COMPARISON OF COLLIMATOR WAKEFIELS FORMULAE

A. M. Toader, R. Barlow, University of Manchester, Manchester, UK

Abstract

There is an extensive literature on transverse wakefield kick factors in collimators. We present a compendium of the formulae and discuss their agreement and disagreement with each other.

INTRODUCTION

Transverse wakefields from collimators placed close to the beam can degrade the high beam quality required by advanced accelerators. A collimator comprises transition from a beam pipe to a smaller aperture and then back again. The wakefields of such collimators are assumed to be separable into two components, a geometric component and a resistive component assuming a uniform beam pipe. In this paper, we summarize the relevant formalism for *near-centre* wakefields (i.e., wakefields effects when the beam offset from the centre of collimator is small compared to the aperture), and compare various analytical formulae for the transverse wake kick for collimators.

GEOMETRIC WAKEFIELDS

We consider a symmetric collimator (both longitudinally and transversely) which tapers from vertical half-gap b to vertical half-gap g, where $g \ll b$ and back again. The taper length is distance L_T, resulting in a taper angle $\alpha = tan^{-1}[(b-g)/L_T] \approx (b-g)/L_T$. The collimator can also have a flat region of length d at the minimum half-gap size. The collimators are of round and rectangular type, where the rectangular type has a gap h much larger than the collimating gap $2g$ as sketched in Fig. 1.

For a high energy electron bunch at a vertical distance $y_0 \ll g$ from the axis, the transverse kick factor $k_\perp$ is defined [1] such that after the passage of the collimator the bunch is deflected by an angle

$$< \Delta y' > = \frac{Nr_e}{\gamma} y_0 k_\perp, \qquad (1)$$

where N is the number of particles in the bunch, γ is the relativistic factor, r_e is the classical electron radius. $k_\perp$ is an average over the whole bunch and is typically reported in $V/pC/mm$. The bunch is assumed to be gaussian, with bunch length σ_z with distribution $f_G = 1/\sqrt{2\pi}e^{-\tau^2/2}$ with $\tau = s/\sigma_z$.

Round Collimator

Analytical formulae for the kick factor, can be found in the limits of small and large α, regimes which we denote with the labels *inductive* and *diffractive*, respectively.
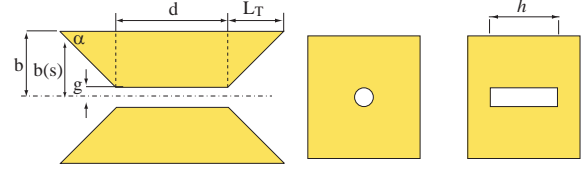
Figure 1: Side view (left) and beam's eye views (right) of a tapered collimator.

Inductive Regime At low frequencies, when $kg \ll 1$ ($k = \omega/c$), the impedance of an obstacle (rectangular with an area $S = hg$, for instance) inside a straight pipe has an inductive character [3]. The inductive regime for small angles (smooth transition) requires both $\alpha \ll \sigma_z/g$ and $\alpha \ll 1$.

According to Yokoya[2], for a general variable beam pipe radius

$$k_\perp = 2I_1 = 2 \int \left(\frac{b'}{b} \right)^2 ds \qquad (2)$$

which for a simple taper is [10]

$$k_\perp = \frac{2\alpha(b-g)}{gb}. \qquad (3)$$

Making the assumption that each electron slice in the bunch acts on only itself this gives

$$\Delta y'(s) = A[2I_1]y_0, \qquad (4)$$

where $A = \frac{Nr_e}{\gamma\sigma_z}f_G(\tau)$, $I_1 = \int \frac{b'^2}{b^2}ds$. Here $b(s)$ is the half-height of the beam pipe as a function of longitudinal coordinate s, and the prime denotes the derivative with respect to s. This gives a kick which varies according to the position of the particle in the bunch; average over the bunch gives $k_\perp = 2I_1$. The condition of applicability of Yokoya's formula [8] is $\alpha kg \ll 1$. The characteristic value of $k(= \omega/c)$ in the beam spectrum is equal to $1/\sigma_z$.

