IMPEDANCE MATRIX — AN UNIFIED APPROACH TO LONGITUDINAL COUPLED–BUNCH FEEDBACKS IN A SYNCHROTRON

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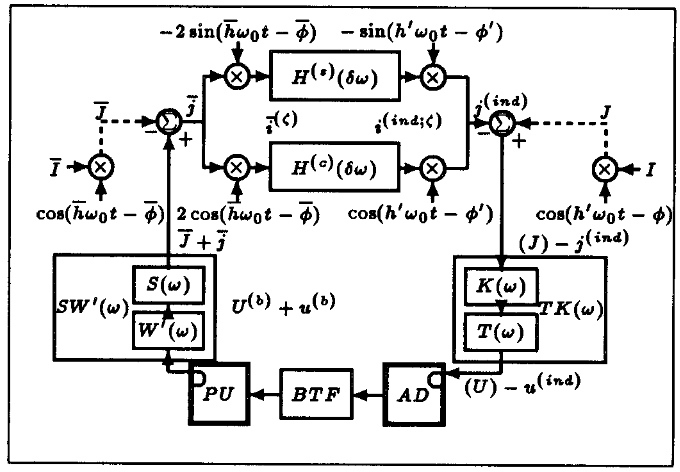
Abstract

Characteristic Eq. of coupled-bunch motion of beam governed by a feedback (FB) is given to find FB's stabilizing effect against coherent instabilities or, say, injection error damping rates. Quite a general FB's schematics is involved: (i) it has two paths, the in-phase and quadrature (or amplitude and phase in a small-signal approach), with unequal gains; (ii) may employ distinct RF-bands to pick-up beam data and feed correction back to the beam. To account for cross-talk between various field and beam current harmonics inflicted by frequency down- and up-mixing, an impedance matrix (with, at most, three non-trivial elements per row) is introduced as a natural concept to gain insight into `FB & beam' dynamics. The important class of FBs to counteract heavy beam loading of accelerating cavities is included into analysis as a particular case.



I. INTRODUCTION

Let $\vartheta = \Theta - \omega_0 t$ be azimuth in a co-rotating frame, where Θ is azimuth around the ring in the laboratory frame, ω_0 is the angular velocity of a reference particle, t is time. The beam current $J(\vartheta, t)$ and longitudinal electric field $E(\vartheta, t)$ are decomposed into $\sum_k J, E_k(\Omega)e^{ik\vartheta - i\Omega t}$ with Ω being the frequency of Fourier transform w.r.t. the co-rotating frame. In the laboratory frame Ω is seen as $\omega = k\omega_0 + \Omega$.

Interacting with passive components inside the vacuum chamber, the beam drives E-field with amplitude

$$E_k(\Omega) = -L^{-1} Z_{kk}(\omega)\, J_k(\Omega), \quad \omega = k\omega_0 + \Omega, \qquad (1)$$

where L is the orbit length, $Z_{kk}(\omega)$, $\mathrm{Re}\, Z_{kk}(\omega) \geq 0$ is the standard longitudinal impedance. Its main-diagonal element is cut from the entire matrix $Z_{kk'}(\omega)$ (it describes the lumped nature of the beam environment) due to a narrow-band response appropriate to, as a matter of fact, slowly perturbed bunched beams,

$$J_{k'}((k-k')\omega_0 + \Omega) \simeq J_k(\Omega)\, \delta_{kk'}, \quad |\Omega| \ll \omega_0, \qquad (2)$$

with $\delta_{kk'}$ being the Kronecker's delta-symbol.

II. FEEDBACK

A. Circuitry with PU $\neq$ AD

To simplify the matters, let a Pick-Up unit and an Acting Device of the FB in question be cavity-like resonant objects which excite longitudinal E-field

$$E^{(a)}(\Theta, t) = L^{-1} G^{(a)}(\Theta)\, u_a(t); \quad a = \mathrm{PU}, \mathrm{AD}, \qquad (3)$$

where $u_a(t)$ is voltage across the gap, $G^{(a)}(\Theta)$ specifies the field localization and is normalized as $\int_0^{2\pi} |G^{(a)}(\Theta)| d\Theta = 2\pi$. Its decomposition into $\sum_k G_k^{(a)} e^{ik\Theta}$ provides $G_k^{(a)}$, the complex transit-time factors at $\omega = k\omega_0$ with $|G_k^{(a)}| \leq 1$ and $\arg G_k^{(a)}$ being proportional to $\Theta^{(a)}$, the object's coordinate along the ring.

