Proc. 23rd Int. Workshop on Description Logics (DL2010), CEUR-WS 573, Waterloo, Canada, 2010.

$\mathcal{EL}$-Concepts go Second-Order: Greatest Fixpoints and Simulation Quantifiers

Carsten Lutz[1], Robert Piro[2], and Frank Wolter[2]

[1] Department of Computer Science, University of Bremen, Germany
[2] Department of Computer Science, University of Liverpool, UK
clu@uni-bremen.de,{Robert.Piro,Wolter}@liverpool.ac.uk

1 Introduction

The well-known description logic (DL) $\mathcal{ALC}$ is usually regarded as the *basic DL* that comprises all Boolean concept constructors and from which all expressive DLs are derived by admitting additional concept constructors. The fundamental role of $\mathcal{ALC}$ is largely due to the fact that it is very well-behaved regarding its logical, model-theoretic, and computational properties. This good behavior can, in turn, be explained nicely by the fact that $\mathcal{ALC}$-concepts can be characterized exactly as the bisimulation invariant fragment of first-order logic (FO) in the sense that an FO formula is invariant under bisimulation if, and only if, it is equivalent to an $\mathcal{ALC}$-concept [22, 13, 16]. In particular, invariance under bisimulation explains the tree-model property of $\mathcal{ALC}$ as well as its favorable computational properties [24]. In the mentioned characterization, the condition that $\mathcal{ALC}$ is a fragment of FO is much less important than its bisimulation invariance. In fact, $\mathcal{ALC}\mu$, the extension of $\mathcal{ALC}$ with fixpoint operators, is not a fragment of FO, but inherits almost all important properties of $\mathcal{ALC}$ [8, 12]. Similar to $\mathcal{ALC}$, $\mathcal{ALC}\mu$'s fundamental role (in particular in its formulation as the modal mu-calculus) can be explained by the fact that $\mathcal{ALC}\mu$-concepts can be characterized exactly as the bisimulation invariant fragment of monadic second-order logic (MSO) [14, 8]. Indeed, from a purely theoretical viewpoint it is hard to explain why $\mathcal{ALC}$ rather than $\mathcal{ALC}\mu$ forms the logical underpinning of current ontology language standards; the facts that mu-calculus concepts can be hard to grasp and that, despite the same theoretical complexity, efficient reasoning in $\mathcal{ALC}\mu$ is more challenging than in $\mathcal{ALC}$ are probably the only reasons for the limited interest in $\mathcal{ALC}\mu$ compared to $\mathcal{ALC}$.

In recent years, the development of very large ontologies and the use of ontologies to access instance data has led to a revival of interest in *tractable* DLs. The main examples are $\mathcal{EL}$ [5] and DL-Lite [9], the logical underpinnings of the OWL profiles OWL2 EL and OWL2 QL, respectively. In contrast to $\mathcal{ALC}$, a satisfactory characterization of the expressivity of such DLs is still missing, and a first aim of this paper is to fill this gap for $\mathcal{EL}$. To this end, we characterize $\mathcal{EL}$ as a maximal fragment of FO that is preserved under *simulations* and has *finite minimal models*. Note that preservation under simulations alone would characterize $\mathcal{EL}$ with disjunctions, and the existence of minimal models reflects the "Horn-aspect" of $\mathcal{EL}$.

The second and main aim of this paper, however, is to introduce and investigate two equi-expressive extensions of $\mathcal{EL}$ with greatest fixpoints, $\mathcal{EL}^{\nu}$ and $\mathcal{EL}^{\nu+}$, and to prove that they stand in a similar relationship to $\mathcal{EL}$ as $\mathcal{ALC}\mu$ to $\mathcal{ALC}$. To this end, we prove that $\mathcal{EL}^\nu$ (and therefore also $\mathcal{EL}^{\nu+}$, which admits mutual fixpoints and is exponentially more succinct than $\mathcal{EL}^\nu$) can be characterized as a maximal fragment of MSO that is preserved under *simulations* and has *finite minimal models*. Similar to $\mathcal{ALC}\mu$, $\mathcal{EL}^\nu$ and $\mathcal{EL}^{\nu+}$ inherit many good properties of $\mathcal{EL}$, the most interesting being that *reasoning with general concept inclusions (GCIs) is still tractable* and that the same type of algorithm can be used. Thus, in contrast to $\mathcal{ALC}\mu$, the development of practical decision procedures is no obstacle to using $\mathcal{EL}^\nu$.

Moreover, $\mathcal{EL}^{\nu+}$ has a number of very useful properties that $\mathcal{EL}$ and most of its extensions are lacking. To begin with, we show that in $\mathcal{EL}^{\nu+}$ *least common subsumers (LCS)* w.r.t. general TBoxes always exist and can be computed in polynomial time (for a bounded number of concepts). This result can be regarded as an extension of similar results for least common subsumers w.r.t. *classical* TBoxes in $\mathcal{EL}$ with greatest fixpoint semantics in [1]. Similarly, in $\mathcal{EL}^{\nu+}$ *most specific concepts* always exist and can be computed in linear time; a result which also generalizes [1]. Secondly, we show that $\mathcal{EL}^{\nu+}$ has the *Beth definability property* with explicit definitions being computable in polytime and of polynomial size. It has been convincingly argued in [21, 20] that this property is of great interest for structuring TBoxes and for ontology based data access. Another application of $\mathcal{EL}^{\nu+}$ is demonstrated in [15], where the succinct representations of definitions in $\mathcal{EL}^{\nu+}$ are used to develop polytime algorithms for decomposing certain general $\mathcal{EL}$-TBoxes.

To prove these result and provide a better understanding of the modeling capabilities of $\mathcal{EL}^{\nu+}$ we show that it has the same expressive power as extensions of $\mathcal{EL}$ by means of *simulation quantifiers*, a variant of second-order quantifiers that quantifies "modulo a simulation of the model"; in fact, the relationship between simulation quantifiers and $\mathcal{EL}^{\nu+}$ is somewhat similar to the relationship between $\mathcal{ALC}\mu$ and bisimulation quantifiers [11]. Proofs are omitted for brevity and the reader is referred to www.csc.liv.ac.uk/∼frank/publ/publ.html.

2 Preliminaries

Let $\mathsf{N_C}$ and $\mathsf{N_R}$ be countably infinite and mutually disjoint sets of concept and role names. *$\mathcal{EL}$-concepts* are built according to the rule

$$C := \quad A \quad | \quad \top \quad | \quad \bot \quad | \quad C \sqcap D \quad | \quad \exists r.C,$$

where $A \in \mathsf{N_C}$, $r \in \mathsf{N_R}$, and C, D range over $\mathcal{EL}$-concepts[3]. An *$\mathcal{EL}$-concept inclusion* takes the form $C \sqsubseteq D$, where C, D are $\mathcal{EL}$-concepts. A *general $\mathcal{EL}$-TBox $\mathcal{T}$* is a finite set of $\mathcal{EL}$-concept inclusions. An *ABox assertion* is an expression of the form $A(a)$ or $r(a, b)$, where a, b are from a countably infinite set of individual names $\mathsf{N_I}$, $A \in \mathsf{N_C}$, and $r \in \mathsf{N_R}$. An *ABox* is a finite set of ABox assertions. By $\mathsf{Ind}(\mathcal{A})$ we denote the set of individual names in $\mathcal{A}$. An $\mathcal{EL}$-knowledge base (KB) is a pair $(\mathcal{T}, \mathcal{A})$ that consists of an $\mathcal{EL}$-TBox $\mathcal{T}$ and an ABox $\mathcal{A}$.

