AN ANALYSIS OF LUMPED CIRCUIT EQUATION FOR SIDE COUPLED LINAC (SCL)

A. D'Elia, M. R. Masullo, V. G. Vaccaro*, INFN and University of Naples - ITALY

Abstract

The behaviour of an SCL module is generally described resorting to an equation system borrowed from lumped circuit theories. This description holds for a narrow frequency band (mono-modal cavity behaviour). A milestone in this field is represented by the classical analysis made by Nagle & alii where they introduced an equation system allowing for the resonant frequencies of the cavities and the first and second order coupling constants. Eigenvalues and eigenvectors (resonant frequencies of the system and relevant current amplitudes) are also given. For the case of half-cell termination and non zero second order coupling constants, we show that the system, even if it is correct for on axis cavities, should not be taken unwisely. If not so, troubles will come from the first two and last two equations of the system. Due to the relevance of this formulation and of the case treated, we give an interpretation of the physical meaning of these equations and suggest the correct use of above mentioned system. We will suggest a measurement (numerical and/or experimental) procedure which may give accurate evaluation of the parameters of the lumped circuit representation. By doing so the use of the circuit representation will be put on firm basis.

A thorough investigation was made on the accuracy of the parameters estimation due to measurement errors and fabrication tolerances.

INTRODUCTION

A Side Coupled Linac (SCL) is formed by a certain number of Accelerating Cavities (AC) on axis with the traveling particles coupled with a certain number of off-axis Coupling Cavities (CC). When the cavities are assembled, they loose their individuality and the whole system can resonate at frequencies each one characterized by its own field phase advance from cavity to cavity. A simple analogy was drawn [1] between a chain of coupled cavities and a chain of masses supported by springs or of atoms bound in a lattice: in the case of N identical coupled resonator chain, the whole system exhibits a comb of N resonant frequencies, each characterized by its own phase advance (modes). These modes have the same field configuration and this behaviour will allow to deal the system by means of a lumped circuit model. In the case of an SCL it is possible to represent the total structure as a biperiodic chain of resonant circuits. The first chain is formed by the AC cavities, cou-

pled with the nearest neighbors (CC) and with the second nearest neighbors (AC); the second chain is formed by the CC cavities coupled with the nearest neighbors (AC) and with the second nearest neighbors (CC).

The lumped circuit representation for such a structures is very powerful since it gives a full and exact characterization of the behaviour of a coupled system without using the more complex field equations. This allows for building home-made codes able to study and to characterize in a simpler and faster way these devices. These codes becomes very useful tools in the SCL designing, testing and tuning phases [2, 3, 4].

THE COUPLED RESONATOR MODEL FOR A BIPERIODIC CHAIN

Let us examine the circuit equation in ref. [1] and its solutions (26) and (27) in the lossless case for the half-cell terminated chain. The above mentioned solutions, when introduced into the circuit equations, lead the well known dispersion relation (28) of the above quoted paper.

Let us explicitly re-write the equation for $n = 0, 1$

$$\left(1 - \frac{f_a^2}{F^2}\right) X_0 + \frac{k_1}{2}\left(X_{-1} + X_1\right) + \frac{k_a}{2}\left(X_{-2} + X_2\right) = 0$$

$$\left(1 - \frac{f_c^2}{F^2}\right) X_1 + \frac{k_1}{2}\left(X_0 + X_2\right) + \frac{k_c}{2}\left(X_{-1} + X_3\right) = 0$$

(1)

where the quantities X's are the currents in the equivalent circuit. Even if the circuit stops at the half cell $n = 0$, the currents X_{-1} and X_{-2} have a deep meaning. Indeed the half-cell termination is realized by means of a perfect conducting plate which acts as a mirror of the whole chain. This means that one should take into account also the virtual currents which are the images of the real ones. Because of the boundary conditions, these currents are just equal to the ones with the index of positive sign: $X_{-1} = X_1$ and $X_{-2} = X_2$. So that the equations becomes

$$\left(1 - \frac{f_a^2}{F^2}\right) X_0 + k_1 X_1 + k_a X_2 = 0$$

$$\left(1 - \frac{f_c^2}{F^2} + \frac{k_c}{2}\right) X_1 + \frac{k_1}{2}\left(X_0 + X_2\right) + \frac{k_c}{2} X_3 = 0.$$