Quite a general coupled-bunch FB circuit employing filter methods is shown in the above Fig., Ref.[1]. The circuitry extracts beam data as a band-pass signal at $\omega \simeq \pm\overline{h}\omega_0$, processes it at IF $\omega = 0$ after frequency down-mixing, and then feeds an up-mixed band-pass correction back to the beam at $\omega \simeq \pm h'\omega_0$. Here $\overline{h}, h'$ are integers, and, generally, $\overline{h} \neq h'$; $\overline{h}, h' \neq h$ where h is the main RF harmonic number. The FB has the in-phase (c) and quadrature (s) paths with unequal gains. Treated in a small-signal approach near the FB's set-point, the former one controls an amplitude, while the latter — a phase, of the accelerating voltage seen by the beam. Either of the paths may be switched off altogether, say, $H^{(c)} = 0$ for an injection error damping system, or in case of a dedicated phase control loop.

On neglecting the PU's (small) impact on the beam, the net voltage imposed by the FB can be put down as

$$u_{\mathrm{AD}}^{(tot)}(t) = u_{\mathrm{AD}}^{(b)}(t) - u_{\mathrm{AD}}^{(ind)}(t) \qquad (4)$$

where (b) and (ind) denote beam-excited and FB-induced voltages, correspondingly; $u_{\mathrm{AD}}^{(ind)}(t)$ is a linear functional of $u_{\mathrm{PU}}^{(b)}(t')$ taken at $t' \leq t$ due to casualty.

Let $\delta\omega$ be a frequency deviation with $|\delta\omega| \ll (\overline{h}, h')\omega_0$. Whenever $H^{(c,s)}(\pm 2\overline{h}\omega_0 + \delta\omega) = 0$, the state of the system is given by 2-D column-vectors

$$\vec{u}_{\mathrm{PU}}(\delta\omega) = \left(u(\overline{h}\omega_0 + \delta\omega);\ u(-\overline{h}\omega_0 + \delta\omega)\right)_{\mathrm{PU}}^T, \qquad (5)$$

$$\vec{u}_{\mathrm{AD}}(\delta\omega) = \left(u(h'\omega_0 + \delta\omega);\ u(-h'\omega_0 + \delta\omega)\right)_{\mathrm{AD}}^T. \qquad (6)$$

The in-out gain through the linear FB is

$$\vec{u}_{\mathrm{AD}}^{(ind)}(\delta\omega) = \widehat{\chi}(\delta\omega)\ \vec{u}_{\mathrm{PU}}^{(b)}(\delta\omega) \qquad (7)$$

where $\widehat{\chi}(\delta\omega)$ is a 2×2 FB's `susceptibility' matrix,

$$\begin{aligned}\chi_{11}(\delta\omega) &= 0.25\,TK(h'\omega_0+\delta\omega)\,S(\overline{h}\omega_0+\delta\omega)\times \quad (8)\\ &\times\left(H^{(c)}(\delta\omega)+H^{(s)}(\delta\omega)\right)\,\mathrm{e}^{i(\phi'-\overline{\phi})};\\ \chi_{12}(\delta\omega) &= 0.25\,TK(h'\omega_0+\delta\omega)\,S(-\overline{h}\omega_0+\delta\omega)\times \quad (9)\\ &\times\left(H^{(c)}(\delta\omega)-H^{(s)}(\delta\omega)\right)\,\mathrm{e}^{i(\phi'+\overline{\phi})};\\ \chi_{21}(\delta\omega) &= \chi_{12}(-\delta\omega^*)^*;\ \chi_{22}(\delta\omega) = \chi_{11}(-\delta\omega^*)^*.\end{aligned}$$

Carrier phases $\overline{\phi}$, ϕ' of to beam and accelerating voltage so as to comply with the FB's particular purpose and its layout along the ring.