[3] In the literature, $\mathcal{EL}$ is typically defined without $\bot$. The sole purpose of including $\bot$ here is to simplify the formulation of some results.

The semantics of $\mathcal{EL}$ is based on interpretations $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$, where the *domain* $\Delta^{\mathcal{I}}$ is a non-empty set, and $\cdot^{\mathcal{I}}$ is a function mapping each concept name A to a subset $A^{\mathcal{I}}$ of $\Delta^{\mathcal{I}}$, each role name r to a binary relation $r^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$, and each individual name a to an element $a^{\mathcal{I}}$ of $\Delta^{\mathcal{I}}$. The interpretation $C^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$ of $\mathcal{EL}$-concepts C in an interpretation $\mathcal{I}$ is defined in the standard way [6]. We will often make use of the fact that $\mathcal{EL}$-concepts can be regarded as formulas in FO (and, therefore, MSO) with unary predicates from $\mathsf{N_C}$, binary predicates from $\mathsf{N_R}$, and exactly one free variable [6]. We will often not distinguish between $\mathcal{EL}$-concepts and their translations into FO/MSO.

We now introduce $\mathcal{EL}^{\nu}$, the extension of $\mathcal{EL}$ with greatest fixpoints and the main language studied in this paper. *$\mathcal{EL}^{\nu}$-concepts* are defined like $\mathcal{EL}$-concepts, but additionally allow the greatest fixpoint constructor $\nu X.C$, where X is from a countably infinite set of *(concept) variables* $\mathsf{N_V}$ and C an $\mathcal{EL}^{\nu}$-concept. A variable is *free* in a concept C if it occurs in C at least once outside the scope of any ν-constructor that binds it. An $\mathcal{EL}^{\nu}$-concept is *closed* if it does not contain any free variables. An *$\mathcal{EL}^{\nu}$-concept inclusion* takes the form $C \sqsubseteq D$, where C, D are are closed $\mathcal{EL}^{\nu}$-concepts. The semantics of the greatest fixpoint constructor is as follows, where $\mathcal{V}$ is an *assignment* that maps variables to subsets of $\Delta^{\mathcal{I}}$ and $\mathcal{V}[X \mapsto W]$ denotes $\mathcal{V}$ modified by setting $\mathcal{V}(X) = W$:

$$(\nu X.C)^{\mathcal{I},\mathcal{V}} = \bigcup\{W \subseteq \Delta^{\mathcal{I}} \mid W \subseteq C^{\mathcal{I},\mathcal{V}[X\mapsto W]}\}$$

We will also consider an extended version of the ν-constructor that allows to capture mutual recursion. It has been considered e.g. in [10, 23] and used in a DL context in [19]; it can be seen as a variation of the fixpoint equations considered in [8]. The constructor has the form $\nu_i X_1 \cdots X_n.C_1, \ldots, C_n$ where $1 \leq i \leq n$. The semantics is defined by setting $(\nu_i X_1 \cdots X_n.C_1, \ldots, C_n)^{\mathcal{I},\mathcal{V}}$ to

$$\bigcup\{W_i \mid \exists W_1, \ldots, W_{i-1}, W_{i+1}, \ldots, W_n \text{ s.t. for } 1 \leq j \leq n\text{:} \\ W_j \subseteq C_j^{\mathcal{I},\mathcal{V}[X_1\mapsto W_1,\ldots,X_n\mapsto W_n]}\}$$

We use $\mathcal{EL}^{\nu+}$ to denote $\mathcal{EL}$ extended with this mutual greatest fixpoint constructor. Clearly, $\nu X.C \equiv \nu_1 X.C$, thus every $\mathcal{EL}^{\nu}$-concept is equivalent to an $\mathcal{EL}^{\nu+}$-concept. Conversely, we have the following result, where the first part follows from [8]. The length of a concept C is defined as the number of occurrences of symbols in it.

Proposition 1. *For every $\mathcal{EL}^{\nu+}$-concept, one can construct an equivalent $\mathcal{EL}^{\nu}$-concept of at most exponential size. Moreover, there is a sequence of $\mathcal{EL}^{\nu+}$-concepts $C_0, C_1, \ldots$ such that C_i is of length $p(i)$, p a polynomial, whereas the shortest $\mathcal{EL}^{\nu}$-concept equivalent to C_i is of length at least 2^i.*

By extending the translation of $\mathcal{EL}$-concepts into FO in the obvious way, one can translate closed $\mathcal{EL}^{\nu+}$-concepts into an MSO formula with one free first-order variable. We will often not distinguish between $\mathcal{EL}^{\nu+}$-concepts and their translation into MSO.

3 Characterizing $\mathcal{EL}$ using simulations

The purpose of this section is to provide a model-theoretic characterization of $\mathcal{EL}$ as a fragment of FO that is similar in spirit to the well-known characterization of $\mathcal{ALC}$

as the bisimulation-invariant fragment of FO. To this end, we first characterize $\mathcal{EL}^{\sqcup}$, the extension of $\mathcal{EL}$ with the disjunction constructor $\sqcup$, as the fragment of FO that is preserved under simulation. Then we characterize the fragment $\mathcal{EL}$ of $\mathcal{EL}^{\sqcup}$ using, in addition, the existence of minimal models. A *pointed interpretation* is a pair $(\mathcal{I}, d)$ consisting of an interpretation $\mathcal{I}$ and $d \in \Delta^{\mathcal{I}}$. A *signature* Σ is a set of concept and role names.

Definition 1 (Simulations). Let $(\mathcal{I}_1, d_1)$ and $(\mathcal{I}_2, d_2)$ be pointed interpretations and Σ a signature. A relation $S \subseteq \Delta^{\mathcal{I}_1} \times \Delta^{\mathcal{I}_2}$ is a *Σ-simulation* between $(\mathcal{I}_1, d_1)$ and $(\mathcal{I}_2, d_2)$, in symbols $S : (\mathcal{I}_1, d_1) \leq_\Sigma (\mathcal{I}_2, d_2)$, if $(d_1, d_2) \in S$ and the following conditions hold:

1. for all concept names $A \in \Sigma$ and all $(e_1, e_2) \in S$, if $e_1 \in A^{\mathcal{I}_1}$ then $e_2 \in A^{\mathcal{I}_2}$;
2. for all role names $r \in \Sigma$, all $(e_1, e_2) \in S$, and all $e_1' \in \Delta^{\mathcal{I}_1}$ with $(e_1, e_1') \in r^{\mathcal{I}_1}$, there exists $e_2' \in \Delta^{\mathcal{I}_2}$ such that $(e_2, e_2') \in r^{\mathcal{I}_2}$ and $(e_1', e_2') \in S$.

If such an S exists, then we also say that $(\mathcal{I}_2, d_2)$ *Σ-simulates* $(\mathcal{I}_1, d_1)$ and write $(\mathcal{I}_1, d_1) \leq_\Sigma (\mathcal{I}_2, d_2)$.

