LUMINOSITY IN $\mathrm{e}^{+}\mathrm{e}^{-}$ RING COLLIDERS

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Abstract

Luminosity is one of the key concepts for accelerator design. In this paper we give a fairly accurate expressions for luminosity based on the Gaussian approximation of the beam distribution function. All kinds of couplings are considered.

1 INTRODUCTION

The luminosity L is one of the most important parameters for colliders. The simplest approximate expression of L is

$$L_0 = \frac{N_+ N_- f_0}{4\pi\sigma_x(0)\sigma_y(0)}, \tag{1}$$

where $N^{\pm}$ is the number of particles in the $e^{\pm}$ colliding bunches, f_0 is the collision frequency and $\sigma_{x(y)}(0)$ is the horizontal (vertical) beam size at the interaction point (IP). This expression is too naive for designing modern high-luminosity colliders: does not include effects as the finite bunch length and/or the crossing angle and it is based on the assumption that the distribution function is Gaussian without correlation between the betatron and synchrotron degrees of freedom (see eq. (17) below). Due to the dynamical effects of monochromatization, such as crossing angle and other possible sources in the arc, however, the distribution function can have a large correlation and the luminosity can be a little more complicated.

Additionally, in simulating the beam-beam effects, L can be calculated directly from the particle distribution of both bunches. This calculation is too-time consuming and is suitable only for linear colliders. For storage rings, we need something less time-consuming. We will give the expressions of L, based on the assumption that the distribution functions of both beams are Gaussian, but with general correlations, which can be used in the presence of general coupling in the 6D sense [1].

2 GENERAL EXPRESSION OF LUMINOSITY

We assume that the distribution function is Gaussian:

$$\psi^{\pm}(\mathbf{x};s) = \frac{\exp\left[-\frac{\sigma^{\pm}_{\mu\nu}(s)^{-1}}{2}(x-X^{\pm}(s))_\mu (x-X^{\pm}(s))_\nu\right]}{\sqrt{(2\pi)^6 \det\sigma^{\pm}(s)}} \tag{2}$$

s is the observation position, $\mathbf{x} = (x, y, p_x, p_y, \varepsilon, z)^t \equiv (x_\mu)$ $(\mu, \nu = 1, ..6)$ the phase space variable (x, y being the horizontal and vertical coordinates, $z = s - ct$, p_x and p_y the momenta normalized to p_0, $\varepsilon = (E - E_0)/E_0$, p_0 and E_0 being referred to the reference particle), $X^{\pm}_\mu(s) = \langle x_\mu \rangle^{\pm}$ and $\sigma^{\pm}_{\mu\nu} = \langle (x - X^{\pm}(s))_\mu (x - X^{\pm}(s))_\nu \rangle$ are the first- and second-order momenta, where $\langle\ \rangle$ is the average with respect to $\psi^{\pm}$. For later convenience, we set $z = x_6$.

For achieving a good accuracy, in the equilibrium, $\psi^{\pm}$ is considered to be a Gaussian as long as the optics are linear. Even with nonlinear forces such as the beam-beam interaction, the Gaussian approximation is still useful [2] in many cases.

In the head-on collision the barycenters of both bunches in the interaction region travel on parallel trajectories with opposite velocities and collide with a possible offset. We decompose both bunches into longitudinal slices and think the bunch-bunch collision as a series of collisions between all pairs of slices from both beams [3]. We "factorize" the distribution function $\psi^{\pm}(\mathbf{x}; s)$ as

$$\psi^{\pm}(\mathbf{x}; s) = \tilde{\psi}^{\pm}(\tilde{\mathbf{x}}; z^{\pm}, s)\, \rho^{\pm}(z^{\pm}; s), \tag{3}$$

where $\tilde{\mathbf{x}} = (x, y, p_x, p_y, \varepsilon)^t$,

$$\rho^{\pm}(z^{\pm}; s) = \int dx\, dy\, dp_x\, dp_y\, d\varepsilon\, \psi^{\pm}(\mathbf{x}; s), \tag{4}$$

is the longitudinal distribution density and $\tilde{\psi}^{\pm}$ is the reduced 5D Gaussian distribution and represents the distribution of the longitudinal slice at $z^{\pm}$

$$\tilde{\psi}^{\pm}(\tilde{\mathbf{x}}; z_{\pm}, s) = \frac{\exp\left[-\frac{\tilde{\sigma}^{\pm}_{ij}(s)^{-1}}{2}(x-\tilde{X}^{\pm}(s))_i (x-\tilde{X}^{\pm}(s))_j\right]}{\sqrt{(2\pi)^5 \det\tilde{\sigma}^{\pm}(s)}} \tag{5}$$

with the reduced momenta $\tilde{X}^{\pm}_i = X^{\pm}_i + \sigma^{\pm}_{i6}/\sigma^{\pm}_{66} z^{\pm}$, $\tilde{\sigma}^{\pm}_{ij} = \sigma^{\pm}_{ij} - \sigma^{\pm}_{i6}\sigma^{\pm}_{j6}/\sigma^{\pm}_{66}$, $(i, j = 1, ..5)$ different from the momenta of $\psi^{\pm}$ [4].