The beam-excited voltages at the PU and AD are

$$u_a^{(b)}(\omega)=-\begin{pmatrix}W'(\omega)\\ T'(\omega)\end{pmatrix}\sum\nolimits_{k=-\infty}^{\infty}G_{-k}^{(a)}\,J_k(\omega-k\omega_0) \quad (10)$$

where $W', T'(\omega)$ are the gap-voltage responses to the beam current of PU and AD, respectively. Generally, the response of AD to external RF-drive $T(\omega)\neq T'(\omega)$.

Insert Eqs.10 into Eqs.7,4 and extract synchronous-to-beam E-field harmonics from Eq.3. Use Eq.2 to truncate $\sum_k$. Then, to generalize the commonly used impedance concept introduced by Eq.1, the FB can be treated as imposing the E-field harmonics

$$\begin{aligned}E_k^{(fb)}(\Omega) &= -L^{-1}\big(Z_{kk}(\omega)\,J_k(\Omega)+ \quad (11)\\ &+Z^{(fb)}_{k,k-h'+\overline{h}}(\omega)\,J_{k-h'+\overline{h}}(\Omega)\;+\;Z^{(fb)}_{k,k-h'-\overline{h}}(\omega)\,J_{k-h'-\overline{h}}(\Omega)\big)\end{aligned}$$

through coupling impedances

$$\begin{aligned}Z_{kk}(\omega) &= T'(\omega)|G_k^{(\mathrm{AD})}|^2, \quad (12)\\ Z^{(fb)}_{k,k-h'+\overline{h}}(\omega) &= -\chi_{11}(\omega-h'\omega_0)\times \quad (13)\\ &\times\; W'(\omega-h'\omega_0+\overline{h}\omega_0)\,G_k^{(\mathrm{AD})}G^{(\mathrm{PU})}_{-k+h'-\overline{h}},\\ Z^{(fb)}_{k,k-h'-\overline{h}}(\omega) &= -\chi_{12}(\omega-h'\omega_0)\times \quad (14)\\ &\times\; W'(\omega-h'\omega_0-\overline{h}\omega_0)\,G_k^{(\mathrm{AD})}G^{(\mathrm{PU})}_{-k+h'+\overline{h}}.\end{aligned}$$

Here $\omega=k\omega_0+\Omega$, $k\sim h'>0$, $|\Omega|\ll\omega_0$. The negative-frequency domain of $k\sim -h'<0$ is arrived at with the reflection property $Z_{-k,-k'}(-\omega^*)^*=Z_{kk'}(\omega)$.

Eq.12 yields the coupling impedance of AD itself treated as a passive device in line with Eq.1. Eqs.13,14 represent an active response of the FB and account for cross-talk between harmonics E_k, $J_{k'}$ with $k\neq k'$ caused by down- and up-mixing of frequencies. Impedances $Z^{(fb)}_{kk'}(\omega)$ are no longer subject to restriction $\mathrm{Re}\,Z^{(fb)}_{kk'}(\omega)\geq 0$, which is to introduce damping into the beam motion. The balance $H^{(c)}(\delta\omega)=H^{(s)}(\delta\omega)$ of the FB's path gains results in matrix $\widehat{\chi}$ becoming diagonal, and in $Z^{(fb)}_{kk'}(\omega)$ with $|k-k'|=h'+\overline{h}$ vanishing. In injection error damping systems, the FB's path gains and, hence, $Z^{(fb)}_{kk'}(\omega)$ may be scaled reciprocally to, say, the average beam current J_0.

