Longitudinal beam distribution in a storage ring with pure inductance described by the LambertW function

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Abstract

The Haissinski equation, accounting for the static distribution of an electron beam circulating in a Storage-Ring and subject to a purely inductive field has an analytical solution, which is a specific form of the LambertW function. We show how the use of this analytical tool allows a better understanding of the validity of the numerical solutions and the distribution normalization properties.

1 INTRODUCTION

The Haissinski equation [1] describes the stationary solution for the longitudinal distribution of electrons in a storage ring. The form of this equation depends on the form of the wake field, which is given by the Fourier transform of the storage ring impedance. The case of the purely inductive impedance has been observed in storage rings like the SLC damping rings [2] and at KEK [3].
Approximated solutions of the Haissinski equation have been extensively investigated numerically [2]-[4]. We note that an analytical solution of this equation exists. This solution is given by a particular expression of the so-called LambertW function [5], which appears frequently in applied mathematics and has important application in many other fields [6]. In this paper we present the LambertW function and analyze the nature of the analytical solution of the Haissinski equation in the case of a purely inductive impedance.

2 THE ANALYTICAL SOLUTION IN THE PURE INDUCTIVE CASE

The Haissinski equation in the pure inductive case can be writen in the form

$$\rho' = -\frac{\xi}{1 - S\rho}\rho, \tag{1}$$

where ρ represents the beam distribution and the derivative is taken with respect to ξ, linked to the position *z* of the electron with respect to the synchronous particle by

$$\xi = \frac{\omega_s}{\alpha_c c \sigma_\epsilon} z, \tag{2}$$

with ω_s the synchrotron frequency, α_c the momentum compaction factor, c the speed of light, and σ_ϵ the natural energy spread of the beam.
The parameter *S* is specified in terms of the inductance *L* of the wake field, the revolution period T_0, the nominal energy of the particles E_0, the number of particles N, the synchrotron frequency, the natural energy spread, and the elementary charge e, as

$$|S| = \frac{e^2 L N \omega_s}{\alpha_c^2 \sigma_\epsilon^2 T_0 E_0}. \tag{3}$$

Equation (1) can be rewritten in the following more convenient form

$$\ln(\rho) - S\rho = -\frac{\xi^2}{2} + \ln(A), \tag{4}$$

where *A* is the normalization constant defined by

$$A = \ln \rho_0 - S\rho_0. \tag{5}$$

As mentioned above, equations of the type (1) have a natural analytical solution in terms of the so-called LambertW function [5]. This function, $W(z)$, is implicitly defined as the root of the following equation

$$W(z)\exp(W(z)) = z, \tag{6}$$

and explicitly by the series expansion

$$W(z) = \sum_{n=0}^{\infty} \frac{(-n)^{n-1}}{n!} z^n, \tag{7}$$

converging for $|z| < \frac{1}{e}$.
The analytical solution of (1) using expressions (4) and (6) is

$$\rho = -\frac{W(-AS\exp(-\frac{\xi^2}{2}))}{S}. \tag{8}$$

3 SINGULARITIES FOR NEGATIVE MOMENTUM COMPACTION FACTOR

It is well known and evident that if S is negative, equation (1) has no singularity and there is always a unique continuous solution. Here we remark that the sign of S depends on the opposite sign of the momentum compaction factor α_c. According to equation (7) the solution of equation (1) can be written as an infinite sum of Gaussians, namely

$$\rho = \frac{1}{S}\sum_{n=1}^{\infty} \frac{n^{n-1}}{n!}(AS)^n \exp(-n\frac{\xi^2}{2}). \tag{9}$$

In the case of positive S, negative α_c, the solution exists, but the presence of a singularity point limits the validity of the solution to a restricted range of S-values. The convergence of the series (8), see the previous section, depends on the values of the constants *A* and *S*, fulfilling the inequality

$$AS \leq \frac{1}{e}. \tag{10}$$

The upper limit, associated with a branch point of the LambertW function, clarifies the role of the singularity.
We can now exploit equation (8) to specify the normalization of the distribution ρ, and the role of *A* and *S*.
It is evident that from equation (8) the normalized distribution is given by the value *A*, for a given *S*:

$$\begin{aligned} S &= \int_{-\infty}^{\infty} W(-SA\exp(-\frac{\xi^2}{2})) \\ &= \sqrt{2\pi}\sum_{n=1}^{\infty}\frac{n^{\frac{2n-3}{2}}}{n!}(AS)^n. \end{aligned} \tag{11}$$

