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Unique longest increasing subsequences in 132-avoiding permutations

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Abstract

The topic of longest increasing subsequences in permutations has long been of interest to combinatorialists. An adjacent, but relatively unknown problem is that of permutations with unique longest increasing subsequences, where there is only one maximal increasing subsequence. We answer a question of Bóna and DeJonge that has been open for several years. Namely, we provide a simple injective proof that the number of 132-avoiding permutations with a unique longest increasing subsequence is at least as large as the number of 132-avoiding permutations without a unique longest increasing subsequence.

Introduction

Let $p = p_1p_2\dots p_n$ be a permutation. An *increasing subsequence* in p is i a subset of (not necessarily consecutive) entries $p_{j_1} < p_{j_2} < \cdots < p_{j_k}$ such that $j_1 < j_2 < \cdots < j_k$.

We say that p has a *unique longest increasing subsequence*, or ULIS, if p has an increasing subsequence that is longer than all other increasing subsequences. For example 34256178 has a ULIS, namely 345678, but 32456178 does not, since 345678 and 245678 are both increasing subsequences of maximal length 6 in p.

Finding the number of all permutations of length n that have a unique longest increasing subsequence has proven to be a difficult problem. They are documented for $n \leq 15$ in Sequence A167995 in the Online Encyclopedia of Integer Sequences [3].

An approach taken to the problem in 2020 by Bóna and DeJonge [1] considered permutations avoiding a single pattern of length 3. It is well known that the number of such permutations for any length 3 pattern is $C_n = \binom{2n}{n}/(n+1)$, the n^{th} Catalan number. Adopting their terminology, they showed, as a corollary of a theorem of [4], that if $u_n(132)$ is the number of permutations with a ULIS avoiding 132, then $\lim_{n\to\infty} u_n(132)/C_n = 0.5$. As a follow-up question, they asked if $u_n(132)/C_n \geq 0.5$ for all n. Finding a simple injective proof of this has been an open problem for four years. In this paper we provide such a proof.

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1 Preliminary Lemmas

We begin with two preliminary lemmas, that will prove essential to our approach:

Lemma 1.1. *Let $p = p_1p_2 \dots p_n$ avoid 132. Then for all $i \in [n]$, there is a unique longest increasing subsequence in p that begins at p_i.*

Note we are not concerned with the subsequence being longest with respect to p, but rather with respect to starting at p_i.

Proof. Assume otherwise, that there exists $i \in p$ such that $p_ia_2a_3 \dots a_k$ and $p_ib_2b_3 \dots b_k$ are both maximal length increasing subsequences starting at p_i. Let j be the smallest natural number such that $a_j \neq b_j$, and assume without loss of generality $a_j < b_j$. If a_j comes before b_j, then $p_ib_2 \dots b_{j-1}a_jb_j \dots b_k$ is a longer increasing subsequence beginning at p_i. If instead b_j comes before a_j, then $p_ib_ja_j$ is a 132 pattern. Either way, we get a contradiction. □

If the length of the longest increasing subsequence starting at p_i is k, then we say p_i has *rank* k. Using this, we can define the rank function $r(p_i)$ on entries, and $R(p) = r(p_1), r(p_2), \dots, r(p_n)$, where $r(p_i)$ is the rank of p_i.

From observation, we can see that $R(p)$ is a sequence of positive integers such that $r(p_i)-r(p_{i+1}) \leq 1$ for all $i \in [n-1]$, and that $R(p)$ must end in 1. If $r(p_i)-r(p_{i+1}) > 1$, then the longest subsequence starting at p_i must not include p_{i+1}. Then the second entry of said subsequence, a, must be smaller than p_{i+1}. Then there must be a 132 pattern formed by p_i, p_{i+1}, a.

Let $\mathcal{S}_n$ be the set of all sequences of length n satisfying those conditions. Then we have the following lemma:

Lemma 1.2. *Let $Av_n(132)$ be the set of 132-avoiding permutations of length n. Then $R : Av_n(132) \to \mathcal{S}_n$ is a bijection.*

Proof. First note $|\mathcal{S}_n| = C_n$, via (6.19u) in [2], although those sequences instead start at 0 and have each entry being at most one greater than the previous. The equivalence can be seen by reversing all sequences and then adding 1 to every entry.

Thus if we can show that R is an injection, we have a bijection. We do this by starting with an element from $\mathcal{S}_n$, and finding the inverse of R.

First, clearly n must be placed at the leftmost 1. Now for $(n-1)$ we have two cases, either there is a 2 to the left of the 1, or there is not. In the former case, $(n-1)$ must be placed at the leftmost 2, as otherwise whatever entry is placed in that position will either form a 132 pattern (if $(n-1)$ is placed after n) or a subsequence of length 3 with $(n-1)$ and n (if $(n-1)$ is placed before n). In the latter case, $(n-1)$ must be placed at the leftmost remaining 1.

This process continues similarly. For each m from n down to 1, after each m placed at rank k, either $m-1$ is placed at the leftmost rank $k+1$ (if it preceeds the placement of m) or at the leftmost remaining location of highest rank at most k (if there is no $k+1$ to the left of k). Any other placement of $m-1$ will give either a longer subsequence in some location, or a 132 pattern.

Since we have a unique inverse, as at each step we only had a single choice, we have an injection, and since $|Av_n(132)| = |\mathcal{S}_n|$, we have a bijection. □

2 Result

It becomes clear from the two lemmas that for $p \in Av_n(132)$, p has a ULIS if and only if $R(p)$ has a maximum value. To show that $u_n(132)/C_n \geq 0.5$, we show an injection from permutations without a ULIS to permutations with one.

Let $\mathcal{T}_n$ be the set of elements of $\mathcal{S}_n$ without a unique maximum element, and $\mathcal{T}'_n$ be $\mathcal{S}_n - \mathcal{T}_n$. We define the function $f : \mathcal{T}_n \to \mathcal{T}'_n$, which we will later claim is an injection, as follows:

Let $T = t_1 t_2 \dots t_n \in \mathcal{T}_n$, with maximal element k. Let i, j be the final two positions of T such that $t_i = t_j = k$, with $i < j$. We increase all values on the range $[i, j)$ by one.

This clearly maps from $\mathcal{T}_n \to \mathcal{T}'_n$, as by choice of i, j, only one value on the range $[i, j)$ has maximal value, and that value becomes a maximum of $k + 1$.

Using this, we can define an injection g from permutations of length n without a ULIS to permutations with a ULIS:

Theorem 2.1. *Let $U_n(132)$ be the set of 132-avoiding permutations with a ULIS, and $V_n(132)$ be the set of 132-avoiding permutations without a ULIS. Then $g : V_n(132) \to U_n(132)$ defined by $g = R^{-1} \circ f \circ R$ is an injection. Thus, $|U_n(132)| \geq |V_n(132)|$.*

Proof. First, note that g clearly maps into $U_n(132)$, as by choice of i, j, only one element on $[i, j)$ is of maximal rank k, and following the transformation we have only one element of rank $k + 1$.

As R, R^{-1} are bijections, it suffices to show that the middle step is an injection. For two elements of $\mathcal{T}_n$ to map into the same element of $\mathcal{T}'_n$, they would need to have the same k, i, j, and values of ranks at all positions, but then the two elements are clearly the same. Thus, we have an injection. □

References

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