem 5.10 we will need the following observation.

Lemma 5.11. *Let* $A, B_0, \ldots, B_k$ *be c.e. sets such that* A *is* 2-*scattered and* $B_i <_{\mathrm{cl}} A$ *for* $i \in \{0, \ldots, k\}$. *Then for* $k' > k$ *there are c.e. sets* $\hat{B}_0, \ldots \hat{B}_k$ *such that*

$$\hat{B}_i \subseteq k' \cdot \mathbb{N} + i = \{k' \cdot n + i : n \in \mathbb{N}\}$$

and $B_i \leq_{\mathrm{cl}} \hat{B}_i <_{\mathrm{cl}} A$ *for* $i \in \{0, \ldots, k\}$.

Moreover, the sets $\hat{B}_i$ *can be chosen in such a way that there is a splitting* $A = A_0 \cup A_1$ *of* A *and a splitting* $\hat{B}_i = \hat{B}_{i0} \dot\cup \hat{B}_{i1}$ *of* $\hat{B}_i$ *with* $\hat{B}_{i0} \leq_{\mathrm{cl}} B_i$ *and* $\hat{B}_{i1} = (A_1)_{f_i}$ *for some unbounded computable shift* f_i.

Proof. W.l.o.g. we may assume that $1, 2, \ldots, 2^{3k'-1} \notin A$ and (by replacing B_i by some bounded shift $B_i + p$ with $B_i \leq_{(\mathrm{i+p})\mathrm{bT}} A$) that $B_i \leq_{\mathrm{ibT}} A$ for $i \in \{0, \ldots, k\}$.

For each $i \in \{0, \ldots, k\}$ do the following:

If B_i is computable, then we can just choose $\hat{B}_i = k' \cdot \mathbb{N} + i$.

Otherwise, using Lemma 2.7, we may assume that there are enumeration functions a of A and b_i of B_i for $i \in \{0, \ldots, k\}$, such that $a(n) \leq b_i(n)$ for all $n \geq 0$.

Fix a computable function $low_i : \mathbb{N} \to \mathbb{N}$ such that, for $m \geq 3k'$ and $y \in [2^m, 2^{m+1})$, $low_i(y) \in [2^m, 2^{m+1} - k') \cap (k' \cdot \mathbb{N} + i)$, and $|low_i(y) - y| \leq 2k'$. Such a function exists, because for $y \in [2^m, 2^{m+1} - k')$, $m \geq 3k'$, at least one of the sets $\{y, y+1, \ldots, y+k'-1\}$ and $\{y, y-1, \ldots, y-k'+1\}$ is a subset of $[2^m, 2^{m+1}-k')$ (since $|[2^m, 2^{m+1}-k')| = 2^{m+1}-2^m-k' = 2^m - k' \geq 2^{3k'} - k' \geq 3k' - k' = 2k'$), and both sets contain a number from $k' \cdot \mathbb{N}$; and for $y \in [2^{m+1} - k', 2^{m+1})$ the set $[2^{m+1} - 2k', 2^{m+1} - k')$ is a subset of $[2^m, 2^{m+1} - k')$ and contains a number from $k' \cdot \mathbb{N}$.

Also fix a computable function $up_i : \mathbb{N} \to \mathbb{N}$ such that, for $m \geq 3k'$ and $y \in [2^m, 2^{m+1})$, $up_i(y)$ is the unique number from $[2^{m+1} - k', 2^{m+1})$ which is in $k' \cdot \mathbb{N} + i$.

Then let $\hat{B}_i = \{\hat{b}_i(n) : n \geq 0\}$ for the computable function $\hat{b}_i$ defined by

$$\hat{b}_i(n) = \begin{cases} low_i(b_i(n)) & \text{if } a(n) \leq b_i(n) < 2a(n) \\ up_i(2a(n) - 1) & \text{otherwise (i.e., if } 2a(n) \leq b_i(n)). \end{cases}$$

Obviously, $\hat{B}_i$ is c.e. and $\hat{B}_i \subseteq k' \cdot \mathbb{N} + i$. Moreover, since A is 2-scattered and $a(n) \geq 2^{3k'}$ for all n, the function $\hat{b}_i$ is one-to-one and $a(n) \leq \hat{b}_i(n) \leq b_i(n) + 2k'$. So $B_i \leq_{\mathrm{cl}} \hat{B}_i \leq_{\mathrm{cl}} A$.