It is worth mentioning that $\tilde{X}^{\pm}_i$ and $\tilde{\sigma}^{\pm}_{ij}$ transform as tensors in the reduced 5D space: when the 6D vector $\mathbf{x}$ obeys a linear transformation $x_\mu \longrightarrow x'_\mu = A_{\mu\nu}\, x_\nu$,

$$A = \left(\begin{array}{c|c} \tilde{A} & \mathbf{o} \\ \hline \mathbf{o}^t & 1 \end{array}\right), \tag{6}$$

where $\tilde{A}$ is a 5×5 matrix and $\mathbf{o}$ is a 5D null vector, then, $\tilde{X}^{\pm}_i$ and $\tilde{\sigma}^{\pm}_{ij}$ obey $\tilde{X}^{\pm}_i \longrightarrow \tilde{A}_{ij}\, \tilde{X}^{\pm}_j$ and $\tilde{\sigma}^{\pm}_{ij} \longrightarrow \tilde{A}_{ik}\, \tilde{A}_{jl}\, \tilde{\sigma}^{\pm}_{kl}$. We call $\tilde{X}^{\pm}_i(z^{\pm}, s)$, $\tilde{\sigma}^{\pm}_{ij}(s)$ momenta of the slice at $z^{\pm} = (t \pm 2s)/2$ or equivalently $s = (z^+ - z^-)/2$, $t = z^+ + z^-$. Here s is the position in the ring where the collision between slices at z^+ and z^- occurs and t represents the order of collision: the heads collide first ($t > 0$) and the tails collide last ($t < 0$).

The luminosity can be expressed as follows:

$$L = N_+ N_- f_0 \int dx dy dz^+ dz^- \rho^+(z^+)\rho^-(z^-)\tilde{\rho}^+\tilde{\rho}^-, \quad (7)$$

with

$$\tilde{\rho}^\pm(x, y; z^\pm, s) = \int dp_x dp_y d\varepsilon \tilde{\psi}^\pm(\tilde{\mathbf{x}}; z^\pm, s). \quad (8)$$

The integrations over x and y are trivial and we obtain

$$L = \int dz^+ dz^- \rho^+(z^+)\rho^-(z^-)\tilde{L}(z^+, z^-), \quad (9)$$

where

$$\tilde{L}(z^+, z^-) = N_+ N_- f_0 \int dx\, dy\, \tilde{\rho}^+\tilde{\rho}^-$$

$$\tilde{L}(z^+, z^-) = \frac{N_+ N_- f_0 \exp\left[-\frac{1}{2}\tilde{\Sigma}^{-1}_{IJ}(s)\tilde{D}_I\tilde{D}_J\right]}{2\pi\sqrt{\det\tilde{\Sigma}(s)}} \quad (10)$$

and $\tilde{D}_{I,J} = (\tilde{X}^+(s, z^+) - \tilde{X}^-(s, z^-))_{I,J}$ $(I, J = 1, 2)$. Equation (9) gives the luminosity when the bunch length is zero for both bunches, and they collide at s. Here and hereafter we use the expression

$$\Sigma_{\mu\nu} = \sigma^+_{\mu\nu} + \sigma^-_{\mu\nu}, \qquad \tilde{\Sigma}_{ij} = \tilde{\sigma}^+_{ij} + \tilde{\sigma}^-_{ij}$$

. The quantities $\tilde{\Sigma}(s)$ and $\tilde{X}^\pm_I(s, z^\pm)$ are related to those at $s = 0$ as follows:

$$\begin{aligned} \tilde{X}_1(s, z) &= \tilde{X}_1(0, z) + \tilde{X}_3(0, z)s, \\ \tilde{X}_2(s, z) &= \tilde{X}_2(0, z) + \tilde{X}_4(0, z)s, \end{aligned} \quad (11)$$

and

$$\begin{aligned} \tilde{\sigma}_{11}(s) &= \tilde{\sigma}_{11}(0) + 2\tilde{\sigma}_{13}(0)s + \tilde{\sigma}_{33}(0)s^2, \\ \tilde{\sigma}_{12}(s) &= \tilde{\sigma}_{12}(0) + \tilde{\sigma}_{14}(0)s + \tilde{\sigma}_{23}(0)s + \tilde{\sigma}_{34}(0)s^2, \\ \tilde{\sigma}_{22}(s) &= \tilde{\sigma}_{22}(0) + 2\tilde{\sigma}_{24}(0)s + \tilde{\sigma}_{44}(0)s^2. \end{aligned} \quad (12)$$

Here the $\pm$ superfixes have been omitted for notational ease. Equation (9), together with eqs. (10), (11) and (12), gives the most general expression of luminosity within the Gaussian approximation.

