Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

New Hermite–Jensen–Mercer-type inequalities via *k*-fractional integrals

Saad Ihsan Butt[1], Muhammad Umar[1], Saima Rashid[2], Ahmet Ocak Akdemir[3] and Yu-Ming Chu[4,5*]

*Correspondence:
chuyuming2005@126.com
[4]Department of Mathematics, Huzhou University, Huzhou, China
[5]Hunan Provincial Key Laboratory of Mathematical Modeling and Analysis in Engineering, Changsha University of Science & Technology, Changsha, China
Full list of author information is available at the end of the article

Abstract

In the article, we establish serval novel Hermite–Jensen–Mercer-type inequalities for convex functions in the framework of the *k*-fractional conformable integrals by use of our new approaches. Our obtained results are the generalizations, improvements, and extensions of some previously known results, and our ideas and methods may lead to a lot of follow-up research.

MSC: 26E60

Keywords: Convex functions; Hermite–Hadamard inequality; Jensen inequality; Jensen–Mercer inequality; New conformable *k*-fractional integrals

1 Introduction

Convex function [1–20] is an important concept that has come to the fore among many other function classes with its many features and areas of use. Giving the definition as an inequality containing linear combinations has helped in using convex functions for classical inequalities. Jensen inequality [21, 22] is one of these inequalities for convex functions, which can be stated as follows.

Let $(\mu_1, \mu_2, \ldots, \mu_n) \in [0,1]^n$ with $\sum_{i=1}^{n} \mu_i = 1$ and τ be a convex function on the interval $[\theta, \vartheta]$. Then the inequality

$$\tau\left(\sum_{i=1}^{n} \mu_i x_i\right) \leq \left(\sum_{i=1}^{n} \mu_i \tau(x_i)\right) \tag{1.1}$$

holds for all $x_i \in [\theta, \vartheta]$ $(i = 1, 2, \ldots, n)$.

Another important inequality for convex functions is the Hermite–Hadamard inequality [23, 24], which has been proved by numerous ways and has many generalizations and extensions [25–29]. This inequality can generate bounds on the average value of convex functions to reveal its functionality with applications to numerical analysis and error calculation formulas such as trapezoidal and midpoint quadrature formulas. Now, we recall the Hermite–Hadamard inequality as follows.

Let $\tau : J \subseteq R \to R$ be a convex function. Then the Hermite–Hadamard inequality

$$\tau\left(\frac{\theta + \vartheta}{2}\right) \leq \frac{1}{\vartheta - \theta} \int_{\theta}^{\vartheta} \tau(\lambda)\, d\lambda \leq \frac{\tau(\theta) + \tau(\vartheta)}{2}$$

holds for all $\theta, \vartheta \in J$ with $\theta \neq \vartheta$. If τ is a concave function on J, then the above inequality is reversed.

There are many interesting studies in the literature for the Jensen inequality, for example, the Jensen–Mercer inequality is a new variant of the Jensen inequality given by Mercer in [30]. Later, Matković et al. [31] generalized the Jensen–Mercer inequality to operators and gave its many applications. Recently, the Jensen–Mercer inequality has been the subject of intensive research.

The following Theorem 1.1 for convex functions can be found in [32].

Theorem 1.1 ([32]) *Let τ be a convex function defined on $[\theta, \vartheta]$. Then the inequality*

$$\tau\left(\theta + \vartheta - \sum_{i=1}^{n} \mu_i x_i\right) \leq \tau(\theta) + \tau(\vartheta) - \sum_{i=1}^{n} \mu_i \tau(x_i) \tag{1.2}$$

holds for all $x_i \in [\theta, \vartheta]$ and $\mu_i \in [0,1]$ with $\sum_{i=1}^{n} x_i = 1$.

Next, we recall the definitions of the Euler Gamma $\Gamma(\cdot)$ and Beta $B(\cdot,\cdot)$ functions, which will be used in the article:

$$\Gamma(\theta) = \int_0^\infty e^{-\lambda} \lambda^{\theta-1}\, d\lambda, \qquad B(r,s) = \int_0^1 \lambda^{r-1}(1-\lambda)^{s-1}\, d\lambda.$$

The concept of fractional order derivative and integral [33–40] that will shed light on some unknown points about differential equations and solutions of some fractional order differential equations, which proved to be useful for their solution, is a novelty in applied sciences as well as in mathematics. New derivatives and integrals contribute to the solution of differential equations that are expressed and solved in classical analysis, as well as using fractional order derivatives and integrals. In addition, it has increased its contribution to the literature with applications in areas such as engineering, biostatistics, and mathematical biology. Fractional derivative and integral operators not only differ from each other in terms of singularity, locality, and kernels, but also brought innovations to fractional analysis in terms of their usage areas and spaces. The new integral operators put forward by the researchers working in the field of fractional analysis led to new approaches, results, and methods in applied mathematics, engineering, and many other fields, and they have found the expected response in inequality theory. Many new integral inequalities and bounds to known inequalities have been found by using new integral operators. The new trends, improvements, and advances on fractional calculus and real world applications can be found in the literature [41–60]. Now let us remember some integral operators that are well known to be useful in fractional analysis.

Definition 1.2 ([61]) Let $\alpha > 0$, $0 \leq \theta < \vartheta$, and $\tau \in [\theta, \vartheta]$. Then the Riemann–Liouville integrals $J^{\alpha}_{\theta^+}\tau$ and $J^{\alpha}_{\vartheta^-}\tau$ of order α are defined by

$$\left(J^{\alpha}_{\theta^+}\right)\tau(y) = \frac{1}{\Gamma(\alpha)} \int_\theta^y (y-\lambda)^{\alpha-1}\tau(\lambda)\, d\lambda \quad (y > \theta) \tag{1.3}$$

and

$$\left(J^{\alpha}_{\vartheta-}\right)\tau(y) = \frac{1}{\Gamma(\alpha)}\int_{y}^{\vartheta}(\lambda - y)^{\alpha-1}\tau(\lambda)\,d\lambda \quad (y < \vartheta), \tag{1.4}$$

respectively, where $(J^{0}_{\theta+})\tau(y) = (J^{0}_{\vartheta-})\tau(y) = \tau(y)$.

In [62], Jarad et al. defined the new fractional integral operators as follows:

$${}^{\beta}_{\theta}J^{\alpha}\tau(y) = \frac{1}{\Gamma(\beta)}\int_{\theta}^{y}\left(\frac{(y-\theta)^{\alpha} - (\lambda-\theta)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{\tau(\lambda)}{(\lambda-\theta)^{1-\alpha}}\,d\lambda \tag{1.5}$$

and

$${}^{\beta}J^{\alpha}_{\vartheta}\tau(y) = \frac{1}{\Gamma(\beta)}\int_{y}^{\vartheta}\left(\frac{(\vartheta-y)^{\alpha} - (\vartheta-\lambda)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{\tau(\lambda)}{(\vartheta-\lambda)^{1-\alpha}}\,d\lambda. \tag{1.6}$$

Remark 1.3 From (1.5) and (1.6) we clearly see that

(i) If $\theta = 0$ and $\alpha = 1$, then (1.5) reduces to the Riemann–Liouville operator given in (1.3).

(ii) If $\theta = 0$ and $\alpha \to 0$, then the new conformable fractional integral coincides with the generalized fractional integral (see [63]).

(iii) Furthermore, (1.6) becomes the Riemann–Liouville operator if we set $\vartheta = 0$ and $\alpha = 1$. It also corresponds the Hadamard fractional integral [63] once we take $\vartheta = 0$ and $\alpha \to 0$ in the generalized fractional integral.

The generalized k-fractional conformable integrals [64] are defined by

$${}^{\beta}_{k}J^{\alpha}_{\theta+}\tau(y) = \frac{1}{k\Gamma_{k}(\beta)}\int_{\theta}^{y}\left(\frac{(y-\theta)^{\alpha} - (\lambda-\theta)^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}-1}\frac{\tau(\lambda)}{(\lambda-\theta)^{1-\alpha}}\,d\lambda \tag{1.7}$$

and

$${}^{\beta}_{k}J^{\alpha}_{\vartheta-}\tau(y) = \frac{1}{k\Gamma_{k}(\beta)}\int_{y}^{\vartheta}\left(\frac{(\vartheta-y)^{\alpha} - (\vartheta-\lambda)^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}-1}\frac{\tau(\lambda)}{(\vartheta-\lambda)^{1-\alpha}}\,d\lambda. \tag{1.8}$$

If $k > 0$, then the k-Gamma function Γ_{k} is defined as

$$\Gamma_{k}(\alpha) = \lim_{m\to\infty}\frac{m!k^{m}(mk)^{\frac{\alpha}{k}-1}}{(\alpha)_{m,k}}. \tag{1.9}$$

If $\mathrm{Re}(\alpha) > 0$, then the k-Gamma function in integral form is defined as

$$\Gamma_{k}(\alpha) = \int_{0}^{\infty} e^{-\frac{\mu^{k}}{k}}\mu^{\alpha-1}\,d\mu \tag{1.10}$$

with $\alpha\Gamma_{k}(\alpha) = \Gamma_{k}(\alpha + k)$.

The main purpose of the article is to reveal new and more general Hermite–Jensen–Mercer-type inequalities for convex functions with the help of k-fractional integral op-

erator. For this purpose, Hölder inequality and its variants have been used in addition to various analysis processes. With the special versions of the main findings, many inequalities in the literature were obtained and the importance of the results was emphasized.

2 New Hermite–Jensen–Mercer type inequalities

Theorem 2.1 *Let $\alpha, \beta > 0$ and $\tau : [\theta, \vartheta] \to R$ be a convex mapping. Then the inequality*

$$\begin{aligned}\tau\left(\theta+\vartheta-\frac{x+y}{2}\right) &\leq \frac{2^{\alpha\frac{\beta}{k}-1}\alpha^{\frac{\beta}{k}}\Gamma_k(\beta+k)}{(y-x)^{\alpha\frac{\beta}{k}}} \\ &\quad\times\left\{{}^{\beta}_{k}J^{\alpha}_{(\theta+\vartheta-\frac{x+y}{2})^+}\tau(\theta+\vartheta-x)+{}^{\beta}_{k}J^{\alpha}_{(\theta+\vartheta-\frac{x+y}{2})^-}\tau(\theta+\vartheta-y)\right\} \\ &\leq \tau(\theta)+\tau(\vartheta)-\left(\frac{\tau(x)+\tau(y)}{2}\right) \end{aligned} \tag{2.1}$$

holds for all $x, y \in [\theta, \vartheta]$.

Proof Since τ is convex, to prove the first inequality, we write

$$\begin{aligned}\tau\left(\theta+\vartheta-\frac{x_1+y_1}{2}\right) &= \tau\left(\frac{\theta+\vartheta-x_1+\theta+\vartheta-y_1}{2}\right) \\ &\leq \frac{\tau(\theta+\vartheta-x_1)+\tau(\theta+\vartheta-y_1)}{2}\end{aligned}$$

for all $x_1, y_1 \in [\theta, \vartheta]$.

Let $x_1 = \frac{\lambda}{2}x + \frac{2-\lambda}{2}y$ and $y_1 = \frac{2-\lambda}{2}x + \frac{\lambda}{2}y$. Then for $x, y \in [\theta, \vartheta]$ and $\lambda \in [0,1]$, we have

$$\begin{aligned}2\tau\left(\theta+\vartheta-\frac{x+y}{2}\right) &\leq \tau\left(\theta+\vartheta-\left(\frac{\lambda}{2}x+\frac{2-\lambda}{2}y\right)\right) \\ &\quad+\tau\left(\theta+\vartheta-\left(\frac{2-\lambda}{2}x+\frac{\lambda}{2}y\right)\right).\end{aligned} \tag{2.2}$$

Multiplying both sides of (2.2) by $(\frac{1-(1-\lambda)^{\alpha}}{\alpha})^{\frac{\beta}{k}-1}(1-\lambda)^{\alpha-1}$ and integrating the obtained inequality with respect to λ over $[0,1]$, and then combining the resulting inequality with the definition of the integral operator gives

$$\begin{aligned}&2\tau\left(\theta+\vartheta-\frac{x+y}{2}\right)\int_0^1\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}-1}(1-\lambda)^{\alpha-1}\,d\lambda \\ &\leq \int_0^1\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}-1}(1-\lambda)^{\alpha-1} \\ &\quad\times\left(\tau\left(\theta+\vartheta-\left(\frac{\lambda}{2}x+\frac{2-\lambda}{2}y\right)\right)+\tau\left(\theta+\vartheta-\left(\frac{2-\lambda}{2}x+\frac{\lambda}{2}y\right)\right)\right)d\lambda \\ &= \int_{\theta+\vartheta-y}^{\theta+\vartheta-\frac{x+y}{2}}\left(\frac{1-(\frac{(\theta+\vartheta-\frac{x+y}{2})-\lambda_1}{\frac{y-x}{2}})^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}-1}\left(\frac{(\theta+\vartheta-\frac{x+y}{2})-\lambda_1}{\frac{y-x}{2}}\right)^{\alpha-1}\tau(\lambda_1)\frac{2}{y-x}\,d\lambda_1\end{aligned}$$

$$+\int_{\theta+\vartheta-x}^{\theta+\vartheta-\frac{x+y}{2}}\left(\frac{1-(\frac{\lambda_2-(\theta+\vartheta-\frac{x+y}{2})}{\frac{y-x}{2}})^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}-1}\left(\frac{\lambda_2-(\theta+\vartheta-\frac{x+y}{2})}{\frac{y-x}{2}}\right)^{\alpha-1}\tau(\lambda_2)\frac{2}{y-x}\,d\lambda_2$$

$$=\left(\frac{2}{y-x}\right)^{\alpha\frac{\beta}{k}}\left\{\Gamma_k(\beta)_k^{\beta}J^{\alpha}_{(\theta+\vartheta-\frac{x+y}{2})^+}\tau(\theta+\vartheta-y)+\Gamma_k(\beta)_k^{\beta}J^{\alpha}_{(\theta+\vartheta-\frac{x+y}{2})^-}\tau(\theta+\vartheta-x)\right\}.$$

Note that

$$\int_0^1\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}-1}(1-\lambda)^{\alpha-1}\,d\lambda=\frac{1}{\frac{\beta}{k}\alpha^{\frac{\beta}{k}}}. \tag{2.3}$$

Therefore,

$$2\tau\left(\theta+\vartheta-\frac{x+y}{2}\right)\frac{1}{\frac{\beta}{k}\alpha^{\frac{\beta}{k}}}$$

$$\le\left(\frac{2}{y-x}\right)^{\alpha\frac{\beta}{k}}\left\{\Gamma_k(\beta)_k^{\beta}J^{\alpha}_{(\theta+\vartheta-\frac{x+y}{2})^+}\tau(\theta+\vartheta-y)+\Gamma_k(\beta)_k^{\beta}J^{\alpha}_{(\theta+\vartheta-\frac{x+y}{2})^-}\tau(\theta+\vartheta-x)\right\}.$$

This completes the proof of the first inequality of (2.1).