It remains to show that $A \not\leq_{\mathrm{cl}} \hat{B}_i$. For a contradiction assume $A \leq_{\mathrm{cl}} \hat{B}_i$. Split A, B_i, and $\hat{B}_i$ into the c.e. sets

$$A_0 = \{a(n) : a(n) \leq b_i(n) < 2a(n)\} \text{ and } A_1 = \{a(n) : 2a(n) \leq b_i(n)\},$$

$$B_{i0} = \{b_i(n) : a(n) \leq b_i(n) < 2a(n)\} \text{ and } B_{i1} = \{b_i(n) : 2a(n) \leq b_i(n)\},$$

and

$$\hat{B}_{i0} = \{\hat{b}_i(n) : a(n) \leq b_i(n) < 2a(n)\} \text{ and } \hat{B}_{i1} = \{\hat{b}_i(n) : 2a(n) \leq b_i(n)\},$$

respectively. Note that, by the fact that any interval $[2^m, 2^{m+1}-1)$ contains at most one element of B_{i0} and by definition of $low(y)$, $\hat{B}_{i0} \equiv_{\mathrm{cl}} B_{i0}$. Moreover, $\hat{B}_{i1} = (A_1)_{f_i}$ for the computable unbounded shift f_i defined by

$$f_i(x) = \begin{cases} x, & \text{if } x < 2^{3k'} \\ up_i(2^{m+1} - 1) + q, & \text{if } x = 2^m + q \text{ with } m \geq 3k' \text{ and } 0 \leq q < 2^m. \end{cases}$$

Note that f_i is indeed strictly increasing, i.e. $f_i(x) < f_i(x+1)$ for all x. For $x < 2^{3k'} - 1$ and for $x = 2^m + q$ with $m \geq 3k'$ and $q < 2^m - 1$ this is clear; for $x = 2^{3k'} - 1$ we have $f_i(x) = 2^{3k'} - 1$ and $f_i(x+1) = up_i(2^{3k'+1} - 1) \geq 2^{3k'+1} - k' = 2^{3k'} + 2^{3k'} - k' \geq 2^{3k'} = f_i(x) + 1$; and for $x = 2^m + (2^m - 1)$ with $m \geq 3k'$ it holds that $f_i(x) = up_i(2^{m+1} - 1) + (2^m - 1) < 2^{m+1} + 2^m - 1 \leq 2^{m+1} + 2^m + 2^m - m \leq 2^{m+1} + 2^{m+1} - k' = 2^{m+2} - k' \leq up_i(2^{m+2} - 1) = f_i(x+1)$.

By the above observation and by assumption,

$$A \leq_{\mathrm{cl}} \hat{B}_i = \hat{B}_{i0} \mathbin{\dot{\cup}} \hat{B}_{i1} \leq_{\mathrm{cl}} \hat{B}_{i0} \mathbin{\dot{\cup}} (A_1)_{f_i}.$$

Hence $A \leq_{\mathrm{cl}} \hat{B}_{i0}$ by the Computable Shift Lemma 2.6. By $\hat{B}_{i0} \equiv_{\mathrm{cl}} B_{i0}$ and $B_{i0} \leq_{\mathrm{cl}} B_i$ this implies $A \leq_{\mathrm{cl}} B_i$ contrary to the choice of A and B_i.

□

Proof of Theorem 5.10. Fix c.e. sets A and B in $\mathbf{a}$ and $\mathbf{b}$, respectively, such that A is 2-scattered. By replacing B by a bounded shift $B+k$, we may assume that $B \leq_{\mathrm{ibT}} A$. Note that A is noncomputable because $B <_{\mathrm{cl}} A$. If B is computable, we may substitute B by some noncomputable set $B' <_{\mathrm{cl}} A$, for example $B' = 2A$. Hence by Lemma 2.7 we can assume that both A and B are noncomputable and that there are enumeration functions a of A and b of B, such that $a(n) \leq b(n)$ for all $n \geq 0$.