There are various other expressions for $k_\perp$ in literature. Stupakov[4] replaces $k_\perp$ in Eq. (1) with Eq. (5) in the case $g \ll b$. Zagorodnov[5] gives Eq. (6) and Tenenbaum[6] gives same equation Eq. (5) as Stupakov, but later in [7] gives Eq. (7). Although both authors are quoting Yokoya[8] and Stupakov[4][1] one can see there are differences.

$$k_\perp = \frac{Z_0 c\alpha}{4\pi^{3/2}\sigma_z} \frac{1}{g} \qquad (5)$$

$$k_\perp = \frac{Z_0 c\alpha}{2\pi^{3/2}\sigma_z} \left(\frac{1}{g} - \frac{1}{b} \right) \qquad (6)$$

$$k_\perp = \frac{Z_0 c\alpha}{2\pi^{3/2}g^2} \left(1 - \frac{g}{b} \right) \qquad (7)$$

Diffractive Regime At high frequencies, the step regime can be geometrically approximated by a periodic stack of perfectly conducting half-planes, spaced at distance g from each other. Then the theory of plane wave diffraction can be used to evaluate the impedance [3].

The diffractive regime requires $\alpha \gg \sigma_z/g$. Analytical formulae exist in the limits of *short* and *long* collimators which we distinguish by the length of the flat region at minimum aperture ($d \rightarrow 0$ and $d \rightarrow \infty$, respectively). In all cases the kick factor is independent of σ_z.

For a short collimator the dipole kick factor is given by Zagorodnov[5] and Tenenbaum[7] see Eq. (8). For a long collimator Stupakov[1] gives Eq. (9) in the limit $b \gg g$. Note that Stupakov's result agrees with Zimmermann[9] Eq. (10) and Zagorodnov and Tenenbaum's result Eq. (11) [5] [7] but these later two are different.

$$k_\perp = \frac{Z_0 c}{4\pi}\left(\frac{1}{g^2} - \frac{g^2}{b^4}\right) \tag{8}$$

$$k_\perp = \frac{Z_0 c}{2\pi}\frac{1}{g^2} \tag{9}$$

$$k_\perp = \frac{Z_0 c}{2\pi g^2}\left(1 - \frac{g^4}{b^4}\right) \tag{10}$$

$$k_\perp = \frac{Z_0 c}{2\pi}\left(\frac{1}{g^2} - \frac{1}{b^2}\right) \tag{11}$$

Rectangular Collimator

For a rectangular collimator analytical formulae can be found in the limits where the parameter $\alpha h^2/\sigma_z g$ is either small or large compared to 1. In addition to *inductive* and *diffractive* regime rectangular collimators have an *intermediate* regime.

Inductive Regime The conditions for an inductive regime are fulfilled when both $\alpha \ll \sigma_z g/h^2$ and $\alpha \ll 1$. A generalization of Yokoya's approach for a rectangular collimator of large aspect ration, $h \gg g$, was given by Stupakov [4]. A current theoretical result [10] by Stupakov gives

$$\Delta y'(s) = A[(2\pi h I_2 - 2I_1)y_0 + 2I_1 y] \tag{12}$$

where $I_2 = \int \frac{b'^2}{b^3}ds$. Now a term depending on the position of the test particle y, is present. The term is called the quadrupole wake, though this is not the same as the quadrupole term in the standard angular expansion [11]. Bane[10] rewrites Eq. (12) as

$$\Delta y'(s) = A k_\perp \Delta y \tag{13}$$

The kick factor $k_\perp$ becomes

$$k_\perp = \left(\frac{\pi}{2}\right)\frac{2h\alpha(b-g)}{g^2 b} \tag{14}$$

and differs from one given for round collimator Eq. (3) by $(\pi/2)h/g$.

