EDDY-CURRENT EFFECT ON FIELD MULTIPOLES ARISING IN DIPOLE MAGNETS WITH ELLIPTIC AND RECTANGULAR BEAM PIPE

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Abstract

We present an analytical evaluation of the field-distortion effect from eddy currents induced by the time variation of magnetic field of dipole magnets in the elliptic or rectangular beam pipe of finite conductivity. The pipe sizes and aspect are arbitrary except that for practical reasons we assume the pipe wall thickness to be small as compared to the skin depth. Handy formulas are presented for the field multipoles arising from non-round shape of the beam pipe.

ELLIPTIC CONDUCTING BEAM PIPE

Consider a cylindrical pipe of elliptic cross-section whose (uniform) thickness d is negligible compared with its major and minor semi-axes, a and b, as well as the skin depth δ: $d \ll a,\, b,\, \delta$. The pipe is put into a homogeneous AC magnetic field perpendicular to the pipe axis directed along the coordinate z-axis, the field amplitude is taken as unity, and its oscillation frequency is ω, so that $\delta = c/\sqrt{2\pi\sigma\omega}$, where σ is the pipe wall conductivity. Find the eddy-current distribution over the pipe circumference and modification of the field due to eddy-currents.

Elliptic Coordinates

We use the elliptic cylindrical coordinates $\mu,\,\theta$ with the focal parameter $2f$, which corresponds to the distance between the foci of the elliptic and hyperbolic coordinate lines in the coordinate plane $x0y$. Their connection to the Cartesian coordinates $x,\,y$ reads:

$$\cosh(\mu + i\theta) = (x + iy)/f\,, \tag{1}$$

$$x = f\cosh\mu\,\cos\theta\,, \qquad y = f\sinh\mu\,\sin\theta\,, \tag{2}$$

$\mu \geq 0,\ -\pi \leq \theta \leq \pi$, and the Lamé factors are

$$h_\mu = h_\theta = h = f\sqrt{\cosh^2\mu - \cos^2\theta}. \tag{3}$$

The foci position is $y = 0,\ x = \pm f$, and the parameter f is found from the pipe sizes $a\,, b$: $f = \sqrt{a^2 - b^2}$.

The elliptic coordinates can be expressed via the Cartesian coordinates using the distances $r_{1,2}$ to the foci,

$$r_{1,2} = \sqrt{(x \pm f)^2 + y^2} = f(\cosh\mu \pm \cos\theta),$$

whence

$$\cosh\mu = (r_1 + r_2)/2f\,, \qquad \cos\theta = 2x/(r_1 + r_2). \tag{4}$$

Vector Potential

We take the vector potential parallel to the z-axis: $\mathbf{A} = (0,\,0,\,A(\mu,\theta))$. In free space $\Delta_\perp A = 0$,

$$\frac{1}{h^2}\left(\frac{\partial^2}{\partial\mu^2} + \frac{\partial^2}{\partial\theta^2}\right)A = 0\,.$$

The Laplacian operator is separable in the elliptic coordinates, thus the suitable form of the vector potential in free space compatible with the asymptotic homogeneous vertical field (unity apmplitude) reads

$$A \sim -x + \sum\nolimits_q (C_1(q)\cosh q\mu + C_2(q)\sinh q\mu)\cos q\theta, \tag{5}$$

where q is the separation constant which should be integer for the field to be invariant under the transformation $\theta \to \theta + 2\pi$, while $C_{1,2}$ are to be found from the boundary conditions.

In the inner domain, inside the pipe, $\mu < \mu_0$, where $\tanh\mu_0 = b/a$, the fields are regular at $\mu \to 0$, so $C_2 = 0$. In the outer domain, $\mu > \mu_0$, we should have $A < +\infty$ when $\mu \to \infty$, therefore $C_2 = -C_1$ $(q > 0)$. Thus,

$$\begin{aligned} A \;=\; & -f\cosh\mu\,\cos\theta \\ & + f\sum_{n=1}^{\infty}\frac{a_n\left(e^{-(2n-1)|\mu-\mu_0|} + e^{-(2n-1)(\mu+\mu_0)}\right)}{2(2n-1)} \\ & \times\cos(2n-1)\theta\,. \end{aligned} \tag{6}$$