It suffices to define a c.e. set $C \leq_{\mathrm{cl}} A$ such that $C \not\leq_{\mathrm{cl}} B$ and $deg_{\mathrm{cl}}(C)$ does not cup to $deg_{\mathrm{cl}}(A)$. In the following we inductively define an enumeration function for such a set C. We ensure that the function c has the following properties.

$$(\forall n)\ (a(n) \leq c(n) < 2a(n)) \tag{111}$$

$$(\forall n)\ (c(n) \text{ is even}) \tag{112}$$

$$(\forall e)\ (\exists n_e)\ (\forall n \geq n_e)\ (c(n) > a(n) + e) \tag{113}$$

In addition we guarantee that the set $C = \{c(n) : n \geq 0\}$ meets the requirements

$$\mathcal{R}_e : C \neq \tilde{\Phi}_e^B.$$

To show that this guarantees that C has the required properties, note that (111) implies that $C \leq_{\mathrm{cl}} A$ while satisfaction of all requirements $\mathcal{R}_e$ ensures that $C \not\leq_{\mathrm{cl}} B$. It remains to show that $deg_{\mathrm{cl}}(C)$ does not cup to $deg_{\mathrm{cl}}(A)$. For a contradiction assume that there is a c.e. set $D <_{\mathrm{cl}} A$ such that

$$deg_{\mathrm{cl}}(A) = deg_{\mathrm{cl}}(C) \vee deg_{\mathrm{cl}}(D).$$

By Lemma 5.11 (applied to $k = 1$, $B_0 = \emptyset$, $B_1 = D$ and $k' = 2$), w.l.o.g. we may assume that $D \subseteq 2\mathbb{N} + 1$. Since, by (112), $C \subseteq 2\mathbb{N}$, it follows that $A \leq_{\mathrm{cl}} C \cup D$ by the Splitting Lemma. So we may fix e such that $A = \tilde{\Phi}_e^{C \cup D}$. Now in order to get the desired contradiction we show that this reduction can be converted into a cl-self-reduction of A relative to D whence $A \leq_{\mathrm{cl}} D$ contrary to the choice of D. This self-reduction is as follows.

Since A is 2-scattered it suffices to compute $A(x)$ for $x = 2^m$ ($m \geq 0$). In fact, by (113), we may fix a number m_e such that for any n such that $a(n) = 2^m$ for some $m \geq m_e$, $c(n) > a(n) + e = 2^m + e$ and w.l.o.g. we may assume that $m \geq m_e$. So in the computation $\tilde{\Phi}_e^{C \cup D}(x)$ any even query y with $y \geq x$ will be answered negatively since $C \cap [2^m, 2^m + e] = \emptyset$ and $y \leq \tilde{u}_e(x) \leq 2^m + e$. For an even query $y < x$, compute $m' < m$ such that $y \in [2^{m'}, 2^{m'+1})$. Then, by using $A \upharpoonright 2^{m'} + 1$ as an oracle, check whether $2^{m'} \in A$. If $2^{m'} \notin A$ then $y \notin C$ by (111). Otherwise, $y \in C$ if and only if $c(n) = y$ for the unique n such that $a(n) = 2^{m'}$. Odd queries in the computation of $\tilde{\Phi}_e^{C\cup D}(x)$ are simply answered by the oracle D.

Now the enumeration function c of C is inductively defined as follows. Given $s \geq 0$ and $c(0), \dots, c(s-1)$, let $A_{s-1} = \{a(0), \dots, a(s-1)\}$, $B_{s-1} = \{b(0), \dots, b(s-1)\}$ and $C_{s-1} = \{c(0), \dots, c(s-1)\}$. Say that requirement $\mathcal{R}_e$ *requires attention* at stage s if $e \leq s$, $a(s) + 2e < 2a(s)$,

$$C_{s-1} \upharpoonright a(s) + 2e + 1 = \tilde{\Phi}_{e,s-1}^{B_{s-1}} \upharpoonright a(s) + 2e + 1, \tag{114}$$

and $b(s) \geq a(s) + 3e + 1$. If no requirement requires attention then let $c(s) = 2a(s) - 2$. Otherwise, for the least e such that $\mathcal{R}_e$ requires attention, let $c(s) = a(s) + 2e$ and say that requirement $\mathcal{R}_e$ is *active* at stage s.