To prove the second inequality, by a similar discussion, making use of the convexity of τ, for $\lambda\in[0,1]$, we have

$$\tau\left(\theta+\vartheta-\left(\frac{\lambda}{2}x+\frac{2-\lambda}{2}y\right)\right)\le\tau(\theta)+\tau(\vartheta)-\left(\frac{\lambda}{2}\tau(x)+\frac{2-\lambda}{2}\tau(y)\right) \tag{2.4}$$

and

$$\tau\left(\theta+\vartheta-\left(\frac{2-\lambda}{2}x+\frac{\lambda}{2}y\right)\right)\le\tau(\theta)+\tau(\vartheta)-\left(\frac{2-\lambda}{2}\tau(x)+\frac{\lambda}{2}\tau(y)\right). \tag{2.5}$$

Adding (2.4) and (2.5) leads to

$$\tau\left(\theta+\vartheta-\left(\frac{\lambda}{2}x+\frac{2-\lambda}{2}y\right)\right)+\tau\left(\theta+\vartheta-\left(\frac{2-\lambda}{2}x+\frac{\lambda}{2}y\right)\right)$$

$$\le 2\big[\tau(\theta)+\tau(\vartheta)\big]-\big[\tau(x)+\tau(y)\big]. \tag{2.6}$$

Multiplying (2.6) by $(\frac{1-(1-\lambda)^{\alpha}}{\alpha})^{\frac{\beta}{k}-1}(1-\lambda)^{\alpha-1}$ and integrating the obtained inequality with respect to λ over $[0,1]$ gives

$$\int_0^1\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}-1}(1-\lambda)^{\alpha-1}$$

$$\times\left\{\tau\left(\theta+\vartheta-\left(\frac{\lambda}{2}x+\frac{2-\lambda}{2}y\right)\right)+\tau\left(\theta+\vartheta-\left(\frac{2-\lambda}{2}x+\frac{\lambda}{2}y\right)\right)\right\}d\lambda$$

$$\le\left\{2\big[\tau(\theta)+\tau(\vartheta)\big]-\big[\tau(x)+\tau(y)\big]d\lambda\right\}\int_0^1\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}-1}(1-\lambda)^{\alpha-1}\,d\lambda$$

$$\times \left(\frac{2}{y-x}\right)^{\alpha^{\frac{\beta}{k}}} \left\{\Gamma_k(\beta)\,{}^{\beta}_{k}J^{\alpha}_{(\theta+\vartheta-\frac{x+y}{2})^+}\tau(\theta+\vartheta-y)+\Gamma_k(\beta)\,{}^{\beta}_{k}J^{\alpha}_{(\theta+\vartheta-\frac{x+y}{2})^-}\tau(\theta+\vartheta-x)\right\}$$

$$\leq \frac{1}{\frac{\beta}{k}\alpha^{\frac{\beta}{k}}}\left\{2\left[\tau(\theta)+\tau(\vartheta)\right]-\left[\tau(x)+\tau(y)\right]\right\},$$

which completes the proof of the desired inequality. □

Remark 2.2 From Theorem 2.1, we clearly see that:

(i) If we take $k=1$, $x=\theta$, and $y=\vartheta$ in Theorem 2.1, then we get Theorem 2.1 of [65].

(ii) If we take $\alpha=k=1$, $x=\theta$, and $y=\vartheta$ in Theorem 2.1, then we get Theorem 2 of [66].

Theorem 2.3 *Let $\alpha,\beta>0$ and $\tau:[\theta,\vartheta]\to R$ be a convex function. Then the inequalities*

$$\tau\left(\theta+\vartheta-\frac{x+y}{2}\right)\leq \tau(\theta)+\tau(\vartheta)-\frac{\alpha^{\frac{\beta}{k}}\Gamma_k(\beta+k)}{2(y-x)^{\alpha^{\frac{\beta}{k}}}}\left\{{}^{\beta}_{k}J^{\alpha}_{x^+}\tau(y)+{}^{\beta}_{k}J^{\alpha}_{y^-}\tau(x)\right\}$$

$$\leq \tau(\theta)+\tau(\vartheta)-\tau\left(\frac{x+y}{2}\right) \tag{2.7}$$

and

$$\tau\left(\theta+\vartheta-\frac{x+y}{2}\right)\leq \frac{\alpha^{\frac{\beta}{k}}\Gamma_k(\beta+k)}{2(y-x)^{\alpha^{\frac{\beta}{k}}}}\left\{{}^{\beta}_{k}J^{\alpha}_{(\theta+\vartheta-x)^+}\tau(\theta+\vartheta-y)+{}^{\beta}_{k}J^{\alpha}_{(\theta+\vartheta-y)^-}\tau(\theta+\vartheta-x)\right\}$$

$$\leq \frac{\tau(\theta+\vartheta-x)+\tau(\theta+\vartheta-y)}{2}\leq \tau(\theta)+\tau(\vartheta)-\frac{\tau(x)+\tau(y)}{2} \tag{2.8}$$

hold for all $x,y\in[\theta,\vartheta]$.

Proof It follows from the Jensen–Mercer inequality that

$$\tau\left(\theta+\vartheta-\frac{x_1+y_1}{2}\right)\leq \tau(\theta)+\tau(\vartheta)-\frac{\tau(x_1)+\tau(y_1)}{2} \tag{2.9}$$

for all $x_1,y_1\in[\theta,\vartheta]$.

By changing the variables $x_1=\lambda x+(1-\lambda)y$ and $y_1=(1-\lambda)x+\lambda y$ for $x,y\in[\theta,\vartheta]$ and $\lambda\in[0,1]$ in (2.9), we get

$$\tau\left(\theta+\vartheta-\frac{x+y}{2}\right)\leq \tau(\theta)+\tau(\vartheta)-\frac{\tau(\lambda x+(1-\lambda)y)+\tau((1-\lambda)x+\lambda y)}{2}. \tag{2.10}$$

Multiplying (2.10) by $(\frac{1-(1-\lambda)^{\alpha}}{\alpha})^{\frac{\beta}{k}-1}(1-\lambda)^{\alpha-1}$ and integrating the obtained inequality with respect to λ over $[0,1]$ leads to the conclusion that

$$\tau\left(\theta+\vartheta-\frac{x+y}{2}\right)\int_0^1\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}}(1-\lambda)^{\alpha-1}\,d\lambda$$

$$\leq \int_0^1\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}}(1-\lambda)^{\alpha-1}$$

$$\times\left\{\tau(\theta)+\tau(\vartheta)-\frac{\tau(\lambda x+(1-\lambda)y)+\tau((1-\lambda)x+\lambda y)}{2}\right\}d\lambda,$$

that is,

$$\tau\left(\theta+\vartheta-\frac{x+y}{2}\right)\leq\tau(\theta)+\tau(\vartheta)-\frac{\alpha^{\frac{\beta}{k}}\Gamma_k(\beta+k)}{2(y-x)^{\alpha\frac{\beta}{k}}}\left\{{}_k^{\beta}J_{x^+}^{\alpha}\tau(y)+{}_k^{\beta}J_{y^-}^{\alpha}\tau(x)\right\},\tag{2.11}$$

which completes the proof of the first inequality of (2.7).

To prove the second inequality of (2.7), from the convexity of τ, for $\lambda\in[0,1]$ we obtain

$$\begin{aligned}\tau\left(\frac{x+y}{2}\right)&=\tau\left(\frac{\lambda x+(1-\lambda)y+(1-\lambda)x+\lambda y}{2}\right)\\&\leq\frac{\tau(\lambda x+(1-\lambda)y)+\tau((1-\lambda)x+\lambda y)}{2}.\end{aligned}\tag{2.12}$$

Multiplying (2.12) by $(\frac{1-(1-\lambda)^{\alpha}}{\alpha})^{\frac{\beta}{k}-1}(1-\lambda)^{\alpha-1}$ and then by using integration with respect to λ over $[0,1]$, we have

$$\begin{aligned}&\tau\left(\frac{x+y}{2}\right)\int_0^1\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}}(1-\lambda)^{\alpha-1}\,d\lambda\\&\leq\int_0^1\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}}(1-\lambda)^{\alpha-1}\left\{\frac{\tau(\lambda x+(1-\lambda)y)+\tau((1-\lambda)x+\lambda y)}{2}\right\}d\lambda,\end{aligned}$$

that is,

$$\begin{aligned}&\tau\left(\frac{x+y}{2}\right)\leq\frac{\alpha^{\frac{\beta}{k}}\Gamma_k(\beta+k)}{2(y-x)^{\alpha\frac{\beta}{k}}}\left\{{}_k^{\beta}J_{x^+}^{\alpha}\tau(y)+{}_k^{\beta}J_{y^-}^{\alpha}\tau(x)\right\},\\&-\tau\left(\frac{x+y}{2}\right)\geq-\frac{\alpha^{\frac{\beta}{k}}\Gamma_k(\beta+k)}{2(y-x)^{\alpha\frac{\beta}{k}}}\left\{{}_k^{\beta}J_{x^+}^{\alpha}\tau(y)+{}_k^{\beta}J_{y^-}^{\alpha}\tau(x)\right\}.\end{aligned}\tag{2.13}$$

Adding $\tau(\theta)+\tau(\vartheta)$ to both sides of (2.13), we obtain

$$\tau(\theta)+\tau(\vartheta)-\tau\left(\frac{x+y}{2}\right)\geq\tau(\theta)+\tau(\vartheta)-\frac{\alpha^{\frac{\beta}{k}}\Gamma_k(\beta+k)}{2(y-x)^{\alpha\frac{\beta}{k}}}\left\{{}_k^{\beta}J_{x^+}^{\alpha}\tau(y)+{}_k^{\beta}J_{y^-}^{\alpha}\tau(x)\right\}.\tag{2.14}$$

Combining (2.11) and (2.14), we get (2.7). To prove inequality (2.8), we use the convexity of τ to get

$$\begin{aligned}\tau\left(\theta+\vartheta-\frac{x_1+y_1}{2}\right)&=\tau\left(\frac{\theta+\vartheta-x_1+\theta+\vartheta-y_1}{2}\right)\\&\leq\frac{\tau(\theta+\vartheta-x_1)+\tau(\theta+\vartheta-y_1)}{2}\end{aligned}\tag{2.15}$$

for all $x_1,y_1\in[\theta,\vartheta]$.

Let $x_1=\lambda x+(1-\lambda)y$ and $y_1=(1-\lambda)x+\lambda y$. Then (2.15) leads to

$$\begin{aligned}&\tau\left(\theta+\vartheta-\frac{x+y}{2}\right)\\&\leq\left\{\frac{\tau(\theta+\vartheta-(\lambda x+(1-\lambda)y))+\tau(\theta+\vartheta-((1-\lambda)x+\lambda y))}{2}\right\}.\end{aligned}\tag{2.16}$$

Multiplying (2.16) by $(\frac{1-(1-\lambda)^{\alpha}}{\alpha})^{\frac{\beta}{k}-1}(1-\lambda)^{\alpha-1}$ and then by integrating the resulting inequality with respect to λ over $[0,1]$, we have

$$\begin{aligned}
&\tau\left(\theta+\vartheta-\frac{x+y}{2}\right)\int_0^1\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}}(1-\lambda)^{\alpha-1}\,d\lambda\\
&\quad\leq\int_0^1\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}}(1-\lambda)^{\alpha-1}\\
&\qquad\times\left\{\frac{\tau(\theta+\vartheta-(\lambda x+(1-\lambda)y))+\tau(\theta+\vartheta-((1-\lambda)x+\lambda y))}{2}\right\}d\lambda,
\end{aligned}$$

which can be rewritten as

$$\begin{aligned}
&\tau\left(\theta+\vartheta-\frac{x+y}{2}\right)\\
&\quad\leq\frac{\alpha^{\frac{\beta}{k}}\Gamma_k(\beta+k)}{2(y-x)^{\alpha\frac{\beta}{k}}}\left\{{}_k^{\beta}J^{\alpha}_{(\theta+\vartheta-x)^+}\tau(\theta+\vartheta-y)+{}_k^{\beta}J^{\alpha}_{(\theta+\vartheta-y)^-}\tau(\theta+\vartheta-x)\right\}.
\end{aligned}\tag{2.17}$$

It follows from the convexity of τ that

$$\tau\big(\lambda(\theta+\vartheta-x)+(1-\lambda)(\theta+\vartheta-y)\big)\leq\lambda\tau(\theta+\vartheta-x)+(1-\lambda)\tau(\theta+\vartheta-y)$$

and

$$\tau\big((1-\lambda)(\theta+\vartheta-x)+\lambda(\theta+\vartheta-y)\big)\leq(1-\lambda)\tau(\theta+\vartheta-x)+\lambda\tau(\theta+\vartheta-y).$$

