A CLT CONCERNING CRITICAL POINTS OF RANDOM FUNCTIONS ON A EUCLIDEAN SPACE

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Abstract. We prove a central limit theorem concerning the number of critical points in large cubes of an isotropic Gaussian random function on a Euclidean space.

Contents

1. Introduction

Throughout this paper $X(\boldsymbol{t})$ denotes a centered, isotropic Gaussian random function on $\mathbb{R}^m$, $m \geq 2$.

Assume that X is a.s. C^1. For any Borel subset $S \subset \mathbb{R}^m$ we denote by $Z(S)$ the number of critical points of X in S. For a positive number L we set

$$Z_L := Z\big([-L, L]^m \big),$$

and we form the new random variable

$$\zeta_L := \frac{1}{(2L)^{m/2}}\Big(Z_L - \boldsymbol{E}\big[\, Z_L \,\big] \Big). \tag{1.1}$$

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In this paper we will prove that, under certain assumptions on $X(\boldsymbol{t})$, the sequence of random variables (ζ_N), converges in distribution as $N \to \infty$ to a Gaussian with mean zero and finite, positive variance.

The proof, inspired from the recent work of Estrade-León [11], uses the Wiener chaos decomposition of Z_L.

Notations.

- $\mathbb{N} := \mathbb{Z}_{>0}$, $\mathbb{N}_0 := \mathbb{Z}_{\geq 0}$.
- For any positive integer n we denote by $\mathbb{1}_n$ the identity map $\mathbb{R}^n \to \mathbb{R}^n$.
- For $\boldsymbol{t} = (t_1, \dots, t_m) \in \mathbb{R}^m$ we set

$$|\boldsymbol{t}| := \sqrt{\sum_{k=1}^m t_k^2}, \ \ |\boldsymbol{t}|_\infty := \max_{1 \leq k \leq m} |t_k|.$$

- For any $v \geq 0$ we denote by $\boldsymbol{\gamma}_v$ the Gaussian measure on $\mathbb{R}$ with mean 0 and variance v.
- If X is a scalar random variable, then we will use the notation $X \in N(0, v)$ to indicate that X is a normal random variable with mean zero and variance v.
- If A is a subset of a given set S, then we denote by $\boldsymbol{I}_A$ the indicator function of A

$$\boldsymbol{I}_A : S \to \{0, 1\}, \ \ \boldsymbol{I}_A(s) = \begin{cases} 1, & s \in A, \\ 0, & s \in S \setminus A. \end{cases}$$

- If X is a random vector, then we denote by $\boldsymbol{E}[X]$ and respectively $\boldsymbol{var}(X)$ the mean and respectively the variance of X.

2. Statement of the main result

Denote by $K(\boldsymbol{t}, \boldsymbol{s})$ the covariance kernel of $X(\boldsymbol{t})$,

$$K(\boldsymbol{s}, \boldsymbol{t}) := \boldsymbol{E}\big[\, X(\boldsymbol{t}) X(\boldsymbol{s}) \,\big], \ \ \boldsymbol{t}, \boldsymbol{s} \in \mathbb{R}^m.$$

The isotropy of X implies that there exists a radially symmetric function $C : \mathbb{R}^m \to \mathbb{R}$ such that $K(\boldsymbol{t}, \boldsymbol{s}) = C(\boldsymbol{t} - \boldsymbol{s})$, $\forall \boldsymbol{t}, \boldsymbol{s} \in \mathbb{R}$. We denote by $\mu(d\boldsymbol{\lambda})$ the spectral measure of X so that $C(\boldsymbol{t})$ is the Fourier transform of μ

$$C(\boldsymbol{t}) = (2\pi)^{-\frac{m}{2}} \int_{\mathbb{R}^m} e^{-\boldsymbol{i}(\boldsymbol{t}, \boldsymbol{\lambda})} \mu(d\boldsymbol{\lambda}). \tag{2.1}$$

2.1. **The setup.** For the claimed central limit result to hold, we need to make certain assumptions on the random function $X(t)$. These assumptions closely mirror the assumptions in [11].

Assumption A1. *The random function $X(t)$ is almost surely C^3.*

To formulate our next assumption we set

$$\psi(\boldsymbol{t}) := \max\big\{\, |\partial_{\boldsymbol{t}}^\alpha C(t)|; \ \ |\alpha| \leq 4 \,\big\}, \ \ \boldsymbol{t} \in \mathbb{R}^m, \tag{2.2}$$

where for any multi-index $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{N}_0^m$ we set

$$|\alpha| := \alpha_1 + \cdots + \alpha_m, \ \ \partial_{\boldsymbol{t}}^\alpha = \partial_{t_1}^{\alpha_1} \cdots \partial_{t_m}^{\alpha_m}.$$

Assumption A2.

$$\lim_{|\boldsymbol{t}| \to \infty} \psi(\boldsymbol{t}) = 0 \ \text{ and } \ \psi \in L^1(\mathbb{R}^m).$$

Our next assumption involves the spectral measure $\mu(d\boldsymbol{\lambda})$ and it states in precise terms that this measure has a continuous density that decays rapidly at ∞.

Assumption A3. *There exists a nontrivial even, continuous function $w:\mathbb{R}\to[0,\infty)$ such that*

$$\mu(d\boldsymbol{\lambda}) = w(|\boldsymbol{\lambda}|)d\boldsymbol{\lambda}.$$

Moreover w has a fast decay at ∞, i.e.,

$$|\lambda|^4 w(|\boldsymbol{\lambda}|) \in L^1(\mathbb{R}^m)\cap L^2(\mathbb{R}^m).$$

Remark 2.1. (a) Let us observe that **A1-A3** imply that

$$\psi \in L^q(\mathbb{R}^m),\ \ \forall q>0.$$

(b) The assumptions **A1-A3** are automatically satisfied if the density w is a Schwartz function on $\mathbb{R}$.

(c) The paper [11] includes one extra assumption on X, namely that the Gaussian vector

$$J_2(X(0)) := \big(\, X(0), \nabla X(0), \nabla^2 X(0)\,\big).$$

is nondegenerate. We do not need this nondegeneracy in this paper, but we want to mention that it is implied by Assumption **A3**; see Proposition A.6. □

Fix real numbers $u\geq 0$ and $v>0$. Denote by $\mathcal{S}_m$ the space of real symmetric $m\times m$ matrices, and by $\mathcal{S}_m^{u,v}$ the space $\mathcal{S}_m$ equipped with the centered Gaussian measure $\boldsymbol{\Gamma}_{u,v}$ uniquely determined by the covariance equalities

$$\boldsymbol{E}[a_{ij}a_{k\ell}] = u\delta_{ij}\delta_{k\ell} + v(\delta_{ik}\delta_{j\ell}+\delta_{i\ell}\delta_{jk}),\ \ \forall 1\leq i,j,.k,\ell\leq m.$$

In particular we have

$$\boldsymbol{E}[a_{ii}^2] = u+2v,\ \ \boldsymbol{E}[a_{ii}a_{jj}] = u,\ \ \ \boldsymbol{E}[a_{ij}^2] = v,\ \ \forall 1\leq i\neq j\leq m, \tag{2.3}$$

while all other covariances are trivial. The ensemble $\mathcal{S}_m^{0,v}$ is a rescaled version of the Gaussian Orthogonal Ensemble (GOE) and we will refer to it as GOE_m^v. As explained in [20, 21], the Gaussian measures $\boldsymbol{\Gamma}_{u,v}$ are invariant with respect to the natural action of $O(m)$ on $\mathcal{S}_m$. Moreover

$$d\boldsymbol{\Gamma}_{0,v}(A) = (2v)^{-\frac{m(m+1)}{4}}\pi^{-\frac{m(m+1)}{4}}\ e^{-\frac{1}{4v}\operatorname{tr}A^2}|dA|. \tag{2.4}$$

The ensemble $\mathcal{S}_m^{u,v}$ can be given an alternate description. More precisely a random $A\in\mathcal{S}_m^{u,v}$ can be described as a sum

$$A = B +\ X\mathbb{1}_m,\ \ B\in\mathrm{GOE}_m^v,\ \ X\in\boldsymbol{N}(0,u),\ \ \ B \text{ and } X \text{ independent}.$$

We write this

$$\mathcal{S}_m^{u,v} = \mathrm{GOE}_m^v\,\hat{+}\boldsymbol{N}(0,u)\mathbb{1}_m, \tag{2.5}$$

where $\hat{+}$ indicates a sum of *independent* variables. We set $\mathcal{S}_m^v := \mathcal{S}_m^{v,v}$. Recall from (2.1) that

$$\boldsymbol{E}\big[\,X(\boldsymbol{t})X(0)\,\big] = C(\boldsymbol{t}) = K(\boldsymbol{t},0) = (2\pi)^{-\frac{m}{2}}\int_{\mathbb{R}^m} e^{-\boldsymbol{i}(\boldsymbol{t},\boldsymbol{\lambda})}w(|\boldsymbol{\lambda}|)d\boldsymbol{\lambda}.$$

Following [23] we define

$$s_m := \frac{1}{(2\pi)^{m/2}}\int_{\mathbb{R}^m} w(|\boldsymbol{x}|)d\boldsymbol{x},\ \ d_m := \frac{1}{(2\pi)^{m/2}}\int_{\mathbb{R}^m} x_1^2 w(|\boldsymbol{x}|)d\boldsymbol{x},$$

$$h_m := \frac{1}{(2\pi)^{m/2}}\int_{\mathbb{R}^m} x_1^2x_2^2 w(|\boldsymbol{x}|)d\boldsymbol{x}.$$

Clearly $s_m, d_m, h_m > 0$. If we set

$$I_k(w) := \int_0^\infty w(r) r^k dr, \tag{2.6}$$

then we have (see [23])

$$\begin{gathered}(2\pi)^{m/2} s_m = \frac{2\pi^{\frac{m}{2}}}{\Gamma(\frac{m}{2})} I_{m-1}(w), \;\; (2\pi)^{m/2} d_m = \frac{2\pi^{\frac{m}{2}}}{m\Gamma(\frac{m}{2})} I_{m+1}(w), \\ (2\pi)^{m/2} h_m = \frac{1}{3}\int_{\mathbb{R}^m} x_1^4 w(|x|) dx = \frac{2\pi^{\frac{m}{2}}}{m(m+2)\Gamma(\frac{m}{2})} I_{m+3}(w). \end{gathered} \tag{2.7}$$

Then we deduce that

$$\boldsymbol{E}\big[\, X(0)\cdot \partial_{t_i} X(0)\,\big] = \boldsymbol{E}\big[\, \partial_{t_i} X(0)\cdot \partial^2_{t_j t_k} X(0)\,\big] = 0, \;\; \forall i,j,k \tag{2.8a}$$

$$\boldsymbol{E}[X(0)^2] = s_m, \;\; \boldsymbol{E}\big[\, \partial_{t_i} X(0)\cdot \partial_{t_j} X(0)\,\big] = d_m \delta_{ij}, \;\; \forall i,j, \tag{2.8b}$$

$$\boldsymbol{E}\big[\, X(0)\cdot \partial^2_{t_i t_j} X(0)\,\big] = -d_m \delta_{ij}, \;\; \forall i,j, \tag{2.8c}$$

$$\boldsymbol{E}\big[\, \partial^2_{t_i t_j} X(0)\cdot \partial^2_{t_k t_\ell} X(0)\,\big] = h_m\big(\,\delta_{ij}\delta_{k\ell} + \delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk}\big), \;\; \forall i,j,k,\ell. \tag{2.8d}$$

The equality (2.8b) shows that $\nabla X(0)$ is a $\mathbb{R}^m$-valued cenetered Gaussian random vector with covariance matrix $d_m \mathbb{1}_m$, while (2.8d) shows that $\nabla^2 X(0) \in \mathcal{S}_m^{h_m}$. □

2.2. **The main result.** We can now state the main result of this paper.

Theorem 2.2. *Suppose that $X(\boldsymbol{t})$ is a centered, stationary, isotropic random function on $\mathbb{R}^m$, $m \geq 2$ satisfying assumptions* **A1**, **A2**, **A3**. *Denote by Z_N the number of critical points of $X(\boldsymbol{t})$ in the cube $C_N := [-N, N]^m$. Then the following hold.*

(i)

$$\boldsymbol{E}\big[\, Z_N\,\big] = C_m(w)(2N)^m, \;\; \forall N, \tag{2.9}$$

where

$$C_m(w) = \left(\frac{h_m}{2\pi d_m}\right)^{\frac{m}{2}} \boldsymbol{E}_{\mathcal{S}_m^1}\big[\, |\det A|\,\big]. \tag{2.10}$$

(ii) *There exists a constant $V_\infty = V_\infty(m, w) > 0$ such that*

$$\boldsymbol{var}(Z_N) \sim V_\infty N^m \;\; \text{as } N \to \infty. \tag{2.11}$$

Moreover, the sequence of random variables

$$\zeta_N = N^{-m/2}\Big(\, Z_N - \boldsymbol{E}\big[\, Z_N\,\big]\,\Big)$$

converges in distribution to a normal random variable ζ_∞ with mean zero an positive variance V_∞.

Remark 2.3. (a) The isotropy condition on $X(\boldsymbol{t})$ may be a bit restrictive, but we believe that the techniques in [11] and this paper extend to the more general case of stationary random functions. However, for the geometric applications we have in mind, the isotropy is a natural assumption. Let us elaborate on this point.