Obviously, the enumeration function $c(n)$ is computable and one-to-one. So it suffices to show that $c(n)$ satisfies (111) to (113) and that, for $C = \{c(n) : n \geq 0\}$, the requirements $\mathcal{R}_e$ are met. Now (111) and (112) are obvious. For a proof of (113) note that, for sufficiently large s, $c(s) \leq a(s) + e$ only if a requirement $\mathcal{R}_{e'}$ with $e' \leq e$ is active at stage s, hence requires attention at stage s. So it suffices to prove the following claim.

Claim. Every requirement $\mathcal{R}_e$ requires attention at most finitely often and is met.

The proof of the claim is by induction on e. Fix e and, by inductive hypothesis, choose a stage $s_{-1} > e$ such that no requirement $\mathcal{R}_{e'}$ with $e' < e$ requires attention after stage s_{-1}. W.l.o.g. we may assume that s_{-1} is sufficiently large such that $a(s) + 2e < 2a(s)$ for all $s \geq s_{-1}$.

Next observe that if $\mathcal{R}_e$ would require attention infinitely often then there were infinitely many stages s such that (114) holds. Since $\lim_{s\to\infty} a(s) = \infty$ and since $\tilde{\Phi}_e$ is a cl-functional this would imply that $C = \tilde{\Phi}_e^B$, i.e., that $\mathcal{R}_e$ is not met. So it suffices to show that $\mathcal{R}_e$ is met.

For a contradiction assume that

$$C = \tilde{\Phi}_e^B. \tag{115}$$

Then, by induction on $m \geq 0$, let s_m be minimal such that $s_m > s_{m-1}$,

$$B_{s_m-1} \upharpoonright 2^m + 3e + 1 = B \upharpoonright 2^m + 3e + 1, \tag{116}$$

and

$$C_{s_m-1} \upharpoonright 2^m + 2e + 1 = \tilde{\Phi}_{e,s_m-1}^{B_{s_m-1}} \upharpoonright 2^m + 2e + 1. \tag{117}$$

We claim that $A(2^m) = A_{s_m-1}(2^m)$ for all m. Since stage s_m can be computed from $B \upharpoonright 2^m + 3e + 1$, it follows that $A \leq_{\mathrm{cl}} B$ contrary to the choice of A and B.

For a contradiction assume that there is a number $m \geq 0$ such that $A(2^m) \neq A_{s_m-1}(2^m)$. Then there is a stage $s^* \geq s_m$ such that $a(s^*) = 2^m$. It follows by (116) that

$$b(s^*) \geq a(s^*) + 3e + 1. \tag{118}$$

Moreover, by (117), (116) and $\tilde{u}_e(x) \leq x + e$, and by (115) it holds that

$$\begin{aligned} C_{s_m-1} \upharpoonright 2^m + 2e + 1 &= \tilde{\Phi}^{B_{s_m-1}}_{e,s_m-1} \upharpoonright 2^m + 2e + 1 \\ &= \tilde{\Phi}^{B}_{e} \upharpoonright 2^m + 2e + 1 \\ &= C \upharpoonright 2^m + 2e + 1. \end{aligned} \tag{119}$$

Since $s^* \geq s_m$, it follows that $C_{s^*} \upharpoonright 2^m + 2e + 1 = C \upharpoonright 2^m + 2e + 1$ and $\tilde{\Phi}^{B_{s^*}}_{e,s^*} \upharpoonright 2^m + 2e + 1 = \tilde{\Phi}^{B}_{e} \upharpoonright 2^m + 2e + 1$.

Now, since $a(s^*) = 2^m$ and $s^* \geq s_m \geq s_{-1}$, it follows from (118) and (117) that $\mathcal{R}_e$ requires attention and becomes active at stage s^*. So $c(s^*) = a(s^*) + 2e = 2^m + 2e$ is enumerated into C at stage s^*, i.e., $2^m + 2e \in C_{s^*} \setminus C_{s^*-1}$. But this contradicts (119).