The r.h.s of equation (11) is a series converging within the same range imposed by (10).
By taking for AS the upper limit of convergence and by using the Stirling approximation, $n! \simeq \sqrt{2\pi n}n^n e^{-n}$, we find the maximum value of *S*

$$S^* \simeq \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}. \tag{12}$$

This is a rough approximation and the exact computation (given by the computing algebra software Maple), even though less appealing from the formal point of view, yields

$$S^* \simeq 1.550608..., \tag{13}$$

which seems to be more accurate than previous values given in the literature [8]. We also remark that the value found in [9] is very close to S^*.
It is important to emphasize that the distribution (8) exhibits an r.m.s value below $\sigma_\xi < 1$. In physical terms, this means that an electron bunch experiencing a purely inductive wake with $S > 0$, due to e.g. a negative momentum compaction factor, may have a length lower than the "natural" value, provided by $\sigma_z = \frac{c\alpha_c}{\omega_s}\sigma_\epsilon$.
The behavior of σ_ξ as function of AS is given in figure 1 and the limiting value, calculated with the Stirling approximation, but very close to the exact value, is

$$\sigma_\xi^* = \sqrt{\frac{6}{\pi^2}\sum_{n=1}^{\infty}n^{-3}} \simeq 0.854846.... \tag{14}$$

The results obtained so far, apart from providing an analytical solution for equation (1) have clarified the nature of the singularity associated with the limits of validity of the Taylor expansion of the LambertW function and the range of values of S ($S < S^*$) for which equation (1) admits a normalizable solution.

Before concluding this section let us note that the solution (8) can be extended to negatives values of *S* too, provided that $SA < \frac{1}{e}$. An idea of the behavior of the solution, for $AS \to \frac{1}{e}$ and for $AS \to \frac{-1}{e}$, is given is figure 2. As is evident, for positive *AS* values the distribution is similar to a gaussian distribution with an r.m.s. value slightly larger than the natural value. On the contrary for negative *AS* values the shape is significantly different from a gaussian shape.

4 LARGE CURRENT AND POSITIVE MOMENTUM COMPACTION FACTOR

In the previous section we have explored the solution of the Haissinski equation for a purely inductive wake using the Taylor expansion of the LambertW function which has a limited convergence radius. In this section we will see how a different expansion, admitting a larger radius of convergence, can be exploited to get a non trivial and useful form of solution valid for negative *S*-values. To this aim we note that, for the present problem, a natural alternative to the Taylor expansion, is provided by

$$\begin{aligned} W(\exp(z)) = \quad & 1 + \frac{1}{2}(z-1) + \frac{1}{16}(z-1)^2 \\ & -\frac{1}{192}(z-1)^3 - \frac{1}{3072}(z-1)^4 \\ & +\frac{13}{61440}(z-1)^5 + O((z-1)^6) \end{aligned}, \tag{15}$$

whose radius of convergence is $\sqrt{4+\pi^2}$.
According to the previous relation we can write the solution of (1) in the form ($\Lambda = -S$)

$$\begin{aligned} W(A\Lambda\exp(-\frac{\xi^2}{2})) = \quad & 1 + \frac{1}{2}(-\xi^2/2 + \ln\frac{A\Lambda}{e}) \\ & +\frac{1}{16}(-\xi^2/2 + \ln\frac{A\Lambda}{e})^2 \\ & -\frac{1}{192}(-\xi^2/2 + \ln\frac{A\Lambda}{e})^3 \\ & -\frac{1}{3072}(-\xi^2/2 + \ln\frac{A\Lambda}{e})^4 \\ & +\frac{13}{61440}(-\xi^2/2 + \ln\frac{A\Lambda}{e})^5 \\ & +O((-\xi^2/2 + \ln\frac{A\Lambda}{e})^6) \end{aligned}. \tag{16}$$