Suppose that (M, g) is a compact m-dimensional Riemannian manifold, such that $\mathrm{vol}_g(M) = 1$. Denote by ρ the injectivity radius of g. For $\varepsilon > 0$ we denote by g_ε the rescaled metric $g_\varepsilon := \varepsilon^{-2} g$. Intuitively, as $\varepsilon \to 0$, the metric g_ε becomes flatter and flatter. Denote by Δ_g the Laplacian of g and by Δ_ε of g_ε. Let

$$\lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots$$

be the eigenvalues of Δ_g, multiplicities included. Fix an orthonormal basis of $L^2(M, dV_g)$ consisting of eigenfunctions Ψ_k of Δ_g,

$$\Delta_g \Psi_k = \lambda_k \Psi_k$$

Then the eigenvalues of Δ_ε are $\lambda_k(\varepsilon) = \varepsilon^2 \lambda_k$ with corresponding eigenfunctions $\Psi_k^\varepsilon := \varepsilon^{\frac{m}{2}} \Psi_k$.

We now define the random function $Y_\varepsilon(\boldsymbol{p})$ on M,

$$Y_\varepsilon(\boldsymbol{p}) = \sum_{k=0}^{\infty} w\big(\sqrt{\lambda_k(\varepsilon)}\,\big)^{\frac{1}{2}} Z_k \Psi_k^\varepsilon(\boldsymbol{p}),$$

where $(Z_k)_{k \geq 0}$ is a sequence of independent standard normal random variables.

Fix a point $\boldsymbol{p}_0 \in M$ and denote by $\exp_\varepsilon$ the exponential map $T_{\boldsymbol{p}_0} M \to M$ defined by the metric g_ε. (This is a diffeomorphism onto when restricted to the ball of radius $\varepsilon^{-1}\rho$ of the tangent space $T_{\boldsymbol{p}_0} M$ equipped with the metric g_ε.) Denote by $\mathcal{R}_\varepsilon$ the rescaling map

$$T_{\boldsymbol{p}_0} M \to T_{\boldsymbol{p}_0} M, \;\; \boldsymbol{v} \mapsto \varepsilon \boldsymbol{v}.$$

This map is an isometry $(T_{\boldsymbol{p}_0} M, g) \to (T_{\boldsymbol{p}_0} M, g_\varepsilon)$. We denote by X_ε the random function on the Euclidean space $(T_{\boldsymbol{p}_0} M, g)$ obtained by pulling back Y_ε via the map $\exp_\varepsilon \circ \mathcal{R}_\varepsilon$. The random function $X_\varepsilon(t)$ is Gaussian and its covariance kernel converges in the C^∞ topology as $\varepsilon \to 0$ to the covariance kernel of random function X we are investigating in this paper. Denote by $N(X_{\boldsymbol{e}}, B_r)$ the number critical points of Y_ε in a g-ball of radius $r < \rho$ on M, and by $N(X, B_R)$ the number of critical points of X in a ball of radius R. In [23] we have shown that

$$\boldsymbol{E}\big[\, N(Y_\varepsilon, B_r) \,\big] \sim \boldsymbol{E}\big[\, N(X, B_{r/\varepsilon} \,\big] = \mathit{const.}\, \varepsilon^{-m} \;\; \text{as } \varepsilon \to 0.$$

Additionally, in [22] we looked at the special case when M is a flat m-dimensional torus and we showed that the variances random variables

$$\varepsilon^{-m/2} \Big(N(Y_\varepsilon, B_r) - \boldsymbol{E}\big[\, N(Y_\varepsilon, B_r) \,\big] \Big), \;\; \varepsilon^{-m/2} \Big(N(X, B_{r/\varepsilon}) - \boldsymbol{E}\big[\, N(X, B_{r/\varepsilon}) \,\big] \Big)$$

have the identical *finite* limits as $\varepsilon > 0$. In [22] we were not able to prove that this common limit is nonzero, but Theorem 2.2 shows this to be the case.

These facts suggest that the random variable $N(Y_\varepsilon, B_r)$ may satisfy a central limit theorem of the type proved in [5, 13]. We will pursue this line of investigation elsewhere. □

2.3. **Organization of the paper.** The strategy of proof owes a great deal to [11]. In Subsections 3.1 and 3.2 we describe the Wiener chaos decomposition of the random variable Z_N in the Gaussian Hilbert space generated by the Gaussian family

$$\big(\, X(\boldsymbol{t}), \nabla X(\boldsymbol{t}), \nabla^2 X(\boldsymbol{t}) \,\big), \;\; \boldsymbol{t} \in \mathbb{R}^m.$$

In the first half of Subsection 3.3 we show that $\boldsymbol{var}(\zeta_N)$ has a finite limit V_∞ as $N \to \infty$. In the second half of this subsection we prove that $V_\infty > 0$. The central limit theorem is then obtained using the Breuer-Major type central limit theorem in [11]. Appendix A contains estimates of the lower order terms in the Hermite polynomial decomposition of $|\det A|$ where $A \in \mathcal{S}_m^v$, $m \gg 1$. These estimates can be used to produce explicit lower bounds for V_∞ for large m.

2.4. **Related work.** Central limit theorems concerning crossing counts of random functions go back a while, e.g. T. Malevich [18] (1969) and J. Cuzik [9] (1976).

The idea to use Wiener chaos decompositions to establish such central limit theorems is more recent, late 80s early 90s and we want to mention here the pioneering contributions of Chambers and Slud [8], Slud [27, 28] and Kratz and León [15].

This topic was further elaborated by Kratz and León in [16] where they also proved a central limit theorem concerning the length of the zero set of a random function of two variables. We refer to [6] for particularly nice discussion of these developments.

Azaïs and León [5] used the technique of Wiener chaos decomposition to give a shorter and more conceptual proof to a central limit theorem due to Granville and Wigman [13] concerning the number of zeros of random trigonometric polynomials of large degree. Recently, Adler and Naitzat [1] used Hermite decompositions to prove a CLT concerning Euler integrals of random functions.

Acknowledgments. I want to thank Yan Fyodorov for sharing with me the tricks in Lemma A.5.

3. Proof of the main result

The random variables $X(\boldsymbol{t})$, $\boldsymbol{t}\in\mathbb{R}^m$ are defined on a common probability space $(\Omega,\mathcal{O},\boldsymbol{P})$. We denote by $\mathcal{O}_X$ the σ-subalgebra of $\mathcal{O}$ generated by the variables $X(t)$, $t\in\mathbb{R}^m$. For simplicity we set $L^2(\Omega):=L^2(\Omega,\mathcal{O}_X,\boldsymbol{P})$.

As detailed in e.g. [14, 17, 26], the space $L^2(\Omega)$ admits an orthogonal Wiener chaos decomposition

$$L^2(\Omega)=\overline{\bigoplus_{q=0}^{\infty}L^2(\Omega)_q},$$

where $L^2(\Omega)_q$ denotes the q-th chaos component. We let $\mathcal{P}_q: L^2(\Omega)\to L^2(\Omega)_q$ denote the orthogonal projection on the q-th chaos component.

Let T denote a compact parallelipiped

$$T:=[a_1,b_1]\times\cdots\times[a_m,b_m],\;\; a_i<b_i,\;\;\forall i=1,\dots,m.$$

From [2, Thm.11.3.1], we deduce that X is a.s. a Morse function on T. In particular, almost surely there are no critical points on the boundary of T.

3.1. **Chaos decompositions of functionals of random symmetric matrices.** The dual space $(\mathcal{S}_m^v)^*=\operatorname{Hom}(\mathcal{S}_m^v,\mathbb{R})$ is a finite dimensional Gaussian linear space in the sense of [14] spanned by the entries $(a_{ij})_{1\leq i\leq j\leq m}$ of a random matrix $A\in\mathcal{S}_m^v$. Its Fock space is the space $L^2(\mathcal{S}_m,\boldsymbol{\Gamma}_{v,v})$ and admits an orthogonal chaos decomposition,

$$L^2(\mathcal{S}_m^v)=\overline{\bigoplus_{q\geq 0}L^2(\mathcal{S}_m^v)_q}.$$

We recall that

$$\widehat{\mathcal{P}}_{q,v}:=\bigoplus_{k=0}^{q}L^2(\mathcal{S}_m^v)_k$$

is the subspace of $L^2(\mathcal{S}_m^v)$ spanned by polynomials of degree $\leq n$ in the entries of $A\in\mathcal{S}_m^v$, and $L^2(\mathcal{S}_m^v)_q$ is the orthogonal complement of $\widehat{\mathcal{P}}_{q-1,v}$ in $\widehat{\mathcal{P}}_{q,v}$. The summand $L^2(\mathcal{S}_m^v)_q$ is called the *q-th chaos component* of $L^2(\mathcal{S}_m^v,)$.

The chaos decomposition construction is equivariant with respect to the action of $O(m)$ on $\mathcal{S}_m^v$. In particular, the chaos components $L^2(\mathcal{S}_m^v)_k$ are $O(m)$-invariant subspaces. If we denote by $L^2(\mathcal{S}_m^v)^{inv}$ the subspace of $L^2(\mathcal{S}_m^v)$ consisting of $O(m)$-invariant functions, then we obtain an orthogonal decomposition

$$L^2\big(\,\mathcal{S}_m^v\,\big)^{inv} = \overline{\bigoplus_{k\geq 0} L^2(\mathcal{S}_m^v)_k^{inv}}, \tag{3.1}$$

where $L^2(\mathcal{S}_m^v)_k^{inv}$ consists of the subspace of $L^2\big(\,\mathcal{S}_m^v\,\big)_k$ where $O(m)$ acts trivially. In particular, we deduce that $L^2(\mathcal{S}_m^v)_k^{inv}$ consists of polynomials in the variables $\operatorname{tr} A, \operatorname{tr} A^2, \dots, \operatorname{tr} A^m$.

We define the $O(m)$-invariant functions

$$p, q, f : \mathcal{S}_m \to \mathbb{R},\ \ p(A) = (\operatorname{tr} A)^2,\ \ q(A) := \operatorname{tr} A^2,\ \ f(A) = |\det A|.$$

A basis of $\widehat{\mathcal{P}}_{2,v}^{inv}$ is given by the polynomials 1, $\operatorname{tr} A$, $p(A)$, $q(A)$. Clearly, since $\operatorname{tr} A$ is an odd function of A, it is orthogonal to the even polynomials $1, p(A), q(A)$. We have

$$\boldsymbol{E}_{\mathcal{S}_m^v}\big[\,p(A)\,\big] = \sum_{i=1}^n \boldsymbol{E}[a_{ii}^2] + 2\sum_{i<j}\boldsymbol{E}[a_{ii}a_{jj}] = 3mv + m(m-1)v = m(m+2)v. \tag{3.2}$$

We have

$$\boldsymbol{E}_{\mathcal{S}_m^v}[q(A)] = \sum_{i=1}^m \boldsymbol{E}[a_{ii}^2] + 2\sum_{i<j}\boldsymbol{E}[a_{ij}^2] = 3m + m(m-1) = m(m+2)v. \tag{3.3}$$

We deduce that the polynomials

$$\begin{aligned} \bar{p}(A) &= p(A) - \boldsymbol{E}_{\mathcal{S}_m^v}[p(A)] = p(A) - m(m+2)v, \\ \bar{q}(A) &= q(A) - \boldsymbol{E}_{\mathcal{S}_m^v}[q_A)] = q(A) - m(m+2)v \end{aligned} \tag{3.4}$$

form a (non-orthonormal) basis of $L^2(\mathcal{S}_m^v)_2^{inv}$.

3.2. **Hermite decomposition of $Z(T)$.** For $\varepsilon > 0$ define

$$\delta_\varepsilon : \mathbb{R}^m \to \mathbb{R},\ \ \delta_\varepsilon = (2\varepsilon)^{-m}\boldsymbol{I}_{[-\varepsilon,\varepsilon]^m}.$$

Observe that the family (δ_ε) approximates the Dirac distribution δ_0 on $\mathbb{R}^m$ as $\varepsilon \searrow 0$. We recall [11, Prop. 1.2] which applies with no change to the setup in this paper.

Proposition 3.1 (Estrade-León)**.** *The random variable $Z(T)$ belongs to $L^2(\Omega)$. Moreover*

$$Z(T) = \lim_{\varepsilon\searrow 0}\int_T \big|\det \nabla^2 X(\boldsymbol{t})\,\big|\delta_\varepsilon\big(\,\nabla X(\boldsymbol{t})\,\big)d\boldsymbol{t}$$

almost surely and in $L^2(\Omega)$. □

The above nontrivial result implies that the random variable $Z(T)$ has finite variance and admits a chaos decomposition as elaborated for example in [14, 17, 26].

Recall that an orthogonal basis of $L^2\big(\mathbb{R}, \boldsymbol{\gamma}_1(dx)\big)$ is given by the Hermite polynomials, [14, Ex. 3.18], [19, V.1.3],

$$H_n(x) := (-1)^n e^{\frac{x^2}{2}}\frac{d^n}{dx^n}\Big(\,e^{-\frac{x^2}{2}}\,\Big) = n!\sum_{r=0}^{\lfloor\frac{n}{2}\rfloor}\frac{(-1)^r}{2^r r!(n-2r)!}x^{n-2r}. \tag{3.5}$$

In particular

$$H_n(0)=\begin{cases}0, & n\equiv 1 \bmod 2,\\ (-1)^r\frac{(2r)!}{2^r r!}, & n=2r.\end{cases} \tag{3.6}$$

For every multi-index $\alpha=(\alpha_1,\alpha_2,\dots)\in\mathbb{N}_0^{\mathbb{N}}$ such that all but finitely many α_k-s are nonzero, and any

$$\underline{x}=(x_1,x_2,\dots)\in\mathbb{R}^{\mathbb{N}}$$

we set

$$|\alpha|:=\sum_k\alpha_k,\;\;\alpha!:=\prod_k\alpha_k!,\;\;H_\alpha(\underline{x}):=\prod_k H_{\alpha_k}(x_k).$$

To simplify the notation we set

$$U(\boldsymbol{t}):=\frac{1}{\sqrt{d_m}}\nabla X(\boldsymbol{t}),\;\;A(\boldsymbol{t}):=\nabla^2X(\boldsymbol{t}).$$

Thus $U(\boldsymbol{t})$ is a $\mathbb{R}^m$-valued Gaussian random vector with covariance matrix $\mathbb{1}_m$ while $A(t)$ is a Gaussian random symmetric matrix in the ensemble $\mathcal{S}_m^{h_m}$.

