PHASE SPACE ELLIPSE TRANSPORT USING TWO RAY VECTORS*

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Abstract

The method described by Lobb for calculating beam envelopes using selected initial ray vectors may be extended by applying the formalism of Norman and Moore to include a generalized description of the entire phase space ellipse (or ellipsoid) using two (or three) initial rays. All relevant information describing the ellipse is contained in a simple matrix and is accessible anywhere in the course of calculation. Existing computer codes as well as other linear ray tracing methods which do not presently exploit the power and conciseness of phase space ellipse transport may easily be adapted to this technique.

Introduction

Lobb[1] has shown that three rays which represent the ends of the semi-axes of an upright phase space ellipsoid contain correlated information which can be used to obtain maximum projections of the ellipsoid after a linear transformation has taken place. This permits determination of the spatial projection or "beam envelope" using only two rays to represent a phase space ellipse or three for the ellipsoid. Other very practical applications exist for these particular rays which actually contain complete information about the phase space ellipsoid.

An explicit description for phase space ellipses using parametric equations and matrix representation has been developed recently by Norman and Moore.[2] This method is an improvement over the σ-matrix approach by Brown and Howry[3] because it conveys the same information without recourse to a congruence transformation. The ellipse matrices of Norman and Moore transform with the simplicity of individual ray vectors. Rays specified as starting conditions by Lobb describe an upright ellipsoid in the matrix representation.

The purpose of this paper is to draw attention to these important developments and further expand on them. Using two ray vectors to represent a phase space ellipse the author has been able to: calculate separately conditions for beam waists, beam diameters and beam minima; plot outlines of the ellipse and beam envelope; approximate the acceptance of apertures distributed throughout a problem; and determine time dispersion in the beam. These procedures are incorporated in a computer program[4] which performs the operations automatically; however, special computer programs are not necessary. Every linear beam transport or ray tracing technique can perform phase space ellipse transport.

The Ellipse Matrix

An arbitrary phase space ellipsoid is described completely by the parametric equations

$$x = e_{11}\cos\theta\,\cos\varphi + e_{12}\sin\theta\,\cos\varphi + e_{13}\sin\varphi + d_1, \qquad (1)$$

$$x' = e_{21}\cos\theta\,\cos\varphi + e_{22}\sin\theta\,\cos\varphi + e_{23}\sin\varphi + d_2, \qquad (2)$$

$$\Delta p/p = e_{31}\cos\theta\,\cos\varphi + e_{32}\sin\theta\,\cos\varphi + e_{33}\sin\varphi + d_3, \qquad (3)$$

where the e_{ij} are constants of the ellipsoid, the d_i are displacements of the center of the ellipsoid along the coordinate axes, x is the projection on the spatial axis, x' is the projection on the divergence axis, $\Delta p/p$ is the projection on the momentum axis and θ and φ are dummy variables which generate the ellipsoid as they vary between 0 and 2π. Unfortunately, the general equations are somewhat cumbersome for purposes of illustration; moreover, most practitioners probably are more familiar with phase space ellipses. Examples which follow will concentrate on a simple phase space ellipse. Extension to other configurations should be reasonably apparent.

Consider a phase space ellipse centered on (but not necessarily aligned with) the coordinate axes; then,

$$x = e_{11}\cos\theta + e_{12}\sin\theta, \qquad (4)$$

$$x' = e_{21}\cos\theta + e_{22}\sin\theta. \qquad (5)$$

The nature of these equations becomes apparent when they are restructured in the following way:

$$x = (e_{11}^2 + e_{12}^2)^{\frac{1}{2}}\cos(\theta + \alpha),\ \alpha = \tan^{-1}(-e_{12}/e_{11}), \qquad (6)$$

$$x' = (e_{21}^2 + e_{22}^2)^{\frac{1}{2}}\sin(\theta + \beta),\ \beta = \tan^{-1}(+e_{21}/e_{22}). \qquad (7)$$

The phase shifts α and β were chosen to simplify the circular functions. The maximum spatial projection is obviously $(e_{11}^2 + e_{12}^2)^{\frac{1}{2}}$; this represents the beam envelope. The maximum angular divergence is $(e_{21}^2 + e_{22}^2)^{\frac{1}{2}}$. The locus of points (x,x') whose projections perform simple harmonic motion on their respective axes as θ varies between 0 and 2π is an ellipse.

Elimination of the parameter θ reduces eqs. (4) and (5) to the quadratic form

$$\sum_i \sum_j a_{ij}\, x_i\, x_j = 1, \qquad (8)$$

which may be compared directly with eq. (3) of Lobb.[1] Relationships between the σ_{ij} (elements of the σ-matrix), e_{ij} and a_{ij} are as follows:

$$\sigma_{11} = (e_{11}^2 + e_{12}^2) = D^2 a_{22}, \qquad (9)$$

$$\sigma_{22} = (e_{21}^2 + e_{22}^2) = D^2 a_{11}, \qquad (10)$$

$$\sigma_{12} = \sigma_{21} = (e_{11}e_{21} + e_{12}e_{22}) = -D^2 a_{12} = -D^2 a_{21}, \qquad (11)$$

$$D = e_{11}e_{22} - e_{12}e_{21}. \qquad (12)$$

The inverse transformation from σ_{ij} (or a_{ij}) to e_{ij} is not unique because the e_{ij} include a phase relationship which specifies where $\theta = 0$ on the ellipse. This phase is of no particular relevance for beam transport; consequently, an arbitrary condition such as $e_{21} = 0$ may be made for convenience in such instances.