B. Circuitry with PU = AD

Take $h',\overline{h}=h$, $W'(\omega)=T'(\omega)$ with PU and AD being merged into a single device AC, an Accelerating Cavity. This particular case represents an RF FB around the final power amplifier which is responsible for the reduction of periodic beam-loading transients and coupled-bunch instability damping, Ref.[2]. Now, Eq.4 is kept intact while the PU detects both, the beam-imposed and correction signals. Therefore, Eq.7 have to undergo an essential modification:

$$\vec{u}^{\,(ind)}_{\mathrm{AC}}(\delta\omega)=\widehat{\chi}(\delta\omega)\;\vec{u}^{\,(tot)}_{\mathrm{AC}}(\delta\omega) \quad (15)$$

due to which the coupling impedances to enter Eq.11 acquire the form other than that given by Eqs.12–14,

$$\begin{aligned}Z_{kk}(\omega)+Z^{(fb)}_{kk}(\omega) &= \varepsilon^{-1}_{11}(\omega-h\omega_0)\times \quad (16)\\ &\times\; T'(\omega)\,|G_k^{(\mathrm{AC})}|^2,\\ Z^{(fb)}_{k,k-2h}(\omega) &= \varepsilon^{-1}_{12}(\omega-h\omega_0)\times \quad (17)\\ &\times\; T'(\omega-2h\omega_0)\,G_k^{(\mathrm{AC})}G^{(\mathrm{AC})}_{-k+2h}\end{aligned}$$

where $\omega=k\omega_0+\Omega$, $k\sim h>0$, $|\Omega|\ll\omega_0$ and

$$\widehat{\varepsilon}(\delta\omega)=\widehat{I}+\widehat{\chi}(\delta\omega), \quad (18)$$

$$\widehat{\varepsilon}^{-1}(\delta\omega)=\frac{1}{\mathrm{Det}\,\widehat{\varepsilon}(\delta\omega)}\begin{pmatrix}1+\chi_{22}(\delta\omega) & -\chi_{12}(\delta\omega)\\ -\chi_{21}(\delta\omega) & 1+\chi_{11}(\delta\omega)\end{pmatrix}. \quad (19)$$

Here $\widehat{I}$, $\widehat{\varepsilon}(\delta\omega)$ and $\widehat{\varepsilon}^{-1}(\delta\omega)$ are 2×2 matrix unit, FB's `permeability' matrix and its inverse, correspondingly.

This FB may turn self-excited, which is avoided technically by putting zeros of $\mathrm{Det}\,\widehat{\varepsilon}(\delta\omega)$ into the lower half-plane $\mathrm{Im}\,\delta\omega<0$ through a proper tailoring of $H^{(c,s)}(\delta\omega)$.

It is evident hereof that by substituting Eqs.16–17 for Eqs.12–14 the formulae to follow can be extended to treat the important case of $h',\overline{h}=h$; PU, AD = AC as well.

III. CHARACTERISTIC EQUATION

A. General Case

The total E-field at the orbit is a sum of two terms

$$E_k^{(tot)}(\Omega)=E_k^{(ext)}(\Omega)+E_k^{(fb)}(\Omega). \quad (20)$$

The former one, (ext) is imposed from the outside, say, by an external RF drive. The latter, (fb) is the induced response of the environment to the coherent motion of the beam: its perturbed current harmonics $J_k(\Omega)$ drive the FBs, both unintentional (Eq.1) and issued (Eq.11 with its negative-frequency counterpart), to yield

$$E_k^{(fb)}(\Omega)=-L^{-1}\sum\nolimits_{k'=-\infty}^{\infty}z_{kk'}(k\omega_0+\Omega)\,J_{k'}(\Omega), \quad (21)$$

$z_{kk'}(\omega)$ having, at most, three non-trivial elements per row:

$$\begin{aligned}z_{kk'}(\omega) &= Z_{kk'}(\omega)\,\delta_{k'k}+ \quad (22)\\ &+\;Z^{(fb)}_{kk'}(\omega)\big(\delta_{k',k-(h'-\overline{h})k/|k|}+\delta_{k',k-(h'+\overline{h})k/|k|}\big).\end{aligned}$$

The first member in r.h.s. of Eq.22 incorporates effect of all the passive devices available.