Recall that $f(A)=|\det A|$. We have $f\in L^2(\mathcal{S}_m^{h_m})^{inv}$ and we denote by $f_n(A)$ the component of $f(A)$ in the n-th chaos summand of the chaos decomposition (3.1). Since f is an even function we deduce that $f_n(A)=0$ for odd n. Note also that

$$f_0(A)=\boldsymbol{E}_{\mathcal{S}_m^{h_m}}\big[\,|\det A|\,\big]\neq 0.$$

Following [11, Eq.(5)] we define for every $\alpha\in\mathbb{N}_0^m$ the quantity

$$d(\alpha):=\frac{1}{\alpha!}(2\pi d_m)^{-\frac{m}{2}}H_\alpha(0). \tag{3.7}$$

Arguing exactly as in the proof of [11, Prop. 1.3] we deduce the following result.

Proposition 3.2. *The following expansion holds in* $L^2(\Omega)$

$$Z(T)=\sum_{q=0}^\infty Z_q(T),$$

where

$$Z_q(T)=P_qZ(T)=\sum_{\substack{\alpha\in\mathbb{N}_0^m,\;n\in\mathbb{N}_0,\\ |\alpha|+n=q}}d(\alpha)\int_T H_\alpha\big(U(t)\big)f_n\big(A(\boldsymbol{t})\big)d\boldsymbol{t}.$$ □

Observe that the expected number of critical points of X on T is given by

$$\boldsymbol{E}\big[\,Z(T)\,\big]=\boldsymbol{E}\big[\,Z(T)\,\big]=d(0)\int_T\boldsymbol{E}\big[H_0\big(U(\boldsymbol{t})\big)f_0\big(A(\boldsymbol{t})\big)\big]d\boldsymbol{t}$$

(use the stationarity of $X(\boldsymbol{t})$)

$$=(2\pi d_m)^{-\frac{m}{2}}\int_T\boldsymbol{E}_{\mathcal{S}_m^{h_m}}\big[\,f(A(\boldsymbol{t}))\,\big]d\boldsymbol{t}=(2\pi d_m)^{-\frac{m}{2}}\boldsymbol{E}_{\mathcal{S}_m^{h_m}}\big[\,|\det A|\,\big]\operatorname{vol}(T)$$

$$=\left(\frac{h_m}{2\pi d_m}\right)^{\frac{m}{2}}\boldsymbol{E}_{\mathcal{S}_m^1}\big[\,|\det A|\,\big]\operatorname{vol}(T).$$

This proves (2.9) and (2.10).

3.3. **Asymptotic variance of $Z(T)$.** Denote by C_N the cube $[-N,N]^m$ and by B the cube $[0,1]^m$. For any Borel measurable subset $S \subset \mathbb{R}^m$ such that $\operatorname{vol}(S) \neq 0$ we set

$$\zeta(S) := \frac{1}{\sqrt{\operatorname{vol}(S)}}\big(\, Z(S) - \boldsymbol{E}\big[\, Z(S)\,\big]\,\big).$$

Thus, $\zeta_N = \zeta(C_N)$. Since $\zeta_N \in L^2(\Omega)$ we deduce $\boldsymbol{var}(\zeta_N) < \infty$.

Proposition 3.3. *There exists $V_\infty \in (0,\infty)$ such that*

$$\lim_{N\to\infty} \boldsymbol{var}(\zeta_N) = V_\infty.$$

Proof. To prove that the above limit exists we follow closely the proof of [11, Prop. 2.1]. We have

$$\zeta_N = \zeta(C_N) = (2N)^{-\frac{m}{2}} \sum_{q\geq 1} Z_q(C_N),$$

$$V_N := \boldsymbol{var}\big(\,\zeta(C_N)\,\big) = \sum_{q\geq 1} \underbrace{(2N)^{-m}\boldsymbol{E}\big[Z_q(C_N)^2\,\big]}_{=:V_{q,N}}.$$

To estimate $V_{q,N}$ we write

$$Z_q(T) = \int_T \rho_q(\boldsymbol{t})d\boldsymbol{t},$$

where

$$\rho_q(\boldsymbol{t}) = \sum_{\substack{\alpha\in\mathbb{N}_0^m,\, n\in\mathbb{N}_0\\ |\alpha|+n=q}} d(\alpha)H_\alpha\big(\,U(\boldsymbol{t})\,\big)f_n\big(\,A(\boldsymbol{t})\,\big). \tag{3.8}$$

Then

$$V_{q,N} = (2N)^{-m}\int_{C_N\times C_N} \boldsymbol{E}\big[\,\rho_q(\boldsymbol{s})\rho_q(\boldsymbol{t})\,\big]d\boldsymbol{s}d\boldsymbol{t}$$

(use the stationarity of $X(t)$)

$$= (2N)^{-m}\int_{C_N\times C_N} \boldsymbol{E}\big[\,\rho_q(0)\rho_q(\boldsymbol{t}-\boldsymbol{s})\,\big]d\boldsymbol{s}d\boldsymbol{t} = \int_{T_{2N}} \boldsymbol{E}\big[\,\rho_q(0)\rho_q(\boldsymbol{u})\,\big]\prod_{k=1}^m\left(1-\frac{|u_k|}{2N}\right)d\boldsymbol{u}.$$

The last equality is obtained by integrating along the fibers of the map

$$C_N\times C_N \ni (\boldsymbol{s},\boldsymbol{t}) \mapsto \boldsymbol{t}-\boldsymbol{s} \in T_{2N}.$$

To estimate the last integral, we fix an orthonormal basis $(b_{ij})_{1\leq i\leq j\leq m}$ of the Gaussian Hilbert space $\operatorname{Hom}(\mathcal{S}_n^{h_m},\mathbb{R})$. We denote by B the vector $(b_{ij})_{1\leq i\leq j\leq m}$, by A the vector $(a_{ij})_{1\leq i\leq j\leq m}$ both viewed as column vectors of size

$$\nu(m) := \dim \mathcal{S}_m = \frac{m(m+1)}{2}.$$

There exists a nondegenerate deterministic matrix Λ of size $\nu(m)\times\nu(m)$, relating A and B, $A = \Lambda B$. We now have an orthogonal decomposition

$$f_n(A) = \sum_{\beta\in\mathbb{N}_0^{\nu(m)},\ |\beta|=n} c(\beta)H_\beta(B).$$

Let us set

$$\mathcal{J}_m := \big(\mathbb{N}_0^m\big)\times\big(\mathbb{N}_0^{\nu(m)}\big).$$

We deduce

$$\rho_q(\boldsymbol{t}) = \sum_{(\alpha,\beta)\in\mathcal{I}_m} d(\alpha)c(\beta)H_\alpha(\, U(t)\,)c(\beta)H_\beta(\, B(\boldsymbol{t})\,).$$

We can further simplify this formula if we introduce the vector

$$Y(\boldsymbol{t}) := \big(\, U(\boldsymbol{t}), B(\boldsymbol{t})\,\big),\;\; B(t) = \Lambda^{-1}A(\boldsymbol{t}).$$

For $\gamma = (\alpha,\beta) \in \mathcal{I}_m$ we set

$$\boldsymbol{a}(\gamma) := d(\alpha)c(\beta)\;\; H_\gamma\big(\, Y(\boldsymbol{t})\,\big) := H_\alpha\big(\, U(\boldsymbol{t})\,\big)H_\beta\big(\, B(\boldsymbol{t})\,\big).$$

Then

$$\rho_q(\boldsymbol{t}) = \sum_{\gamma\in\mathcal{I}_m,\; |\gamma|=q} \boldsymbol{a}(\gamma)H_\gamma\big(\, Y(\boldsymbol{t})\,\big), \tag{3.9}$$

$$\boldsymbol{E}\big[\, \rho_q(0)\rho_q(u)\,\big] = \sum_{\substack{\gamma,\gamma'\in\mathcal{I}_m\\ |\gamma|=|\gamma'|=q}} \boldsymbol{a}(\gamma)\boldsymbol{a}(\gamma')\boldsymbol{E}\big[\, H_\gamma(\, Y(0)\,)H_{\gamma'}(\, Y(\boldsymbol{u})\,)\,\big].$$

We set $\omega(m) := m + \nu(m)$, and we denote by $Y_i(\boldsymbol{t})$, $1 \leq i \leq \omega(m)$, the components of $Y(\boldsymbol{t})$ labelled so that $Y_i(\boldsymbol{t}) = U_i(\boldsymbol{t})$, $\forall 1 \leq i \leq m$. For $\boldsymbol{u} \in \mathbb{R}^m$ and $1 \leq i,j \leq \omega(m)$ we define the covariances

$$\Gamma_{ij}(\boldsymbol{u}) := \boldsymbol{E}\big[\, Y_i(0)Y_j(\boldsymbol{u})\,\big].$$

Observe that there exists a positive constant K such that

$$\big|\, \Gamma_{i,j}(\boldsymbol{u})\,\big| \leq K\psi(\boldsymbol{u}),\;\; \forall i,j = 1,\dots,\omega(m),\;\; \boldsymbol{u}\in\mathbb{R}^m, \tag{3.10}$$

where ψ is the function defined in (2.2).

Using the Diagram Formula (see e.g.[17, Cor. 5.5] or [14, Thm. 7.33]) we deduce that for any $\gamma,\gamma' \in \mathcal{I}_m$ such that $|\gamma| = |\gamma'| = q$ there exists a universal homogeneous polynomial of degree q, $P_{\gamma,\gamma'}$ in the variables $\Gamma_{ij}(\boldsymbol{u})$ such that

$$\boldsymbol{E}\big[\, H_\gamma(\, Y(0)\,)H_{\gamma'}(\, Y(\boldsymbol{u})\,)\,\big] = P_{\gamma,\gamma'}\big(\, \Gamma_{ij}(\boldsymbol{u})\,\big).$$

Hence

$$V_{q,N} = \sum_{\substack{\gamma,\gamma'\in\mathcal{I}_m\\ |\gamma|=|\gamma'|=q}} \boldsymbol{a}(\gamma)\boldsymbol{a}(\gamma')\; \underbrace{\int_{T_{2N}} P_{\gamma,\gamma'}\big(\, \Gamma_{ij}(\boldsymbol{u})\,\big)\prod_{k=1}^m\left(1 - \frac{|u_k|}{2N}\right)d\boldsymbol{u}}_{=:R_N(\boldsymbol{\gamma},\boldsymbol{\gamma}')}. \tag{3.11}$$

From (3.10) we deduce that for any $\gamma,\gamma' \in \mathcal{I}_m$ such that $|\gamma| = |\gamma'| = q$ there exists a caonstant $C_{\gamma,\gamma'} > 0$ such that

$$\big|\, P_{\gamma,\gamma'}\big(\, \Gamma_{ij}(\boldsymbol{u})\,\big)\,\big| \leq C_{\gamma,\gamma'}\psi(\boldsymbol{u})^q,\;\; \forall\boldsymbol{u}\in\mathbb{R}^m.$$

Since $\psi \in L^p(\mathbb{R}^m)$, $\forall p > 1$, we deduce from the dominated convergence theorem that

$$\lim_{N\to\infty} R_N(\boldsymbol{\gamma},\boldsymbol{\gamma}') = R_\infty(\gamma,\gamma') := \int_{\mathbb{R}^m} P_{\gamma,\gamma'}\big(\, \Gamma_{ij}(\boldsymbol{u})\,\big)d\boldsymbol{u}, \tag{3.12}$$

and thus

$$\lim_{N\to\infty} V_{q,N} = V_{q,\infty} := \sum_{\substack{\gamma,\gamma'\in\mathcal{I}_m\\ |\gamma|=|\gamma'|=q}} \boldsymbol{a}(\gamma)\boldsymbol{a}(\gamma')R_\infty(\gamma,\gamma') = \int_{\mathbb{R}^m} \boldsymbol{E}\big[\, \rho_q(0)\rho_q(\boldsymbol{u})\,\big]d\boldsymbol{u}. \tag{3.13}$$

Since $V_{q,N} \geq 0$, $\forall q,N$, we have

$$V_{q,\infty} \geq 0,\;\; \forall q.$$

Lemma 3.4. *For any positive integer Q we set*

$$V_{>Q,N} := \sum_{q>Q} V_{q,N}.$$

Then

$$\lim_{Q\to\infty}\big(\sup_N V_{>Q,N}\big) = 0, \tag{3.14}$$

the series

$$\sum_{q\geq 1} V_{q,\infty}$$

is convergent and, if V_∞ is its sum, then

$$V_\infty = \lim_{N\to\infty} V_N = \lim_{N\to\infty}\sum_{g\geq 1} V_{q,N}. \tag{3.15}$$

Proof. For $\boldsymbol{s}\in\mathbb{R}^m$ we denote by $\theta_{\boldsymbol{s}}$ the shift operator associated with the field X, i.e.,

$$\theta_{\boldsymbol{s}} X(\bullet) = X(\bullet + \boldsymbol{s}).$$

This extends to a unitary map $L^2(\Omega)\to L^2(\Omega)$ that commutes with the chaos decomposition of $L^2(\Omega)$. Moreover, for any parallelipiped T we have

$$Z(T+\boldsymbol{s}) = \theta_{\boldsymbol{s}} Z(T).$$

If we denote by $\mathcal{L}_N$ the set of lattice points

$$\mathcal{L}_N := [-N,N)^m\cap\mathbb{Z}^m$$

then we deduce

$$\zeta(C_N) = (2N)^{-m/2}\sum_{\boldsymbol{s}\in\mathcal{L}_m}\theta_{\boldsymbol{s}}\zeta\big(B\big),\ \ B=[0,1]^m.$$