Eq. (13) has been implemented in the tracking code PLACET by G. Rumolo[12] as

$$\Delta y' = A\left[\left(\pi h\frac{(b-g)^2(b+g)}{g^2 b^2 L_T} - 2\frac{(b-g)^2}{gbL_T}\right)y_0 \right.$$
$$\left. + 2\frac{(b-g)^2}{gbL_T}y\right] \tag{15}$$

Considering only the dipole contribution, the kick factor calculated by Stupakov[1] in the case $b \gg g$ is given in Eq. (16), Zagorodnov[5] in Eq. (17) and Tenenbaum[7] in Eq. (18) as

$$k_\perp = \frac{Z_0 c}{4\sqrt{\pi}}\frac{\alpha h}{2\sigma_z g^2} \tag{16}$$

$$k_\perp = \frac{Z_0 c}{4\sqrt{\pi}}\frac{\alpha h}{\sigma_z}\left(\frac{1}{g^2} - \frac{1}{b^2}\right) \tag{17}$$

$$k_\perp \approx \frac{Z_0 c}{4\sqrt{\pi}}\frac{\alpha h}{g^3}. \tag{18}$$

Again, these expressions are different. But, in order-of-magnitude terms, one can see that the kick from a swallow tapered rectangular collimator, see Eq. (18), is larger than the kick from an equivalent round one, Eq. (7) by a factor $\pi h/2g$ in the limit $g \ll b$.

Diffractive Regime In the diffractive regime in the limit $\alpha \geq \pi^2 \sigma_z g/h^2$, the dipole kick factor for a rectangular collimator compares to a round collimator and is given by Eq. (8) and Eq. (11) by Tenenbaum and Zagorodnov and by Eq. (9) multiplied by one half by Stupakov.

Taking into account both dipole and the quadrupole terms as defined in Eq. (12), Rumolo[12] gives

$$\Delta y'(s) = A\frac{\sigma_z 2\sqrt{2}\pi}{g^2}\left[1 - \frac{g^4}{b^4}\right]\left(Y_D y_0 + Y_Q y\right) \tag{19}$$

where $Y_D \approx \pi^2/12$ and $Y_Q \approx \pi^2/24$ are called the *Yokoya factors* associated to dipole and quadrupole wake fields.

Intermediate Regime If $\alpha \ll \sigma_z/g$ but $\alpha \geq \pi^2 \sigma_z g/h^2$ then a rectangular collimator is in the intermediate regime. In this case, considering $b \gg g$, the dipole kick factor is given by: Stupakov[1] Eq. (20), Zagorodnov[5] Eq. (21), Tenenbaum[7] Eq. (22)

$$k_\perp \approx (2.7)\frac{Z_0 c}{4\pi}\sqrt{\frac{\alpha}{\sigma_z g^3}} \tag{20}$$

$$k_\perp = C\frac{Z_0 c}{4\pi}\sqrt{\frac{\alpha}{\sigma_z g^3}}, \tag{21}$$

with $C = 1$ for a long collimator ($d \rightarrow \infty$) and $C = 1/2$ for a short collimator ($d \rightarrow 0$)

$$k_\perp \approx (1.35)\frac{Z_0 c}{4\pi g^2}\sqrt{\alpha} \tag{22}$$

Note that the kick factor given by Tenenbaum, Eq. (22) is a factor 2 different than the one given by Stupakov in

Eq. (20). Furthermore, Eq. (22) considers only the dipole impedance and and neglects the quadrupole impedance. The kick factors given by Stupakov and Zagorodnov differ by a factor C but show same dependence of $(\sigma_z g^3)^{-1/2}$. In PLACET Rumolo[12] uses Eq. (20) to define the kick factor $k_\perp$ in Eq. (13).

RESISTIVE WAKEFIELDS

The theory of wakefields due to finite resistive vacuum chambers has been developed by Piwinsky[13] and amplified by Bane[14]. The scale for the problem is given by

$$s_0 = \left(\frac{cg^2}{2\pi\sigma}\right)^{1/3}_{Gaussian} \quad or \quad s_0 = \left(\frac{2g^2}{Z_0\sigma}\right)^{1/3}_{SI} \quad (23)$$

with c the speed of light, g the tube radius, $Z_0 \equiv 377\Omega$ the impedance of free space and σ the conductivity of the metal walls. The resistive wakefields can be classified in two different regimes.

Long-range Regime

The long range resistive wall wakefields due to a point charge moving at the speed of light in a cylindrical tube had been given by Chao[11].