(2)

One can verify that the above representation of the circuit behaviour satisfies the dispersion relation (28) of the ref. [1]. These equations are certainly correct for a structure with all the cavities on axis. For a Side Coupled Linac formed by coupling cavities off axis with alternating offset

* vaccaro@na.infn.it

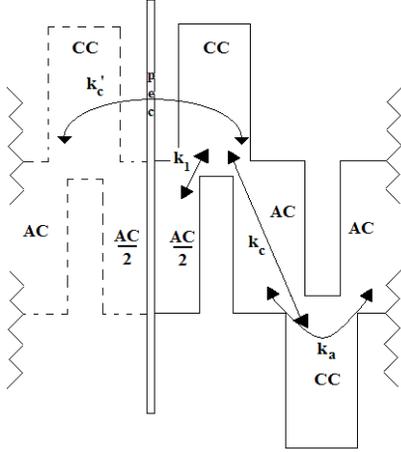one has to carefully consider the mirror images. Indeed,

Figure 1: SCL formed by coupling cavities off axis with alternating offset.

if the half cell configuration stops with an halved accelerating cavity, the mirror image is not the correct continuation of the cavity chain because, as one may see from Fig. 1, the periodicity of the structure is broken. This has the important consequence that the second order coupling coefficients between cell number 1 and its image cell, number -1, is no longer equal to the constant k_c, which was defined for the periodic structure, but it is transformed in a new constant that we name k_c'. Its value will be closer to k_a rather than to k_c; namely, may be, almost one order of magnitude larger. The second line of the equation (2) becomes then

$$\left(1 - \frac{f_a^2}{F^2} + \frac{k_c'}{2}\right) X_1 + \frac{k_1}{2} (X_0 + X_2) + \frac{k_c}{2} X_3 = 0. \quad (3)$$

From this we may infer that the dispersion relation eq. (28) of ref. [1] is no longer satisfied as well as the eigenvectors do not satisfy the eq (27) of the same reference. In order to get the real dispersion relation we just have to resort to the system determinant equated to zero.

This situation does not happen if the p.e.c. mirror is placed in the CC's. Indeed, in this case, the replica does not break the periodicity. Therefore we may state that a half cell structure with CC's in the ends is representative of an infinite structure. Indeed this structure will exhibit the values of f_a, f_c, k_1, k_a and k_c as an infinite system.

USEFUL APPLICATIONS

The considerations, done in the previous paragraph, have extremely important consequences for numerical and experimental measurements.

Let us consider a structure as in Fig. 2 with two half AC end cells. One can easily realize that, because of the additional mirroring the currents X_3 and X_4 are equal to X_1 and X_0 respectively. Furthermore also the constant k_c of the eq. (2) goes to k_c'. Therefore the lumped circuit
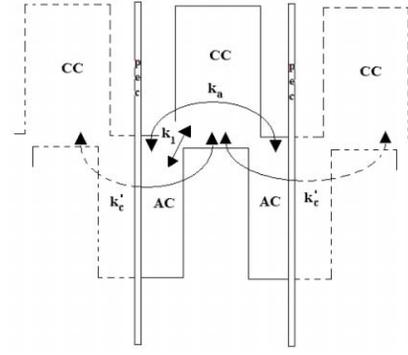

Figure 2: Three cell structure half AC cell ended.