We denote by $\mathcal{P}_{>Q}$ the projection

$$\mathcal{P}_{>Q} = \sum_{q>Q}\mathcal{P}_q,$$

where we recall that $\mathcal{P}_q$ denotes the projection on the q-th chaos component of $L^2(\Omega)$. We have

$$\mathcal{P}_{>Q}\zeta(C_N) = (2N)^{-m/2}\sum_{\boldsymbol{s}\in\mathcal{L}_m}\theta_{\boldsymbol{s}}\mathcal{P}_{>Q}\zeta\big(B\big).$$

Using the stationarity of X we deduce

$$V_{>Q,N} = \boldsymbol{E}\Big[\,\big|\mathcal{P}_{>Q}\zeta(C_N)\big|^2\,\Big] = (2N)^{-m}\sum_{\boldsymbol{s}\in\mathcal{L}_{2N}}\nu(\boldsymbol{s},N)\boldsymbol{E}\big[\,\mathcal{P}_{>Q}\zeta\big(B\big)\cdot\theta_{\boldsymbol{s}}\mathcal{P}_{>Q}\zeta\big(B\big)\,\big], \tag{3.16}$$

where $\nu(\boldsymbol{s},N)$ denotes the number of lattice points $\boldsymbol{t}\in\mathcal{L}$ such that $\boldsymbol{t}-\boldsymbol{s}\in\mathcal{L}_N$. Clearly

$$\nu(\boldsymbol{s},N)\leq (2N)^m. \tag{3.17}$$

With K denoting the positive constant in (3.10) we choose positive numbers a,ρ such that

$$\psi(\boldsymbol{s})\leq\rho<\frac{1}{K},\ \ \forall|\boldsymbol{s}|_\infty>a.$$

We split $V_{>Q,N}$ into two parts

$$V_{>Q,N} = V'_{>Q,N} + V''_{>Q,N},$$

where $V'_{>Q,N}$ is made up of the terms in (3.16) corresponding to lattice points $\boldsymbol{s}\in\mathcal{L}_{2N}$ such that $|\boldsymbol{s}|_\infty < a+1$, while $V''_{>Q,N}$ corresponds to lattice points $\boldsymbol{s}\in\mathcal{L}_{2N}$ such that $|\boldsymbol{s}|_\infty\geq a+1$.

We deduce from (3.17) that for $2M > a+1$ we have

$$\Big|\, V'_{>Q,N} \,\Big| \leq (2N)^{-m}(2a+2)^m(2N)^m \boldsymbol{E}\Big[\, \big| \mathcal{P}_{>Q}\zeta(B)\big|^2\,\Big].$$

As $Q\to\infty$, the right-hand side of the above inequality goes to 0 uniformly with respect to N.

To estimate $V''_{>Q,N}$ observe that for $\boldsymbol{s}\in\mathcal{L}_{2N}$ such that $|\boldsymbol{s}|_\infty > a+1$ we have

$$\boldsymbol{E}\big[\,\mathcal{P}_{>Q}\zeta\big(B\big)\cdot\theta_{\boldsymbol{s}}\mathcal{P}_{>Q}\zeta\big(B\big)\,\big] = \sum_{q>Q}\int_B\int_B \boldsymbol{E}\big[\,\rho_q(\boldsymbol{t})]\rho_q(\boldsymbol{u}+\boldsymbol{s})\,\big]d\boldsymbol{t}d\boldsymbol{u}, \tag{3.18}$$

where we recall from (3.9) that

$$\rho_q(\boldsymbol{t}) = \sum_{\gamma\in\mathcal{I}_m,\,|\gamma|=q}\boldsymbol{a}(\gamma)H_\gamma\big(\,Y(\boldsymbol{t})\,\big),\;\; \mathcal{I}_m := \mathbb{N}_0^m\times\mathbb{N}_0^{\nu(m)},\;\; \nu(m)=\frac{m(m+1)}{2}.$$

Thus

$$\boldsymbol{E}\big[\,\rho_q(\boldsymbol{t})\rho_q(\boldsymbol{u}+\boldsymbol{s})\,\big] = \boldsymbol{E}\Big[\,\Big(\sum_{\gamma\in\mathcal{I}_m,\,|\gamma|=q}\boldsymbol{a}(\gamma)H_\gamma\big(\,Y(\boldsymbol{t})\,\big)\Big)\Big(\sum_{\gamma\in\mathcal{I}_m,\,|\gamma|=q}\boldsymbol{a}(\gamma)H_\gamma\big(\,Y(\boldsymbol{s}+\boldsymbol{u})\,\big)\Big)\Big]$$

Arcones' inequality [4, Lemma 1] implies that

$$\boldsymbol{E}\big[\,\rho_q(\boldsymbol{t})\rho_q(\boldsymbol{u}+\boldsymbol{s})\,\big] \leq K^q\psi^q(\boldsymbol{s}+\boldsymbol{u}-\boldsymbol{t})^q\sum_{\gamma\in\mathcal{I}_m,\,|\gamma|=q}|\boldsymbol{a}(\gamma)|^2\gamma!. \tag{3.19}$$

The series $\sum_{\gamma\in\mathcal{I}_m}|\boldsymbol{a}(\gamma)|^2\gamma!$ is divergent because the series $\sum_{\gamma\in\mathcal{I}_m}\boldsymbol{a}(\gamma)H_\gamma(Y)$, $Y=(U,B)$, is the Hermite series decomposition of the distribution $\delta_0(\sqrt{d_m}U)|\det A|$.

On the other hand, for $\gamma=(\alpha,\beta)\in\mathcal{I}_m$ we have $\boldsymbol{a}(\gamma)=d(\alpha)c(\beta)$, where, according to (3.7) we have $d(\alpha)=\frac{1}{\alpha!}(2\pi d_m)^{-\frac{m}{2}}H_\alpha(0)$. Recalling that

$$H_{2r}(0) = (-1)^r\frac{(2r)!}{2^r r!},\;\; H_{2r+1}(0)=0.$$

we deduce that

$$(2r)!\Big|\frac{1}{(2r)!}H_{2r}(0)\Big|^2 = \frac{1}{2^{2r}}\binom{2r}{r}\leq 1,$$

and

$$d(\alpha)^2\alpha!\leq C = \frac{1}{(2\pi d_m)^{m/2}}.$$

This allows us to conclude that

$$\sum_{\gamma\in\mathcal{I}_m,\,|\gamma|=q}|\boldsymbol{a}(\gamma)|^2\gamma! \leq (2\pi d_m)^{-m/2}q^m\sum_{\beta\in\mathbb{N}_0^{\nu(m)},|\beta|\leq q}c(\beta)^2\beta! \leq Cq^m\boldsymbol{E}_{\mathcal{S}_m^{hm}}\big[\,|\det A|^2\,\big].$$

Using this in (3.18) and (3.19) we deduce

$$\boldsymbol{E}\big[\,\mathcal{P}_{>Q}\zeta\big(B\big)\cdot\theta_{\boldsymbol{s}}\mathcal{P}_{>Q}\zeta\big(B\big)\,\big]$$
$$\leq \underbrace{C\boldsymbol{E}_{\mathcal{S}_m^{hm}}\big[\,|\det A|^2\,\big]}_{=:C'}\sum_{q>Q}q^mK^q\int_B\int_B\psi(\boldsymbol{s}+\boldsymbol{u}-\boldsymbol{t})^q d\boldsymbol{u}d\boldsymbol{t}$$

Hence

$$\big|\,V''_{>Q,N}\big| \leq C'\Big(\sum_{q>Q}q^mK^q\rho^{q-1}\Big)\Big(\sum_{\boldsymbol{s}\in\mathcal{L}_{2N};\,|\boldsymbol{s}|_\infty>a+1}\int_B\int_B\psi(\boldsymbol{s}+\boldsymbol{u}-\boldsymbol{t})d\boldsymbol{u}d\boldsymbol{t}\Big),$$

Where we have used the fact that for $|s|_\infty \geq a+1$, $|\boldsymbol{u}|, |\boldsymbol{t}| \leq 1$ we have $\psi(\boldsymbol{s}+\boldsymbol{u}-\boldsymbol{t}) < \rho$.

Since $\rho < \frac{1}{K}$, the sum

$$\sum_{q>Q} q^m K^q \rho^{q-1}$$

is the tail of a convergent power series. On the other hand,

$$\sum_{\boldsymbol{s}\in\mathcal{L}_{2N};\,|\boldsymbol{s}|_\infty > a+1} \int_B \int_B \psi(\boldsymbol{s}+\boldsymbol{u}-\boldsymbol{t}) d\boldsymbol{u} d\boldsymbol{t} \leq \sum_{\boldsymbol{s}\in\mathcal{L}_{2n}} \int_{[-1,1]^m} \psi(\boldsymbol{s}+\boldsymbol{u}) \leq 2\int_{\mathbb{R}^m} \psi(\boldsymbol{u}) d\boldsymbol{u} < \infty.$$

This proves that $\sup_N |V''_{>Q,N}|$ goes to zero as $Q\to\infty$ and completes the proof of (3.14). The claim (3.15) follows immediately from (3.14). □

Lemma 3.5. *The asymptotic variance V_∞ is positive. More precisely,*

$$V_{2,\infty} > 0. \tag{3.20}$$

Proof. From (3.13) we deduce

$$V_{2,\infty} = \int_{\mathbb{R}^m} \boldsymbol{E}\big[\,\rho_q(0)\rho_q(\boldsymbol{u})\,\big] d\boldsymbol{u}, \tag{3.21}$$

where, according to (3.8) we have

$$\rho_2(\boldsymbol{t}) = \sum_{\substack{\alpha\in\mathbb{N}_0^m,\, n\in\mathbb{N}_0\\ |\alpha|+n=2}} d(\alpha) H_\alpha\big(\,U(\boldsymbol{t})\,\big) f_n\big(\,A(\boldsymbol{t})\,\big).$$

The second chaos decomposition $f_2(A)$ is a linear combination of the polynomials $\bar{p}(A)$ and $\bar{q}(A)$ defined in (3.4) where $v = h_m$.

In the above sum the only nontrivial terms correspond to $\alpha = 0$ or $\alpha = (2\delta_{i1}, 2\delta_{i2}, \dots, 2\delta_{im})$, $i = 1,\dots,m$. In each of these latter cases we deduce from (3.6) that

$$d(\alpha) := -\frac{1}{2} d(0),\;\; d(0) \overset{(3.7)}{=} (2\pi d_m)^{-\frac{m}{2}},$$

and we conclude that

$$\rho_2(\boldsymbol{t}) = d(0)\Big(f_2(\,A(\boldsymbol{t})\,) - \frac{f_0(A)}{2}\sum_{i=1}^m H_2(\,U_i(\boldsymbol{t})\,)\Big) = d(0)\Big(x\bar{p}(A(\boldsymbol{t})) + y\bar{q}(A(\boldsymbol{t})) - \frac{f_0(A)}{2}\sum_{i=1}^m H_2(\,U_i(\boldsymbol{t})\,)\Big).$$

For uniformity we set

$$z := -\frac{f_0(A)}{2}$$

so that

$$\rho_2(\boldsymbol{t}) = d(0)\Big(\,x\bar{p}(A) + y\bar{q}(A) + z\sum_{i=1}^m H_2(U_i)\,\Big).$$

We first express the polynomials $\bar{p}(A)$ and $\bar{q}(A)$ in terms of Hermite polynomials. We set

$$\widehat{a}_{ij} := \begin{cases} \frac{1}{\sqrt{3h_m}} a_{ii}, & i = j,\\ \frac{1}{\sqrt{h_m}} a_{ij}, & i \neq j.\end{cases}$$

We have

$$(\operatorname{tr} A)^2 = 3h_m\Big(\sum_i \widehat{a}_{ii}\Big)^2 = 3h_m \sum_i \widehat{a}_{ii}^2 + 6h_m \sum_{i<j} \widehat{a}_{ii}\widehat{a}_{jj}$$

$$= 3h_m \sum_i \big(H_2(\widehat{a}_{ii}) + 1 \big) + 6h_m \sum_{i<j} H_1(\widehat{a}_{ii}) H_1(\widehat{a}_{jj})$$

$$\bar{p}(A) = (\operatorname{tr} A)^2 - m(m+2)h_m = h_m \left(3 \sum_i H_2(\widehat{a}_{ii}) + 6 \sum_{i<j} H_1(\widehat{a}_{ii}) H_1(\widehat{a}_{jj}) - m(m-1) \right).$$

We have

$$\operatorname{tr} A^2 = 3h_m \sum_i \widehat{a}_{ii}^2 + 2h_m \sum_{i<j} \widehat{a}_{ij}^2 = 3h_m \sum_i H_2(\widehat{a}_{ii}) + 2h_m \sum_{i<j} H_2(\widehat{a}_{ij}) + m(m+2)h_m,$$

$$\bar{q}(A) = h_m \left(3 \sum_i H_2(\widehat{a}_{ii}) + 2 \sum_{i<j} H_2(\widehat{a}_{ij}) \right).$$

Define

$$F_0(\boldsymbol{t}) = \sum_i H_2\big(U_i(\boldsymbol{t})\big), \;\; F_1(\boldsymbol{t}) = \sum_i H_2\big(\widehat{a}_{ii}(\boldsymbol{t})\big), \;\; F_2(\boldsymbol{t}) = \sum_{i<j} H_2\big(\widehat{a}_{ij}(\boldsymbol{t})\big)$$

$$F_3(\boldsymbol{t}) = \sum_{i<j} H_1\big(\widehat{a}_{ii}(\boldsymbol{t})\big) H_1\big(\widehat{a}_{jj}(\boldsymbol{t})\big).$$