Adding the above two inequalities and using the Jensen–Mercer inequality gives

$$\begin{aligned}
&\tau\big(\lambda(\theta+\vartheta-x)+(1-\lambda)(\theta+\vartheta-y)\big)+\tau\big((1-\lambda)(\theta+\vartheta-x)+\lambda(\theta+\vartheta-y)\big)\\
&\quad\leq\tau(\theta+\vartheta-x)+\tau(\theta+\vartheta-y)\leq2\big(\tau(\theta)+\tau(\vartheta)\big)-\big(\tau(x)+\tau(y)\big).
\end{aligned}\tag{2.18}$$

Multiplying (2.18) by $(\frac{1-(1-\lambda)^{\alpha}}{\alpha})^{\frac{\beta}{k}-1}(1-\lambda)^{\alpha-1}$ and then by using integration with respect to λ over $[0,1]$, we have

$$\begin{aligned}
&\int_0^1\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}}(1-\lambda)^{\alpha-1}\\
&\qquad\times\left\{\tau\big(\lambda(\theta+\vartheta-x)+(1-\lambda)(\theta+\vartheta-y)\big)+\tau\big((1-\lambda)(\theta+\vartheta-x)+\lambda(\theta+\vartheta-y)\big)\right\}d\lambda\\
&\quad\leq2\big(\tau(\theta)+\tau(\vartheta)\big)-\big(\tau(x)+\tau(y)\big)\int_0^1\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}}(1-\lambda)^{\alpha-1}\,d\lambda,
\end{aligned}$$

that is,

$$\begin{aligned}
&\frac{\alpha^{\frac{\beta}{k}}\Gamma_k(\beta+k)}{2(y-x)^{\alpha\frac{\beta}{k}}}\left\{{}_k^{\beta}J^{\alpha}_{(\theta+\vartheta-x)^+}\tau(\theta+\vartheta-y)+{}_k^{\beta}J^{\alpha}_{(\theta+\vartheta-y)^-}\tau(\theta+\vartheta-x)\right\}\\
&\quad\leq\big(\tau(\theta)+\tau(\vartheta)\big)-\left(\frac{\tau(x)+\tau(y)}{2}\right).
\end{aligned}\tag{2.19}$$

Combining (2.17) and (2.19) leads to (2.8). □

Remark 2.4 Let $\alpha = \beta = k = 1$. Then Theorem 2.3 leads to the conclusion that

$$\begin{aligned}\text{(i)}\quad \tau\left(\theta+\vartheta-\frac{x+y}{2}\right) &\le \tau(\theta)+\tau(\vartheta)-\int_0^1 \tau\big(\lambda x+(1-\lambda)y\big)\,d\lambda \\ &\le \tau(\theta)+\tau(\vartheta)-\tau\left(\frac{x+y}{2}\right)\end{aligned}$$

and

$$\begin{aligned}\text{(ii)}\quad \tau\left(\theta+\vartheta-\frac{x+y}{2}\right) &\le \frac{1}{y-x}\int_x^y \tau(\theta+\vartheta-\lambda)\,d\lambda \\ &\le \tau(\theta)+\tau(\vartheta)-\frac{\tau(x)+\tau(y)}{2},\end{aligned}$$

which was also proved in Theorem 2.1 of [67].

Lemma 2.5 *Let $\alpha,\beta>0$, $\theta<\vartheta$ and $\tau:[\theta,\vartheta]\to R$ be a differentiable mapping such that $\tau'\in L[\theta,\vartheta]$. Then the inequality*

$$\begin{aligned}&\frac{2^{\alpha\frac{\beta}{k}-1}\alpha^{\frac{\beta}{k}}\Gamma_k(\beta+k)}{(y-x)^{\alpha\frac{\beta}{k}}}\left\{{}_k^{\beta}J^{\alpha}_{(\theta+\vartheta-\frac{x+y}{2})^+}\tau(\theta+\vartheta-x)+{}_k^{\beta}J^{\alpha}_{(\theta+\vartheta-\frac{x+y}{2})^-}\tau(\theta+\vartheta-y)\right\}\\ &\quad-\tau\left(\theta+\vartheta-\frac{x+y}{2}\right)\\ &=\frac{(y-x)\alpha^{\frac{\beta}{k}}}{4}\int_0^1\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}}\\ &\quad\times\left\{\tau'\left(\theta+\vartheta-\left(\frac{2-\lambda}{2}x+\frac{\lambda}{2}y\right)\right)-\tau'\left(\theta+\vartheta-\left(\frac{\lambda}{2}x+\frac{2-\lambda}{2}y\right)\right)\right\}d\lambda\end{aligned}\tag{2.20}$$

holds for all $x,y\in[\theta,\vartheta]$.

Proof Let

$$I=\frac{y-x}{4}\alpha^{\frac{\beta}{k}}\{I_1-I_2\},\tag{2.21}$$

where

$$I_1=\int_0^1\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}}\tau'\left(\theta+\vartheta-\left(\frac{2-\lambda}{2}x+\frac{\lambda}{2}y\right)\right)d\lambda$$

and

$$I_2=\int_0^1\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}}\tau'\left(\theta+\vartheta-\left(\frac{\lambda}{2}x+\frac{2-\lambda}{2}y\right)\right)d\lambda.$$

Then integrating by parts, we get

$$
\begin{aligned}
&= -\frac{2}{\alpha^{\frac{\beta}{k}}(y-x)}\tau\left(\theta+\vartheta-\frac{x+y}{2}\right)\\
&\quad+\frac{2\alpha^{\frac{\beta}{k}}}{\alpha^{\frac{\beta}{k}}(y-x)}\int_0^1\left(1-(1-\lambda)^{\alpha}\right)^{\frac{\beta}{k}-1}(1-\lambda)^{\alpha-1}\tau\left(\theta+\vartheta-\left(\frac{2-\lambda}{2}x+\frac{\lambda}{2}y\right)\right)d\lambda\\
&= -\frac{2}{\alpha^{\frac{\beta}{k}}(y-x)}\tau\left(\theta+\vartheta-\frac{x+y}{2}\right)\\
&\quad+\frac{2^{\alpha\frac{\beta}{k}}\frac{\beta}{k}}{\alpha^{\frac{\beta}{k}-1}(y-x)^{\alpha\frac{\beta}{k}+1}}\int_{\theta+\vartheta-\frac{x+y}{2}}^{\theta+\vartheta-x}\left(\left(\frac{y-x}{2}\right)^{\alpha}-\left(\lambda_1-\left(\theta+\vartheta-\frac{x+y}{2}\right)\right)^{\alpha}\right)^{\frac{\beta}{k}-1}\\
&\quad\times\frac{\tau(\lambda_1)}{(\lambda_1-(\theta+\vartheta-\frac{x+y}{2}))^{1-\alpha}}d\lambda_1\\
&= -\frac{2}{\alpha^{\frac{\beta}{k}}(y-x)}\tau\left(\theta+\vartheta-\frac{x+y}{2}\right)\\
&\quad+\left(\frac{2}{y-x}\right)^{\alpha\frac{\beta}{k}+1}\frac{\Gamma_k(\beta+k)}{\alpha^{\frac{\beta}{k}-1}}{}_k^{\beta}J^{\alpha}_{(\theta+\vartheta-\frac{x+y}{2})^+}\tau(\theta+\vartheta-x).
\end{aligned}
\tag{2.22}
$$

Similarly, we have

$$
\begin{aligned}
I_2 &= \int_0^1\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}}\tau'\left(\theta+\vartheta-\left(\frac{\lambda}{2}x+\frac{2-\lambda}{2}y\right)\right)d\lambda\\
&= \frac{2}{\alpha^{\frac{\beta}{k}}(y-x)}\tau\left(\theta+\vartheta-\frac{x+y}{2}\right)\\
&\quad-\left(\frac{2}{y-x}\right)^{\alpha\frac{\beta}{k}+1}\frac{\Gamma_k(\beta+k)}{\alpha^{\frac{\beta}{k}-1}}{}_k^{\beta}J^{\alpha}_{(\theta+\vartheta-\frac{x+y}{2})^-}\tau(\theta+\vartheta-y).
\end{aligned}
\tag{2.23}
$$

Therefore, inequality (2.20) follows from (2.21)–(2.23). □

Remark 2.6 Lemma 2.5 leads to the conclusion that:

(i) If we take $k=1$, $x=\theta$, and $y=\vartheta$, then we can get Lemma 3.1 of [65].

(ii) If we take $\alpha=k=1$, $x=\theta$, and $y=\vartheta$, then Lemma 2.5 reduces to Lemma 1.1 of [68].

Lemma 2.7 *Let $\alpha,\beta>0$, $\theta<\vartheta$ and $\tau:[\theta,\vartheta]\to R$ be a differentiable mapping such that $\tau'\in L[\theta,\vartheta]$. Then the identity*

$$
\begin{aligned}
&\frac{\tau(\theta+\vartheta-x)+\tau(\theta+\vartheta-y)}{2}-\frac{\alpha^{\frac{\beta}{k}}\Gamma_k(\beta+k)}{2(y-x)^{\alpha\frac{\beta}{k}}}\\
&\quad\times\left\{{}_k^{\beta}J^{\alpha}_{(\theta+\vartheta-y)^+}\tau(\theta+\vartheta-x)+{}_k^{\beta}J^{\alpha}_{(\theta+\vartheta-x)^-}\tau(\theta+\vartheta-y)\right\}\\
&= \frac{(y-x)\alpha^{\frac{\beta}{k}}}{2}\int_0^1\left[\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}}-\left(\frac{1-\lambda^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}}\right]\\
&\quad\times\tau'\big(\theta+\vartheta-\big(\lambda x+(1-\lambda)y\big)\big)\,d\lambda
\end{aligned}
\tag{2.24}
$$

holds for all $x,y\in[\theta,\vartheta]$.

Proof Let

$$I = \frac{(y-x)\alpha^{\frac{\beta}{k}}}{2}\int_0^1 \left[\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}} - \left(\frac{1-\lambda^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}}\right]\tau'\big(\theta+\vartheta-\big(\lambda x+(1-\lambda)y\big)\big)\,d\lambda.$$
$$= \frac{(y-x)\alpha^{\frac{\beta}{k}}}{2}\{I_1 - I_2\}. \tag{2.25}$$

Then we clearly see that

$$\begin{aligned} I_1 &= \int_0^1 \left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}} \tau'\big(\theta+\vartheta-\big(\lambda x+(1-\lambda)y\big)\big)\,d\lambda \\ &= \frac{1}{\alpha^{\frac{\beta}{k}}}\frac{\tau(\theta+\vartheta-x)}{y-x} - \frac{\frac{\beta}{k}}{y-x}\int_0^1 \left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}-1} \\ &\quad \times (1-\lambda)^{\alpha-1}\tau\big(\theta+\vartheta-\big(\lambda x+(1-\lambda)y\big)\big)\,d\lambda \\ &= \frac{1}{\alpha^{\frac{\beta}{k}}}\frac{\tau(\theta+\vartheta-x)}{y-x} - \frac{\Gamma_k(\beta+k)}{(y-x)^{\alpha\frac{\beta}{k}+1}}\big\{{}_k^{\beta}J^{\alpha}_{(\theta+\vartheta-x)^+}\tau(\theta+\vartheta-y)\big\} \end{aligned} \tag{2.26}$$

and

$$\begin{aligned} I_2 &= \int_0^1 \left(\frac{1-\lambda^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}} \tau'\big(\theta+\vartheta-\big(\lambda x+(1-\lambda)y\big)\big)\,d\lambda \\ &= -\frac{1}{\alpha^{\frac{\beta}{k}}}\frac{\tau(\theta+\vartheta-y)}{y-x} + \frac{\frac{\beta}{k}}{y-x}\int_0^1 \left(\frac{1-\lambda^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}-1} \lambda^{\alpha-1}\tau\big(\theta+\vartheta-\big(\lambda x+(1-\lambda)y\big)\big)\,d\lambda \\ &= -\frac{1}{\alpha^{\frac{\beta}{k}}}\frac{\tau(\theta+\vartheta-y)}{y-x} + \frac{\Gamma_k(\beta+k)}{(y-x)^{\alpha\frac{\beta}{k}+1}}\big\{{}_k^{\beta}J^{\alpha}_{(\theta+\vartheta-y)^-}\tau(\theta+\vartheta-x)\big\}. \end{aligned} \tag{2.27}$$

Therefore, identity (2.24) follows from (2.25)–(2.27). □

Corollary 2.8 *If we take* $\alpha = \beta = k = 1$, *then Lemma* 2.7 *leads to the equality*

$$\frac{\tau(\theta+\vartheta-x)+\tau(\theta+\vartheta-y)}{2} - \frac{1}{y-x}\int_{\theta+\vartheta-y}^{\theta+\vartheta-x}\tau(\lambda)\,d\lambda$$
$$= \frac{y-x}{2}\int_0^1 (2\lambda-1)\tau'\big(\theta+\vartheta-\big(\lambda x+(1-\lambda)y\big)\big)\,d\lambda. \tag{2.28}$$

Remark 2.9 If we take $x = \theta$ and $y = \vartheta$ in Corollary 2.8, then equality (2.28) becomes the equality

$$\frac{\tau(\theta)+\tau(\vartheta)}{2} - \frac{1}{\vartheta-\theta}\int_{\theta}^{\vartheta}\tau(\lambda)\,d\lambda = \frac{\vartheta-\theta}{2}\int_0^1 (2\lambda-1)\tau'\big((1-\lambda)\theta+\lambda\vartheta\big)\,d\lambda,$$

which was proved in Lemma 2.1 of [69].