equation system is the following:

$$\left(1 - \frac{f_a^2}{F^2}\right) X_0 + k_1 X_1 + k_a X_2 = 0$$

$$\left(1 - \frac{f_c^2}{F^2} + k_c'\right) X_1 + \frac{k_1}{2} (X_0 + X_2) = 0 \quad (4)$$

$$\left(1 - \frac{f_a^2}{F^2}\right) X_2 + k_1 X_1 + k_a X_0 = 0.$$

The dispersion relation is given by solving the following equation:

$$\left\| \begin{matrix} \frac{\alpha_a}{2} & \frac{k_1}{2} & \frac{k_a}{2} \\ \frac{k_1}{2} & \alpha_c + k_c' & \frac{k_1}{2} \\ \frac{k_a}{2} & \frac{k_1}{2} & \frac{\alpha_a}{2} \end{matrix} \right\| = 0 \quad (5)$$

where $\alpha_{a,c} \equiv \left[1 - \left(\frac{f_{a,c}}{F}\right)^2\right]$ with $f_{a,c}$ stands for the resonant frequency of the AC and CC cells.

Let us define the following quantities

$$\begin{cases} \tilde{f}_a^2 = f_a^2/(1 + k_a) \\ \tilde{f}_c^2 = f_c^2/(1 + k_c') \\ \tilde{k} = k_1^2/[(1 + k_a)(1 + k_c')] \end{cases} \quad (6)$$

The solutions of the above eq. (5) are

$$\begin{cases} \frac{1}{F_0^2} = \frac{1 - k_a}{f_a^2} \\ \frac{1}{F_\mp^2} = \frac{1}{2}(S \pm \sqrt{D^2 + 4\tilde{k}\tilde{f}_a^{-2}\tilde{f}_c^{-2}}) \end{cases} \quad (7)$$

with $S = \tilde{f}_c^{-2} + \tilde{f}_a^{-2}$ while $D = \tilde{f}_c^{-2} - \tilde{f}_a^{-2}$.

It is worth noting that the frequencies F_-, F_0 and F_+ are known quantities by means of real or numerical measurements, while f_a, f_c, k_1, k_a and k_c' are the unknowns. The only way to act, in order to find the unknown terms, is to make a parametric study of the system as done in ref. [4]. In this case by varying the value of f_c, for instance by means of the insertion of some pins in order to vary its inductance, it is possible to find out the wanted quantities.

Let us define the variable y and x

$$y \equiv \left(F_-^{-2} - F_+^{-2}\right)^2 = D + 4\tilde{k}^2\tilde{f}_a^{-2}\tilde{f}_c^{-2}$$
$$\quad (8)$$
$$x \equiv \left(F_-^{-2} + F_+^{-2}\right) = S.$$

Some interesting properties can be drawn from the eq. (8): the variable y as a function of x is a parabola, its minimum coordinates (x_{min}, y_{min}) may give useful information on the lumped circuit parameters

$$\tilde{f_a}^{-2} = \tfrac{1}{2}\left(y_{min} + x_{min}^2\right)/x_{min}$$

$$k_1^2 = \tfrac{1}{2}(1 + k_c')f_a^2\, y_{min}/x_{min}. \qquad (9)$$

From the above we readly get k_a and f_a, but the structure intrinsically does not allow to get k_c'. This means that the exact value of k_1 cannot be obtained. However as said beforehand, setting $k_c' = k_a$, we may get a good estimation of k_1.

Therefore one can realize the following measuring procedure:

1. vary the CC frequency by means of tuner insertion [1];

2. measure the mode frequencies;

3. plot the points in a cartesian plane: in the ordinate $y \equiv \left(F_-^{-2} - F_+^{-2}\right)^2$ and in abscissa $x \equiv F_-^{-2} + F_+^{-2}$;

4. find the best fit of the parabola and evaluate the minimum coordinates.

An identical procedure can be set-up for the complementary configuration of a central AC and two half CC's off axis with opposite offset. In this case the mirroring due to the p.e.c. plates does not break the periodicity of the system, and the second order coupling constants are not perturbed. The dispersion relations is always given by the eq. (5) where it is only to exchange the index a with c and, in this case it is $k_c' \equiv k_c$. A precise evaluation of k_c can be obtained and, combining the two procedures, one can cross check the value of k_1.

NUMERICAL MEASUREMENTS

Two numerical experiments were done.