Thus

$$\rho_2(\boldsymbol{t}) = d(0) \Big(x h_m \big(3F_1(\boldsymbol{t}) + 6F_3(\boldsymbol{t}) - m(m-1) \big) + y h_m \big(3F_1(\boldsymbol{t}) + 2F_2(\boldsymbol{t}) \big) + z F_0(\boldsymbol{t}) \Big).$$

$$\boldsymbol{E}[F_1(0)] = \boldsymbol{E}[F_2(0)] = 0,$$

$$\boldsymbol{E}[F_3(\boldsymbol{t})] = \boldsymbol{E}[F_3(0)] = \frac{m(m-1)}{2} \boldsymbol{E}[\widehat{a}_{11}(0)\widehat{a}_{22}(0)] = \frac{m(m-1)}{6}$$

We set

$$\widehat{F}_3(\boldsymbol{t}) = F_3(\boldsymbol{t}) - \boldsymbol{E}[F_3(\boldsymbol{t})].$$

Then

$$\boldsymbol{E}[F_0(0)\widehat{F}_3(\boldsymbol{t})] = \boldsymbol{E}[F_0(0)F_3(\boldsymbol{t})], \;\; \boldsymbol{E}[F_1(0)\widehat{F}_3(\boldsymbol{t})] = \boldsymbol{E}[F_1(0)F_3(\boldsymbol{t})],$$

$$\boldsymbol{E}[F_2(0)\widehat{F}_3(\boldsymbol{t})] = \boldsymbol{E}[F_2(0)F_3(\boldsymbol{t})],$$

$$\rho_2(\boldsymbol{t}) = d(0) \Big(3x h_m \underbrace{\big(F_1(\boldsymbol{t}) + 2\widehat{F}_3(\boldsymbol{t}) \big)}_{=:\mathcal{Z}_1(\boldsymbol{t})} + y h_m \underbrace{\big(3F_1(\boldsymbol{t}) + 2F_2(\boldsymbol{t}) \big)}_{=:\mathcal{Z}_2(\boldsymbol{t})} + z F_0(\boldsymbol{t}) \Big). \tag{3.22}$$

To estimate $\boldsymbol{E}\big[\,\rho_2(0)\rho_2(\boldsymbol{u})\,\big]$ we will rely on the following consequences of the Diagram Formula [7], [14, Thm. 3.12].

Lemma 3.6. *Suppose that X_1, X_2, X_3, X_4 are centered Gaussian random variables such that*

$$\boldsymbol{E}[X_i^2] = 1, \;\; \boldsymbol{E}[X_i X_j] = c_{ij}, \;\; \forall i,j = 1,2,4, \;\; i \neq j.$$

Then

$$\boldsymbol{E}\big[\, H_1(X_1) H_1(X_2) \,\big] = c_{12}, \tag{3.23a}$$

$$\boldsymbol{E}\big[\, H_2(X_1) H_2(X_2) \,\big] = 2c_{12}^2, \;\; \boldsymbol{E}\big[\, H_2[X_1] H_1(X_2) H_1(X_3) \,\big] = 2c_{12}c_{13}, \tag{3.23b}$$

$$\boldsymbol{E}\big[H_1(X_1) H_1(X_2) H_1(X_3) H_1(X_4) \,\big] = c_{12}c_{34} + c_{13}c_{24} + c_{14}c_{23}. \tag{3.23c}$$

□

To compute the expectations involved in (3.21) we need to know the covariances between $\widehat{a}_{ij}(0)$ and $\widehat{a}_{jk}(\boldsymbol{t})$. These are determined by the covariance kernel

$$C(\boldsymbol{t}) = \mathcal{F}[\mu(\boldsymbol{\lambda})],$$

where $\mathcal{F}$ denotes the Fourier transform. For any $i_1, \dots, i_k \in \{1, \dots m\}$ we set

$$C_{i_1 \dots i_k}(\boldsymbol{t}) := \partial^k_{t_{i_1} \dots t_{i_k}} C(\boldsymbol{t}), \;\; \mathcal{F}\big[\, (-\boldsymbol{i})^k \lambda_{i_1} \cdots \lambda_{i_k} \mu(\boldsymbol{\lambda})\,\big].$$

We have

$$\boldsymbol{E}\big[\partial^k_{t_{i_1} \dots t_{i_k}} X(0) \partial^\ell_{t_{j_1} \dots t_{j_\ell}} X(\boldsymbol{t})\,\big] = (-1)^k C_{i_1 \dots i_k j_1 \dots j_\ell}(\boldsymbol{t})$$

$$U_i(\boldsymbol{t}) = \frac{1}{\sqrt{d_m}} \partial_{t_i} X(t), \;\; \widehat{a}_{ii}(\boldsymbol{t}) = \frac{1}{\sqrt{3h_m}} \partial^2_{t_i} X(\boldsymbol{t}), \;\; \widehat{a}_{ij}(\boldsymbol{t}) = \frac{1}{\sqrt{h_m}} \partial^2_{t_i t_j} X(t).$$

Recalling that the spectral measure $\mu(d\boldsymbol{\lambda})$ has the form

$$\mu(\boldsymbol{\lambda}) = w(|\boldsymbol{\lambda}|) d\boldsymbol{\lambda},$$

we introduce the functions

$$M_{i_1 \dots i_k}(\boldsymbol{\lambda}) := \lambda_{i_1} \cdots \lambda_{i_k} w(|\boldsymbol{\lambda}|)$$

and denote by $\mathcal{F}_{i_1 \dots i_k}$ their Fourier transforms

$$\mathcal{F}_{i_1 \dots i_k}(\boldsymbol{t}) = (2\pi)^{-m/2} \int_{\mathbb{R}^m} e^{-\boldsymbol{i}(\boldsymbol{t}, \boldsymbol{\lambda})} M_{i_1 \dots i_k}(\boldsymbol{\lambda}) d\boldsymbol{\lambda}.$$

Then

$$\boldsymbol{E}\big[U_i(0) U_j(\boldsymbol{t})\,\big] = -\frac{1}{d_m} C_{ij} = \frac{1}{d_m} \mathcal{F}_{ij}(t), \tag{3.24a}$$

$$\boldsymbol{E}\big[U_i(0) \widehat{a}_{jj}(\boldsymbol{t})\big] = -\boldsymbol{E}\big[\widehat{a}_{jj}(0) U_i(\boldsymbol{t})\big] = -\frac{1}{\sqrt{3h_m d_m}} C_{ijj}(\boldsymbol{t}) = \frac{\boldsymbol{i}}{\sqrt{3h_m d_m}} \mathcal{F}_{ijj}(\boldsymbol{t}), \;\; \forall i, j, \tag{3.24b}$$

$$\begin{aligned} \boldsymbol{E}\big[U_i(0) \widehat{a}_{jk}(\boldsymbol{t})\big] = -\boldsymbol{E}\big[\widehat{a}_{jk}(0) U_i(\boldsymbol{t})\big] &= -\frac{1}{\sqrt{h_m d_m}} C_{ijk}(\boldsymbol{t}) \\ &= \frac{\boldsymbol{i}}{\sqrt{h_m d_m}} \mathcal{F}_{ijk}(\boldsymbol{t}) \;\; \forall i, j, k, \;\; j \neq k, \end{aligned} \tag{3.24c}$$

$$\boldsymbol{E}\big[\widehat{a}_{ii}(0) \widehat{a}_{jj}(\boldsymbol{t})\big] = \frac{1}{3h_m} C_{iijj}(\boldsymbol{t}) = \frac{1}{3h_m} \mathcal{F}_{iijj}(\boldsymbol{t}), \tag{3.24d}$$

$$\begin{aligned} \boldsymbol{E}\big[\widehat{a}_{ii}(0) \widehat{a}_{jk}(\boldsymbol{t})\big] = \boldsymbol{E}\big[\widehat{a}_{jk}(0) \widehat{a}_{ii}(\boldsymbol{t})\big] &= \frac{1}{h_m \sqrt{3}} C_{iijk}(\boldsymbol{t}) \\ &= \frac{1}{h_m \sqrt{3}} \mathcal{F}_{iijk}(\boldsymbol{t}), \;\; \forall i, j, k, \;\; j \neq k, \end{aligned} \tag{3.24e}$$

$$\begin{aligned} \boldsymbol{E}\big[\widehat{a}_{ij}(0) \widehat{a}_{k\ell}(\boldsymbol{t})\big] = \boldsymbol{E}\big[\widehat{a}_{k\ell}(0) \widehat{a}_{ij}(\boldsymbol{t})\big] &= \frac{1}{h_m} C_{iijk}(\boldsymbol{t}) \\ &= \frac{1}{h_m} \mathcal{F}_{ijk\ell}(\boldsymbol{t}), \;\; \forall i, j, k, \ell, \;\; i \neq j, \;\; k \neq \ell. \end{aligned} \tag{3.24f}$$

We have

$$\boldsymbol{E}[F_0(0) F_0(\boldsymbol{t})] = \sum_{i,j} \boldsymbol{E}[H_2(U_i(0) H_2(U_j(\boldsymbol{t}))] = \frac{2}{d_m^2} \sum_{i,j} \mathcal{F}_{ij}(\boldsymbol{t})^2.$$

Using the fact that the Fourier transform is an isometry and the equality $M_{ij}^2 = M_{ii} M_{jj}$ we deduce

$$\int_{\mathbb{R}^m} \mathcal{F}_{ij}(\boldsymbol{t})^2 d\boldsymbol{t} = \int_{\mathbb{R}^m} M_{ij}(\boldsymbol{\lambda})^2 d\boldsymbol{\lambda} = \int_{\mathbb{R}^m} M_{ii}(\boldsymbol{\lambda}) M_{jj}(\boldsymbol{\lambda}) d\boldsymbol{\lambda}$$

and thus,

$$\int_{\mathbb{R}^m} \boldsymbol{E}[F_0(0)F_0(\boldsymbol{t})]d\boldsymbol{t} = \left\langle \frac{\sqrt{2}}{d_m}\sum_i M_{ii}, \frac{\sqrt{2}}{d_m}\sum_j M_{jj} \right\rangle_{L^2} \tag{3.25}$$

$$\boldsymbol{E}\big[F_0(0)F_1(\boldsymbol{t})] = \sum_{i,j} \boldsymbol{E}[H_2(U_i(0))H_2(\widehat{a}_{jj}(t))] = \frac{2}{3h_m d_m}\sum_{i,j}\big(\boldsymbol{i}\mathcal{F}_{ijj}(\boldsymbol{t})\,\big)^2.$$

Since $M_{ijj}^2 = M_{ii}M_{jjjj}$ we deduce

$$\int_{\mathbb{R}^m} \boldsymbol{E}\big[F_0(0)F_1(\boldsymbol{t})]d\boldsymbol{t} = \frac{2}{3h_m d_m}\sum_{i,j}\|M_{ijj}\|^2_{L^2} = \left\langle \frac{\sqrt{2}}{d_m}\sum_i M_{ii}, \frac{\sqrt{2}}{3h_m}\sum_j M_{jjjj} \right\rangle_{L^2}. \tag{3.26}$$

Arguing similary we deduce

$$\int_{\mathbb{R}^m} \boldsymbol{E}\big[F_0(0)F_1(\boldsymbol{t})]d\boldsymbol{t} = \int_{\mathbb{R}^m} \boldsymbol{E}\big[F_0(\boldsymbol{t})F_1(0)]d\boldsymbol{t}.$$

$$\boldsymbol{E}[\,F_0(0)F_2(\boldsymbol{t})] = \sum_{i,j<k} H_2(U_i(0))H_2(\widehat{a}_{jk}(\boldsymbol{t})) = 2\sum_{i,j<k}\frac{\boldsymbol{i}}{\sqrt{d_m h_m}}\mathcal{F}_{ijk}(\boldsymbol{t}),$$

$$\int_{\mathbb{R}^m} \boldsymbol{E}[\,F_0(0)F_2(\boldsymbol{t})]d\boldsymbol{t} = \left\langle \frac{\sqrt{2}}{d_m}\sum_i M_{ii}, \frac{\sqrt{2}}{h_m}\sum_{j<k} M_{jjkk} \right\rangle_{L^2} = \int_{\mathbb{R}^m} \boldsymbol{E}[\,F_0(\boldsymbol{t})F_2(0)]d\boldsymbol{t}. \tag{3.27}$$

$$\boldsymbol{E}[F_0(0)F_3(\boldsymbol{t})] = \sum_{i,j<k} \boldsymbol{E}\big[H_2(U_i(0))H_1(\widehat{a}_{jj}(\boldsymbol{t})H_1(\widehat{a}_{kk}(\boldsymbol{t}))\,\big] = -2\sum_{i,j<k}\frac{1}{3d_m h_m}\mathcal{F}_{ijj}(\boldsymbol{t})(\boldsymbol{t})\mathcal{F}_{ikk}(\boldsymbol{t}),$$

$$\begin{split}\int_{\mathbb{R}^m} \boldsymbol{E}[F_0(0)F_3(\boldsymbol{t})]d\boldsymbol{t} &= \frac{2}{3h_m d_m}\sum_{i,j<k}\langle M_{ijj}, M_{ikk}\rangle_{L^2}\\ = \left\langle \frac{\sqrt{2}}{d_m}\sum_i M_{ii}, \frac{\sqrt{2}}{3h_m}\sum_{j<k} M_{jjkk} \right\rangle_{L^2} &= \int_{\mathbb{R}^m} \boldsymbol{E}[F_0(\boldsymbol{t})F_3(0)]d\boldsymbol{t}.\end{split} \tag{3.28}$$