If $\Sigma = \mathsf{N_C} \cup \mathsf{N_R}$, then we omit Σ and use the term *simulation* to denote Σ-simulations and $(\mathcal{I}_1, d_1) \leq (\mathcal{I}_2, d_2)$ stands for $(\mathcal{I}_1, d_1) \leq_\Sigma (\mathcal{I}_2, d_2)$. It is well-known that the description logic $\mathcal{EL}$ is intimately related to the notion of a simulation, see for example [4, 17]. In particular, $\mathcal{EL}$-concepts are preserved under simulations in the sense that if $d \in C^{\mathcal{I}}$ for an $\mathcal{EL}$-concept C and $(\mathcal{I}_1, d_1) \leq_\Sigma (\mathcal{I}_2, d_2)$, then $d_2 \in C^{\mathcal{I}_2}$. This observation, which clearly generalizes to $\mathcal{EL}^{\sqcup}$, illustrates the (limitations of the) modeling capabilities of $\mathcal{EL}$/$\mathcal{EL}^{\sqcup}$. We now strengthen it to an exact characterization of the expressive power of these logics relative to FO.

Let $\varphi(x)$ be an FO-formula (or, later, MSO-formula) with one free variable x. We say that $\varphi(x)$ is *preserved under simulations* if, and only if, for all $(\mathcal{I}_1, d_1)$ and $(\mathcal{I}_2, d_2)$, $\mathcal{I}_1 \models \varphi[d_1]$ and $(\mathcal{I}_1, d_1) \leq (\mathcal{I}_2, d_2)$ implies $\mathcal{I}_2 \models \varphi[d_2]$.

Theorem 1. *An FO-formula $\varphi(x)$ is preserved under simulations if, and only if, it is equivalent to an $\mathcal{EL}^{\sqcup}$-concept.*

To characterize $\mathcal{EL}$, we add a central property of Horn-logics on top of preservation under simulations. Let $\mathcal{L}$ be a set of FO (or, later, MSO) formulas, each with one free variable. We say that $\mathcal{L}$ *has (finite) minimal models* if, and only if, for every $\varphi(x) \in \mathcal{L}$ there exists a (finite) pointed interpretation $(\mathcal{I}, d)$ such that for all $\psi(x) \in \mathcal{L}$, we have $\mathcal{I} \models \psi[d]$ if, and only if, $\forall x.(\varphi(x) \rightarrow \psi(x))$ is a tautology.

Theorem 2. *The set of $\mathcal{EL}$-concepts is a maximal set of FO-formulas that is preserved under simulations and has minimal models (equivalently: has finite minimal models): if $\mathcal{L}$ is a set of FO-formulas that properly contains all $\mathcal{EL}$-concepts, then either it contains a formula not preserved under simulations or it does not have (finite) minimal models.*

We note that de Rijke and Kurtonina have given similar characterizations of various non-Boolean fragments of $\mathcal{ALC}$. In particular, Theorem 1 is rather closely related to results proved in [16] and would certainly have been included in the extensive list of characterizations given there had $\mathcal{EL}$ already been as popular as it is today. In contrast, the novelty of Theorem 2 is that it makes the Horn character of $\mathcal{EL}$ explicit through minimal models while the characterizations of disjunction-free languages in [16] are based on simulations that take sets (rather than domain-elements) as arguments.

4 Simulation quantifiers and $\mathcal{EL}^\nu$

To understand and characterize the expressive power and modeling capabilities of $\mathcal{EL}^\nu$, we introduce three distinct types of simulation quantifiers and show that, in each case, the resulting language has the same expressive power as $\mathcal{EL}^\nu$.

Simulating interpretations. The first language $\mathcal{EL}^{si}$ extends $\mathcal{EL}$ by the concept constructor $\exists^{sim}(\mathcal{I}, d)$, where $(\mathcal{I}, d)$ is a finite pointed interpretation in which only finitely many $\sigma \in \mathsf{N_C} \cup \mathsf{N_R}$ have a non-empty interpretation $\sigma^\mathcal{I} \subseteq \Delta^\mathcal{I}$. The semantics of $\exists^{sim}(\mathcal{I}, d)$ is defined by setting for all interpretations $\mathcal{J}$ and $e \in \Delta^\mathcal{J}$,

$$e \in (\exists^{sim}(\mathcal{I}, d))^\mathcal{J} \text{ iff } (\mathcal{I}, d) \leq (\mathcal{J}, e).$$

Example 1. Let $\mathcal{I}$ consist of one point d such that $(d, d) \in r^\mathcal{I}$. Then $e \in (\exists^{sim}(\mathcal{I}, d))^\mathcal{J}$ iff there is an infinite r-chain starting at e in $\mathcal{I}$, i.e., there exist $e_0, e_1, e_2, \ldots$ such that $e = e_0$ and $(e_i, e_{i+1}) \in r^\mathcal{J}$ for all $i \geq 0$.

To attain a better understanding of the constructor $\exists^{sim}$, it is interesting to observe that *every* $\mathcal{EL}^{si}$-concept is equivalent to a concept of the form $\exists^{sim}(\mathcal{I}, d)$.

Lemma 1. *For every $\mathcal{EL}^{si}$-concept C one can construct, in linear time, an equivalent concept of the form $\exists^{sim}(\mathcal{I}, d)$.*

Proof. By induction on the construction of C. If $C = A$ for a concept name A, then let $\mathcal{I} = (\{d\}, \cdot^\mathcal{I})$, where $A^\mathcal{I} = \{d\}$ and $\sigma^\mathcal{I} = \emptyset$ for all symbols distinct from A. Clearly, A and $\exists^{sim}(\mathcal{I}, d)$ are equivalent. For $C_1 = \exists^{sim}(\mathcal{I}_1, d_1)$ and $C_2 = \exists^{sim}(\mathcal{I}_2, d_2)$ assume that $\Delta^{\mathcal{I}_1} \cap \Delta^{\mathcal{I}_2} = \{d_1\} = \{d_2\}$. Then $\exists^{sim}(\mathcal{I}_1 \cup \mathcal{I}_2, d_1)$ is equivalent to $C_1 \sqcap C_2$. For $C = \exists r.\exists^{sim}(\mathcal{I}, d)$ construct a new interpretation $\mathcal{I}'$ by adding a new node e to $\Delta^\mathcal{I}$ and setting $(e, d) \in r^{\mathcal{I}'}$. Then $\exists^{sim}(\mathcal{I}', e)$ and C are equivalent. $\square$

We will show that there are polynomial translations between $\mathcal{EL}^{si}$ and $\mathcal{EL}^{\nu+}$. When using $\mathcal{EL}^\nu$ in applications and to provide a translation from $\mathcal{EL}^{\nu+}$ to $\mathcal{EL}^{si}$, it is convenient to have available a "syntactic" simulation operator.

Simulating models of TBoxes. The second language $\mathcal{EL}^{st}$ extends $\mathcal{EL}$ by the concept constructor $\exists^{sim}\Sigma.(\mathcal{T}, C)$, where Σ is a finite signature, $\mathcal{T}$ a general TBox, and C a concept. To admit nestings of $\exists^{sim}$, the concepts of $\mathcal{EL}^{st}$ are defined by simultaneous induction; namely, $\mathcal{EL}^{st}$-concepts, concept inclusions, and general TBoxes are defined as follows:

- every $\mathcal{EL}$-concept, concept inclusion, and general TBox is an $\mathcal{EL}^{st}$-concept, concept inclusion, and general TBox, respectively;
- if $\mathcal{T}$ is a general $\mathcal{EL}^{st}$-TBox, C an $\mathcal{EL}^{st}$-concept, and Σ a finite signature, then $\exists^{sim}\Sigma.(\mathcal{T}, C)$ is an $\mathcal{EL}^{st}$-concept;
- if C, D are $\mathcal{EL}^{st}$-concepts, then $C \sqsubseteq D$ is a $\mathcal{EL}^{st}$-concept inclusion;
- a general $\mathcal{EL}^{st}$-TBox is a finite set of $\mathcal{EL}^{st}$-concept inclusions.