Mixed Boundary Condition

The series coefficients in Eq. (6) are written so as to have a continuous vector potential at the boundary $\mu = \mu_0$, and thus the field $H_\mu = \frac{1}{h}\frac{\partial A}{\partial\theta}$ normal to the pipe wall, will be continuous. Discontinuity in the tangential component $H_\theta = -\frac{1}{h}\frac{\partial A}{\partial\mu}$ is related to the eddy-current surface density i,

$$H_\theta|_{\mu=\mu_0+0} - H_\theta|_{\mu=\mu_0-0} = \frac{4\pi}{c}i\,. \tag{7}$$

For a very thin wall we can write

$$i \approx d\,j = d\sigma E_z = -\frac{1}{c}d\sigma\dot{A} = \frac{i\omega}{c}d\sigma A\,. \tag{8}$$

Using Eqs. (7,8) we arrive at the boundary condition for the vector potential which has the mixed form: it relates the normal derivative of A, Eq. (6), with its value on the boundary $\mu = \mu_0 = \mathrm{arctanh}\, b/a$,

$$\left.\frac{\partial A}{\partial\mu}\right|_{\mu=\mu_0-0} - \left.\frac{\partial A}{\partial\mu}\right|_{\mu=\mu_0+0} = \frac{2id}{\delta^2}\,hA|_{\mu=\mu_0} \tag{9}$$

Coupling of Harmonics

Term-by-term equating of the Fourier series in θ will be possible in Eq. (9) if we express the periodic factor $h/f = \sqrt{\cosh^2\mu_0 - \cos^2\theta}$ via its Fourier harmonics h_{2j} available in terms of the hypergeometric function [1],

$$\begin{aligned} h_{2j} &= -(2-\delta_{j0})\frac{(2j-3)!!}{2^{3j}\, j!\, \cosh^j\mu_0} \\ &\quad \times {}_2F_1(j-\frac{1}{2},\, j+\frac{1}{2}; 2j+1; \mathrm{sech}^2\mu_0). \end{aligned} \quad (10)$$

The result can also be expressed in terms of the complete elliptic integrals $\mathbf{K} \equiv \mathbf{K}(\mathrm{sech}^2\mu_0)$ and $\mathbf{E} \equiv \mathbf{E}(\mathrm{sech}^2\mu_0)$, $\mathrm{sech}^2\mu_0 = 1 - b^2/a^2$,

$$\begin{aligned} h_0 &= \frac{2}{\pi}\cosh\mu_0\mathbf{E}\,, \\ h_2 &= \frac{4}{3\pi}\cosh\mu_0\left((\cosh 2\mu_0 - 1)\mathbf{K} - \cosh 2\mu_0\mathbf{E}\right)\,, \\ h_4 &= \frac{4}{15\pi}\cosh\mu_0(4\cosh 2\mu_0(\cosh 2\mu_0 - 1)\mathbf{K} \\ &\quad -(2\cosh 4\mu_0 - 1)\mathbf{E}),\ \text{etc.} \end{aligned}$$

Multiplying two series on the RHS of Eq. (9) we obtain a linear combination of different Fourier coefficients a_n at each $\cos(2n-1)\theta$ term. For convenience of computation, Eq. (9) can be rewritten in the form of an infinite set of linear equations with respect to $a_n,\ n,\ n' = 1,\ 2,\ 3\ldots$:

$$a_n = \frac{idf}{\delta^2}H_{nn'}(-\cosh\mu_0\,\delta_{n'1} + D_{n'n''}a_{n''}), \quad (11)$$

where we introduced two infinite matrices $\mathbf{H}$ and $\mathbf{D}$; the form of the diagonal matrix $\mathbf{D}$ follows from Eq. (9),

$$\mathbf{D} = \mathrm{diag}\left\{\frac{1+e^{-2(2n-1)\mu_0}}{2(2n-1)}\right\},\ n = 1,\ 2,\ 3\ldots \quad (12)$$

and the symmetric harmonic-coupling matrix $\mathbf{H}$ can be found by decomposition of the cosine products in Eq. (9) into cosine sums,

$$\mathbf{H} = \begin{pmatrix} 2h_0+h_2 & h_2+h_4 & h_4+h_6 & \ldots \\ h_2+h_4 & 2h_0+h_6 & h_2+h_8 & \ldots \\ h_4+h_6 & h_2+h_8 & 2h_0+h_{10} & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{pmatrix} \quad (13)$$