We have

$$\boldsymbol{E}\big[\,F_1(0)F_1(\boldsymbol{t})\,\big] = \sum_{i,j} \boldsymbol{E}[\,H_2(\widehat{a}_{ii}(0))H_2(\widehat{a}_{jj}(t))\,\big] = \frac{2}{9h_m^2}\sum_{i,j}\mathcal{F}_{iijj}(t)^2$$

$$\int_{\mathbb{R}^m} \boldsymbol{E}\big[\,F_1(0)F_1(\boldsymbol{t})\,\big]d\boldsymbol{t} = \left\langle \frac{\sqrt{2}}{3h_m}\sum_i M_{iiii}, \frac{\sqrt{2}}{3h_m}\sum_i M_{iiii} \right\rangle. \tag{3.29}$$

$$\boldsymbol{E}\big[\,F_1(0)F_2(\boldsymbol{t})\,\big] = \sum_{i,j<k} \boldsymbol{E}[\,H_2(\widehat{a}_{ii}(0))H_2(\widehat{a}_{jk}(t))\,\big] = \frac{2}{3h_m^2}\sum_{i,j<k}\mathcal{F}_{iijk}(\boldsymbol{t})^2,$$

$$\int_{\mathbb{R}^m} \boldsymbol{E}\big[\,F_1(0)F_2(\boldsymbol{t})\,\big]d\boldsymbol{t} = \left\langle \frac{\sqrt{2}}{3h_m}\sum_i M_{iiii}, \frac{\sqrt{2}}{h_m}\sum_{j<k} M_{jkjk} \right\rangle_{L^2}. \tag{3.30}$$

$$\boldsymbol{E}\big[\,F_1(0)F_3(\boldsymbol{t})\,\big] = \sum_{i,j<k} \boldsymbol{E}[\,H_2(\widehat{a}_{ii}(0))H_1(\widehat{a}_{jj}(\boldsymbol{t}))H_1(\widehat{a}_{kk}(\boldsymbol{t})\,\big] = \frac{2}{9h_m^2}\sum_{i,j<k}\mathcal{F}_{iijj}(t)\mathcal{F}_{iikk}(\boldsymbol{t}).$$

$$\int_{\mathbb{R}^m} \boldsymbol{E}\big[\,F_1(0)F_3(\boldsymbol{t})\,\big]d\boldsymbol{t} = \left\langle \frac{\sqrt{2}}{3h_m}\sum_i M_{iiii}, \frac{\sqrt{2}}{3h_m}\sum_{j<k} M_{jjkk} \right\rangle_{L^2}. \tag{3.31}$$

$$\boldsymbol{E}\big[\,F_2(0)F_2(\boldsymbol{t})\,\big]=\sum_{i<j,k<\ell}\boldsymbol{E}\big[\,H_2(\widehat{a}_{ij}(0))H_2(\widehat{a}_{k\ell}(\boldsymbol{t})\,\big]=2\sum_{i<j,k<\ell}\mathcal{F}_{ijk\ell}(t)^2,$$

$$\int_{\mathbb{R}^m}\boldsymbol{E}\big[\,F_2(0)F_2(\boldsymbol{t})\,\big]d\boldsymbol{t}=\left\langle\frac{\sqrt{2}}{h_m}\sum_{i<j}M_{ijij},\frac{\sqrt{2}}{h_m}\sum_{i<j}M_{ijij}\right\rangle_{L^2}. \tag{3.32}$$

$$\boldsymbol{E}\big[\,F_2(0)F_3(\boldsymbol{t})\,\big]=\sum_{i<j,k<\ell}\boldsymbol{E}\big[\,H_2(\widehat{a}_{ij}(0))H_1(\widehat{a}_{kk}(\boldsymbol{t}))H_1(\widehat{a}_{\ell\ell}(\boldsymbol{t}))\big]=\frac{2}{3h_m^2}\sum_{i<j,k<\ell}\mathcal{F}_{ijkk}(\boldsymbol{t})\mathcal{F}_{ij\ell\ell}(\boldsymbol{t}),$$

$$\int_{\mathbb{R}^m}\boldsymbol{E}\big[\,F_2(0)F_3(\boldsymbol{t})\,\big]=\left\langle\frac{\sqrt{2}}{h_m}\sum_{i<j}M_{ijij},\frac{\sqrt{2}}{3h_m}\sum_{k<\ell}M_{kk\ell\ell}\right\rangle_{L^2}. \tag{3.33}$$

$$\boldsymbol{E}\big[\,F_3(0)F_3(\boldsymbol{t})\,\big]=\sum_{i<j,k<\ell}\boldsymbol{E}\big[\,H_1(\widehat{a}_{ii}(0))H_1(\widehat{a}_{jj}(0))H_1(\widehat{a}_{kk}(\boldsymbol{t}))H_1(\widehat{a}_{\ell\ell}(\boldsymbol{t}))\big]$$

$$=\sum_{i<j,k<\ell}\boldsymbol{E}[\widehat{a}_{ii}(0)\widehat{a}_{jj}(0)]\boldsymbol{E}[\widehat{a}_{kk}(\boldsymbol{t})\widehat{a}_{\ell\ell}(\boldsymbol{t})]+\frac{1}{9h_m^2}\sum_{i<j,k<\ell}\Big(\mathcal{F}_{iikk}(\boldsymbol{t})\mathcal{F}_{jj\ell\ell}(\boldsymbol{t})+\mathcal{F}_{ii\ell\ell}(\boldsymbol{t})\mathcal{F}_{jjkk}(\boldsymbol{t})\Big)$$

$$=\boldsymbol{E}[F_3(0)]^2+\frac{1}{9h_m^2}\sum_{i<j,k<\ell}\Big(\mathcal{F}_{iikk}(\boldsymbol{t})\mathcal{F}_{jj\ell\ell}(\boldsymbol{t})+\mathcal{F}_{ii\ell\ell}(\boldsymbol{t})\mathcal{F}_{jjkk}(\boldsymbol{t})\Big).$$

$$\boldsymbol{E}\big[\,\widehat{F}_3(0)\widehat{F}_3(\boldsymbol{t})\,\big]=\boldsymbol{E}\big[\,F_3(0)F_3(\boldsymbol{t})\,\big]-\boldsymbol{E}[F_3(0)]^2=\frac{1}{9h_m^2}\sum_{i<j,k<\ell}\Big(\mathcal{F}_{iikk}(\boldsymbol{t})\mathcal{F}_{jj\ell\ell}(\boldsymbol{t})+\mathcal{F}_{ii\ell\ell}(\boldsymbol{t})\mathcal{F}_{jjkk}(\boldsymbol{t})\Big).$$

We deduce

$$\int_{\mathbb{R}^m}\boldsymbol{E}\big[\,\widehat{F}_3(0)\widehat{F}_3(\boldsymbol{t})\,\big]d\boldsymbol{t}=\frac{1}{9h_m^2}\sum_{i<j,k<\ell}\int_{\mathbb{R}^m}\Big(\mathcal{F}_{iikk}(\boldsymbol{t})\mathcal{F}_{jj\ell\ell}(\boldsymbol{t})+\mathcal{F}_{ii\ell\ell}(\boldsymbol{t})\mathcal{F}_{jjkk}(\boldsymbol{t})\Big)d\boldsymbol{t}$$

and we conclude

$$\int_{\mathbb{R}^m}\boldsymbol{E}\big[\,\widehat{F}_3(0)\widehat{F}_3(\boldsymbol{t})\,\big]d\boldsymbol{t}=\left\langle\frac{\sqrt{2}}{3h_m}\sum_{i<j}M_{ijij},\frac{\sqrt{2}}{3h_m}\sum_{k<\ell}M_{kk\ell\ell}\right\rangle_{L^2} \tag{3.34}$$

To put the above equalities in perspective we introduce the functions

$$G_0=\frac{\sqrt{2}}{d_m}\sum_i M_{ii},\;\;G_1=\frac{\sqrt{2}}{3h_m}\sum_i M_{iiii},\;\;G_2=\frac{\sqrt{2}}{h_m}\sum_{j<k}M_{jjkk},\;\;G_3=\frac{1}{3}G_2.$$

The assumption **A3** implies that $G_0,G_1,G_2\in L^2(\mathbb{R}^m,d\boldsymbol{\lambda})$. Using the notation

$$F_i\bullet F_j:=\int_{\mathbb{R}^m}\boldsymbol{E}\big[\,F_i(0),F_j(\boldsymbol{t})\,\big].$$

we can rewrite the equalities (3.25, ..., 3.34) in a more concise form

$$F_i\bullet F_j=F_j\bullet F_i=\langle G_i,G_j\rangle_{L^2},\;\;F_i\bullet\widehat{F}_3=F_i\bullet F_3,\;\;\forall i,j=0,1,2,$$

$$\widehat{F}_3\bullet F_i=F_i\bullet\widehat{F}_3=\langle G_i,G_3\rangle_{L^2},\;\;\forall i=0,\dots,3.$$

From (3.21) and (3.22) we deduce

$$V_{2,\infty}=\int_{\mathbb{R}^m}\rho_2(\boldsymbol{t})d\boldsymbol{t}=d(0)^2\Big(3xh_m\mathcal{Z}_1+yh_m\mathcal{Z}_2+zF_0)\Big)\bullet\Big(3xh_m\mathcal{Z}_1+yh_m\mathcal{Z}_2+zF_0)\Big)$$

$$=d(0)^2\Big\|3xh_m(G_1+2G_3)+yh_M(3G_1+2G_2)+zG_0\Big\|^2_{L^2}$$

$$= (d(0))^2 \Big\| 3xh_m\Big(G_1 + \frac{2}{3}G_3\Big) + 3yh_M\Big(G_1 + \frac{2}{3}G_2\Big) + zG_0 \Big\|^2_{L^2}$$

$$= d(0)^2 \Big\| 3h_m(x+y)\Big(G_1 + \frac{2}{3}G_2\Big) + zG_0 \Big\|^2_{L^2}.$$

The functions $G_1 + \frac{2}{3}G_2$ and G_0 are linearly independent and

$$z = -\frac{1}{2}f_0(A) = -\frac{1}{2}\boldsymbol{E}\big[\,|\det A|\,\big] \neq 0.$$

Hence $V_{2,\infty} > 0$. □

This concludes the proof of Proposition 3.3. □

Remark 3.7. The numbers x, y that describe $f_2(A)$, the 2nd chaos component of $f(A)$ seem hard to compute in general. In Appendix A we describe their large m asymptotics; see (A.16). □

3.4. **Conclusion.** To conclude the proof of Theorem 2.2 we observe that from (3.14) we deduce that

$$\lim_{Q\to\infty}\lim_{N\to\infty}\boldsymbol{var}\Big(\mathcal{P}_{>Q}\zeta_N\Big) = 0.$$

Hence, it suffices to establish the asymptotic normality of the sequence

$$\mathcal{P}_{\leq Q}\zeta_N = \frac{1}{(2N)^{m/2}}\int_{C_N}\sum_{2\leq q\leq Q}\rho_q(\boldsymbol{t})d\boldsymbol{t}.$$

This follows from a Breuer-Major type central limit theorem, [7, 24, 25]. In our instance, we can invoke [11, Prop. 2.4] and its proof to reach the desired conclusion.

Appendix A. Asymptotics of some Gaussian integrals

We want to give an approximate description of the 2nd chaos component of $|\det A|$ when $m \gg 0$.

Observe that if $u : \mathcal{S}_m \to \mathbb{R}$ is a continuous function, homogeneous of degree k, then for any $v > 0$ we have

$$\boldsymbol{E}_{\mathcal{S}_m^v}[u(A)] = (2v)^{\frac{k}{2}}\boldsymbol{E}_{\mathcal{S}_m^{1/2}}[u(A)].$$

Proposition A.1. *Set $\boldsymbol{C}_m := 2^{\frac{3}{2}}\Gamma\left(\frac{m+3}{2}\right)$. We have the following asymptotic estimates as $m \to \infty$*

$$\boldsymbol{E}_{\mathcal{S}_m^{1/2}}\big[\,|\det A|\,\big] \sim \boldsymbol{C}_m\sqrt{\frac{2}{\pi}}\,m^{-\frac{1}{2}}. \tag{A.1}$$

$$\boldsymbol{E}_{\mathcal{S}_m^{1/2}}\big[\,p(A)f(A)\,\big] \sim \frac{2\boldsymbol{C}_m}{\sqrt{\pi}}\,m^{\frac{3}{2}}. \tag{A.2}$$

$$\boldsymbol{E}_S\big[\,q(A)f(A)\,\big] \sim \frac{\boldsymbol{C}_m}{\sqrt{2\pi}}\,m^{\frac{7}{2}}. \tag{A.3}$$

Proof. We need to make a brief detour in the world of random matrices.