The semantics of $\exists^{sim}\Sigma.(\mathcal{T}, C)$ is as follows:

> $d \in (\exists^{sim}\Sigma.(\mathcal{T}, C))^\mathcal{I}$ iff there exists $(\mathcal{J}, e)$ such that $\mathcal{J}$ is a model of $\mathcal{T}$, $e \in C^\mathcal{J}$ and $(\mathcal{J}, e) \leq_\Gamma (\mathcal{I}, d)$, where $\Gamma = (\mathsf{N_C} \cup \mathsf{N_R}) \setminus \Sigma$.

Example 2. Let $\mathcal{T} = \{A \sqsubseteq \exists r.A\}$ and $\Sigma = \{A\}$. Then $\exists^{sim}\Sigma.(\mathcal{T}, A)$ is equivalent to the concept $\exists^{sim}(\mathcal{I}, d)$ defined in Example 1.

We will later exploit the fact that $\exists^{sim}\Sigma.(\mathcal{T}, C)$ is equivalent to $\exists^{sim}\Sigma \cup \{A\}.(\mathcal{T}', A)$, where A is a fresh concept name and $\mathcal{T}' = \mathcal{T} \cup \{A \sqsubseteq C\}$. Another interesting (but subsequently unexploited) observation is that we can w.l.o.g. restrict Σ to singleton sets since

$$\exists^{sim}(\{\sigma\} \cup \Sigma).(\mathcal{T}, C) \equiv \exists^{sim}\{\sigma\}.(\emptyset, \exists^{sim}\Sigma.(\mathcal{T}, C))$$
$$\exists^{sim}\emptyset.(\mathcal{T}, C) \equiv \exists^{sim}\{B\}.(\mathcal{T}, C)$$

where B is a concept name that does not occur in $\mathcal{T}$ and C.

Simulating models of KBs. The third language $\mathcal{EL}^{sa}$ extends $\mathcal{EL}$ by the concept constructor $\exists^{sim}\Sigma.(\mathcal{T}, \mathcal{A}, a)$, where a is an individual name in the ABox $\mathcal{A}$, $\mathcal{T}$ is a TBox, and Σ a finite signature. More precisely, we define $\mathcal{EL}^{sa}$-concepts, concept inclusions, general TBoxes, and KBs, by simultaneous induction as follows:

- every $\mathcal{EL}$-concept, concept inclusion, general TBox, and KB is an $\mathcal{EL}^{sa}$-concept, concept inclusion, general TBox, and KB, respectively;
- if $(\mathcal{T}, \mathcal{A})$ is a general $\mathcal{EL}^{sa}$-KB, a an individual name in $\mathcal{A}$, and Σ a finite signature, then $\exists^{sim}\Sigma.(\mathcal{T}, \mathcal{A}, a)$ is an $\mathcal{EL}^{sa}$-concept;
- if C, D are $\mathcal{EL}^{sa}$-concepts, then $C \sqsubseteq D$ is an $\mathcal{EL}^{sa}$-concept inclusion;
- a general $\mathcal{EL}^{sa}$-TBox is a finite set of $\mathcal{EL}^{sa}$-concept inclusions;
- an $\mathcal{EL}^{sa}$-KB is a pair $(\mathcal{T}, \mathcal{A})$ consisting of a general $\mathcal{EL}^{sa}$-TBox and an ABox.

The semantics of $\exists^{sim}\Sigma.(\mathcal{T}, \mathcal{A}, a)$ is as follows:

> $d \in (\exists^{sim}\Sigma.(\mathcal{T}, \mathcal{A}, a))^{\mathcal{I}}$ iff there exists a model $\mathcal{J}$ of $(\mathcal{T}, \mathcal{A})$ such that $(\mathcal{J}, a^{\mathcal{J}}) \leq_{\Gamma} (\mathcal{I}, d)$, where $\Gamma = (\mathsf{N_C} \cup \mathsf{N_R}) \setminus \Sigma$.

Example 3. Let $\mathcal{T} = \emptyset$, $\mathcal{A} = \{r(a, a)\}$, and $\Sigma = \emptyset$. Then $\exists^{sim}\Sigma.(\mathcal{T}, \mathcal{A}, a)$ is equivalent to the concept $\exists^{sim}(\mathcal{I}, d)$ defined in Example 1.

Let $\mathcal{L}_1, \mathcal{L}_2$ be sets of concepts. We say that $\mathcal{L}_2$ is *polynomially at least as expressive as* $\mathcal{L}_1$, in symbols $\mathcal{L}_1 \leq_p \mathcal{L}_2$, if for every $C_1 \in \mathcal{L}_1$ one can construct in polynomial time a $C_2 \in \mathcal{L}_2$ such that C_1 and C_2 are equivalent. We say that $\mathcal{L}_1, \mathcal{L}_2$ are *polynomially equivalent*, in symbols $\mathcal{L}_1 \equiv_p \mathcal{L}_2$, if $\mathcal{L}_1 \leq_p \mathcal{L}_2$ and $\mathcal{L}_2 \leq_p \mathcal{L}_1$.

Theorem 3. *The languages $\mathcal{EL}^{\nu+}$, $\mathcal{EL}^{si}$, $\mathcal{EL}^{st}$, and $\mathcal{EL}^{sa}$ are polynomially equivalent.*

We provide sketches of proofs of $\mathcal{EL}^{si} \leq_p \mathcal{EL}^{\nu+}$, $\mathcal{EL}^{\nu+} \leq_p \mathcal{EL}^{st}$, $\mathcal{EL}^{st} \leq_p \mathcal{EL}^{sa}$, and $\mathcal{EL}^{sa} \leq_p \mathcal{EL}^{si}$.

$\mathcal{EL}^{si} \leq_p \mathcal{EL}^{\nu+}$. By Lemma 1, considering $\mathcal{EL}^{si}$-concepts of the form $\exists^{sim}(\mathcal{I}, d)$ is sufficient. Each such concept is equivalent to the $\mathcal{EL}^{\nu+}$-concept $\nu_\ell d_1 \cdots d_n.C_1, \ldots, C_n$, where $\Delta^{\mathcal{I}} = \{d_1, \ldots, d_n\}$ is regarded as a set of concept variables, $d = d_\ell$, and

$$C_i = \bigsqcap\{A \mid d_i \in A^{\mathcal{I}}\} \sqcap \bigsqcap\{\exists r.d_j \mid (d_i, d_j) \in r^{\mathcal{I}}\}.$$

$\mathcal{EL}^{\nu+} \leq_p \mathcal{EL}^{st}$. Let C be a closed $\mathcal{EL}^{\nu+}$-concept. An equivalent $\mathcal{EL}^{st}$-concept is constructed by replacing each subconcept of C of the form $\nu_\ell X_1, \ldots, X_n.C_1, \ldots, C_n$ with an $\mathcal{EL}^{st}$-concept, proceeding from the inside out. We assume that for every variable X that occurs in the original $\mathcal{EL}^{\nu+}$-concept C, there is a concept name A_X that does not occur in C. Now $\nu_\ell X_1, \ldots, X_n.C_1, \ldots, C_n$ (which potentially contains free variables) is replaced with the $\mathcal{EL}^{st}$-concept

$$\exists^{sim}\{A_{X_1}, \ldots, A_{X_n}\}.(\{A_{X_i} \sqsubseteq C_i^{\downarrow} \mid 1 \leq i \leq n\}, A_{X_\ell})$$

where $C_i^{\downarrow}$ is obtained from C_i by replacing every variable X with the concept name A_X.