The above solution shows that the charge distribution, in the case of a perfect inductor is symmetric about $\xi = 0$ and tends to a parabolic shape for $A\Lambda \gg 1$.
A comparison between analytical and numerical solution is offered by figure 3 and the agreement is more than satisfactory.
As a further comment we remark that the series converges for $A\Lambda \leq 41.4$, which is a good range for the specific problem we are considering.
It is also worth noting that it can be easily verified that the

normalization constant can be directly inferred and reads ($\delta = A\Lambda$)

$$\tilde{N} = \int_{-\infty}^{\infty} \rho(\xi d\xi) \simeq 3.1\delta^{\frac{1}{3}}, \qquad (17)$$

while the second order normalized momentum can be written as

$$\sigma_\xi^2 = \frac{1}{\tilde{N}} \int_{-\infty}^{\infty} \xi^2 \rho(\xi) d\xi \simeq 1.058(\delta + 5)^{\frac{1}{6}}. \qquad (18)$$

Therefore for large δ, and thus for large current, the r.m.s. value of the distribution scales roughly as $\tilde{N}^{\frac{1}{4}}$.

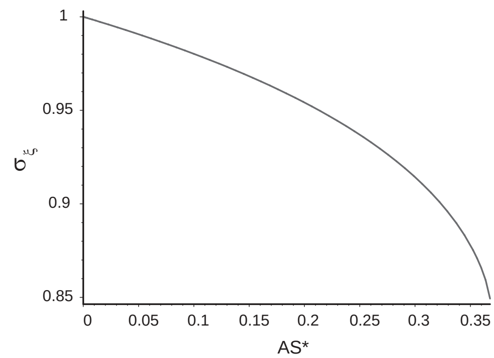

Figure 1: The bunch length, σ_ξ, as function of AS^*.

5 CONCLUSION

The analytical solution of the Haissinski equation for a purely inductive impedance, the LambertW function, has been presented and the analysis confirmed previous numerical studies. The behavior of the bunch distribution changes its equilibrium shape according to the strength of the inductance and the sign of the momentum compaction factor. For a negative sign, the distribution is no more gaussian and its standard deviation, the bunch length, is smaller than the natural one. For a positive sign the distribution becomes more quadratic for large inductance values.
The investigation of the general Haissinski equation, i.e. for a general form of the impedance, and using the LambertW function, will be done at a later stage.

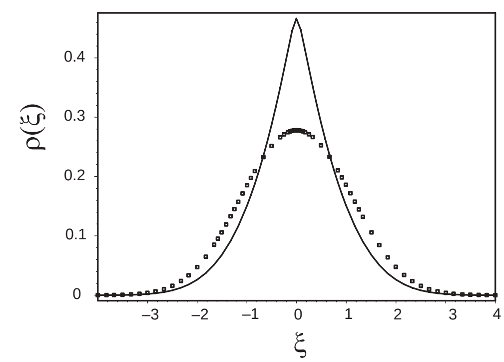

Figure 2: The bunch distribution in the case $AS \simeq \frac{1}{e}$ (point) and in the case $AS \simeq -\frac{1}{e}$ (line).

6 ACKNOWLEDGMENT

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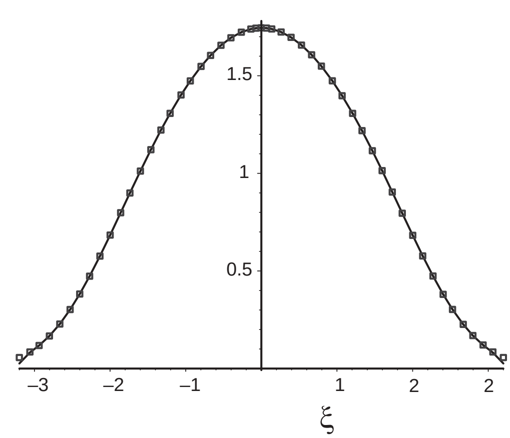

Figure 3: Comparison between analytical and numerical solution of the Haissinski equation.