Theorem 2.10 *Let $\alpha,\beta>0$, $\theta<\vartheta$ and $\tau:[\theta,\vartheta]\to R$ be a differentiable mapping such that $\tau'\in L[\theta,\vartheta]$ and $|\tau'|$ is a convex mapping on $[\theta,\vartheta]$. Then the inequality*

$$\begin{aligned}
&\left|\frac{2^{\alpha\frac{\beta}{k}-1}\alpha^{\frac{\beta}{k}}\Gamma_k(\beta+k)}{(y-x)^{\alpha\frac{\beta}{k}}}\left\{{}_k^{\beta}J^{\alpha}_{(\theta+\vartheta-\frac{x+y}{2})^+}\tau(\theta+\vartheta-x)+{}_k^{\beta}J^{\alpha}_{(\theta+\vartheta-\frac{x+y}{2})^-}\tau(\theta+\vartheta-y)\right\}\right.\\
&\quad\left.-\tau\left(\theta+\vartheta-\frac{x+y}{2}\right)\right|\\
&\le\frac{y-x}{4}\alpha^{\frac{\beta}{k}}\left[\left(\left|\tau'(\theta)\right|+\left|\tau'(\vartheta)\right|\right)\left(\frac{1}{\alpha^{\frac{\beta}{k}+1}}B\left(\frac{\beta}{k}+1,\frac{1}{\alpha}\right)\right)\right.\\
&\quad-\left\{\left|\tau'(x)\right|\left(\frac{1}{2\alpha^{\frac{\beta}{k}+1}}\left(B\left(\frac{\beta}{k}+1,\frac{2}{\alpha}\right)+B\left(\frac{\beta}{k}+1,\frac{1}{\alpha}\right)\right)\right)\right\}\\
&\quad+\left\{\left|\tau'(y)\right|\left(\frac{1}{2\alpha^{\frac{\beta}{k}+1}}\left(B\left(\frac{\beta}{k}+1,\frac{2}{\alpha}\right)-B\left(\frac{\beta}{k}+1,\frac{1}{\alpha}\right)\right)\right)\right\}\\
&\quad+\left(\left|\tau'(\theta)\right|+\left|\tau'(\vartheta)\right|\right)\left(\frac{1}{\alpha^{\frac{\beta}{k}+1}}B\left(\frac{\beta}{k}+1,\frac{1}{\alpha}\right)\right)\\
&\quad-\left\{\left|\tau'(x)\right|\left(\frac{1}{2\alpha^{\frac{\beta}{k}+1}}\left(B\left(\frac{\beta}{k}+1,\frac{1}{\alpha}\right)-B\left(\frac{\beta}{k}+1,\frac{2}{\alpha}\right)\right)\right)\right\}\\
&\quad\left.+\left\{\left|\tau'(y)\right|\left(\frac{1}{2\alpha^{\frac{\beta}{k}+1}}\left(B\left(\frac{\beta}{k}+1,\frac{2}{\alpha}\right)+B\left(\frac{\beta}{k}+1,\frac{1}{\alpha}\right)\right)\right)\right\}\right],
\end{aligned}\tag{2.29}$$

holds for all $x,y\in[\theta,\vartheta]$.

Proof It follows from Lemma 2.5 and Jensen–Mercer inequality using the convexity of $|\tau'|$ that

$$\begin{aligned}
&\left|\frac{2^{\alpha\frac{\beta}{k}-1}\alpha^{\frac{\beta}{k}}\Gamma_k(\beta+k)}{(y-x)^{\alpha\frac{\beta}{k}}}\left\{{}_k^{\beta}J^{\alpha}_{(\theta+\vartheta-\frac{x+y}{2})^+}\tau(\theta+\vartheta-x)+{}_k^{\beta}J^{\alpha}_{(\theta+\vartheta-\frac{x+y}{2})^-}\tau(\theta+\vartheta-y)\right\}\right.\\
&\quad\left.-\tau\left(\theta+\vartheta-\frac{x+y}{2}\right)\right|\\
&\le\frac{y-x}{4}\alpha^{\frac{\beta}{k}}\left\{\int_0^1\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}}\left|\tau'\left(\theta+\vartheta-\left(\frac{2-\lambda}{2}x+\frac{\lambda}{2}y\right)\right)\right|d\lambda\right.\\
&\quad\left.+\int_0^1\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}}\left|\tau'\left(\theta+\vartheta-\left(\frac{\lambda}{2}x+\frac{2-\lambda}{2}y\right)\right)\right|d\lambda\right\}\\
&\le\frac{y-x}{4}\alpha^{\frac{\beta}{k}}\left\{\int_0^1\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}}\left\{\left|\tau'(\theta)\right|+\left|\tau'(\vartheta)\right|-\left(\frac{2-\lambda}{2}\left|\tau'(x)\right|+\frac{\lambda}{2}\left|\tau'(y)\right|\right)\right\}d\lambda\right.\\
&\quad\left.+\int_0^1\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}}\left\{\left|\tau'(\theta)\right|+\left|\tau'(\vartheta)\right|-\left(\frac{\lambda}{2}\left|\tau'(x)\right|+\frac{2-\lambda}{2}\left|\tau'(y)\right|\right)\right\}d\lambda\right\}\\
&\le\frac{y-x}{4}\alpha^{\frac{\beta}{k}}\times\left\{\left(\left|\tau'(\theta)\right|+\left|\tau'(\vartheta)\right|\right)\int_0^1\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}}d\lambda\right.\\
&\quad\left.-\left(\left|\tau'(x)\right|\int_0^1\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}}\frac{2-\lambda}{2}d\lambda+\left|\tau'(y)\right|\int_0^1\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}}\frac{\lambda}{2}d\lambda\right)\right\}
\end{aligned}$$

$$+\left\{\left(\left|\tau'(\theta)\right|+\left|\tau'(\vartheta)\right|\right)\int_0^1\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}}d\lambda$$
$$-\left(\left|\tau'(x)\right|\int_0^1\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}}\frac{\lambda}{2}\,d\lambda+\left|\tau'(y)\right|\int_0^1\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}}\frac{2-\lambda}{2}\,d\lambda\right)\right\}.$$

Therefore, inequality (2.29) can be derived after some simple calculations. □

Remark 2.11 From Theorem 2.10 we clearly see that:

(i) If we take $k=1$, $x=\theta$, and $y=\vartheta$ in Theorem 2.10, then we get Theorem 3.1 of [65].

(ii) If we take $\alpha=k=1$, $x=\theta$, and $y=\vartheta$ in Theorem 2.10, then we obtain Theorem 5 of [68] in the case of $q=1$.

Theorem 2.12 *Let $q>1$, $\alpha,\beta>0$, $\theta<\vartheta$ and $\tau:[\theta,\vartheta]\to R$ be a differentiable mapping such that $\tau'\in L[\theta,\vartheta]$ and $|\tau'|^q$ is a convex mapping on $[\theta,\vartheta]$. Then the inequality*

$$\begin{aligned}
&\left|\frac{2^{\alpha\frac{\beta}{k}-1}\alpha^{\frac{\beta}{k}}\Gamma_k(\beta+k)}{(y-x)^{\alpha\frac{\beta}{k}}}\left\{{}_k^{\beta}J^{\alpha}_{(\theta+\vartheta-\frac{x+y}{2})^+}\tau(\theta+\vartheta-x)+{}_k^{\beta}J^{\alpha}_{(\theta+\vartheta-\frac{x+y}{2})^-}\tau(\theta+\vartheta-y)\right\}\right.\\
&\quad\left.-\tau\left(\theta+\vartheta-\frac{x+y}{2}\right)\right|\\
&\le\frac{y-x}{4}\alpha^{\frac{\beta}{k}}\left(\frac{1}{\alpha^{\frac{\beta}{k}+1}}B\left(\frac{\beta}{k}+1,\frac{1}{\alpha}\right)\right)^{1-\frac{1}{q}}\\
&\quad\times\left\{\left(\left|\tau'(\theta)\right|^q+\left|\tau'(\vartheta)\right|^q\right)\left(\left(\frac{1}{\alpha^{\frac{\beta}{k}+1}}B\left(\frac{\beta}{k}+1,\frac{1}{\alpha}\right)\right)\right)\right.\\
&\quad-\left\{\left|\tau'(x)\right|^q\left(\frac{1}{2\alpha^{\frac{\beta}{k}+1}}\left(B\left(\frac{\beta}{k}+1,\frac{1}{\alpha}\right)-B\left(\frac{\beta}{k}+1,\frac{2}{\alpha}\right)\right)\right)\right.\\
&\quad\left.\left.+\left|\tau'(y)\right|^q\left(\frac{1}{2\alpha^{\frac{\beta}{k}+1}}\left(B\left(\frac{\beta}{k}+1,\frac{2}{\alpha}\right)+B\left(\frac{\beta}{k}+1,\frac{1}{\alpha}\right)\right)\right)\right\}\right.\\
&\quad+\left(\left|\tau'(\theta)\right|^q+\left|\tau'(\vartheta)\right|^q\right)\left(\left(\frac{1}{\alpha^{\frac{\beta}{k}+1}}B\left(\frac{\beta}{k}+1,\frac{1}{\alpha}\right)\right)\right)\\
&\quad-\left\{\left|\tau'(x)\right|^q\left(\frac{1}{2\alpha^{\frac{\beta}{k}+1}}\left(B\left(\frac{\beta}{k}+1,\frac{2}{\alpha}\right)-B\left(\frac{\beta}{k}+1,\frac{1}{\alpha}\right)\right)\right)\right.\\
&\quad\left.\left.+\left|\tau'(y)\right|^q\left(\frac{1}{2\alpha^{\frac{\beta}{k}+1}}\left(B\left(\frac{\beta}{k}+1,\frac{1}{\alpha}\right)+B\left(\frac{\beta}{k}+1,\frac{2}{\alpha}\right)\right)\right)\right\}\right\}^{\frac{1}{q}}
\end{aligned}\tag{2.30}$$

holds for all $x,y\in[\theta,\vartheta]$.

Proof It follows from Lemma 2.5, Jensen–Mercer inequality, power-mean inequality, and the convexity of function $|\tau'|^q$ that

$$\begin{aligned}
&\left|\frac{2^{\alpha\frac{\beta}{k}-1}\alpha^{\frac{\beta}{k}}\Gamma_k(\beta+k)}{(y-x)^{\alpha\frac{\beta}{k}}}\left\{{}_k^{\beta}J^{\alpha}_{(\theta+\vartheta-\frac{x+y}{2})^+}\tau(\theta+\vartheta-x)+{}_k^{\beta}J^{\alpha}_{(\theta+\vartheta-\frac{x+y}{2})^-}\tau(\theta+\vartheta-y)\right\}\right.\\
&\quad\left.-\tau\left(\theta+\vartheta-\frac{x+y}{2}\right)\right|
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{y-x}{4}\alpha^{\frac{\beta}{k}}\left\{\left(\int_0^1\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}}d\lambda\right)^{1-\frac{1}{q}}\right.\\
&\quad\times\left(\int_0^1\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}}\left|\tau'\left(\theta+\vartheta-\left(\frac{\lambda}{2}x+\frac{2-\lambda}{2}y\right)\right)\right|^{q}d\lambda\right)^{\frac{1}{q}}\\
&\quad+\left(\int_0^1\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}}d\lambda\right)^{1-\frac{1}{q}}\\
&\quad\times\left.\left(\int_0^1\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}}\left|\tau'\left(\theta+\vartheta-\left(\frac{2-\lambda}{2}x+\frac{\lambda}{2}y\right)\right)\right|^{q}d\lambda\right)^{\frac{1}{q}}\right\}\\
&\leq \frac{y-x}{4}\alpha^{\frac{\beta}{k}}\left\{\left(\int_0^1\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}}d\lambda\right)^{1-\frac{1}{q}}\left(\int_0^1\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}}\right.\right.\\
&\quad\times\left.\left(\left(\left|\tau'(\theta)\right|^{q}+\left|\tau'(\vartheta)\right|^{q}\right)-\left(\frac{\lambda}{2}\left|\tau'(x)\right|^{q}+\frac{2-\lambda}{2}\left|\tau'(y)\right|^{q}\right)\right)d\lambda\right)^{\frac{1}{q}}\\
&\quad+\left(\int_0^1\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}}d\lambda\right)^{1-\frac{1}{q}}\left(\int_0^1\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}}\right.\\
&\quad\times\left.\left.\left(\left(\left|\tau'(\theta)\right|^{q}+\left|\tau'(\vartheta)\right|^{q}\right)-\left(\frac{2-\lambda}{2}\left|\tau'(x)\right|^{q}+\frac{\lambda}{2}\left|\tau'(y)\right|^{q}\right)\right)d\lambda\right)^{\frac{1}{q}}\right\}. \qquad (2.31)
\end{aligned}
$$

Making simple simplifications, we get (2.30) from (2.31). □

Remark 2.13 Theorem 2.12 leads to the conclusion that:

(i) If we take $k=1$, $x=\theta$, and $y=\vartheta$ in Theorem 2.12, then we get Theorem 3.2 of [65].

(ii) Let $\alpha=k=1$, $x=\theta$, and $y=\vartheta$, then Theorem 2.12 reduces to Theorem 5 of [68].