From now on, one enters a standard route of instability analysis, and via the Vlasov's linearized Eq. finds

$$J_k(\Omega)=L\sum\nolimits_{k'=-\infty}^{\infty}y_{kk'}(\Omega)\,E_{k'}^{(tot)}(\Omega). \quad (23)$$

Here $y_{kk'}(\Omega)$ is the beam `admittance' matrix which, say, for the beam of average current J_0 in $M \le h$ (h/M is an integer) identical and equispaced bunches is equal to

$$y_{kk'}(\Omega) = CJ_0 \, (Y_{kk'}(\Omega)/k') \sum_{l=-\infty}^{\infty} \delta_{k-k',lM}, \quad (24)$$

$$Y_{kk'}(\Omega) = -i \sum_{m=-\infty}^{\infty} \int_0^{\infty} \frac{m}{\Omega - m\Omega_s(\mathcal{J})} \times \quad (25)$$
$$\times \frac{\partial F_0(\mathcal{J})}{\partial \mathcal{J}} \, I_{mk}(\mathcal{J}) \, I^*_{mk'}(\mathcal{J}) \, d\mathcal{J}.$$

Here $(\psi, \mathcal{J})$ are the longitudinal angle-action variables introduced in the phase-plane $(\vartheta, \vartheta' \equiv d\vartheta/dt)$ with the origin $\vartheta = 0$ being put on the reference particle of a bunch; $\Omega_s(\mathcal{J}) = d\psi/dt$ is the non-linear synchrotron frequency; F_0 is unperturbed bunch distribution normalized to unit; functions $I^*_{mk}(\mathcal{J})$ are the coefficients of series which expand a plane wave $\mathrm{e}^{ik\vartheta(\mathcal{J},\psi)}$ into sum over multipoles: $\sum_m I^*_{mk}(\mathcal{J})\mathrm{e}^{im\psi}$. The leftmost factor C in Eq.24 is

$$C = \Omega_0^2 / \left(hV_0 \sin\varphi_s \right), \quad (26)$$

where Ω_0 is the small-amplitude synchrotron frequency (circular), V_0 is the nominal amplitude of accelerating voltage, φ_s is the stable phase angle ($\varphi_s > 0$ below transition, the synchronous energy gain being $\mathrm{e}V_0 \cos\varphi_s$).

Insert Eq.23 into Eq.21 and use Eq.20 to get

$$E_k^{(ext)}(\Omega) = \sum_{k'=-\infty}^{\infty} \epsilon_{kk'}(\Omega) \, E_{k'}^{(tot)}(\Omega), \quad (27)$$

$$\epsilon_{kk'}(\Omega) = \delta_{kk'} + \chi'_{kk'}(\Omega), \quad (28)$$

$$\chi'_{kk'}(\Omega) = \sum_{k''=-\infty}^{\infty} z_{kk''}(k\omega_0 + \Omega) \, y_{k''k'}(\Omega). \quad (29)$$

Here $\chi'_{kk'}(\Omega)$, $\epsilon_{kk'}(\Omega)$ are `susceptibility' and `permeability' matrices of `beam $\&$ FB' medium. Zeros of the characteristic Eq.

$$\mathrm{Det}\, \widehat{\epsilon}(\Omega) = 0 \quad (30)$$

are the eigen-frequencies of beam coherent oscillations which must be located in the lower half-plane $\mathrm{Im}\,\Omega \le -1/\tau_\epsilon < 0$. Here τ_ϵ is the sought-for damping time of beam coherent oscillations which, as well, determines duration of beam injection transients under the FB showing themselves up, mainly, at the dipole side-bands $\Omega \simeq \pm\Omega_0$.