3 CROSSING ANGLE

Let us consider a vertical crossing angle. We evaluate the luminosity in the boosted frame [5], and perform a Lorentz boost in the vertical direction to obtain a head-on collision.

At IP, a boost map is applied to the (physical) particle coordinates $\mathbf{x}(s = 0)$ to perform a Lorentz transformation ($\mathcal{L}$) which makes the collision head-on. After the beam-beam map is applied in the boosted frame, the coordinates are transformed back to the original frame using the inverse boost map ($\mathcal{L}^{-1}$). The boost map is nonlinear but here we use the linear part.

Let us denote coordinates and momenta in the boosted frame with a $*$ superfix:

$$x^*_\mu = \mathcal{L}_{\mu\nu}x_\nu,\ X^{\pm*}_\mu = \mathcal{L}_{\mu\nu}X^\pm_\nu,\ \sigma^{\pm*}_{\mu\nu} = \mathcal{L}_{\mu\alpha}\mathcal{L}_{\nu\beta}\sigma^\pm_{\alpha\beta}. \quad (13)$$

These define the Gaussian distribution $\psi^{\pm*}(\mathbf{x}^*; s^*)$ in the boosted frame. We can then apply the same procedure as that of the head-on collision: factorizing $\psi^{\pm*}$ as $\tilde{\psi}^{\pm*}\rho^{\pm*}$ and applying eq. (9), we get the luminosity L^* in the boosted frame

$$L^* = \int dz^*_+ dz^*_- \rho^{+*}(z^*_+)\rho^{-*}(z^*_-)\tilde{L}^*(z^*_+, z^*_-), \quad (14)$$

where

$$\tilde{L}^*(z^*_+, z^*_-) = \frac{N_+ N_- f^*_0 \exp\left[-\frac{1}{2}\tilde{\Sigma}^{-1*}_{IJ}(s_*)\tilde{D}^*_I\tilde{D}^*_J\right]}{2\pi\sqrt{\det\tilde{\Sigma}^*(s_*)}}, \quad (15)$$

$\tilde{D}^*_{I,J} = (\tilde{X}^{+*}(s_*, z^*_+) - \tilde{X}^{-*}(s_*, z^*_-))_{I,J}$ and f^*_0 is the collision frequency in the boosted frame.

Finally, to obtain the luminosity L in the laboratory frame all we have to do is to let

$$f^*_0 \longrightarrow f_0. \quad (16)$$

4 A SIMPLE CASE

Let us discuss here the simplest example where

$$X^\pm_\mu(0) = 0, \quad \sigma^\pm_{\mu\nu}(0) = \sigma^2_{\mu\mu}\delta_{\mu\nu}, \ (\mu \text{ not summed}), \quad (17)$$

and the bunches collide with a vertical crossing angle ϕ.

The linearized boost map for $\mathbf{x}$ reads as follows:

$$\mathcal{L} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \tan\phi \\ 0 & 0 & \frac{1}{\cos\phi} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\cos\phi} & 0 & 0 \\ 0 & 0 & 0 & -\tan\phi & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\cos\phi} \end{pmatrix}. \quad (18)$$

After introducing the longitudinal slicing, from eq. (12), we obtain

$$\tilde{X}^{\pm*}_1(s_*) = 0, \quad (19)$$

$$\tilde{X}^{+*}_2 - \tilde{X}^{-*}_2 = \frac{\sigma^*_{26}}{\sigma^*_{66}}(z^{+*} - z^{-*}) = \sin\phi(z^{+*} - z^{-*}), \quad (20)$$

$$\tilde{\Sigma}^*_{11}(s_*) = \Sigma_{11}(0) + s^2_*\Sigma_{33}(0)/\cos^2\phi, \quad (21)$$

$$\tilde{\Sigma}^*_{22}(s_*) = \Sigma_{22}(0) + s^2_*\Sigma_{44}(0)/\cos^2\phi, \quad (22)$$

$$\tilde{\Sigma}^*_{12}(s_*) = 0. \quad (23)$$

Here, we use $\tilde{\Sigma}^*_{11} = \Sigma^*_{11} = \Sigma_{11}$, $\tilde{\Sigma}^*_{12} = 0$, $\tilde{\sigma}^{\pm*}_{24} = 0$ and

$$\tilde{\Sigma}^*_{22} = \sigma^{+*}_{22} - \frac{\sigma^{+*}_{26}\sigma^{+*}_{26}}{\sigma^{+*}_{66}} + \sigma^{-*}_{22} - \frac{\sigma^{-*}_{26}\sigma^{-*}_{26}}{\sigma^{-*}_{66}} = \Sigma_{22}. \quad (24)$$

at $s_* = s = 0$.