We allowed for two circuits: the first one as described in Fig. 2 and the second one its complementary configuration. They have the following parameters, inclusive to the reactive impedance due to ohmic losses: $k_1 = 3.406\%$, $k_a = -0.705\%$, $k_c' = -0.5\%$, $k_c = 0.05\%$, $f_a = 3004.57 MHz$, $f_c = 2997.75$ and $Q = 8000$. The frequencies $f_{a,c}$ range, by means of a numerical "tuning", in a interval of 19MHz. The circuit is fed in the first cell and the signal is detected in the last one in a frequency range including all the modes. In this way we "measure" the mode frequency as indicated in the point 2. In order to simulate a real measurement, to each value of the mode system was added a random error in a uniform distribution in the range $[-\Delta F, \Delta F]$. Then we went on according to the procedure described in the previous section. We performed a statistics for a set of N=40 measurements.

The results are given in Fig. 3.

---

[1] Note that f_0 stays unchanged.

ΔF	200kHz	300kHz	400kHz
k_1 (%)	3.406±0.002	3.405±0.003	3.404±0.004
k_a (%)	-0.71±0.02	-0.72±0.04	-0.73±0.06
k_c (%)	0.05±0.01	0.06±0.02	0.06±0.05
f_a (MHz)	3004.5±0.3	3004.7±0.6	3004.7±0.9
f_c (MHz)	2997.5±0.2	2997.4±0.3	2997.4±0.7

Figure 3: Results of the first numerical experiment.

A similar second numerical experiment was done which foresaw an unbalance of δf between the two AC's.

The results are shown in Fig. 4.

	$\delta f = 1MHz$		$\delta f = 2MHz$	
ΔF	200kHz	300kHz	200kHz	300kHz
k_1 (%)	3.405±0.002	3.405±0.002	3.406±0.002	3.406±0.003
k_a (%)	-0.71±0.02	-0.715±0.03	-0.71±0.02	-0.71±0.04
k_c (%)	0.05±0.01	0.05±0.02	0.05±0.01	0.05±0.02
f_a (MHz)	3004.45±0.4	3004.5±0.5	3004.5±0.3	3004.4±0.5
f_c (MHz)	2997.5±0.2	2997.6±0.3	2997.6±0.2	2997.6±0.2

Figure 4: Results of the second numerical experiment.

CONCLUSIONS

The results are very interesting: while the values of k_a and f_a are acceptable till $\Delta F \leq 300 kHz$, the evaluation of k_1 is always very accurate. It is worth nothing that the usual experimental error in a network analyzer measurement is in the order of 200kHz. Furthermore the presence of an asymmetry between the half AC cell frequencies, due to some machining error, gives some values of the "measured" quantities on the limits for $\delta f \leq 2MHz$ and $\Delta F \leq 200 kHz$; anyway the value of k_1 stays "exact" up to values $\Delta F = 300 kHz$ and $\delta f \leq 2MHz$.

The error analysis suggests that this method is largely sufficient to have a good estimation of the lumped circuit parameters.

It is worth noting that the first order coupling constant k_1 is the result of the measurement of a minimum and it will intrinsically have the best accuracy.

REFERENCES

[1] D. E. Nagle, E. A. Knapp and B. C. Knapp, "Coupled Resonator Model for Standing Wave Accelerator Tanks", The Review of Scientific Instruments, vol. 38, nr. 11, pag. 1583-1587, November 1967.

[2] V. G. Vaccaro et al., "Some Relevant Aspects in the Design and Construction of a 30-62 MeV Linac Booster for Proton Therapy", LINAC 2004, Lubeck, Germany (August 2004).

[3] V. G. Vaccaro et al., "A New Tuning Method for Resonant Coupling Structures", PAC05, Knoxville (Tennessee), U.S.A, 16-20 May 2005.

[4] V. G. Vaccaro et al., "A Rationale to Design Side Coupled Linac (SCL): A Faster and More Reliable Tool", EPAC 2006, Edinburgh, UK, 26-30 June 2006.