This completes the proof of the claim and the proof of the theorem. □

Corollary 5.12. *The first order theory* $\mathrm{Th}(\mathcal{R}_{\mathrm{ibT}})$ *of the partial ordering of the c.e.* ibT*-degrees and the first order theory* $\mathrm{Th}(\mathcal{R}_{\mathrm{cl}})$ *of the partial ordering of the c.e.* cl*-degrees are different.*

Proof. This follows from Theorem 5.3 and Theorem 5.10, because for the theorem σ of first-order predicate logic stating that for all degrees $\mathbf{a} \neq \mathbf{0}$ the set of the c.e. $\mathbf{a}$-noncuppable degrees has an upper bound $\mathbf{b}$ less than $\mathbf{a}$ it holds that

$$\mathcal{R}_{\mathrm{ibT}} \models \sigma$$

and

$$\mathcal{R}_{\mathrm{cl}} \not\models \sigma.$$

□

Note that the theorem

$$\begin{aligned} \sigma \equiv & (\forall a)(\exists b)(\forall c)(\forall d)(\exists f) \\ & (a \neq 0 \to b < a \wedge (c < a \wedge (d < a \to c \leq f \wedge d \leq f \wedge a \not\leq f) \to c \leq b)) \end{aligned}$$

in the proof above is a Π_4-statement. It is an open question whether $\mathcal{R}_{\mathrm{ibT}}$ and $\mathcal{R}_{\mathrm{cl}}$ satisfy the same theorems of first-order predicate logic with 2 or 3 quantifier changes; it is not hard to see that they satisfy the same Σ_1-theorems, since every finite partial order can be embedded into $\mathcal{R}_{\mathrm{ibT}}$ and $\mathcal{R}_{\mathrm{cl}}$.

By Theorem 5.10 for a 2-scattered cl-degree $\mathbf{a}$, the set $\mathbf{NCu}_{\mathrm{cl}}(\mathbf{a})$ has no greatest element. At the end of this chapter we want to extend this result and show that $\mathbf{NCu}_{\mathrm{cl}}(\mathbf{a})$ does not even have *maximal* elements.

Lemma 5.13. *Let* $\mathbf{a}$ *be 2-scattered. Then* $\mathbf{NCu}_{\mathrm{cl}}(\mathbf{a})$ *has no maximal elements.*

Proof. Let $\mathbf{b} \leq \mathbf{a}$ with $\mathbf{b} \in \mathbf{NCu}_{\mathrm{cl}}(\mathbf{a})$. By Theorem 5.10 there is a c.e. cl-degree $\mathbf{c} \in \mathbf{NCu}_{\mathrm{cl}}(\mathbf{a})$ such that $\mathbf{c} \not\leq \mathbf{b}$. Now if $\mathbf{b} < \mathbf{c}$, then clearly $\mathbf{b}$ is not maximal in $\mathbf{NCu}_{\mathrm{cl}}(\mathbf{a})$. Hence assume that $\mathbf{b}$ and $\mathbf{c}$ are incomparable. We claim that there is some c.e. cl-degree $\mathbf{d}$ with $\mathbf{b}, \mathbf{c} \leq \mathbf{d}$ and $\mathbf{d} \in \mathbf{NCu}_{\mathrm{cl}}(\mathbf{a})$; since $\mathbf{c}$ is incomparable to $\mathbf{b}$ but not to $\mathbf{d}$, necessarily $\mathbf{b} \neq \mathbf{d}$. Hence $\mathbf{b} < \mathbf{d}$ and $\mathbf{b}$ is not maximal in $\mathbf{NCu}_{\mathrm{cl}}(\mathbf{a})$ again.