Theorem 2.14 *Let $\alpha,\beta>0$, $p,q>1$ with $1/p+1/q=1$, $\theta<\vartheta$ and $\tau:[\theta,\vartheta]\to R$ be a differentiable mapping such that $\tau'\in L[\theta,\vartheta]$ and $|\tau'|^{q}$ is a convex mapping on $[\theta,\vartheta]$. Then one has*

$$
\begin{aligned}
&\left|\frac{2^{\alpha\frac{\beta}{k}-1}\alpha^{\frac{\beta}{k}}\Gamma_k(\beta+k)}{(y-x)^{\alpha\frac{\beta}{k}}}\left\{{}_k^{\beta}J^{\alpha}_{(\theta+\vartheta-\frac{x+y}{2})^{+}}\tau(\theta+\vartheta-x)+{}_k^{\beta}J^{\alpha}_{(\theta+\vartheta-\frac{x+y}{2})^{-}}\tau(\theta+\vartheta-y)\right\}\right.\\
&\quad\left.-\tau\left(\theta+\vartheta-\frac{x+y}{2}\right)\right|\\
&\leq \frac{y-x}{4}\alpha^{\frac{\beta}{k}}\left(\frac{1}{\alpha^{\frac{\beta}{k}p+1}}B\left(\frac{\beta}{k}p+1,\frac{1}{\alpha}\right)\right)^{\frac{1}{p}}\\
&\quad\times\left\{\left(\left|\tau'(\theta)\right|^{q}+\left|\tau'(\vartheta)\right|^{q}-\left(\frac{3|\tau'(x)|^{q}+|\tau'(y)|^{q}}{4}\right)\right)^{\frac{1}{q}}\right.\\
&\quad+\left.\left(\left|\tau'(\theta)\right|^{q}+\left|\tau'(\vartheta)\right|^{q}-\left(\frac{|\tau'(x)|^{q}+3|\tau'(y)|^{q}}{4}\right)\right)^{\frac{1}{q}}\right\} \qquad (2.32)
\end{aligned}
$$

for all $x,y\in[\theta,\vartheta]$.

Proof By using Lemma 2.5, and the Jensen–Mercer and Hölder integral inequalities, we obtain

$$\begin{aligned}
&\left|\frac{2^{\alpha\frac{\beta}{k}-1}\alpha^{\frac{\beta}{k}}\Gamma_k(\beta+k)}{(y-x)^{\alpha\frac{\beta}{k}}}\left\{{}_k^{\beta}J^{\alpha}_{(\theta+\vartheta-\frac{x+y}{2})^+}\tau(\theta+\vartheta-x)+{}_k^{\beta}J^{\alpha}_{(\theta+\vartheta-\frac{x+y}{2})^-}\tau(\theta+\vartheta-y)\right\}\right.\\
&\quad\left.-\tau\left(\theta+\vartheta-\frac{x+y}{2}\right)\right|\\
&\le\frac{y-x}{4}\alpha^{\frac{\beta}{k}}\left(\int_0^1\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}p}d\lambda\right)^{\frac{1}{p}}\\
&\quad\times\left\{\left(\int_0^1\left|\tau'\left(\theta+\vartheta-\left(\frac{2-\lambda}{2}x+\frac{\lambda}{2}y\right)\right)\right|^q\right)^{\frac{1}{q}}d\lambda\right.\\
&\quad\left.+\left(\int_0^1\left|\tau'\left(\theta+\vartheta-\left(\frac{\lambda}{2}x+\frac{2-\lambda}{2}y\right)\right)\right|^q\right)^{\frac{1}{q}}d\lambda\right\}.
\end{aligned}$$

It follows from the convexity of $|\tau'|^q$ that

$$\begin{aligned}
&\left|\tau'\left(\theta+\vartheta-\left(\frac{2-\lambda}{2}x+\frac{\lambda}{2}y\right)\right)\right|^q\\
&\le\left|\tau'(\theta)\right|^q+\left|\tau'(\vartheta)\right|^q-\left(\frac{2-\lambda}{2}\left|\tau'(x)\right|^q+\frac{\lambda}{2}\left|\tau'(y)\right|^q\right)\\
&\le\frac{y-x}{4}\alpha^{\frac{\beta}{k}}\left(\int_0^1\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}p}d\lambda\right)^{\frac{1}{p}}\\
&\quad\times\left\{\left(\int_0^1\left(\left|\tau'(\theta)\right|^q+\left|\tau'(\vartheta)\right|^q-\left(\frac{2-\lambda}{2}\left|\tau'(x)\right|^q+\frac{\lambda}{2}\left|\tau'(y)\right|^q\right)\right)\right)^{\frac{1}{q}}d\lambda\right.\\
&\quad\left.+\left(\int_0^1\left(\left|\tau'(\theta)\right|^q+\left|\tau'(\vartheta)\right|^q-\left(\frac{\lambda}{2}\left|\tau'(x)\right|^q+\frac{2-\lambda}{2}\left|\tau'(y)\right|^q\right)\right)\right)^{\frac{1}{q}}d\lambda\right\}\\
&\le\frac{y-x}{4}\alpha^{\frac{\beta}{k}}\left(\frac{1}{\alpha^{\frac{\beta}{k}p+1}}B\left(\frac{\beta}{k}p+1,\frac{1}{\alpha}\right)\right)^{\frac{1}{p}}\\
&\quad\times\left\{\left(\left|\tau'(\theta)\right|^q+\left|\tau'(\vartheta)\right|^q-\left(\frac{3|\tau'(x)|^q+|\tau'(y)|^q}{4}\right)\right)^{\frac{1}{q}}\right.\\
&\quad\left.+\left(\left|\tau'(\theta)\right|^q+\left|\tau'(\vartheta)\right|^q-\left(\frac{|\tau'(x)|^q+3|\tau'(y)|^q}{4}\right)\right)^{\frac{1}{q}}\right\},
\end{aligned}$$

which completes the proof. □

Corollary 2.15 *Let $\alpha = k = 1$. Then Theorem 2.14 leads to*

$$\begin{aligned}
&\left|\frac{1}{y-x}\int_{\theta+\vartheta-y}^{\theta+\vartheta-x}\tau(\lambda)\,d\lambda-\tau\left(\theta+\vartheta-\frac{x+y}{2}\right)\right|\\
&\quad\le\frac{1}{2^{\frac{1}{p}}}\times\left\{\left(\left|\tau'(\theta)\right|^q+\left|\tau'(\vartheta)\right|^q-\left(\frac{3|\tau'(x)|^q+|\tau'(y)|^q}{4}\right)\right)^{\frac{1}{q}}\right.
\end{aligned}$$

$$+\left(\left|\tau'(\theta)\right|^q+\left|\tau'(\vartheta)\right|^q-\left(\frac{|\tau'(x)|^q+3|\tau'(y)|^q}{4}\right)\right)^{\frac{1}{q}}\Bigg\}.$$

Theorem 2.16 *Let $\alpha,\beta>0$, $p,q>1$ with $1/p+1/q=1$, $\theta<\vartheta$ and $\tau:[\theta,\vartheta]\to R$ be a differentiable mapping such that $\tau'\in L[\theta,\vartheta]$ and $|\tau'|^q$ is a convex mapping on $[\theta,\vartheta]$. Then the inequality*

$$\begin{aligned}
&\left|\frac{2\alpha^{\frac{\beta}{k}-1}\alpha^{\frac{\beta}{k}}\Gamma_k(\beta+k)}{(y-x)^{\alpha\frac{\beta}{k}}}\left\{{}_k^{\beta}J^{\alpha}_{(\theta+\vartheta-\frac{x+y}{2})^+}\tau(\theta+\vartheta-x)+{}_k^{\beta}J^{\alpha}_{(\theta+\vartheta-\frac{x+y}{2})^-}\tau(\theta+\vartheta-y)\right\}\right.\\
&\quad\left.-\tau\left(\theta+\vartheta-\frac{x+y}{2}\right)\right|\\
&\le\frac{(y-x)\alpha^{\frac{\beta}{k}}}{4}\left\{\left(\left|\tau'(\theta)\right|^q+\left|\tau'(\vartheta)\right|^q\right)\left(\frac{B(\frac{\beta}{k}+1,\frac{1}{\alpha})}{\alpha^{\frac{\beta}{k}+1}}\right)\right.\\
&\quad-\left(\left(\frac{B(\frac{\beta}{k}+1,\frac{2}{\alpha})+B(\frac{\beta}{k}+1,\frac{1}{\alpha})}{2\alpha^{\frac{\beta}{k}+1}}\right)\left|\tau'(x)\right|^q\right.\\
&\quad\left.+\left(\frac{B(\frac{\beta}{k}+1,\frac{1}{\alpha})-B(\frac{\beta}{k}+1,\frac{2}{\alpha})}{2\alpha^{\frac{\beta}{k}+1}}\right)\left|\tau'(y)\right|^q\right)^{\frac{1}{q}}\\
&\quad+\left(\left|\tau'(\theta)\right|^q+\left|\tau'(\vartheta)\right|^q\right)\left(\frac{B(\frac{\beta}{k}+1,\frac{1}{\alpha})}{\alpha^{\frac{\beta}{k}+1}}\right)-\left(\left(\frac{B(\frac{\beta}{k}+1,\frac{1}{\alpha})-B(\frac{\beta}{k}+1,\frac{2}{\alpha})}{2\alpha^{\frac{\beta}{k}+1}}\right)\left|\tau'(x)\right|^q\right.\\
&\quad\left.\left.+\left(\frac{B(\frac{\beta}{k}+1,\frac{2}{\alpha})+B(\frac{\beta}{k}+1,\frac{1}{\alpha})}{2\alpha^{\frac{\beta}{k}+1}}\right)\left|\tau'(y)\right|^q\right)^{\frac{1}{q}}\right\}
\end{aligned}\tag{2.33}$$

holds for all $x,y\in[\theta,\vartheta]$.

Proof It follows from Lemma 2.5, Jensen–Mercer inequality, convexity of $|\tau'|^q$, and Hölder integral inequality that

$$\begin{aligned}
&\left|\frac{2\alpha^{\frac{\beta}{k}-1}\alpha^{\frac{\beta}{k}}\Gamma_k(\beta+k)}{(y-x)^{\alpha\frac{\beta}{k}}}\left\{{}_k^{\beta}J^{\alpha}_{(\theta+\vartheta-\frac{x+y}{2})^+}\tau(\theta+\vartheta-x)+{}_k^{\beta}J^{\alpha}_{(\theta+\vartheta-\frac{x+y}{2})^-}\tau(\theta+\vartheta-y)\right\}\right.\\
&\quad\left.-\tau\left(a+b-\frac{x+y}{2}\right)\right|\\
&\le\frac{(y-x)\alpha^{\frac{\beta}{k}}}{4}\left\{\left(\int_0^1 1\,d\lambda\right)^{\frac{1}{p}}\right.\\
&\quad\times\left(\int_0^1\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}}\left|\tau'\left(\theta+\vartheta-\left(\frac{2-\lambda}{2}x+\frac{\lambda}{2}y\right)\right)\right|^q d\lambda\right)^{\frac{1}{q}}\\
&\quad\left.+\left(\int_0^1 1\,d\lambda\right)^{\frac{1}{p}}\times\left(\int_0^1\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}}\left|\tau'\left(\theta+\vartheta-\left(\frac{\lambda}{2}x+\frac{2-\lambda}{2}y\right)\right)\right|^q d\lambda\right)^{\frac{1}{q}}\right\}\\
&\le\frac{(y-x)\alpha^{\frac{\beta}{k}}}{4}\left\{\left(\int_0^1\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}}\right.\right.
\end{aligned}$$

$$\times \left[\left|\tau'(\theta)\right|^q + \left|\tau'(\vartheta)\right|^q - \left(\frac{2-\lambda}{2} \left|\tau'(x)\right|^q + \frac{\lambda}{2} \left|\tau'(y)\right|^q \right) \right] \Bigg)^{\frac{1}{q}}$$

$$+ \left(\int_0^1 \left(\frac{1-(1-\lambda)^\alpha}{\alpha} \right)^{\frac{\beta}{k}} \left[\left|\tau'(\theta)\right|^q + \left|\tau'(\vartheta)\right|^q - \left(\frac{\lambda}{2} \left|\tau'(x)\right|^q + \frac{2-\lambda}{2} \left|\tau'(y)\right|^q \right) \right] \right)^{\frac{1}{q}} \Bigg\}.$$

By making necessary changes, we get (2.33). □

Theorem 2.17 *Let $\theta < \vartheta$ and $\tau : [\theta, \vartheta] \to R$ be a differentiable mapping such that $\tau' \in L[\theta, \vartheta]$ and $|\tau'|$ is a convex mapping on $[\theta, \vartheta]$. Then one has*

$$\begin{aligned}
&\left| \frac{\tau(\theta+\vartheta-x)+\tau(\theta+\vartheta-y)}{2} - \frac{\alpha^{\frac{\beta}{k}}\Gamma_k(\beta+k)}{2(y-x)^{\alpha\frac{\beta}{k}}} \right. \\
&\quad \left. \times \left\{ {}_k^\beta J^\alpha_{(\theta+\vartheta-y)^+}\tau(\theta+\vartheta-x) + {}_k^\beta J^\alpha_{(\theta+\vartheta-x)^-}\tau(\theta+\vartheta-y) \right\} \right| \\
&\leq \frac{(y-x)\alpha^{\frac{\beta}{k}}}{2} \left[\left\{ \left(\left|\tau'(\theta)\right| + \left|\tau'(\vartheta)\right| \right) \frac{B(\frac{1}{\alpha}, \frac{\beta}{k}+1)}{\alpha^{\frac{\beta}{k}+1}} \right. \right. \\
&\quad - \frac{|\tau'(x)|}{\alpha^{\frac{\beta}{k}+1}} \left\{ B_{\frac{1}{2^\alpha}}\left(\frac{1}{\alpha}, \frac{\beta}{k}+1 \right) + B\left(\frac{2}{\alpha}, \frac{\beta}{k}+1 \right) - B\left(\frac{1}{\alpha}, \frac{\beta}{k}+1 \right) \right\} \\
&\quad \left. - \frac{|\tau'(y)|}{\alpha^{\frac{\beta}{k}+1}} \left\{ B_{\frac{1}{2^\alpha}}\left(\frac{1}{\alpha}, \frac{\beta}{k}+1 \right) - B\left(\frac{2}{\alpha}, \frac{\beta}{k}+1 \right) \right\} \right\} \\
&\quad + \left\{ \left(\left|\tau'(\theta)\right| + \left|\tau'(\vartheta)\right| \right) \frac{B(\frac{1}{\alpha}, \frac{\beta}{k}+1)}{\alpha^{\frac{\beta}{k}+1}} \right. \\
&\quad - \frac{|\tau'(x)|}{\alpha^{\frac{\beta}{k}+1}} \left\{ B_{\frac{1}{2^\alpha}}\left(\frac{1}{\alpha}, \frac{\beta}{k}+1 \right) - B\left(\frac{2}{\alpha}, \frac{\beta}{k}+1 \right) \right\} \\
&\quad \left. \left. - \frac{|\tau'(y)|}{\alpha^{\frac{\beta}{k}+1}} \left\{ B_{\frac{1}{2^\alpha}}\left(\frac{1}{\alpha}, \frac{\beta}{k}+1 \right) + B\left(\frac{2}{\alpha}, \frac{\beta}{k}+1 \right) - B\left(\frac{1}{\alpha}, \frac{\beta}{k}+1 \right) \right\} \right\} \right]
\end{aligned} \tag{2.34}$$

for all $x, y \in [\theta, \vartheta]$.