$\mathcal{EL}^{st} \leq_p \mathcal{EL}^{sa}$. Let C be an $\mathcal{EL}^{st}$-concept. As already observed, we may assume that D is a concept name in all subconcepts $\exists^{sim}\Sigma.(\mathcal{T}, D)$ of C. Now replace each $\exists^{sim}\Sigma.(\mathcal{T}, A)$ in C, proceeding from the inside out, by $\exists^{sim}\Sigma.(\mathcal{T}, \mathcal{A}, a)$, where $\mathcal{A} = \{A(a)\}$. The resulting concept is equivalent to C.

$\mathcal{EL}^{sa} \leq_p \mathcal{EL}^{si}$. To prove this inclusion, we make use of *canonical models* for $\mathcal{EL}^{sa}$-KBs, similar to those used for $\mathcal{EL}$ in [5]. In particular, canonical models for $\mathcal{EL}^{sa}$ can be constructed by an extension of the algorithm given in [5], see the full version for details.

Theorem 4 (Canonical model). *For every satisfiable $\mathcal{EL}^{sa}$-KB $(\mathcal{T}, \mathcal{A})$, one can construct in polynomial time a model $\mathcal{I}_{\mathcal{T},\mathcal{A}}$ of $(\mathcal{T}, \mathcal{A})$ with $|\Delta^{\mathcal{I}_{\mathcal{T},\mathcal{A}}}|$ bounded by twice the size of $(\mathcal{T}, \mathcal{A})$ and such that for every model $\mathcal{J}$ of $(\mathcal{T}, \mathcal{A})$, we have $(\mathcal{I}_{\mathcal{T},\mathcal{A}}, a^{\mathcal{I}_{\mathcal{T},\mathcal{A}}}) \leq (\mathcal{J}, a^{\mathcal{J}})$ for all $a \in \mathsf{Ind}(\mathcal{A})$.*

To prove $\mathcal{EL}^{sa} \leq_p \mathcal{EL}^{si}$, it suffices to show that any outermost occurrence of a concept of the form $\exists^{sim}\Sigma.(\mathcal{T}, \mathcal{A}, a)$ in an $\mathcal{EL}^{sa}$-concept C can be replaced with the equivalent $\mathcal{EL}^{si}$-concept $\exists^{sim}(\mathcal{I}^{\Sigma}_{\mathcal{T},\mathcal{A}}, a)$, where $\mathcal{I}^{\Sigma}_{\mathcal{T},\mathcal{A}}$ denotes $\mathcal{I}_{\mathcal{T},\mathcal{A}}$ except that all $\sigma \in \Sigma$ are interpreted as empty sets. First let $d \in (\exists^{sim}\Sigma.(\mathcal{T}, \mathcal{A}, a))^{\mathcal{J}}$. Then there is a model $\mathcal{I}'$ of $(\mathcal{T}, \mathcal{A})$ such that $(\mathcal{I}', a^{\mathcal{I}'}) \leq_\Sigma (\mathcal{J}, d)$. By Theorem 4, $(\mathcal{I}_{\mathcal{T},\mathcal{A}}, a^{\mathcal{I}_{\mathcal{T},\mathcal{A}}}) \leq (\mathcal{I}', a^{\mathcal{I}'})$. Thus, by closure of simulations under composition, $(\mathcal{I}^{\Sigma}_{\mathcal{T},\mathcal{A}}, a) \leq_\Sigma (\mathcal{J}, d)$ as required. The converse direction follows from the condition that $\mathcal{I}_{\mathcal{T},\mathcal{A}}$ is a model of $(\mathcal{T}, \mathcal{A})$. This finishes our proof sketch for Theorem 3.

It is interesting to note that, as a consequence of the proofs of Theorem 3, for every $\mathcal{EL}^{\nu+}$-concept there is an equivalent $\mathcal{EL}^{\nu+}$-concept of polynomial size in which the greatest fixpoint constructor is not nested, and similarly for $\mathcal{EL}^{st}$, $\mathcal{EL}^{sa}$. An important consequence of the existence of canonical models, as granted by Theorem 4, is that reasoning in our family of extensions of $\mathcal{EL}$ is tractable.

Theorem 5 (Tractable reasoning). *Let $\mathcal{L}$ be any of the languages $\mathcal{EL}^{\nu}$, $\mathcal{EL}^{\nu+}$, $\mathcal{EL}^{si}$, $\mathcal{EL}^{st}$, or $\mathcal{EL}^{sa}$. Then KB consistency, subsumption w.r.t. TBoxes, and the instance problem can be decided in* PTIME.

Proof. By Theorem 3, it suffices to concentrate on $\mathcal{L} = \mathcal{EL}^{sa}$. Consistency can be decided in PTIME by the algorithm that constructs the canonical model. Subsumption can be polynomially reduced in the standard way to the instance problem. Finally, by Theorem 4, we can decide the instance problem as follows: to decide whether $(\mathcal{T}, \mathcal{A}) \models C(a)$, where we can w.l.o.g. assume that $C = A$ for a concept name A, we check whether $(\mathcal{T}, \mathcal{A})$ is inconsistent or $a^{\mathcal{I}_{\mathcal{T},\mathcal{A}}} \in A^{\mathcal{I}_{\mathcal{T},\mathcal{A}}}$. Both can be done in PTIME. □

5 Characterizing $\mathcal{EL}^\nu$ using simulations

When characterizing $\mathcal{EL}$ as a fragment of first-order logic in Theorem 2, our starting point was the observation that $\mathcal{EL}$-concepts are preserved under simulations and that $\mathcal{EL}$ is a Horn logic, thus having finite minimal models. The same is true for $\mathcal{EL}^\nu$: first, $\mathcal{EL}^\nu$-concepts are preserved under simulations, as $\mathcal{EL}^{si}$ is obviously preserved under simulations and, by Theorem 3, every $\mathcal{EL}^\nu$-concept is equivalent to an $\mathcal{EL}^{si}$-concept. And second, a finite minimal model of an $\mathcal{EL}^\nu$-concept C can be constructed by taking the canonical model $\mathcal{I}_{\mathcal{T},\mathcal{A}}$ from Theorem 4 for $\mathcal{T} = \{A \sqsubseteq C\}$ and $\mathcal{A} = \{A(a)\}$. As required, we then have $\models C \sqsubseteq D$ iff $(\mathcal{T}, \mathcal{A}) \models D(a)$ iff $a \in D^{\mathcal{I}_{\mathcal{T},\mathcal{A}}}$, for all $\mathcal{EL}^\nu$-concepts D. However, $\mathcal{EL}^\nu$ is clearly not a fragment of FO. Instead, it relates to *MSO* in exactly the way that $\mathcal{EL}$ related to FO.