To prove that such $\mathbf{d}$ exists, choose c.e. sets $A \in \mathbf{a}$, $B \in \mathbf{b}$ and $C \in \mathbf{c}$ such that A is 2-scattered. Then $B <_{\mathrm{cl}} A$ and $C <_{\mathrm{cl}} A$. By Lemma 5.11 (applied to $k = 1$, $B_0 = B$, $B_1 = C$ and $k' = 3$) there are c.e. sets $\hat{B} \subseteq 3\mathbb{N}$ and $\hat{C} \subseteq 3\mathbb{N} + 1$ such that $B \leq_{\mathrm{cl}} \hat{B} <_{\mathrm{cl}} A$ and $C \leq_{\mathrm{cl}} \hat{C} <_{\mathrm{cl}} A$ and such that there are splittings $A = A_0 \dot\cup A_1$ and $A = A_0' \dot\cup A_1'$ of A, $\hat{B} = \hat{B}_0 \dot\cup \hat{B}_1$ of $\hat{B}$ and $\hat{C} = \hat{C}_0 \dot\cup \hat{C}_1$ of $\hat{C}$ and unbounded computable shifts f_B and f_C with $\hat{B}_0 \leq_{\mathrm{cl}} B$, $\hat{B}_1 = (A_1)_{f_B}$, $\hat{C}_0 \leq_{\mathrm{cl}} B$ and $\hat{C}_1 = (A_1)_{f_C}$.

Let $\mathbf{d} = deg_{\mathrm{cl}}(\hat{B} \cup \hat{C})$. By the Splitting Lemma, $\mathbf{d} \geq deg_{\mathrm{cl}}(\mathbf{b})$ and $\mathbf{d} \geq deg_{\mathrm{cl}}(\mathbf{c})$. Hence it suffices to prove that $\mathbf{d} \in \mathbf{NCu}_{\mathrm{cl}}(\mathbf{a})$.

For a contradiction assume that there is some c.e. cl-degree $\mathbf{e} < \mathbf{a}$ such that $\mathbf{d} \vee \mathbf{e} = \mathbf{a}$. Let E be a c.e. set such that $E \in \mathbf{e}$. Without loss of generality we may assume that $E \subseteq 3\mathbb{N} + 2$, because otherwise using Lemma 5.11 (applied to $k = 2$, $B_0 = B_1 = \emptyset$, $B_2 = E$ and $k' = 3$) we can replace E by a set $\hat{E}$ with this property and such that $deg_{\mathrm{cl}}(E) \leq deg_{\mathrm{cl}}(\hat{E}) < \mathbf{a}$, and replace $\mathbf{e}$ by $deg_{\mathrm{cl}}(\hat{E})$.

Now by the Splitting Lemma again, since $\hat{B}$, $\hat{C}$ and E are pairwise disjoint,

$$deg_{\mathrm{cl}}(\hat{B} \cup \hat{C} \cup E) = deg_{\mathrm{cl}}(\hat{B} \cup \hat{C}) \vee \deg_{\mathrm{cl}}(E) = \mathbf{d} \vee \mathbf{e} = \mathbf{a}.$$

Hence

$$\hat{B}_1 \cup \hat{C}_1 \cup \hat{B}_0 \cup \hat{C}_0 \cup E \equiv_{\mathrm{cl}} A.$$

Since $\hat{B}_1$ is disjoint from $\hat{C}_1 \cup \hat{B}_0 \cup \hat{C}_0 \cup E$ and $\hat{B}_1 = (A_1)_{f_B}$, it follows by the Computable Shift Lemma 2.6 that

$$\hat{C}_1 \cup \hat{B}_0 \cup \hat{C}_0 \cup E \equiv_{\mathrm{cl}} A.$$

By similar reasoning for $\hat{C}_1$ instead of $\hat{B}_1$ we conclude that

$$\hat{B}_0 \cup \hat{C}_0 \cup E \equiv_{\mathrm{cl}} A.$$

Now since $\hat{B}_0 \leq_{\mathrm{cl}} B$ and $deg_{\mathrm{cl}}(B) = \mathbf{b} \in \mathbf{NCu}_{\mathrm{cl}}(\mathbf{a})$, it follows that

$$\hat{C}_0 \cup E \equiv_{\mathrm{cl}} A.$$

Finally, since $\hat{C}_0 \leq_{\mathrm{cl}} C$ and $deg_{\mathrm{cl}}(C) = \mathbf{c} \in \mathbf{NCu}_{\mathrm{cl}}(\mathbf{a})$, we see that $E \equiv_{\mathrm{cl}} A$, contradicting $\mathbf{e} < \mathbf{a}$.

□

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