We have a *Weyl integration formula* [3] which states that if $f : \mathcal{S}_m \to \mathbb{R}$ is a measurable function which is invariant under conjugation, then the value $f(A)$ at $A \in \mathcal{S}_m$ depends only on the eigenvalues $\lambda_1(A) \leq \cdots \leq \lambda_n(A)$ of A and we have

$$\boldsymbol{E}_{\mathrm{GOE}_m^v}\big(\, f(X)\,\big) = \frac{1}{\boldsymbol{Z}_m(v)} \int_{\mathbb{R}^m} f(\lambda_1, \dots, \lambda_m) \underbrace{\left(\prod_{1\leq i<j\leq m} |\lambda_i - \lambda_j| \right) \prod_{i=1}^m e^{-\frac{\lambda_i^2}{4v}}}_{=:Q_{m,v}(\lambda)} |d\lambda_1 \cdots d\lambda_m|, \tag{A.4}$$

where $\boldsymbol{Z}_m(v)$ can be computed via Selberg integrals, [3, Eq. (2.5.11)], and we have

$$\boldsymbol{Z}_m(v) = (2v)^{\frac{m(m+1}{4}} \boldsymbol{Z}_m, \;\; \boldsymbol{Z}_m = (2\pi)^{\frac{m}{2}} m! \prod_{j=1}^m \frac{\Gamma(\frac{j}{2})}{\Gamma(\frac{1}{2})} = 2^{\frac{m}{2}} m! \prod_{j=1}^m \Gamma\left(\frac{j}{2}\right). \tag{A.5}$$

For any positive integer n we define the *normalized* 1-point correlation function $\rho_{n,v}(x)$ of GOE_n^v to be

$$\rho_{n,v}(x) = \frac{1}{\boldsymbol{Z}_n(v)} \int_{\mathbb{R}^{n-1}} Q_{n,v}(x, \lambda_2, \dots, \lambda_n) d\lambda_1 \cdots d\lambda_n.$$

For any Borel measurable function $f : \mathbb{R} \to \mathbb{R}$ we have [10, §4.4]

$$\frac{1}{n} \boldsymbol{E}_{\mathrm{GOE}_n^v}\big(\, \mathrm{tr}\, f(X)\,\big) = \int_{\mathbb{R}} f(\lambda) \rho_{n,v}(\lambda) d\lambda. \tag{A.6}$$

The equality (A.6) characterizes $\rho_{n,v}$. Let us observe that for any constant $c > 0$, if

$$A \in \mathrm{GOE}_n^v \Longleftrightarrow cA \in \mathrm{GOE}_n^{c^2 v}.$$

Hence for any Borel set $B \subset \mathbb{R}$ we have

$$\int_{cB} \rho_{n,c^2v}(x) dx = \int_B \rho_{n,v}(y) dy,$$

and we conclude that

$$c\rho_{n,c^2v}(cy) = \rho_{n,v}(y), \;\; \forall n, c, y. \tag{A.7}$$

The behavior of $\rho_{n,v}$ for large n is described the the celebrated *Wigner semicircle theorem.*

Theorem A.2 (Wigner)**.** *As $n \to \infty$, the probability measures*

$$\rho_{n,vn^{-1}}(\lambda)|d\lambda| = n^{1/2} \rho_{n,v}\big(\, n^{1/2}\lambda\,\big)|d\lambda|$$

converge weakly to the semicircle distribution

$$\rho_{\infty,v}(\lambda)|d\lambda| = \boldsymbol{I}_{|\lambda|\leq 2\sqrt{v}} \frac{1}{2\pi v} \sqrt{4v - \lambda^2} |d\lambda|.$$

□

We have the following result of Y. Fyodorov [12]; see also [20, Lemmas C.1, C2.].

Lemma A.3. *Suppose $v > 0$. Then for any $\lambda \in \mathbb{R}$ we have*

$$\boldsymbol{E}_{\mathrm{GOE}_m^v}\big(\, |\det(\lambda + B|\,\big) = (2v)^{\frac{m+1}{2}} \boldsymbol{C}_m e^{\frac{c^2}{4v}} \rho_{m+1,v}(\lambda), \;\; \boldsymbol{C}_m := 2^{\frac{3}{2}} \Gamma\left(\frac{m+3}{2} \right). \tag{A.8}$$

$$\begin{aligned} \boldsymbol{E}_{\mathcal{S}_m^v}\big(\, |\det(A)|\,\big) &= (2v)^{\frac{m+1}{2}} \frac{\boldsymbol{C}_m}{\sqrt{2\pi v}} \int_{\mathbb{R}} \boldsymbol{E}_{\mathrm{GOE}_m^v}\big(\, |\det(\lambda + B|\,\big) e^{-\frac{\lambda^2}{2v}} d\lambda \\ &= (2v)^{\frac{m+1}{2}} \frac{\boldsymbol{C}_m}{\sqrt{2\pi v}} \int_{\mathbb{R}} \rho_{m+1,v}(\lambda) e^{-\frac{x^2}{4v}} d\lambda. \end{aligned} \tag{A.9}$$

□

We will also need the following asymptotic estimates.

Lemma A.4. *Let k be a nonnegative integer. Then*

$$\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\rho_{n,\frac{1}{2}}(\lambda)\lambda^{2k}e^{-\frac{\lambda^2}{2v}}d\lambda \sim \frac{v^k\sqrt{2v}(2k-1)!!}{\sqrt{\pi n}}\ \text{ as } n\to\infty \tag{A.10}$$

Proof. Consider the function

$$w(\lambda)=\frac{\lambda^{2k}}{v^{\frac{2k+1}{2}}(2k-1)!!}\frac{e^{-\frac{\lambda^2}{2v}}}{\sqrt{2\pi}}.$$

Then

$$\int_{\mathbb{R}}w(\lambda)d\lambda=1.$$

We set $w_n(\lambda):=\sqrt{n}w(\sqrt{n}\lambda)$. The probability measures $w_n(\lambda)d\lambda$ converge to δ_0 and we have

$$\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\rho_{n,\frac{1}{2}}(\lambda)\lambda^{2k}e^{-\frac{\lambda^2}{2v}}d\lambda=\frac{v^{\frac{2k+1}{2}}(2k-1)!!}{\sqrt{n}}\int_{\mathbb{R}}\rho_{n,\frac{1}{2n}}(\lambda)w_n(\lambda)d\lambda.$$

Arguing exactly as in [21, Sec. 4.6] we deduce

$$\lim_{n\to\infty}\int_{\mathbb{R}}\rho_{n,\frac{1}{2n}}(\lambda)w_n(\lambda)d\lambda=\rho_{\infty,\frac{1}{2}}(0)=\sqrt{\frac{2}{\pi}}.$$

□

The estimate (A.1) follows from (A.9) and (A.10).

To simplify the notation we set

$$\boldsymbol{E}_G:=\boldsymbol{E}_{\mathrm{GOE}_m^{1/2}},\quad \boldsymbol{E}_S:=\boldsymbol{E}_{\mathcal{S}_m^{1/2}}.$$

Let us observe that the equality (2.5) implies that

$$\boldsymbol{E}_S\big[\,p(A)f(A)\,\big]=\frac{1}{\sqrt{\pi}}\int_{\mathbb{R}}\boldsymbol{E}_G\big[\,p(\lambda+B)f(\lambda+B)\,\big]e^{-\lambda^2}d\lambda, \tag{A.11a}$$

$$\boldsymbol{E}_S\big[\,q(A)f(A)\,\big]=\frac{1}{\sqrt{\pi}}\int_{\mathbb{R}}\boldsymbol{E}_G\big[\,q(\lambda+B)f(\lambda+B)\,\big]e^{-\lambda^2}d\lambda. \tag{A.11b}$$

To estimate $\boldsymbol{E}_S\big[\,p(A)f(A)\,\big]$ and $\boldsymbol{E}_S\big[\,q(A)f(A)\,\big]$ for m large we use a nice trick we learned from Yan Fyodorov. Introduce the functions

$$\Phi,\Psi:\mathbb{R}\times(-\infty,1)\to\mathbb{R},$$

$$\Phi(\lambda,z):=\boldsymbol{E}_G\big[\,|\det(\lambda+B)|e^{z\,\mathrm{tr}(\lambda+B)}\,\big],\quad \Psi(\lambda,z):=\boldsymbol{E}_G\big[\,|\det(\lambda+B)|e^{\frac{z}{2}\,\mathrm{tr}(\lambda+B)^2}\,\big].$$

Obviously

$$\Phi(\lambda,0)=\Psi(\lambda,0)=\boldsymbol{E}_G\big[\,|\det(\lambda+B)|\,\big]$$

so both $\Psi(\lambda,0)$ and $\Psi(\lambda,0)$ can be determined using (A.8).

Observe next that

$$\Phi''_{zz}(\lambda,0)=\boldsymbol{E}_G\Big[\,\big(\,\mathrm{tr}(\lambda+B)\,\big)^2\,|\det(\lambda+B)|\,\Big]=\boldsymbol{E}_G\big[\,p(\lambda+B)f(\lambda+B)\,\big], \tag{A.12a}$$

$$2\Psi''_{zz}(\lambda,0)=\boldsymbol{E}_G\big[\,(\mathrm{tr}(\lambda+B)^2\,|\det(\lambda+B)|\,\big]=\boldsymbol{E}_G\big[\,q(\lambda+B)f(\lambda+B)\,\big]. \tag{A.12b}$$

We have the following key observation.

Lemma A.5 (Y. Fyodorov)**.**

$$\Phi(\lambda,z)=e^{m(\frac{z^2}{2}+z\lambda)}\Phi(\lambda+z,0)=e^{-\frac{m\lambda^2}{2}}e^{\frac{m}{2}(\lambda+z)^2}\Phi(\lambda+z,0), \tag{A.13a}$$

$$\Psi(\lambda,z)=\frac{e^{\frac{m\lambda^2 z}{2(1-z)}}}{(1-z)^{\frac{m(m+3)}{4}}}\Psi\left(\frac{\lambda}{\sqrt{1-z}},0\right). \tag{A.13b}$$

Proof. Using (2.4) we deduce

$$\Phi(\lambda,z)=\boldsymbol{K}_m\int_{\mathcal{S}_m}|\det(\lambda+B)|e^{z\operatorname{tr}(\lambda+B)-\frac{1}{2}\operatorname{tr}B^2}dB=e^{m(\frac{z^2}{2}+\lambda z)}\boldsymbol{K}_m\int_{\mathcal{S}_m}e^{-\frac{1}{2}\operatorname{tr}(B-z)^2}dB$$

(make the change in variables $C:=B-z$)

$$=e^{m(\frac{z^2}{2}+\lambda z)}\boldsymbol{K}_m\int_{\mathcal{S}_m}|\det(\lambda+z+B)|e^{-\frac{1}{2}\operatorname{tr}C^2}dB$$

$$=e^{m(\frac{z^2}{2}+\lambda z)}\boldsymbol{E}_G\big[\,|\det(\lambda+z+C)|\,\big]=e^{m(\frac{z^2}{2}+\lambda z)}\Phi(\lambda+z,0).$$

Similarly, for $z<1$ we have

$$\Psi(\lambda,z)=\boldsymbol{K}_m\int_{\mathcal{S}_m}|\det(\lambda+B)|e^{\frac{z}{2}\operatorname{tr}(\lambda+B)^2-\frac{1}{2}\operatorname{tr}B^2}dB$$

$$=e^{\frac{mz\lambda^2}{2}}\int_{\mathcal{S}_m}|\det(\lambda+B)|e^{\frac{\lambda z}{2}\operatorname{tr}B-\frac{1}{z}2\operatorname{tr}B^2}dB.$$

Making the change in variables $B=(1-z)^{-1/2}C$ so that

$$dB=(1-z)^{-\frac{m(m+1)}{4}}dC\quad\det(\lambda+B)=(1-v)^{-\frac{m}{2}}\det(\lambda\sqrt{1-z}+C).$$

We deduce

$$\Psi(\lambda,z)=\frac{e^{\frac{mz\lambda^2}{2}}}{(1-z)^{\frac{m(m+3)}{4}}}\boldsymbol{K}_m\int_{\mathcal{S}_m}|\det(\lambda\sqrt{1-z}+C)|e^{-\frac{1}{2}\operatorname{tr}(C-\frac{\lambda z}{\sqrt{1-z}})^2}dC$$

$(C-\frac{\lambda z}{\sqrt{1-z}}\to B)$

$$=\frac{e^{\frac{mz\lambda^2}{2}}}{(1-z)^{\frac{m(m+3)}{4}}}\boldsymbol{K}_m\int_{\mathcal{S}_m}|\det(\lambda\sqrt{1-z}+\frac{\lambda z}{\sqrt{1-z}}+B)|e^{-\frac{1}{2}\operatorname{tr}B^2}dB$$

$$=\frac{e^{\frac{mz\lambda^2}{2}}}{(1-z)^{\frac{m(m+3)}{4}}}\Psi\left(\frac{\lambda}{\sqrt{1-z}},0\right).$$

□

The asymptotic behavior of $\boldsymbol{E}_S\big[\,p(A)f(A)\,\big]$. Using (A.13a) we deduce

$$\Phi''_{zz}(\lambda,0)=e^{-\frac{m\lambda^2}{2}}\partial^2_{zz}\big|_{z=0}\Big(e^{\frac{m}{2}(\lambda+z)^2}\Phi(\lambda+z,0)\Big)=e^{-\frac{m\lambda^2}{2}}\frac{d^2}{d\lambda^2}\big(e^{\frac{m\lambda^2}{2}}\Phi(\lambda,0)\big).$$

Using (A.11a) and (A.12a) we deduce

$$\boldsymbol{E}_S\big[\,p(A)f(A)\,\big]=\frac{1}{\sqrt{\pi}}\int_{\mathbb{R}}e^{-\frac{m\lambda^2}{2}}\frac{d^2}{d\lambda^2}\big(e^{\frac{m\lambda^2}{2}}\Phi(\lambda,0)\big)e^{-\lambda^2}d\lambda$$

$$= \frac{1}{\sqrt{\pi}}\int_{\mathbb{R}} \frac{d^2}{d\lambda^2}\big(\, e^{\frac{m\lambda^2}{2}}\Phi(\lambda,0)\,\big)e^{-\frac{m+2}{2}\lambda^2}d\lambda.$$

Since $\Phi(\lambda,0)$ has polynomial growth in λ, we can integrate by parts in the above equality and we deduce

$$\boldsymbol{E}_S\big[\,p(A)f(A)\,\big] = \frac{1}{\sqrt{\pi}}\int_{\mathbb{R}} e^{\frac{m\lambda^2}{2}}\Phi(\lambda,0)\frac{d^2}{d\lambda^2}\big(\,e^{-\frac{m+2}{2}\lambda^2}\,\big)d\lambda$$

$$\stackrel{(A.8)}{=} \frac{\boldsymbol{C}_m}{\sqrt{\pi}}\int_{\mathbb{R}} e^{\frac{m\lambda^2}{2}}e^{\frac{\lambda^2}{2}}\rho_{m+1,\frac{1}{2}}(\lambda)\frac{d^2}{d\lambda^2}\big(\,e^{-\frac{m+2}{2}\lambda^2}\,\big)d\lambda$$

$$= \frac{\boldsymbol{C}_m}{\sqrt{\pi}}\int_{\mathbb{R}} e^{\frac{(m+1)\lambda^2}{2}}\rho_{m+1,\frac{1}{2}}(\lambda)\frac{d^2}{d\lambda^2}\big(\,e^{-\frac{m+2}{2}\lambda^2}\,\big)d\lambda$$

$$= \frac{\boldsymbol{C}_m(m+2)\sqrt{2}}{\sqrt{2\pi}}\int_{\mathbb{R}}\rho_{m+1,\frac{1}{2}}(\lambda)\big(\,(m+2)\lambda^2-1\,\big)e^{-\frac{\lambda^2}{2}}d\lambda \stackrel{(A.10)}{\sim} \frac{\boldsymbol{C}_m(m+2)\sqrt{2}}{\sqrt{m+1}}\big(\,m+1\,\big)\rho_{\infty,\frac{1}{2}}(0).$$

We have thus proved that

$$\boldsymbol{E}_S\big[\,p(A)f(A)\,\big] \sim \sqrt{\frac{2}{m+1}}\boldsymbol{C}_m(m+2)(m+1)\rho_{\infty,\frac{1}{2}}(0)\ \text{ as } m\to\infty. \tag{A.14}$$

This proves (A.2).