Note that by eq. (20), $\tilde{L}^*(z^+, z^-)$ depends on $z^*_\pm$ only through s^*.
Then we obtain the luminosity reduction factor $R_L = L/L_0$

$$R_L = \int_{-\infty}^{\infty} ds_* \frac{\exp\{-s_*^2\left[1+\Phi^2/(1+s_*^2/(2a_y^{*2}))\right]\}}{\sqrt{\pi}\sqrt{1+s_*^2/(2a_x^{*2})}\sqrt{1+s_*^2/(2a_y^{*2})}}, \quad (25)$$

where Φ is the normalized crossing angle

$$\Phi = \sqrt{\frac{\sigma_{26}^*}{\sigma_{22}^*(0)}} = \sqrt{\frac{\sigma_{66}}{\sigma_{22}(0)}} \tan\phi, \quad (26)$$

and

$$a_x = \frac{\sigma_x(0)\cos\phi}{\sqrt{2}\sigma_{p_x}(0)\sigma_z(0)} \quad , \quad a_y = \frac{\sigma_y(0)\cos\phi}{\sqrt{2}\sigma_{p_y}(0)\sigma_z(0)} \quad . \quad (27)$$

If the beams are very flat and $a_x \gg 1$, as it usually happens in electron rings, the luminosity reduction factor,

$$R_L = \int_{-\infty}^{\infty} ds_* \frac{\exp\{-s_*^2\left[1+\Phi^2/(1+s_*^2/(2a_y^2))\right]\}}{\sqrt{\pi}\sqrt{1+s_*^2/(2a_y^2)}}, \quad (28)$$

is a function of a_y and Φ and is plotted in Fig. 1.

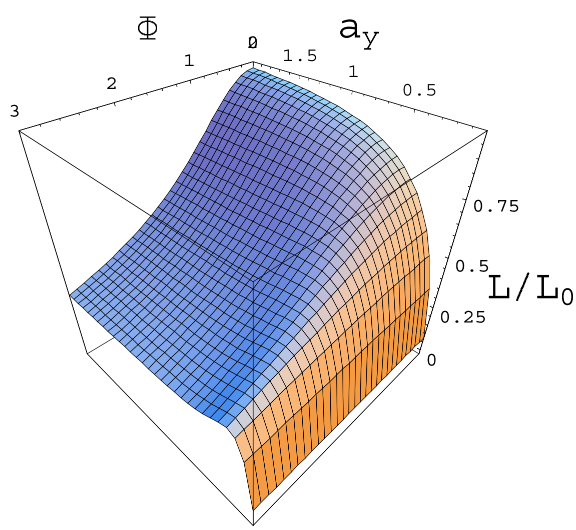

Figure 1: $R_h = L/L_0$ as a function of a_y and Φ for flat beams colliding with vertical crossing .

On the other hand, if $a_y \gg 1$ and a_x is not necessarily large, we have

$$R_L = \sqrt{\frac{2}{\pi}} a_x e^{b_x} K_0(b_x), \quad (29)$$

where

$$b_x = a_x^2(1+\Phi^2). \quad (30)$$

It is easily seen that for the horizontal crossing with very flat beam, the luminosity is given by eq. (29) but with x and y exchanged [6].

5 DISCUSSION

We gave fairly accurate formulae for the luminosity based on the longitudinal slicing. Simple expressions were given for the luminosity with a crossing angle.

Luminosity has been discussed by many authors [7, 8, 9]. In this paper, we have stressed the bunch length effects using the longitudinal slicing and introducing the reduced distribution function $\tilde{\psi}$. It should be noted that the longitudinal slicing fits well with the application of the symplectic 6D mapping for the beam-beam interaction (synchrobeam mapping) [3].

For very high-luminosity colliders of the future, apart from monochromatization, we require more detailed kinematic and dynamical analysis for luminosity, energy resolution and other features.

6 REFERENCES

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[6] The expression was given in ref. [5]. There should be a slight modification, though the difference between $\tilde{\sigma}$ and σ was not made so that the final expression was not accurate. The difference appears in the presence of $\cos\phi$ factors in a_x and a_y. Numerically, however, the difference in L is very small for the usual machine parameters.

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