Proof By using Lemma 2.7 and similar arguments as in the the proofs the previous theorems, we get

$$\begin{aligned}
&\left| \frac{\tau(\theta+\vartheta-x)+\tau(\theta+\vartheta-y)}{2} - \frac{\alpha^{\frac{\beta}{k}}\Gamma_k(\beta+k)}{2(y-x)^{\alpha\frac{\beta}{k}}} \right. \\
&\quad \left. \times \left\{ {}_k^\beta J^\alpha_{(\theta+\vartheta-y)^+}\tau(\theta+\vartheta-x) + {}_k^\beta J^\alpha_{(\theta+\vartheta-x)^-}\tau(\theta+\vartheta-y) \right\} \right| \\
&\leq \frac{(y-x)\alpha^{\frac{\beta}{k}}}{2} \int_0^1 \left| \left(\frac{1-(1-\lambda)^\alpha}{\alpha} \right)^{\frac{\beta}{k}} - \left(\frac{1-\lambda^\alpha}{\alpha} \right)^{\frac{\beta}{k}} \right| \left| \tau'\big(\theta+\vartheta-\big(\lambda x+(1-\lambda)y\big)\big) \right| d\lambda \\
&\leq \frac{(y-x)\alpha^{\frac{\beta}{k}}}{2} \int_0^1 \left| \left(\frac{1-(1-\lambda)^\alpha}{\alpha} \right)^{\frac{\beta}{k}} - \left(\frac{1-\lambda^\alpha}{\alpha} \right)^{\frac{\beta}{k}} \right| \\
&\quad \times \left\{ \left|\tau'(\theta)\right| + \left|\tau'(\vartheta)\right| - \left(\lambda\left|\tau'(x)\right| + (1-\lambda)\left|\tau'(y)\right| \right) \right\} d\lambda \\
&\leq \frac{(y-x)\alpha^{\frac{\beta}{k}}}{2} \left[\int_0^{\frac{1}{2}} \left[\left(\frac{1-\lambda^\alpha}{\alpha} \right)^{\frac{\beta}{k}} - \left(\frac{1-(1-\lambda)^\alpha}{\alpha} \right)^{\frac{\beta}{k}} \right] \right.
\end{aligned}$$

$$\times \left\{ \left|\tau'(\theta)\right| + \left|\tau'(\vartheta)\right| - \left(\lambda\left|\tau'(x)\right| + (1-\lambda)\left|\tau'(y)\right|\right)\right\} d\lambda$$
$$+ \int_{\frac{1}{2}}^{1} \left[\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha} \right)^{\frac{\beta}{k}} - \left(\frac{1-\lambda^{\alpha}}{\alpha} \right)^{\frac{\beta}{k}} \right]$$
$$\times \left\{ \left|\tau'(\theta)\right| + \left|\tau'(\vartheta)\right| - \left(\lambda\left|\tau'(x)\right| + (1-\lambda)\left|\tau'(y)\right|\right)\right\} d\lambda \Bigg].$$

This completes the proof. □

3 New inequalities via improved Hölder inequality

Theorem 3.1 *Let $\alpha, \beta > 0$, $p, q > 1$ with $1/p + 1/q = 1$, $\theta < \vartheta$ and $\tau : [\theta, \vartheta] \to R$ be a differentiable mapping such that $\tau' \in L[\theta, \vartheta]$ and $|\tau'|^q$ is a convex mapping on $[\theta, \vartheta]$. Then one has*

$$\begin{aligned}
&\left| \frac{2^{\alpha\frac{\beta}{k}-1}\alpha^{\frac{\beta}{k}}\Gamma_k(\beta+k)}{(y-x)^{\alpha\frac{\beta}{k}}} \left\{ {}_k^{\beta}J^{\alpha}_{(\theta+\vartheta-\frac{x+y}{2})^+}\tau(\theta+\vartheta-x) + {}_k^{\beta}J^{\alpha}_{(\theta+\vartheta-\frac{x+y}{2})^-}\tau(\theta+\vartheta-y) \right\} \right. \\
&\quad \left. - \tau\left(\theta+\vartheta-\frac{x+y}{2}\right) \right| \\
&\leq \frac{(y-x)\alpha^{\frac{\beta}{k}}}{2} \left[\left\{ \left(\frac{B(\frac{2}{\alpha}, \frac{\beta}{k}p+1)}{\alpha^{\frac{\beta}{k}p+1}} \right)^{\frac{1}{p}} \right. \right. \\
&\quad \times \left(\frac{|\tau'(\theta)|^q + |\tau'(\vartheta)|^q}{2} - \left(\frac{5}{12}\left|\tau'(x)\right|^q + \frac{1}{12}\left|\tau'(y)\right|^q \right) \right)^{\frac{1}{q}} \\
&\quad + \left(\frac{B(\frac{1}{\alpha}, \frac{\beta}{k}p+1) - B(\frac{2}{\alpha}, \frac{\beta}{k}p+1)}{\alpha^{\frac{\beta}{k}p+1}} \right)^{\frac{1}{p}} \\
&\quad \left. \times \left(\frac{|\tau'(\theta)|^q + |\tau'(\vartheta)|^q}{2} - \left(\frac{1}{3}\left|\tau'(x)\right|^q + \frac{1}{6}\left|\tau'(y)\right|^q \right) \right)^{\frac{1}{q}} \right\} \\
&\quad + \left\{ \left(\frac{B(\frac{2}{\alpha}, \frac{\beta}{k}p+1)}{\alpha^{\frac{\beta}{k}p+1}} \right)^{\frac{1}{p}} \left(\frac{|\tau'(\theta)|^q + |\tau'(\vartheta)|^q}{2} - \left(\frac{1}{12}\left|\tau'(x)\right|^q + \frac{5}{12}\left|\tau'(y)\right|^q \right) \right)^{\frac{1}{q}} \right. \\
&\quad + \left(\frac{B(\frac{1}{\alpha}, \frac{\beta}{k}p+1) - B(\frac{2}{\alpha}, \frac{\beta}{k}p+1)}{\alpha^{\frac{\beta}{k}p+1}} \right)^{\frac{1}{p}} \\
&\quad \left. \left. \times \left(\frac{|\tau'(\theta)|^q + |\tau'(\vartheta)|^q}{2} - \left(\frac{1}{6}\left|\tau'(x)\right|^q + \frac{1}{3}\left|\tau'(y)\right|^q \right) \right)^{\frac{1}{q}} \right\} \right]
\end{aligned} \tag{3.1}$$

for all $x, y \in [\theta, \vartheta]$.

Proof It follows from Lemma 2.5, Jensen–Mercer inequality, the convexity of $|\tau'|^q$, and Hölder–İşcan integral inequality given in Theorem 1.4 of [70] that

$$\begin{aligned}
&\left| \frac{2^{\alpha\frac{\beta}{k}-1}\alpha^{\frac{\beta}{k}}\Gamma_k(\beta+k)}{(y-x)^{\alpha\frac{\beta}{k}}} \left\{ {}_k^{\beta}J^{\alpha}_{(\theta+\vartheta-\frac{x+y}{2})^+}\tau(\theta+\vartheta-x) + {}_k^{\beta}J^{\alpha}_{(\theta+\vartheta-\frac{x+y}{2})^-}\tau(\theta+\vartheta-y) \right\} \right. \\
&\quad \left. - \tau\left(a+b-\frac{x+y}{2}\right) \right|
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{(y-x)\alpha^{\frac{\beta}{k}}}{4}\Bigg[\Bigg\{\left(\int_0^1(1-\lambda)\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}p}d\lambda\right)^{\frac{1}{p}}\\
&\quad\times\left(\int_0^1(1-\lambda)\left|\tau'\left(\theta+\vartheta-\left(\frac{2-\lambda}{2}x+\frac{\lambda}{2}y\right)\right)\right|^{q}d\lambda\right)^{\frac{1}{q}}\\
&\quad\times\left(\int_0^1\lambda\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}p}d\lambda\right)^{\frac{1}{p}}\left(\int_0^1\lambda\left|\tau'\left(\theta+\vartheta-\left(\frac{2-\lambda}{2}x+\frac{\lambda}{2}y\right)\right)\right|^{q}d\lambda\right)^{\frac{1}{q}}\Bigg\}\\
&\quad+\Bigg\{\left(\int_0^1(1-\lambda)\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}p}d\lambda\right)^{\frac{1}{p}}\\
&\quad\times\left(\int_0^1(1-\lambda)\left|\tau'\left(\theta+\vartheta-\left(\frac{\lambda}{2}x+\frac{2-\lambda}{2}y\right)\right)\right|^{q}d\lambda\right)^{\frac{1}{q}}\\
&\quad\times\left(\int_0^1\lambda\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}p}d\lambda\right)^{\frac{1}{p}}\\
&\quad\times\left(\int_0^1\lambda\left|\tau'\left(\theta+\vartheta-\left(\frac{\lambda}{2}x+\frac{2-\lambda}{2}y\right)\right)\right|^{q}d\lambda\right)^{\frac{1}{q}}\Bigg\}\Bigg]\\
&\leq \frac{(y-x)\alpha^{\frac{\beta}{k}}}{4}\Bigg[\Bigg\{\left(\int_0^1(1-\lambda)\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}p}d\lambda\right)^{\frac{1}{p}}\\
&\quad\times\left(\int_0^1(1-\lambda)\left[\left|\tau'(\theta)\right|^{q}+\left|\tau'(\vartheta)\right|^{q}-\left(\frac{2-\lambda}{2}\left|\tau'(x)\right|^{q}+\frac{\lambda}{2}\left|\tau'(y)\right|^{q}\right)\right]d\lambda\right)^{\frac{1}{q}}\\
&\quad\times\left(\int_0^1\lambda\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}p}d\lambda\right)^{\frac{1}{p}}\\
&\quad\times\left(\int_0^1\lambda\left[\left|\tau'(\theta)\right|^{q}+\left|\tau'(\vartheta)\right|^{q}-\left(\frac{2-\lambda}{2}\left|\tau'(x)\right|^{q}+\frac{\lambda}{2}\left|\tau'(y)\right|^{q}\right)\right]d\lambda\right)^{\frac{1}{q}}\Bigg\}\\
&\quad+\Bigg\{\left(\int_0^1(1-\lambda)\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}p}d\lambda\right)^{\frac{1}{p}}\\
&\quad\times\left(\int_0^1(1-\lambda)\left[\left|\tau'(\theta)\right|^{q}+\left|\tau'(\vartheta)\right|^{q}-\left(\frac{\lambda}{2}\left|\tau'(x)\right|^{q}+\frac{2-\lambda}{2}\left|\tau'(y)\right|^{q}\right)\right]d\lambda\right)^{\frac{1}{q}}\\
&\quad\times\left(\int_0^1\lambda\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}p}d\lambda\right)^{\frac{1}{p}}\\
&\quad\times\left(\int_0^1\lambda\left[\left|\tau'(\theta)\right|^{q}+\left|\tau'(\vartheta)\right|^{q}-\left(\frac{\lambda}{2}\left|\tau'(x)\right|^{q}+\frac{2-\lambda}{2}\left|\tau'(y)\right|^{q}\right)\right]d\lambda\right)^{\frac{1}{q}}\Bigg\}\Bigg].
\end{aligned}
$$