With these definitions, the linear equation set, Eq. (11), takes the form

$$\left(\mathbf{I} - \frac{idf}{\delta^2}\mathbf{H}\cdot\mathbf{D}\right)\cdot\mathbf{a} = -\frac{idf}{\delta^2}\cosh\mu_0\,\mathbf{H}\cdot\mathbf{1}, \quad (14)$$

where we introduced the identity matrix $\mathbf{I}$ and vectors $\mathbf{a}, \mathbf{1}$:

$$\begin{aligned} \mathbf{a} &= (a_1,\, a_2,\, a_3, \ldots)\,, \\ \mathbf{1} &= (1,\, 0,\, 0,\, \ldots)\,. \end{aligned} \quad (15)$$

Now the inverse matrix,

$$\mathbf{R} = \left(\mathbf{I} - \frac{idf}{\delta^2}\mathbf{H}\cdot\mathbf{D}\right)^{-1}, \quad (16)$$

yields the solution to Eq. (14),

$$\mathbf{a} = -\frac{idf}{\delta^2}\cosh\mu_0\,\mathbf{R}\cdot\mathbf{H}\cdot\mathbf{1}. \quad (17)$$

Truncating the matrices and vectors to a finite dimension, we can find the series coefficients in Eq. (6) with any needed accuracy. For a weak shielding effect, we neglect the self-consistent part in the fields and obtain from Eqs. (16,17)

$$\mathbf{R} \approx \mathbf{I}\,,\ \mathbf{a} \approx -\frac{2ida}{\delta^2}\mathbf{H}\cdot\mathbf{1},\ \text{for}\ \frac{df}{\delta^2} \ll 1\,.$$

Multipole Expansion of the Field

In the central part of the chamber aperture we can expand the found fields into a power series in terms of $x + iy = re^{i\alpha}$, where $r,\ \alpha$ are the polar coordinates,

$$A = \sum_{l=1}^{\infty}\frac{c_{2l-1}}{(2l-1)!}r^{2l-1}\cos(2l-1)\alpha\,, \quad (18)$$

where we retained only terms with the dipole symmetry.

We observe that the coordinate dependence for $\mu < \mu_0$ in Eq. (6) can be recast into a sum of powers of $re^{i\alpha}$,

$$\cosh(2n-1)\mu\cos(2n-1)\theta = \mathrm{Re}[T_{2n-1}(\frac{r}{f}e^{i\alpha})] \quad (19)$$

where T_{2n-1} is the Chebyshev polynomial. Using explicit formula for its coefficients [2], substituting Eq. (19) into Eq. (18), and rearranging summation, we find the multipole coefficients $c_{2l-1},\ l = 1, 2,\ 3\ldots$ in Eq. (18),

$$\begin{aligned} c_{2l-1} &= -\delta_{l1} + (-1)^l\left(\frac{2}{f}\right)^{2(l-1)} \\ &\quad \times\sum_{n=l}^{\infty}(-1)^n\frac{(n+l-2)!}{(n-l)!}a_n e^{-(2n-1)\mu_0}, \end{aligned} \quad (20)$$

where δ_{l1} is the Kronecker symbol, and coefficients a_n are given by Eq. (17).

RECTANGULAR CONDUCTING VACUUM CHAMBER

Consider a window-frame dipole magnet where the infinitely-thin current sheets on the sides generate a quasi-static AC magnetic field. In our simplified two-dimensional model, a rectangular window with the size $2a \times 2b$ is surrounded with a perfect non-conducting magnetic medium ($\mu \to \infty$). Without a vacuum chamber, the uniform current density would have produced a uniform field. Our goal is to find the distribution of eddy-currents over the rectangular conducting vacuum chamber with thin walls lining the window from inside, as well as to calculate distortion of the uniform field caused by eddy-currents.

We consider the case where the plane conducting walls are modelled by ”infinitely thin” eddy-current sheets with coordinates $y = \pm b$, $-a \leq x \leq a$, and $x = \pm a$, $-b \leq y \leq b$, for the horizontal and vertical walls, respectively. This assumption holds if the wall thickness d is negligible as compared with the window sizes a, b, and with the effective skin depth δ: $d \ll a,\ b,\ \delta$.

Horizontal Conducting Plates

We characterize the magnetic field at $\mathbf{r} = (x,\ y,\ z)$ by the vector potential $\mathbf{A} = (0,\ 0,\ A)$ and assume the harmonic time-dependent factor $e^{-i\omega t}$ in all the field components. In the rectangular domain of free space, $|x| < a$, $|y| < b$, the field is Laplacian,

$$A = -x + \sum_{n=0}^{\infty} \frac{a_n}{k_n} \sin k_n x \, \frac{\cosh k_n y}{\sinh k_n b}. \qquad (21)$$

The first term stands for the oscillating uniform vertical driving field with unity amplitude, and the Fourier-series with a dipole symmetry represents the contribution from eddy-currents induced in the horizontal conducting plates.