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Equations (4) and (5) may also be expressed in matrix notation:

$$\vec{X}_o = \begin{pmatrix} x \\ x' \end{pmatrix}_o = \begin{pmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{pmatrix}_o \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} = E_o \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}. \quad (13)$$

The matrix E_o represents a phase space ellipse. For each value of θ, $\vec{X}_o$ corresponds to a ray vector which, when operated on by any linear transformation T, transforms into a new vector

$$\vec{X} = T\,\vec{X}_o. \quad (14)$$

Substitution of eq. (14) into eq. (13) reveals that

$$\vec{X} = E\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} = T\,\vec{X}_o = T \times E_o \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}.$$

As expected, the original ellipse transforms linearly, point by point, into a new ellipse. More significantly, however,

$$E = T \times E_o, \quad (16)$$

that is, the matrix E_0, representing the original ellipse, evolves under linear transformation into a new ellipse E which is equivalent to continuous mapping by individual ray vectors.

The general ellipsoid is described by a matrix having the following characteristic format:

$$E = \begin{pmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ 0 & 0 & e_{33} \end{pmatrix} \quad \begin{array}{l} \underline{\text{Row entries}} \\ x \text{ - horizontal distance} \\ x' \text{- divergence "angle"} \\ \Delta p/p \text{ - momemtum fraction} \end{array} \quad (17)$$

Programs which combine x- and y-transport in a single matrix will require additional rows for y, y', and $(\Delta p/p)_y$. The number and ordering of rows will depend on the particular program being used.

Displacements of the ellipsoid from the center of coordinates, corresponding to d_1, d_2 and d_3 in eqs. (1)-(3), can be formally introduced within the beam transport matrices.[2] When this feature is available, a row containing a constant (ordinarily unity) is added to each vector; this can be done inside the program without perturbing the format of the input vectors.

Spatial extremes of the general ellipsoid are obtained formally by taking partial derivatives of eq. (1) with respect to θ and φ then setting these to zero. When substituted back into eq. (1) these give

$$x_{max/min} = d_1 \pm (e_{11}^2 + e_{12}^2 + e_{13}^2)^{\frac{1}{2}}. \quad (18)$$

The result is anticipated from eq. (6). If the ellipsoid is centered, d_1 goes to zero and the result is identical to that obtained by Lobb.[1] Maximum projections on the other axes are obtained using analogous procedures.

Ellipse Transport

All linear beam transport calculations are based on some variation of eq. (14). The most popular method involves matrix manipulation by digital computer but graphical and analog techniques also exist. All of these are equally capable, without modification, of solving eq. (16) to achieve phase space ellipse (or ellipsoid) transport. It is only necessary to interpret the individual columns of the matrices E and E_o as ray vectors for this purpose.

Two vectors are required to describe the ellipse; three describe an ellipsoid. Lobb[1] found one of the diagonal representations for an upright ellipsoid to be

$$\begin{array}{cc} & \begin{array}{ccc} \underline{\text{Ray1}} & \underline{\text{Ray2}} & \underline{\text{Ray3}} \end{array} \\ E_o = & \begin{pmatrix} \bar{x} & 0 & 0 \\ 0 & \bar{x}' & 0 \\ 0 & 0 & \overline{\Delta p/p} \end{pmatrix}_o . \end{array} \quad (19)$$

Here $\bar{x}$, $\bar{x}'$, and $\overline{\Delta p/p}$ represent the semi-axes of the ellipsoid in space, divergence and momentum, respectively.

Probably the most convenient representation of the non-upright ellipse is obtained by operating on an upright ellipse with a drift length L such that

$$E = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \bar{x} & 0 \\ 0 & \bar{x}' \end{pmatrix} = \begin{pmatrix} x & L\bar{x}' \\ 0 & \bar{x}' \end{pmatrix}.$$

The resultant ellipse E is tangent to the line $x = \bar{x}'$ at $x = L\bar{x}'$. Because the four elements e_{11}, e_{12}, e_{21} and e_{22} describing any ellipse contain irrelevant phase information it is always possible, using the following expressions, to transform an ellipse matrix so that it appears like the right hand side of eq.(20):

$$e'_{21} = 0, \quad (21)$$

$$e'_{22} = (e_{21}^2 + e_{22}^2)^{\frac{1}{2}}, \quad (22)$$

$$e'_{11} = (e_{11}e_{22} - e_{12}e_{21})/(e_{21}^2 + e_{22}^2)^{\frac{1}{2}}, \quad (23)$$

$$e'_{12} = (e_{11}e_{21} + e_{12}e_{22})/(e_{21}^2 + e_{22}^2)^{\frac{1}{2}}, \quad (24)$$

If the numerator in eq. (24) is zero, the ellipse is upright with semi-axes e_{11} and e'_{22}. If the ellipse is not upright,

$$L = e'_{12}/e'_{22} = (e_{11}e_{21} + e_{12}e_{22})/(e_{21}^2 + e_{22}^2) \quad (25)$$

is the drift distance required to return it to the upright position. This distance is equivalent to the "skew ratio" mentioned by Bassetti et al.[5]

References

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