By making use of some computations, one can get the required result. □

Theorem 3.2 *Let $\alpha,\beta>0$, $p,q>1$ with $1/p+1/q=1$, $\theta<\vartheta$ and $\tau:[\theta,\vartheta]\to R$ ba a differentiable mapping such that $\tau'\in L[\theta,\vartheta]$ and $|\tau'|^{q}$ is a convex mapping on $[\theta,\vartheta]$. Then the inequality*

$$
\begin{aligned}
&\left|\frac{2^{\alpha\frac{\beta}{k}-1}\alpha^{\frac{\beta}{k}}\Gamma_k(\beta+k)}{(y-x)^{\alpha\frac{\beta}{k}}}\left\{{}_k^{\beta}J^{\alpha}_{(\theta+\vartheta-\frac{x+y}{2})^{+}}\tau(\theta+\vartheta-x)+{}_k^{\beta}J^{\alpha}_{(\theta+\vartheta-\frac{x+y}{2})^{-}}\tau(\theta+\vartheta-y)\right\}\right.\\
&\quad\left.-\tau\left(\theta+\vartheta-\frac{x+y}{2}\right)\right|
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{(y-x)\alpha^{\frac{\beta}{k}}}{4}\Bigg[\Bigg\{\left(\frac{B(\frac{2}{\alpha},\frac{\beta}{k}+1)}{\alpha^{\frac{\beta}{k}+1}}\right)^{1-\frac{1}{q}}\\
&\quad\times\left(\left(\left|\tau'(\theta)\right|^q+\left|\tau'(\vartheta)\right|^q\right)\left(\frac{B(\frac{2}{\alpha},\frac{\beta}{k}+1)}{\alpha^{\frac{\beta}{k}+1}}\right)\right.\\
&\quad-\left(\frac{1}{2\alpha^{\frac{\beta}{k}+1}}\left|\tau'(x)\right|^q\left(B\left(\frac{2}{\alpha},\frac{\beta}{k}+1\right)+B\left(\frac{3}{\alpha},\frac{\beta}{k}+1\right)\right)\right.\\
&\quad\left.\left.+\frac{1}{2\alpha^{\frac{\beta}{k}+1}}\left|\tau'(y)\right|^q\left(B\left(\frac{2}{\alpha},\frac{\beta}{k}+1\right)-B\left(\frac{3}{\alpha},\frac{\beta}{k}+1\right)\right)\right)\right)^{\frac{1}{q}}\\
&\quad+\left(\frac{B(\frac{1}{\alpha},\frac{\beta}{k}+1)-B(\frac{2}{\alpha},\frac{\beta}{k}+1)}{\alpha^{\frac{\beta}{k}+1}}\right)^{1-\frac{1}{q}}\\
&\quad\times\left(\left(\left|\tau'(\theta)\right|^q+\left|\tau'(\vartheta)\right|^q\right)\left(\frac{B(\frac{1}{\alpha},\frac{\beta}{k}+1)-B(\frac{2}{\alpha},\frac{\beta}{k}+1)}{\alpha^{\frac{\beta}{k}+1}}\right)\right.\\
&\quad-\left(\frac{1}{2\alpha^{\frac{\beta}{k}+1}}\left|\tau'(x)\right|^q\left(B\left(\frac{1}{\alpha},\frac{\beta}{k}+1\right)-B\left(\frac{3}{\alpha},\frac{\beta}{k}+1\right)\right)\right.\\
&\quad\left.\left.\left.+\frac{1}{2\alpha^{\frac{\beta}{k}+1}}\left|\tau'(y)\right|^q\left(B\left(\frac{1}{\alpha},\frac{\beta}{k}+1\right)-2B\left(\frac{2}{\alpha},\frac{\beta}{k}+1\right)+B\left(\frac{3}{\alpha},\frac{\beta}{k}+1\right)\right)\right)\right)^{\frac{1}{q}}\Bigg\}\\
&\quad+\left\{\left(\frac{B(\frac{2}{\alpha},\frac{\beta}{k}+1)}{\alpha^{\frac{\beta}{k}+1}}\right)^{1-\frac{1}{q}}\left(\left(\left|\tau'(\theta)\right|^q+\left|\tau'(\vartheta)\right|^q\right)\left(\frac{B(\frac{2}{\alpha},\frac{\beta}{k}+1)}{\alpha^{\frac{\beta}{k}+1}}\right)\right.\right.\\
&\quad-\left(\frac{1}{2\alpha^{\frac{\beta}{k}+1}}\left|\tau'(x)\right|^q\left(B\left(\frac{2}{\alpha},\frac{\beta}{k}+1\right)-B\left(\frac{3}{\alpha},\frac{\beta}{k}+1\right)\right)\right.\\
&\quad\left.\left.\left.+\frac{1}{2\alpha^{\frac{\beta}{k}+1}}\left|\tau'(y)\right|^q\left(B\left(\frac{2}{\alpha},\frac{\beta}{k}+1\right)+B\left(\frac{3}{\alpha},\frac{\beta}{k}+1\right)\right)\right)\right)^{\frac{1}{q}}\\
&\quad+\left(\frac{B(\frac{1}{\alpha},\frac{\beta}{k}+1)-B(\frac{2}{\alpha},\frac{\beta}{k}+1)}{\alpha^{\frac{\beta}{k}+1}}\right)^{1-\frac{1}{q}}\\
&\quad\times\left(\left(\left|\tau'(\theta)\right|^q+\left|\tau'(\vartheta)\right|^q\right)\left(\frac{B(\frac{1}{\alpha},\frac{\beta}{k}+1)-B(\frac{2}{\alpha},\frac{\beta}{k}+1)}{\alpha^{\frac{\beta}{k}+1}}\right)\right.\\
&\quad-\left(\frac{1}{2\alpha^{\frac{\beta}{k}+1}}\left|\tau'(x)\right|^q\left(B\left(\frac{1}{\alpha},\frac{\beta}{k}+1\right)-2B\left(\frac{2}{\alpha},\frac{\beta}{k}+1\right)+B\left(\frac{3}{\alpha},\frac{\beta}{k}+1\right)\right)\right.\\
&\quad\left.\left.\left.+\frac{1}{2\alpha^{\frac{\beta}{k}+1}}\left|\tau'(y)\right|^q\left(B\left(\frac{1}{\alpha},\frac{\beta}{k}+1\right)-B\left(\frac{3}{\alpha},\frac{\beta}{k}+1\right)\right)\right)\right)^{\frac{1}{q}}\Bigg\}\Bigg],
\end{aligned}
\tag{3.2}
$$

holds for all $x,y\in[\theta,\vartheta]$.

Proof It follows from Lemma 2.5, Jensen–Mercer inequality, the convexity of $|\tau'|^q$, and the improved power-mean integral inequality given in Theorem 1.5 of [70] that

$$
\begin{aligned}
&\left|\frac{2^{\alpha\frac{\beta}{k}-1}\alpha^{\frac{\beta}{k}}\Gamma_k(\beta+k)}{(y-x)^{\alpha\frac{\beta}{k}}}\left\{{}_k^{\beta}J^{\alpha}_{(\theta+\vartheta-\frac{x+y}{2})^+}\tau(\theta+\vartheta-x)+{}_k^{\beta}J^{\alpha}_{(\theta+\vartheta-\frac{x+y}{2})^-}\tau(\theta+\vartheta-y)\right\}\right.\\
&\quad\left.-\tau\left(\theta+\vartheta-\frac{x+y}{2}\right)\right|
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{(y-x)\alpha^{\frac{\beta}{k}}}{4}\Bigg[\Bigg\{\left(\int_0^1 (1-\lambda)\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}} d\lambda\right)^{1-\frac{1}{q}} \\
&\quad\times\left(\int_0^1 (1-\lambda)\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}}\left|\tau'\left(\theta+\vartheta-\left(\frac{2-\lambda}{2}x+\frac{\lambda}{2}y\right)\right)\right|^{q} d\lambda\right)^{\frac{1}{q}} \\
&\quad+\left(\int_0^1 \lambda\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}} d\lambda\right)^{1-\frac{1}{q}} \\
&\quad\times\left(\int_0^1 \lambda\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}}\left|\tau'\left(\theta+\vartheta-\left(\frac{2-\lambda}{2}x+\frac{\lambda}{2}y\right)\right)\right|^{q} d\lambda\right)^{\frac{1}{q}}\Bigg\} \\
&\quad+\Bigg\{\left(\int_0^1 (1-\lambda)\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}} d\lambda\right)^{1-\frac{1}{q}} \\
&\quad\times\left(\int_0^1 (1-\lambda)\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}}\left|\tau'\left(\theta+\vartheta-\left(\frac{\lambda}{2}x+\frac{2-\lambda}{2}y\right)\right)\right|^{q} d\lambda\right)^{\frac{1}{q}} \\
&\quad+\left(\int_0^1 \lambda\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}} d\lambda\right)^{1-\frac{1}{q}} \\
&\quad\times\left(\int_0^1 \lambda\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}}\left|\tau'\left(\theta+\vartheta-\left(\frac{\lambda}{2}x+\frac{2-\lambda}{2}y\right)\right)\right|^{q} d\lambda\right)^{\frac{1}{q}}\Bigg\}\Bigg] \\
&\leq \frac{(y-x)\alpha^{\frac{\beta}{k}}}{4}\Bigg[\Bigg\{\left(\int_0^1 (1-\lambda)\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}} d\lambda\right)^{1-\frac{1}{q}}\left(\int_0^1 (1-\lambda)\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}}\right. \\
&\quad\left.\times\left[\left|\tau'(\theta)\right|^{q}+\left|\tau'(\vartheta)\right|^{q}-\left(\frac{2-\lambda}{2}\left|\tau'(x)\right|^{q}+\frac{\lambda}{2}\left|\tau'(y)\right|^{q}\right)\right] d\lambda\right)^{\frac{1}{q}} \\
&\quad+\left(\int_0^1 \lambda\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}} d\lambda\right)^{1-\frac{1}{q}}\left(\int_0^1 \lambda\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}}\right. \\
&\quad\left.\times\left[\left|\tau'(\theta)\right|^{q}+\left|\tau'(\vartheta)\right|^{q}-\left(\frac{2-\lambda}{2}\left|\tau'(x)\right|^{q}+\frac{\lambda}{2}\left|\tau'(y)\right|^{q}\right)\right] d\lambda\right)^{\frac{1}{q}}\Bigg\} \\
&\quad+\Bigg\{\left(\int_0^1 (1-\lambda)\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}} d\lambda\right)^{1-\frac{1}{q}}\left(\int_0^1 (1-\lambda)\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}}\right. \\
&\quad\left.\times\left[\left|\tau'(\theta)\right|^{q}+\left|\tau'(\vartheta)\right|^{q}-\left(\frac{\lambda}{2}\left|\tau'(x)\right|^{q}+\frac{2-\lambda}{2}\left|\tau'(y)\right|^{q}\right)\right] d\lambda\right)^{\frac{1}{q}} \\
&\quad+\left(\int_0^1 \lambda\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}} d\lambda\right)^{1-\frac{1}{q}}\left(\int_0^1 \lambda\left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}}\right. \\
&\quad\left.\times\left[\left|\tau'(\theta)\right|^{q}+\left|\tau'(\vartheta)\right|^{q}-\left(\frac{\lambda}{2}\left|\tau'(x)\right|^{q}+\frac{2-\lambda}{2}\left|\tau'(y)\right|^{q}\right)\right] d\lambda\right)^{\frac{1}{q}}\Bigg\}\Bigg].
\end{aligned}
$$

By computing the above integrals, one can obtain the required result. □

4 Conclusions

The Hermite–Kadamard inequality is one of the most important inequalities for convex functions and in the theory of inequalities, while the Hermite–Jensen–Mercer inequality is a variant of the Hermite–Kadamard inequality which has attracted the attention of many researchers in recently years due to its many applications in pure and applied mathemat-

ics, as well as in physics. Therefore, it is important to further generalize and improve the Hermite–Jensen–Mercer inequality. In the article, we have found new methods to generalize the Hermite–Jensen–Mercer inequality to the fractional integrals, established several novel Hermite–Jensen–Mercer-type inequalities for convex functions in the framework of the k-fractional conformable integrals, generalized and improved many previously known results in the literature. The ideas and techniques we put forward are likely to open new research directions in this field and lead to a large number of follow-up studies.

Acknowledgements
The authors would like to express their sincere thanks to the support of National Natural Science Foundation of China. The research of the first author has been fully supported by H.E.C. Pakistan under NRPU project 7906.

Funding
This work was supported by the National Natural Science Foundation of China (Grant No. 61673169).

Availability of data and materials
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Author details
[1]Department of Mathematics, COMSATS University Islamabad, Lahore, Pakistan. [2]Department of Mathematics, Government College University, Faisalabad, Pakistan. [3]Department of Mathematics, Faculty of Science and Letters, Ağrılbrahim Çeçen University, Agrı, Turkey. [4]Department of Mathematics, Huzhou University, Huzhou, China. [5]Hunan Provincial Key Laboratory of Mathematical Modeling and Analysis in Engineering, Changsha University of Science & Technology, Changsha, China.