Theorem 6. *The set of $\mathcal{EL}^\nu$-concepts is a maximal set of MSO-formulas that is preserved under simulations and has finite minimal models: if $\mathcal{L}$ is a set of MSO-formulas that properly contains all $\mathcal{EL}^\nu$-concepts, then either it contains a formula not preserved under simulations or it does not have finite minimal models.*

Proof. Assume that $\mathcal{L} \supseteq \mathcal{EL}^\nu$ is preserved under simulations and has finite minimal models. Let $\varphi(x) \in \mathcal{L}$. We have to show that $\varphi(x)$ is equivalent to an $\mathcal{EL}^\nu$-concept. To this end, take a finite minimal model of φ, i.e., an interpretation $\mathcal{I}$ and a $d \in \Delta^{\mathcal{I}}$ such that for all $\psi(x) \in \mathcal{L}$ we have that $\forall x.(\varphi(x) \rightarrow \psi(x))$ is valid iff $\mathcal{I} \models \psi[d]$. We will show that φ is equivalent to (the MSO translation of) $\exists^{sim}(\mathcal{I}, d)$. We may assume that $\exists^{sim}(\mathcal{I}, d) \in \mathcal{L}$. Since $d \in (\exists^{sim}(\mathcal{I}, d))^{\mathcal{I}}$, we thus have that $\forall x.(\varphi(x) \rightarrow \exists^{sim}(\mathcal{I}, d)(x))$ is valid. Conversely, assume that $d' \in (\exists^{sim}(\mathcal{I}, d))^{\mathcal{J}}$ for some interpretation $\mathcal{J}$. Then $(\mathcal{I}, d) \leq (\mathcal{J}, d')$. We have $(\mathcal{I}, d) \models \varphi[d]$. Thus, by preservation of $\varphi(x)$ under simulations, $\mathcal{J} \models \varphi[d']$. Thus $\forall x.(\exists^{sim}(\mathcal{I}, d)(x) \rightarrow \varphi(x))$ is also valid. □

A number of closely related characterizations remain open. For example, we conjecture that an extension of Theorem 1 holds for $\mathcal{EL}^{\nu,\sqcup}$ and MSO (instead of $\mathcal{EL}$ and FO). Also, it is open whether Theorem 6 still holds if finite minimal models are replaced by arbitrary minimal models.

6 Applications and Logical Properties

The μ-calculus is considered to be extremely well-behaved regarding its expressive power and logical properties. The aim of this section is to take a brief look at the expressive power of its $\mathcal{EL}$-analogues $\mathcal{EL}^\nu$ and $\mathcal{EL}^{\nu+}$. In particular, we show that $\mathcal{EL}^{\nu+}$ is more well-behaved than $\mathcal{EL}$ in a number of respects. Throughout this section, we will not distinguish between the languages previously proved polynomially equivalent.

To begin with, we construct the *least common subsumer* (LCS) of two concepts w.r.t. a general $\mathcal{EL}^{\nu+}$-TBox (the generalization to more than two concepts is straightforward). Given a general $\mathcal{EL}^{\nu+}$-TBox $\mathcal{T}$ and concepts C_1, C_2, a concept C is called the *LCS of C_1, C_2 w.r.t. $\mathcal{T}$ in $\mathcal{EL}^{\nu+}$* if

- $\mathcal{T} \models C_i \sqsubseteq C$ for $i = 1, 2$;

– if $\mathcal{T} \models C_i \sqsubseteq D$ for $i = 1, 2$ and D a $\mathcal{EL}^{\nu+}$-concept, then $\mathcal{T} \models C \sqsubseteq D$.

It is known that, in $\mathcal{EL}$, the LCS does not always exist [1].

Example 4. In $\mathcal{EL}$, the LCS of A, B w.r.t.

$$\mathcal{T} = \{A \sqsubseteq \exists \mathsf{has\_parent}.A, B \sqsubseteq \exists \mathsf{has\_parent}.B\}$$

does not exist. In $\mathcal{EL}^{\nu}$, however, the LCS of A,B w.r.t. $\mathcal{T}$ is given by $\nu X.\exists \mathsf{has\_parent}.X$.

To construct the LCS in $\mathcal{EL}^{\nu+}$, we adopt the product construction used in [1] for the case of classical TBoxes with a fixpoint semantics. For interpretations $\mathcal{I}_1$ and $\mathcal{I}_2$, the *product* $\mathcal{I}_1 \times \mathcal{I}_2$ is defined by setting $\Delta^{\mathcal{I}_1 \times \mathcal{I}_2} = \Delta^{\mathcal{I}_1} \times \Delta^{\mathcal{I}_2}$, $(d_1, d_2) \in A^{\mathcal{I}_1 \times \mathcal{I}_2}$ iff $d_i \in A^{\mathcal{I}_i}$ for $i = 1, 2$, and $((d_1, d_2), (d'_1, d'_2)) \in r^{\mathcal{I}_1 \times \mathcal{I}_2}$ iff $(d_i, d'_i) \in r^{\mathcal{I}_i}$ for $i = 1, 2$.

Theorem 7. *Let $\mathcal{T}$ be a general $\mathcal{EL}^{\nu+}$-TBox and C_1 and C_2 be $\mathcal{EL}^{\nu+}$-concepts. Then $\exists^{sim}(\mathcal{I}_{\mathcal{T},C_1} \times \mathcal{I}_{\mathcal{T},C_2}, (d_{C_1}, d_{C_2}))$ is the LCS of C_1, C_2 w.r.t. $\mathcal{T}$ in $\mathcal{EL}^{\nu}$.*

The same product construction has been used in [1] for the case of classical TBoxes with a fixpoint semantics, which, however, additionally require a notion of conservative extension (see Section 7).

Our second result concerns the most specific concept, which plays an important role in the bottom-up construction of knowledge bases and has received quite a bit of attention in the context of $\mathcal{EL}$ [1, 7]. Formally, a concept C is the *most specific concept (MSC)* for an individual a in a knowledge base $(\mathcal{T}, \mathcal{A})$ in $\mathcal{EL}^{\nu+}$ if

– $(\mathcal{T}, \mathcal{A}) \models C(a)$ and
– for every $\mathcal{EL}^{\nu+}$-concept D with $(\mathcal{T}, \mathcal{A}) \models D(a)$, we have $\mathcal{T} \models C \sqsubseteq D$.

In $\mathcal{EL}$, the MSC need not exist, as is witnessed by the KB $(\emptyset, \{\mathsf{has\_parent}(a, a)\})$, where the MSC for a is non-existent.

Theorem 8. *In $\mathcal{EL}^{\nu+}$, the MSC always exists for any a in any KB $(\mathcal{T}, \mathcal{A})$ and is given as $\exists^{sim}\emptyset.(\mathcal{T}, \mathcal{A}, a)$.*

In [1], the MSC in $\mathcal{EL}$-KBs based on classical TBoxes with a fixpoint semantics is defined. The relationship between $\mathcal{EL}^{\nu+}$ and fixpoint TBoxes is discussed in more detail in Section 7.

We now turn our attention to issues of definability and interpolation. From now on, we use $\mathsf{sig}(C)$ to denote the set of concept and role names used in the concept C. A concept C is a *Σ-concept* if $\mathsf{sig}(C) \subseteq \Sigma$. Let $\mathcal{T}$ be a general $\mathcal{EL}^{\nu+}$-TBox, C an $\mathcal{EL}^{\nu+}$-concept and Γ a finite signature.