The boundary condition for induced fields at $x = \pm a$ reads $1 - H_y = 0$ for any y, and differentiating Eq. (21) over x we deduce: $\cos k_n a = 0$; thus,

$$k_n = \pi(2n+1)/2a. \qquad (22)$$

The boundary condition at $y = \pm b$, $|x| \leq a$, implies vanishing tangential components of magnetic field at the surface of perfect magnetic material, and involves the surface density of eddy-currents, i,

$$H_x|_{y=b} = \frac{4\pi}{c} i\,. \qquad (23)$$

Neglecting non-uniformity of current density j over the conducting plate thickness d, we relate the surface density $i \approx jd$ with the curl electric field $E_z = -\frac{1}{c}\dot{A}$, using Ohm's law, $j = \sigma E_z$, where σ is the wall conductivity, we transform the RHS of Eq. (23):

$$H_x|_{y=b} = \frac{4\pi}{c} i = \frac{4\pi}{c} d\,\sigma E_z = \frac{4\pi i\,\sigma\omega}{c^2} d\ A|_{y=b}\,, \qquad (24)$$

and, finally, the mixed boundary condition reads

$$H_x|_{y=b} = \frac{2id}{\delta^2}\ A|_{y=b}\ ,. \qquad (25)$$

Substituting Eqs. (21) into Eq. (25), we rewrite the boundary condition Eq. (23) in terms of the Fourier series,

$$\sum_{n=0}^{\infty} a_n \sin k_n x = \frac{2id}{\delta^2}\left(-x + \sum_{n=0}^{\infty} \frac{a_n}{k_n} \sin k_n x \coth k_n b\right) \qquad (26)$$

To find the unknown series coefficients a_n, we expand the first term on the RHS of Eq. (26) into the Fourier series on the interval $-a < x < a$,

$$x = \frac{8a}{\pi^2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \sin k_n x\,. \qquad (27)$$

Now, equating the series in Eq. (26) term-by-term, we obtain:

$$\begin{aligned} a_n \quad &= \quad -\frac{2id}{\delta^2} \frac{8a\,(-1)^n}{\pi^2(2n+1)^2} \qquad (28) \\ &\times \left(1 - \frac{2id}{\delta^2} \frac{2\,a}{\pi(2n+1)} \coth \frac{\pi(2n+1)b}{2\,a}\right)^{-1}. \end{aligned}$$

Thus, the found coefficients, Eq. (28), of the Fourier series, Eq. (21), together with the spatial harmonic wavenumbers, Eq. (22), completely determine the magnetic field inside the rectangular domain between the horizontal conducting plates.

Multipole Expansion of the Field

As before, in the central part of the chamber aperture we can expand the found fields into a power series in terms of $x + iy = re^{i\alpha}$, where r, α are the polar coordinates,

$$A = \sum_{l=0}^{\infty} \frac{c_{2l+1}}{(2l+1)!} r^{2l+1} \cos(2l+1)\alpha\,, \qquad (29)$$

where we retained only terms with dipole symmetry.

We observe that the coordinate dependence in Eq. (21) can be recast into a power-series form,

$$\sin k_n x \cosh k_n y = \sum_{l=0}^{\infty} \frac{(-1)^l (k_n r)^{2l+1}}{(2l+1)!} \cos(2l+1)\alpha. \qquad (30)$$

Substituting Eq. (30) into Eq. (21), and rearranging summation, we obtain expressions for the multipole coefficients in Eq. (29),

$$c_{2l+1} = -\delta_{l0} + (-1)^l \sum_{n=0}^{\infty} \frac{a_n k_n^{2l}}{\sinh k_n b}, \quad l = 0,\ 1, 2, \ldots \qquad (31)$$

where δ_{l0} is the Kronecker symbol, and a_n, k_n are given by Eqs. (28,22).

DISCUSSION

The vertical walls of the rectangular beam pipe alone leave the field uniform, and their uniform eddy-currents reduce the field amplitude, i.e., $H_y = 1$ is replaced by

$$H_y = \left(1 - 2id_v a/\delta_v^2\right)^{-1}. \qquad (32)$$

However, the combined effect of vertical and horizontal conducting walls leads to non-uniform eddy-current density in the side walls, its analysis is performed in line with the approaches shown in the above sections. The involved form of the result falls beyond the scope of this paper.

Generalization of the obtained results to the case of quadrupole magnet is straightforward.

REFERENCES

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