Publisher's Note

Received: 23 May 2020 Accepted: 1 November 2020 Published online: 11 November 2020

References

1. Khurshid, Y., Adil Khan, M., Chu, Y.-M.: Conformable fractional integral inequalities for *GG*- and *GA*-convex functions. AIMS Math. **5**(5), 5012–5030 (2020)
2. Chu, Y.-M., Awan, M.U., Javad, M.Z., Khan, A.W.: Bounds for the remainder in Simpson's inequality via n-polynomial convex functions of higher order using Katugampola fractional integrals. J. Math. **2020**, Article ID 4189036 (2020)
3. Yan, P.-Y., Li, Q., Chu, Y.-M., Mukhtar, S., Waheed, S.: On some fractional integral inequalities for generalized strongly modified h-convex function. AIMS Math. **5**(6), 6620–6638 (2020)
4. Adil Khan, M., Hanif, M., Khan, Z.A., Ahmad, K., Chu, Y.-M.: Association of Jensen's inequality for s-convex function with Csiszár divergence. J. Inequal. Appl. **2019**, Article ID 162 (2019)
5. Rashid, S., Ashraf, R., Noor, M.A., Noor, K.I., Chu, Y.-M.: New weighted generalizations for differentiable exponentially convex mapping with application. AIMS Math. **5**(4), 3525–3546 (2020)
6. Ge-JiLe, H., Rashid, S., Noor, M.A., Suhail, A., Chu, Y.-M.: Some unified bounds for exponentially *tgs*-convex functions governed by conformable fractional operators. AIMS Math. **5**(6), 6108–6123 (2020)
7. Zhao, T.-H., He, Z.-Y., Chu, Y.-M.: On some refinements for inequalities involving zero-balanced hypergeometric function. AIMS Math. **5**(6), 6479–6495 (2020)
8. Agarwal, P., Kadakal, M., İşcan, İ., Chu, Y.-M.: Better approaches for n-times differentiable convex functions. Mathematics **8**, Article ID 950 (2020)
9. Wang, M.-K., Chu, H.-H., Li, Y.-M., Chu, Y.-M.: Answers to three conjectures on convexity of three functions involving complete elliptic integrals of the first kind. Appl. Anal. Discrete Math. **14**(1), 255–271 (2020)
10. Shen, J.-M., Yang, Z.-H., Qian, W.-M., Zhang, W., Chu, Y.-M.: Sharp rational bounds for the gamma function. Math. Inequal. Appl. **23**(3), 843–853 (2020)
11. Wang, M.-K., Chu, Y.-M., Li, Y.-M., Zhang, W.: Asymptotic expansion and bounds for complete elliptic integrals. Math. Inequal. Appl. **23**(3), 821–841 (2020)
12. Hai, G.-J., Zhao, T.-H.: Monotonicity properties and bounds involving the two-parameter generalized Grötzsch ring function. J. Inequal. Appl. **2020**, Article ID 66 (2020)
13. Awan, M.U., Akhtar, N., Kashuri, A., Noor, M.A., Chu, Y.-M.: *2D* approximately reciprocal ρ-convex functions and associated integral inequalities. AIMS Math. **5**(5), 4662–4680 (2020)
14. Sun, M.-B., Chu, Y.-M.: Inequalities for the generalized weighted mean values of g-convex functions with applications. Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat. **114**(4), Article ID 172 (2020)

15. Abbas Baloch, I., Mughal, A.A., Chu, Y.-M., Haq, A.U., De La Sen, M.: A variant of Jensen-type inequality and related results for harmonic convex functions. AIMS Math. **5**(6), 6404–6418 (2020)
16. Kalsoom, H., Idrees, M., Baleanu, D., Chu, Y.-M.: New estimates of q_1q_2-Ostrowski-type inequalities within a class of n-polynomial prevexity of function. J. Funct. Spaces **2020**, Article ID 3720798 (2020)
17. Awan, M.U., Akhtar, N., Iftikhar, S., Noor, M.A., Chu, Y.-M.: New Hermite–Hadamard type inequalities for n-polynomial harmonically convex functions. J. Inequal. Appl. **2020**, Article ID 125 (2020)
18. Awan, M.U., Talib, S., Noor, M.A., Chu, Y.-M., Noor, K.I.: Some trapezium-like inequalities involving functions having strongly n-polynomial preinvexity property of higher order. J. Funct. Spaces **2020**, Article ID 9154139 (2020)
19. Zhou, S.-S., Rashid, S., Noor, M.A., Noor, K.I., Safdar, F., Chu, Y.-M.: New Hermite–Hadamard type inequalities for exponentially convex functions and applications. AIMS Math. **5**(6), 6874–6901 (2020)
20. Feng, B.-L., Ghafoor, M., Chu, Y.-M., Qureshi, M.I., Feng, X., Yao, C., Qiao, X.: Hermite–Hadamard and Jensen's type inequalities for modified (p,h)-convex functions. AIMS Math. **5**(6), 6959–6971 (2020)
21. Khan, S., Adil Khan, M., Chu, Y.-M.: Converses of the Jensen inequality derived from the Green functions with applications in information theory. Math. Methods Appl. Sci. **43**(5), 2577–2587 (2020)
22. Adil Khan, M., Pečarić, J., Chu, Y.-M.: Refinements of Jensen's and McShane's inequalities with applications. AIMS Math. **5**(5), 4931–4945 (2020)
23. Awan, M.U., Talib, S., Chu, Y.-M., Noor, M.A., Noor, K.I.: Some new refinements of Hermite–Hadamard-type inequalities involving Ψ_k-Riemann–Liouville fractional integrals and applications. Math. Probl. Eng. **2020**, Article ID 3051920 (2020)
24. Yang, X.-Z., Farid, G., Nazeer, W., Chu, Y.-M., Dong, C.-F.: Fractional generalized Hadamard and Fejér–Hadamard inequalities for m-convex function. AIMS Math. **5**(6), 6325–6340 (2020)
25. Guo, S.-Y., Chu, Y.-M., Farid, G., Mehmood, S., Nazeer, W.: Fractional Hadamard and Fejér–Hadamard inequalities associated with exponentially (s,m)-convex functions. J. Funct. Spaces **2020**, Article ID 2410385 (2020)
26. Iqbal, A., Adil Khan, M., Ullah, S., Chu, Y.-M.: Some new Hermite–Hadamard-type inequalities associated with conformable fractional integrals and their applications. J. Funct. Spaces **2020**, Article ID 9845407 (2020)
27. Khurshid, Y., Adil Khan, M., Chu, Y.-M.: Conformable integral version of Hermite–Hadamard–Fejér inequalities via η-convex functions. AIMS Math. **5**(5), 5106–5120 (2020)
28. Qi, H.-X., Yussouf, M., Mehmood, S., Chu, Y.-M., Farid, G.: Fractional integral versions of Hermite–Hadamard type inequality for generalized exponentially convexity. AIMS Math. **5**(6), 6030–6042 (2020)
29. Iqbal, A., Adil Khan, M., Mohammad, N., Nwaeze, E.R., Chu, Y.-M.: Revisiting the Hermite–Hadamard integral inequality via a Green function. AIMS Math. **5**(6), 6087–6107 (2020)
30. Mercer, A.McD.: A variant of Jensen's inequality. JIPAM. J. Inequal. Pure Appl. Math. **4**(4), Article ID 73 (2003)
31. Matković, A., Pečarić, J., Perić, I.: A variant of Jensen's inequality of Mercer's type for operators with applications. Linear Algebra Appl. **418**(2–3), 551–564 (2006)
32. Jarad, F., Abdeljawad, T., Baleanu, D.: Caputo-type modification of the Hadamard fractional derivatives. Adv. Differ. Equ. **2012**, Article ID 142 (2012)
33. Mohammed, P.O., Abdeljawad, T.: Opial integral inequalities for generalized fractional operators with nonsingular kernel. J. Inequal. Appl. **2020**, Article ID 148 (2020)
34. Yokuş, A., Gülbahar, S.: Numerical solutions with linearization techniques of the fractional Harry Dym equation. Appl. Math. Nonlinear Sci. **4**(1), 35–41 (2019)
35. Al-Ghafri, K.S., Rezazadeh, H.: Solitons and other solutions of (3 + 1)-dimensional space-time fractional modified KdV–Zakharov–Kuznetsov equation. Appl. Math. Nonlinear Sci. **4**(2), 289–304 (2019)
36. Odibat, Z., Baleanu, D.: Numerical simulation of initial value problems with generalized Caputo-type fractional derivatives. Appl. Numer. Math. **156**, 94–105 (2020)
37. Singh, J., Kumar, D., Hammouch, Z., Atangana, A.: A fractional epidemiological model for computer viruses pertaining to a new fractional derivative. Appl. Math. Comput. **316**, 504–515 (2018)
38. Ameen, I., Baleanu, D., Ali, H.M.: An efficient algorithm for solving the fractional optimal control of SIRV epidemic model with a combination of vaccination and treatment. Chaos Solitons Fractals **137**, Article ID 109892 (2020)
39. Hussain, S., Khalid, J., Chu, Y.-M.: Some generalized fractional integral Simpson's type inequalities with applications. AIMS Math. **5**(6), 5859–5883 (2020)
40. Chen, S.-B., Rashid, S., Noor, M.A., Ashraf, R., Chu, Y.-M.: A new approach on fractional calculus and probability function. AIMS Math. **5**(6), 7041–7054 (2020)
41. Chu, Y.-M., Adil Khan, M., Ali, T., Dragomir, S.S.: Inequalities for α-fractional differentiable functions. J. Inequal. Appl. **2017**, Article ID 93 (2017)
42. Wang, L., Dai, L.-Z., Bian, H.-B., Ma, Y.-F., Zhang, J.-R.: Concrete cracking prediction under combined prestress and strand corrosion. Struct. Infrastruct. Eng. **15**(3), 285–295 (2019)
43. Dai, L.-Z., Bian, H.-B., Wang, L., Potier-Ferry, M., Zhang, J.-R.: Prestress loss diagnostics in pretensioned concrete structures with corrosive cracking. J. Struct. Eng. **146**(3), Article ID 04020013 (2020)
44. Yang, A.-M., Zhang, Y.-Z., Cattani, C., Xie, G.-N., Rashidi, M.M., Zhou, Y.-J., Yang, X.-J.: Application of local fractional series expansion method to solve Klein–Gordon equations on Cantor sets. Abstr. Appl. Anal. **2014**, Article ID 372741 (2014)
45. Gao, W., Veeresha, P., Prakasha, D.G., Baskonus, H.M., Yel, G.: New approach for the model describing the deathly disease in pregnant women using Mittag-Leffler function. Chaos Solitons Fractals **134**, Article ID 109696 (2020)
46. Cattani, C., Pierro, G.: On the fractal geometry of DNA by the binary image analysis. Bull. Math. Biol. **75**(9), 1544–1570 (2013)
47. Gao, W., Yel, G., Baskonus, H.M., Cattani, C.: Complex solitons in the conformable (2 + 1)-dimensional Ablowitz–Kaup–Newell–Segur equation. AIMS Math. **5**(1), 507–521 (2020)
48. Cattani, C.: Connection coefficients of Shannon wavelets. Math. Model. Anal. **11**(2), 117–132 (2006)
49. İlhan, E., Kıymaz, İ.O.: A generalization of truncated M-fractional derivative and applications to fractional differential equations. Appl. Math. Nonlinear Sci. **5**(1), 171–188 (2020)
50. Mohammad, M., Cattani, C.: A collocation method via the quasi-affine biorthogonal systems for solving weakly singular type of Volterra–Fredholm integral equations. Alex. Eng. J. **59**, 2181–2191 (2020)
51. Sweilam, N., Al-Mekhlafi, S., Shatta, S., Baleanu, D.: Numerical study for two types variable-order Burgers' equations with proportional delay. Appl. Numer. Math. **156**, 364–376 (2020)

52. Shen, J.-M., Rashid, S., Noor, M.A., Ashraf, R., Chu, Y.-M.: Certain novel estimates within fractional calculus theory on time scales. AIMS Math. **5**(6), 6073–6086 (2020)
53. Rashid, S., Jarad, F., Chu, Y.-M.: A note on reverse Minkowski inequality via generalized proportional fractional integral operator with respect to another function. Math. Probl. Eng. **2020**, Article ID 7630260 (2020)
54. Adil Khan, M., Begum, S., Khurshid, Y., Chu, Y.-M.: Ostrowski type inequalities involving conformable fractional integrals. J. Inequal. Appl. **2018**, Article ID 70 (2018)
55. Xu, L., Chu, Y.-M., Rashid, S., El-Deeb, A.A., Nisar, K.S.: On new unified bounds for a family of functions with fractional q-calculus theory. J. Funct. Spaces **2020**, Article ID 4984612 (2020)
56. Rashid, S., Khalid, A., Rahman, S., Nisar, K.S., Chu, Y.-M.: On new modifications governed by quantum Hahn's integral operator pertaining to fractional calculus. J. Funct. Spaces **2020**, Article ID 8262860 (2020)
57. Rashid, S., Jarad, F., Noor, M.A., Kalsoom, H., Chu, Y.-M.: Inequalities by means of generalized proportional fractional integral operators with respect to another function. Mathematics **7**(12), Article ID 1225 (2019)
58. Kalsoom, H., Idrees, M., Kashuri, A., Awan, M.U., Chu, Y.-M.: Some new (p_1p_2, q_1q_2)-estimates of Ostrowski-type integral inequalities via n-polynomials s-type convexity. AIMS Math. **5**(6), 7122–7144 (2020)
59. Zhao, T.-H., Wang, M.-K., Chu, Y.-M.: A sharp double inequality involving generalized complete elliptic integral of the first kind. AIMS Math. **5**(5), 4512–4528 (2020)
60. Huang, X.-F., Wang, M.-K., Shao, H., Zhao, Y.-F., Chu, Y.-M.: Monotonicity properties and bounds for the complete p-elliptic integrals. AIMS Math. **5**(6), 7071–7086 (2020)
61. Akdemir, A.O., Ekinci, A., Set, E.: Conformable fractional integrals and related new integral inequalities. J. Nonlinear Convex Anal. **18**(4), 661–674 (2017)
62. Jarad, F., Uğurlu, E., Abdeljawad, T., Baleanu, D.: On a new class of fractional operators. Adv. Differ. Equ. **2017**, Article ID 247 (2017)
63. Kilbas, A.A.: Hadamard-type fractional calculus. J. Korean Math. Soc. **38**(6), 1191–1204 (2001)
64. Qi, F., Habib, S., Mubeen, S., Naeem, M.N.: Generalized k-fractional conformable integrals and related inequalities. AIMS Math. **4**(3), 343–368 (2019)
65. Gözpınar, A.: Some Hermite–Hadamard type inequalities for convex functions via new fractional conformable integrals and related inequalities. AIP Conf. Proc. **1991**(1), Article ID 020006 (2018). https://doi.org/10.1063/1.5047879
66. Sarikaya, M.Z., Set, E., Yaldiz, H., Başak, N.: Hermite–Hadamard's inequalities for fractional integrals and related fractional inequalities. Math. Comput. Model. **57**(9), 2403–2407 (2013)
67. Kian, M., Moslehian, M.S.: Refinements of the operator Jensen–Mercer inequality. Electron. J. Linear Algebra **26**, 742–753 (2013)
68. Sarikaya, M.Z., Yildirim, H.: On Hermite–Hadamard type inequalities for Riemann–Liouville fractional integrals. Miskolc Math. Notes **17**(2), 1049–1059 (2016)
69. Dragomir, S.S., Agarwal, R.P.: Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula. Appl. Math. Lett. **11**(5), 91–95 (1998)
70. Özcan, S., İşcan, İ.: Some new Hermite–Hadamard type inequalities for s-convex functions and their applications. J. Inequal. Appl. **2019**, Article ID 201 (2019)