B. Narrow-Band Case

Label the normal coupled-bunch modes by $n = 0, 1, \ldots, M-1$, phase shift between adjacent bunches being $2\pi n/M$. Suppose h'/M and $\overline{h}/M$ be integers, due to which the FB would not couple beam modes whose $n' \ne n$. Let band-width of the FB be $\Delta\omega \ll M\omega_0$. Hence, there would be only four resonant harmonics $J_k(\Omega)$ of beam current perturbation which belong to the given mode n and cross-talk through the FB. Their subscripts are

$$k'_{1,2} = n + l'_{1,2}M \simeq \pm h', \quad \overline{k}_{1,2} = n + \bar{l}_{1,2}M \simeq \pm\overline{h} \quad (31)$$

with $l'_{1,2}, \bar{l}_{1,2}$ the integers. The essential E-field harmonics $E_k(\Omega)$ to occur within $\Delta\omega$ are the two with $k = k'_{1,2}$. In this 2×2 case $\mathrm{Det}\,\widehat{\epsilon}(\Omega)$ can be found, which results in characteristic Eq. to follow,

$$1 + \chi'_{k'_1k'_1}(\Omega) + \chi'_{k'_2k'_2}(\Omega) + \quad (32)$$
$$+ \left[\chi'_{k'_1k'_1}(\Omega)\, \chi'_{k'_2k'_2}(\Omega) - \chi'_{k'_1k'_2}(\Omega)\, \chi'_{k'_2k'_1}(\Omega) \right] \simeq 0.$$

L.h.s. of Eq.32 involves dispersion integrals $Y_{kk'}$ whose subscripts are $k = k'_{1,2}, \overline{k}_{1,2}$ and $k' = k'_{1,2}$. Given $\Delta\omega\Delta\vartheta_0/\omega_0 \ll \pi$, where $\Delta\vartheta_0$ is bunch half-width, $Y_{kk'}$ become slow functions of k, k', which allow substitutions $k'_{1,2} \simeq \pm h'$, $\overline{k}_{1,2} \simeq \pm\overline{h}$ be performed in subscripts of all the essential $Y_{kk'}$ that enter the characteristic Eq.32.

Usually, at $\Omega \simeq m\Omega_0 + i0$ a single resonant term $Y^{(m)}_{kk'}$ dominates in $\sum_m$ of Eq.25. Hereof, one arrives at the reflection properties of $Y_{kk'} \simeq Y^{(m)}_{kk'}$,

$$Y_{-k,k'} \simeq Y_{k,-k'} \simeq (-1)^m Y_{kk'}, \quad Y_{-k,-k'} \simeq Y_{kk'}. \quad (33)$$

Up to these two assumptions, expression in square brackets of Eq.32 vanishes, while the characteristic Eq. itself reduces to much a simpler form

$$1 + CJ_0 \left(\zeta_n(\Omega)\, Y_{h'h'}(\Omega) + \zeta_n^{(fb)}(\Omega)\, Y_{\overline{h}h'}(\Omega) \right) \simeq 0, \quad (34)$$

being put down in terms of the effective, or instability driving, impedances at side-bands $\Omega \simeq m\Omega_0$ of coupled-bunch mode n,

$$\zeta_n(\Omega) \simeq Z_{k'_1k'_1}(k'_1\omega_0 + \Omega)/k'_1 + \ \ldots \quad k'_1 \to k'_2, \quad (35)$$

$$\zeta_n^{(fb)}(\Omega) \simeq Z^{(fb)}_{k'_1, k'_1 - h' + \overline{h}}(k'_1\omega_0 + \Omega)/k'_1 + \quad (36)$$
$$+ (-1)^m Z^{(fb)}_{k'_1, k'_1 - h' - \overline{h}}(k'_1\omega_0 + \Omega)/k'_1 +$$
$$+ \quad \ldots \quad k'_1 \to k'_2, \quad h' \to -h', \quad \overline{h} \to -\overline{h}.$$

Items with $(-1)^m$, if any, are responsible for the intrinsic asymmetry in damping of within-bunch multipole modes m with opposite parity inherent in FBs with the unbalanced path gains, $H^{(c)} \ne H^{(s)}$.

References

[1] F. Pedersen, CERN PS/90–49(AR), CERN, Geneva, 1990.

[2] D. Boussard, CAS Proceed., CERN/87–03, Vol. 2, Geneva, 1987, pp. 626–646.