We are in the long range regime if $\sqrt{\sigma_z \lambda} \ll g \ll (\sigma_z/\lambda)\sqrt{\sigma_z\lambda}$ with $\lambda = 1/Z_0\sigma$ known as the resistive depth. For a rectangular collimator, in the long-range regime in PLACET Rumolo[12] gives the condition for long range regime as: $0.63s_0 \ll z \ll (2g^2 Z_0\sigma)$, with z the distance at which the wake generated by a source charge is calculated. Hence the resistive kick as the sum of a flat contribution Eq. (24) and tapered contribution Eq. (25) using Chao's formulae

$$k_\perp^F = \frac{d}{g^3}\sqrt{\lambda\sigma_z} \quad (24)$$

$$k_\perp^T = \frac{(b+g)L_T}{g^2 b^2}\sqrt{\lambda\sigma_z} \quad (25)$$

Bane et al.[10] gives the resistive-wall kick as the expressions found by Rumolo, with the exception of being multiplied by geometric factor α_R, which is 1 for a round collimator and $\pi^2/8$ for a rectangular collimator.

While Tenenbaum[6] as the sum of Eq. (26) for the flat contribution and Eq. (27) for the tapered contribution, assuming the case $g \ll b$,

$$k_\perp^F = \alpha_R \Gamma(0.25)\frac{d}{g^3}\sqrt{\lambda\sigma_z} \quad (26)$$

$$k_\perp^T = \alpha_R \Gamma(0.25)\frac{1}{\alpha g^2}\sqrt{\lambda\sigma_z} \quad (27)$$

where α is the tapered angle of collimator, α_R is the same as defined by Bane. Note that compared to Bane's expression $k_\perp^F$ has one more factor, $\Gamma(0.25) = 3.62560$, while $k_\perp^T$ is quite different.

Short-range Regime

For short bunches, such that $z < 0.63(2g^2/Z_0\sigma)^{1/3}$ the short-range wake fields for a rectangular collimator have been given by Bane and Sands[14]

$$k_\perp = \int \int E_z^1(z,s)dz'ds \quad (28)$$

with

$$E_z^1(s) = -\frac{16q}{g^2}\left[\frac{1}{3}e^{\frac{s}{s_0}}\cos\left(\frac{\sqrt{3}s}{s_0}\right)\right.$$
$$\left. -\frac{\sqrt{2}}{\pi}\int_0^\infty \frac{x^2 e^{-x^2\frac{z'}{s_0}}}{x^6+8}dx\right] \quad (29)$$

having defined $s_0(s) = (2g(s)^2/Z_0\sigma)^{1/3}$ [14]. This is implemented in PLACET by Rumolo[12].

SUMMARY

There are several different formulae for wakefield kick factors in the the literature. Sometimes differences are due to different regime application, but sometimes there appears to be real disagreement. This may be due to misprints, or error in our understanding, or other reasons. A full comparison will be found on *http://www.hep.manchester.ac.uk/u/adina/* which will be uploaded as such issues are resolved.

REFERENCES

[1] G.Stupakov, in Proceedings of the PAC, 2001

[2] P. Chen, K. Yokoya, Phys.Rev.D38:987, 1988

[3] B.W. Zotter and S. A. Kheifets, "Impedances and Wakes in High-Energy Particle Accelerators" World Scientific, 1998

[4] G. Stupakov, SLAC, Tech. Rep. SLAC-PUB-7167, 1996

[5] I. Zagorodnov and K.L.F. Bane EUROTeV-Rep-2006-074, 2006

[6] P. Tenenbaum, Linear Collider Collaboration Tech Notes, LCC-0101, August 2002

[7] P. Tenenbaum et al. Phys. Rev. ST Accel. Beams **10**, 034401, 2007

[8] K.Yokoya, CERN, Tech. Rep. CERN SL/90-88 (AP), 1990

[9] F. Zimmermann, K. L. Bane and C. K. Ng, in Proc. European Particle Accelerator Conference, Sitges, 1996

[10] K. L. Bane, J. Irwin, T. Raubenheimer, in NLC ZDR, SLAC Report 474, Vol.2, p 594, 1996

[11] A. W. Chao, "Physics of Collective Beam Instabilities In High Enengy Accelerators", NY, 1993

[12] G. Rumolo et al. EUROTeV-Report-2006-026, 2006

[13] A. Piwinski, "Impedance in Lossy Elliptical Vacuum Chamberr", DESY-94-068, 1994

[14] K. L. F. Bane and M. Sands, SLAC-PUB-95-7074, 1995