We start with considering the fundamental notion of a Γ-definition. The question addressed here is whether a given concept can be expressed in an equivalent way by referring only to the symbols in a given signature Γ [21, 20]. Formally, a Γ-concept D is an *explicit Γ-definition* of a concept C w.r.t. a TBox $\mathcal{T}$ if, and only if, $\mathcal{T} \models C \equiv D$ (i.e., C and D are equivalent w.r.t. $\mathcal{T}$). Clearly, explicit Γ-definitions do not always exist in any of the logics studied in this paper: for example, there is no explicit $\{A\}$-definition of B w.r.t. the TBox $\{A \sqsubseteq B\}$. However, it is not hard to show the following using the fact that $\exists^{sim}\Sigma.(\mathcal{T}, C)$ is the most specific Γ-concept that subsumes C w.r.t. $\mathcal{T}$.

Proposition 2. *Let C be an $\mathcal{EL}^{\nu+}$-concept, $\mathcal{T}$ a general $\mathcal{EL}^{\nu+}$-TBox and Γ a signature. There exists an explicit Γ-definition of C w.r.t. $\mathcal{T}$ iff $\exists^{sim}\Sigma.(\mathcal{T}, C)$ is such a definition (for $\Sigma = \mathsf{sig}(\mathcal{T}, C) \setminus \Gamma$).*

It is interesting to note that if $\mathcal{T}$ happens to be a general $\mathcal{EL}$-TBox and C an $\mathcal{EL}$-concept and there exists an explicit Γ-definition of C w.r.t. $\mathcal{T}$, then the concept $\exists^{sim}\Sigma.(\mathcal{T}, C)$ from Proposition 2 is equivalent w.r.t. $\mathcal{T}$ to an $\mathcal{EL}$-concept over Γ. This follows from the fact that $\mathcal{EL}$ has the Beth definability property (see below for a definition) which follows immediately from interpolation results proved for $\mathcal{EL}$ in [15].

The advantage of giving explicit Γ-definitions in $\mathcal{EL}^{\nu+}$ even when $\mathcal{T}$ and C are formulated in $\mathcal{EL}$ is that Γ-definitions in $\mathcal{EL}^{\nu+}$ are of polynomial size while the following example shows that they may be exponentially large in $\mathcal{EL}$.

Example 5. Let $\mathcal{T}$ consist of $A_i \equiv \exists r_i.A_{i+1} \sqcap \exists s_i A_{i+1}$ for $0 \leq i < n$, and $A_n \equiv \top$. Let $\Gamma = \{r_0, \ldots, r_{n-1}, s_0, \ldots, s_{n-1}\}$. Then A_0 has an explicit Γ-definition w.r.t. $\mathcal{T}$ in $\mathcal{EL}$, namely C_0, where $C_i = \exists r_i.C_{i+1} \sqcap \exists s_i.C_{i+1}$ and $C_n = \top$. This definition is of exponential size and it is easy to see that there is no shorter Γ-definition of A_0 w.r.t. $\mathcal{T}$ in $\mathcal{EL}$.

Say that a concept C is *implicitly Γ-defined w.r.t.* $\mathcal{T}$ iff $\mathcal{T} \cup \mathcal{T}_\Gamma \models C \equiv C_\Gamma$, where $\mathcal{T}_\Gamma$ and C_Γ are obtained from $\mathcal{T}$ and C, respectively, by replacing each $\sigma \notin \Gamma$ by a fresh symbol σ'. The Beth definability property, which was studied in a DL context in [21, 20], ensures that explicit Γ-definitions always exist when they possibly can.

Theorem 9. *$\mathcal{EL}^{\nu+}$ has the polynomial Beth definability property: for every general $\mathcal{EL}^{\nu+}$-TBox $\mathcal{T}$, concept C, and signature Γ such that C is implicitly Γ-defined w.r.t. $\mathcal{T}$, there is an explicit Γ-definition w.r.t. $\mathcal{T}$, namely $\exists^{sim}(\mathsf{sig}(\mathcal{T}, C) \setminus \Gamma).(\mathcal{T}, C)$.*

The proof of Theorem 9 relies on $\mathcal{EL}^{\nu}$ having a certain interpolation property. Say that two general TBoxes $\mathcal{T}_1$ and $\mathcal{T}_2$ are *Δ-inseparable* w.r.t. $\mathcal{EL}^{\nu}$ if $\mathcal{T}_1 \models C \sqsubseteq D$ iff $\mathcal{T}_2 \models C \sqsubseteq D$ for all $\mathcal{EL}^{\nu}$-inclusions $C \sqsubseteq D$.

Theorem 10. *Let $\mathcal{T}_1 \cup \mathcal{T}_2 \models C \sqsubseteq D$ and assume that $\mathcal{T}_1$ and $\mathcal{T}_2$ are Δ-inseparable w.r.t. $\mathcal{EL}^{\nu}$ for $\Delta = \mathsf{sig}(\mathcal{T}_1, C) \cap \mathsf{sig}(\mathcal{T}_2, D)$. Then the Δ-concept $F = \exists^{sim}\Sigma.(\mathcal{T}_1, C)$, $\Sigma = \mathsf{sig}(\mathcal{T}_1, C) \setminus \Delta$, is an interpolant of C, D w.r.t. $\mathcal{T}_1, \mathcal{T}_2$; i.e. $\mathcal{T}_1 \models C \sqsubseteq F$ and $\mathcal{T}_2 \models F \sqsubseteq D$.*

We show how Theorem 9 follows from Theorem 10. Assume that $\mathcal{T} \cup \mathcal{T}_\Gamma \models C \equiv C_\Gamma$, where $\mathcal{T}, \mathcal{T}_\Gamma, C, C_\Gamma$ satisfy the conditions of Theorem 9. Then $\mathcal{T}$ and $\mathcal{T}_\Gamma$ are Γ-inseparable and $\Gamma \supseteq \mathsf{sig}(\mathcal{T}, C) \cap \mathsf{sig}(\mathcal{T}_\Gamma, C_\Gamma)$. Thus, by Theorem 10, $\mathcal{T} \models \exists^{sim}\Sigma.(\mathcal{T}_\Gamma, C_\Gamma) \sqsubseteq C$ for $\Sigma = \mathsf{sig}(\mathcal{T}_\Gamma, C_\Gamma) \setminus \Gamma$. Now Theorem 9 follows from the fact that $\exists^{sim}\Sigma.(\mathcal{T}_\Gamma, C_\Gamma)$ is equivalent to $\exists^{sim}\Sigma'.(\mathcal{T}, C)$ for $\Sigma' = \mathsf{sig}(\mathcal{T}, C) \setminus \Gamma$.

In [15], it is shown that $\mathcal{EL}$ also has this interpolation property. However, the advantage of using $\mathcal{EL}^{\nu+}$ is that interpolants are of polynomial size. The decomposition algorithm for $\mathcal{EL}$ given in [15] crucially depends on this property of $\mathcal{EL}^{\nu+}$.

7 Relation to TBoxes with Fixpoint Semantics

There is a tradition of considering DLs that introduce fixpoints at the TBox level instead of at the concept level [18, 19, 2]. In [4], Baader proposes and analyzes such a DL based on $\mathcal{EL}$ and greatest fixpoints. This DL, which we call $\mathcal{EL}^{\mathsf{gfp}}$ from now on, differs from our $\mathcal{EL}^{\nu}$ in that (i) TBoxes are classical TBoxes rather than sets of GCIs (but cycles are allowed) and (ii) the ν-concept constructor is not present; instead, a greatest fixpoint semantics is adopted for the defined concept names.

On the concept level, $\mathcal{EL}^{\nu}$ is clearly strictly more expressive than $\mathcal{EL}^{\mathsf{gfp}}$: since fixpoints are introduced at the TBox level, concepts of $\mathcal{EL}^{\mathsf{gfp}}$ coincide with $\mathcal{EL}$-concepts, and thus there is no $\mathcal{EL}^{\mathsf{gfp}}$-concept equivalent to the $\mathcal{EL}^{\nu}$-concept $\nu X.\exists r.X$. In the following, we show that $\mathcal{EL}^{\nu}$ is also more expressive than $\mathcal{EL}^{\mathsf{gfp}}$ also on the TBox level, even if we restrict $\mathcal{EL}^{\nu}$-TBoxes as in $\mathcal{EL}^{\mathsf{gfp}}$. We use the standard notion of logical equivalence, i.e., two TBoxes $\mathcal{T}$ and $\mathcal{T}'$ are *equivalent* iff $\mathcal{T}$ and $\mathcal{T}'$ have precisely the same models. As observed by Schild in the context of $\mathcal{ALC}$ [19], every $\mathcal{EL}^{\mathsf{gfp}}$-TBox $\mathcal{T} = \{A_1 \equiv C_1, \ldots, A_n \equiv C_n\}$ is equivalent in this sense to the $\mathcal{EL}^{\nu+}$-TBox $\{A_i \equiv \nu_i X_1, \ldots, X_n.C'_1, \ldots, C'_n \mid 1 \leq i \leq n\}$, where each C'_i is obtained from C_i by replacing each A_j with X_j, $1 \leq j \leq n$. Note that since we are using mutual fixpoints the size of the resulting TBox is polynomial in the size of the original one. In the converse direction, there is no equivalence-preserving translation.

Lemma 2. *For each $\mathcal{EL}^{\mathsf{gfp}}$-TBox, there is an equivalent $\mathcal{EL}^{\nu+}$-TBox of polynomial size, but no $\mathcal{EL}^{\mathsf{gfp}}$-TBox is equivalent to the $\mathcal{EL}^{\nu}$-TBox $\{A \equiv P \sqcap \nu X.\exists r.X\}$.*

Proof. (sketch) It is not hard to prove that for every $\mathcal{EL}^{\mathsf{gfp}}$-TBox $\mathcal{T}$, defined concept name A in $\mathcal{T}$, and role name r, one of the following holds:

- there is an $m \geq 0$ such that $A \sqsubseteq \exists r^n.\top$ implies $n \leq m$ or
- $A \sqsubseteq \exists r^n.B$ for some $n > 0$ and defined concept name B.

However, no such TBox can be equivalent to $A \sqsubseteq \exists r^n.B$ since $\mathcal{T} \models \exists r^n.\top$ for all $n > 0$, but there is no $n > 0$ and defined concept name B with $A \sqsubseteq \exists r^n.B$. □

$\mathcal{EL}^{\mathsf{gfp}}$ and $\mathcal{EL}^{\nu}$ become equi-expressive if the strict notion of equivalence used above is replaced with one based on conservative extensions, thus allowing the introduction of new concept names that are suppressed from logical equivalence. However, we believe that not having to deal with conservative extensions is an advantage of $\mathcal{EL}^{\nu}$ over $\mathcal{EL}^{\mathsf{gfp}}$, as conservative extensions tend to make simple definitions somewhat awkward, c.f. the least common subsumers and most specific concepts for $\mathcal{EL}^{\mathsf{gfp}}$ in [3, 4].

References

1. F. Baader. Least common subsumers and most specific concepts in description logics with existential restrictions and terminological cycles. In *Proc. of IJCAI03*, pages 319–324. Morgan Kaufmann, 2003.
2. F. Baader. Using automata theory for characterizing the semantics of terminological cycles. *Annals of Mathematics and Artificial Intelligence*, 18(2–4):175–219, 1996.

3. F. Baader. The instance problem and the most specific concept in the description logic $\mathcal{EL}$ w.r.t. terminological cycles with descriptive semantics. In *Proc. of KI 2003*, volume 2821 of *LNAI*, pages 64–78, 2003. Springer.
4. F. Baader. Terminological cycles in a description logic with existential restrictions. In *Proc. of IJCAI03*, pages 325–330. Morgan Kaufmann, 2003.
5. F. Baader, S. Brandt, and C. Lutz. Pushing the $\mathcal{EL}$ envelope. In *Proc. of IJCAI05*. Morgan Kaufmann, 2005.
6. F. Baader, D. Calvanes, D. McGuiness, D. Nardi, and P. Patel-Schneider. *The Description Logic Handbook: Theory, implementation and applications*. Cambridge Univ. Press, 2003.
7. F. Baader and F. Distel. A finite basis for the set of $\mathcal{EL}$-implications holding in a finite model. In *Proc. of ICFCA8*, volume 4933 of *LNAI*, pages 46–61. Springer, 2008.
8. J. Bradfield and C. Stirling. Modal mu-calculus. In *Handbook of Modal Logic*. Elsevier, 2006.
9. D. Calvanese, G. De Giacomo, D. Lembo, M. Lenzerini, and R. Rosati. Tractable reasoning and efficient query answering in description logics: The *DL-Lite* family. *J. of Automated Reasoning*, 39(3):385–429, 2007.
10. J. W. de Bakker. *Mathematical Theory of Program Correctness*. Prentice-Hall, 1980.
11. T. French. *Bisimulation quantifiers for modal logics*. PhD thesis, University of Western Australia, 2006.
12. G. De Giacomo and M. Lenzerini. Boosting the Correspondence between Description Logics and Propositional Dynamic Logics. In *Proc. of AAAI94*, pages 205-212. AAAI Press, 1994.
13. V. Goranko and M. Otto. Model theory of modal logic. In *Handbook of Modal Logic*. Elsevier, 2007.
14. D. Janin and I. Walukiewicz. On the expressive completeness of the propositional mu-calculus with respect to monadic second order logic. In *Proc. of CONCUR96*, volume 1119 of *PNCS*, pages 263–277. Springer, 1996.
15. B. Konev, C. Lutz, D. Ponomaryov, and F. Wolter. Decomposing description logic ontologies. In *Proceedings of KR10*, 2010.
16. N. Kurtonina and M. de Rijke. Expressiveness of concept expressions in first-order description logics. *Artificial Intelligence*, 107, 1999.
17. C. Lutz and F. Wolter. Deciding inseparability and conservative extensions in the description logic $\mathcal{EL}$. *J. of Symbolic Computation*, 45(2):194–228, 2010.
18. B. Nebel. Terminological cycles: Semantics and computational properties. In *Principles of Semantic Networks*. Morgan Kaufmann, 1991.
19. K. Schild. Terminological cycles and the propositional μ-calculus. In *Proc. of KR'94*, pages 509–520. Morgan Kaufmann, 1994.
20. I. Seylan, E. Franconi, and J. de Bruijn. Effective query rewriting with ontologies over DBoxes. In *Proc. of IJCAI09*, pages 923–925, 2009.
21. B. ten Cate, W. Conradie, M. Marx, and Y. Venema. Definitorially complete description logics. In *Proc. of KR'06*, pages 79–89. AAAI Press, 2006.
22. J. van Benthem. Modal Correspondence Theory. Mathematical Institute, University of Amsterdam, 1976.
23. M. Vardi and P. Wolper. Automata theoretic techniques for modal logics of programs. In *Proc. of STOC84*, pages 446–456. ACM Press, 1984.
24. M. Y. Vardi. Why is modal logic so robustly decidable? In *Descriptive Complexity and Finite Models*, pages 149–184, 1996.