The asymptotic behavior of $\boldsymbol{E}_S\big[\,q(A)f(A)\,\big]$. We set

$$u(z) := \frac{e^{\frac{m\lambda^2 z}{2(1-z)}}}{(1-z)^{\frac{m(m+3)}{4}}} = e^{-\frac{m\lambda^2}{2}}\frac{e^{\frac{m\lambda^2}{2(1-z)}}}{(1-z)^{\frac{m(m+3)}{4}}}.$$

Then

$$\Psi(\lambda,z) = u(z)\Psi\big(\lambda(1-z)^{-1/2},0\big),$$

$$\Psi'_z(\lambda,z) = u'(z)\Psi\big(\lambda(1-z)^{-1/2},0\big) + \frac{\lambda}{2}u(z)(1-z)^{-3/2}\Psi'_\lambda\big(\lambda(1-z)^{-1/2},0\big).$$

$$\Psi''_{zz}(\lambda,z) = u''(z)\Psi\big(\lambda(1-z)^{-1/2},0\big) + \frac{\lambda}{2}u'(z)(1-z)^{-3/2}\Psi'_\lambda\big(\lambda(1-z)^{-1/2},0\big)$$

$$+\frac{\lambda}{2}\frac{d}{dx}\big(\,u(z)(1-z)^{-3/2}\,\big)\Psi'_\lambda\big(\lambda(1-z)^{-1/2},0\big) + \frac{3\lambda^2}{4}u(z)(1-z)^{-4}\Psi''_{\lambda\lambda}\big(\lambda(1-z)^{-1/2},0\big).$$

Thus

$$\Psi''_{zz}(\lambda,0) = u''(0)\Psi\big(\lambda,0\big) + \frac{\lambda}{2}u'(0)\Psi'_\lambda\big(\lambda,0\big)$$

$$+\frac{\lambda}{2}\big(\,u'(0)+\frac{3}{2}u(0)\,\big)\Psi'_\lambda\big(\lambda,0\big) + \frac{3\lambda^2}{4}u(0)\Psi''_{\lambda\lambda}\big(\lambda,0\big)$$

$$= u''(0)\Psi\big(\lambda,0\big) + \frac{\lambda}{2}\Big(\,2u'(0)+\frac{3}{2}u(0)\,\Big)\Psi'_\lambda\big(\lambda,0\big) + \frac{3\lambda^2}{4}u(0)\Psi''_{\lambda\lambda}\big(\lambda,0\big).$$

Setting $\kappa(m)=\frac{m(m+3)}{4}$ we deduce

$$u'(z) = e^{-\frac{m\lambda^2}{2}}\frac{d}{dz}\left(\,e^{\frac{m\lambda^2}{2(1-z)}}(1-z)^{-\kappa(m)}\,\right)$$

$$= e^{-\frac{m\lambda^2}{2}}\left(\frac{m\lambda^2}{2}e^{\frac{m\lambda^2}{2(1-z)}}(1-z)^{-\kappa(m)-2} + \kappa(m)e^{\frac{m\lambda^2}{2(1-z)}}(1-z)^{-\kappa(m)-1}\right).$$

Thus

$$u'(0) = \frac{m\lambda^2}{2} + \kappa(m).$$

We set
$$\frac{1}{2}A_1(\lambda)=\frac{\lambda}{2}\big(\,u'(0)+\frac{3}{2}\big)=\frac{\lambda}{2}\Big(\,\frac{m\lambda^2}{2}+\kappa(m)+\frac{3}{2}\,\Big).$$
Similarly, we deduce
$$\begin{aligned}u''(0)&=\frac{m\lambda^2}{2}\left(\frac{m\lambda^2}{2}+\kappa(m)+2\right)+\kappa(m)\Big(\,\frac{m\lambda^2}{2}+\kappa(m)+1\,\Big)\\&=\underbrace{\frac{m^2\lambda^4}{4}+(\kappa(m)+1)m\lambda^2+\kappa(m)\big(\,\kappa(m)+1\,\big)}_{=:\frac{1}{2}A_0(\lambda)}.\end{aligned}$$
We set $A_2(\lambda):=\frac{3}{2}\lambda^2$. We have
$$2\Psi_{zz}(\lambda,0)=A_2(\lambda)\Psi''_{\lambda\lambda}(\lambda,0)+A_1(\lambda)\Psi'_{\lambda}(\lambda,0)+A_0(\lambda)\Psi(\lambda,0).$$
Using (A.11b) and (A.12b) we deduce
$$\begin{aligned}\boldsymbol{E}_S\big[\,q(A)f(A)\,\big]&=\frac{1}{\sqrt{\pi}}\int_{\mathbb{R}}\Big(\,A_2(\lambda)\Psi''_{\lambda\lambda}(\lambda,0)+A_1(\lambda)\Psi'_{\lambda}(\lambda,0)+A_0(\lambda)\Psi(\lambda,0)\,\Big)e^{-\lambda^2}d\lambda\\&=\frac{1}{\sqrt{\pi}}\int_{\mathbb{R}}\Psi(\lambda,0))\left(\frac{d^2}{d\lambda^2}\big(\,A_2(\lambda)e^{-\lambda^2}\,\big)-\frac{d}{d\lambda}\big(\,A_1(\lambda)e^{-\lambda^2}\,\big)+A_0(\lambda)e^{-\lambda^2}\right)d\lambda.\\&=\frac{1}{\sqrt{\pi}}\int_{\mathbb{R}}P_{4,m}(\lambda)\Phi(\lambda)^2\Psi(\lambda,0)e^{-\lambda^2}\,d\lambda,\end{aligned}$$
where
$$\begin{aligned}P_{4,m}(\lambda)&=A''_2(\lambda)-4\lambda A'_2(\lambda)+4\lambda^2A_2(\lambda)-A'_1(\lambda)+2\lambda A_1(\lambda)+A_0(\lambda)\\&=C_4(m)\lambda^4+C_2(m)\lambda^2+C_0(m),\end{aligned}$$
where the coefficients $C_0(m),C_2(m),C_4(m)$ are polynomials in m. Recalling that
$$\Psi(\lambda,0)=\boldsymbol{E}_G\big[\,|\det(\lambda+B)|\,\big]\boldsymbol{C}_m\rho_{m+1,1/2}(\lambda),$$
we deduce
$$\begin{aligned}\boldsymbol{E}_S\big[\,q(A)f(A)\,\big]=\frac{C_4(m)}{\sqrt{\pi}}\int_{\mathbb{R}}\rho_{m+1,\frac{1}{2}}(\lambda)\lambda^4\frac{e^{-\lambda^2}}{\sqrt{\pi}}d\lambda+\frac{C_2(m)}{\sqrt{\pi}}\int_{\mathbb{R}}\rho_{m+1,\frac{1}{2}}(\lambda)\lambda^2\frac{e^{-\lambda^2}}{\sqrt{\pi}}d\lambda\\+\frac{C_0(m)}{\sqrt{\pi}}\int_{\mathbb{R}}\rho_{m+1,\frac{1}{2}}(\lambda)\frac{e^{-\lambda^2}}{\sqrt{\pi}}d\lambda.\end{aligned}$$
Using (A.10) with $v=1/2$ we deduce that as $m\to\infty$ we have
$$\boldsymbol{E}_S\big[\,q(A)f(A)\,\big]\sim\boldsymbol{C}_mm^{-1/2}2^{\frac{1}{2}}\left(2^{-1}\frac{3C_4(m)}{\sqrt{\pi}}+2^{-1/2}\frac{C_2(m)}{\sqrt{\pi}}+\frac{C_0(m)}{\sqrt{\pi}}\right).$$
Upon investigating the definition of $A_0(\lambda)$, $A_1(\lambda)$, and $A_0(\lambda)$ we see that of the three
$$\deg C_0(m)=4>\deg C_2(m),\deg C_4(m).$$
The degree-4 term in $C_0(m)$ comes from the product
$$2\kappa(m)(\kappa(m)+1)=\frac{m^4}{2}+\text{ lower order terms.}$$
We conclude that as $m\to\infty$ we have
$$\boldsymbol{E}_S\big[\,q(A)f(A)\,\big]\sim\frac{\boldsymbol{C}_m}{\sqrt{2\pi}}\,m^{\frac{7}{2}}.$$

□

To understand the 2nd chaos component of $|\det A|$ we need to also understand the inner product in $L^2(\mathcal{S}_m^v)_2^{inv}$. For simplicity will write $\boldsymbol{E}$ instead of the more precise $\boldsymbol{E}_{\mathcal{S}_m^v}$.

We know that

$$\boldsymbol{E}\big[\,p(A)\,\big]=\big[q(A)\,\big]=m(m+2)v.$$

This implies that

$$\boldsymbol{E}\big[\,p(A)^2\,\big]=\boldsymbol{E}\big[(\operatorname{tr}A)^4\,\big]=3m^2(m+2)^2v^2.$$

To compute $\boldsymbol{E}[\,p(A)q(A)\,]$, $\boldsymbol{E}[\,q(A)^2\,]$ we will use Wick's formula, [14, Thm. 1.28]. We have

$$p(A)q(A)=\left(\sum_i a_{ii}^2+2\sum_{i<j}a_{ii}a_{jj}\right)\left(\sum_k a_{kk}^2+2\sum_{k<\ell}a_{k\ell}^2\right)$$

$$=\underbrace{\sum_i a_{ii}^4}_{S_1}+\underbrace{2\sum_{i<k}a_{ii}^2a_{kk}^2}_{S_2}+\underbrace{2\sum_{i,\,k<\ell}a_{ii}^2a_{k\ell}^2}_{S_3}+\underbrace{2\sum_{k,\,i<j}a_{kk}^2a_{ii}a_{jj}}_{S_4}+\underbrace{4\sum_{i<j,\,k<\ell}a_{ii}a_{jj}a_{k\ell}^2}_{S_5}.$$

We have

$$\boldsymbol{E}[S_1]=\boldsymbol{E}\left[\sum_i a_{ii}^4\right]=m\boldsymbol{E}[a_{11}^4]=27mv^2.$$

$$\boldsymbol{E}[S_3]=\boldsymbol{E}\left[2\sum_{i,\,k<\ell}a_{ii}^2a_{k\ell}^2\right]=m^2(m-1)\boldsymbol{E}[a_{11}^2]\boldsymbol{E}[a_{12}^2]=3m^2(m-1)v^2.$$

$$\boldsymbol{E}[S_5]=\boldsymbol{E}\left[4\sum_{i<j,\,k<\ell}a_{ii}a_{jj}a_{k\ell}^2\right]=m^2(m-1)^2\boldsymbol{E}[a_{11}a_{22}]\boldsymbol{E}[a_{12}^2]=m^2(m-1)^2v^2.$$

$$\boldsymbol{E}[S_2]=\boldsymbol{E}\left[2\sum_{i<k}a_{ii}^2a_{kk}^2\right]=m(m-1)\boldsymbol{E}[a_{11}^2a_{22}^2]$$

Using Wick's formula we deduce

$$\boldsymbol{E}[a_{11}^2a_{22}^2]=\boldsymbol{E}[a_{11}^2]\boldsymbol{E}[a_{22}^2]+2\boldsymbol{E}[a_{11}a_{22}]^2=11v^2. \tag{A.15}$$

Hence

$$\boldsymbol{E}[S_2]=11m(m-1)v^2.$$

$$\boldsymbol{E}[S_4]=\boldsymbol{E}\left[2\sum_{k,\,i<j}a_{kk}^2a_{ii}a_{jj}\right]$$

$$=2\boldsymbol{E}\left[\sum_{i<j}a_{ii}^3a_{jj}\right]+2\boldsymbol{E}\left[\sum_{i<j}a_{ii}a_{jj}^3\right]+2\boldsymbol{E}\left[\sum_{i<j,\ k\neq i,j}a_{kk}^2a_{ii}a_{jj}\right]$$

$$=4\boldsymbol{E}\left[\sum_{i<j}a_{ii}^3a_{jj}\right]+m(m-1)(m-2)\boldsymbol{E}[a_{11}a_{22}a_{33}^2]$$

$$=2m(m-1)\boldsymbol{E}[a_{11}^3a_{22}]++m(m-1)(m-2)\boldsymbol{E}[a_{11}a_{22}a_{33}^2].$$

Using Wick's formula we deduce

$$\boldsymbol{E}[a_{11}^3a_{22}]=3\boldsymbol{E}[a_{11}^2]\boldsymbol{E}[a_{11}a_{22}]=9v^2$$

$$\boldsymbol{E}[a_{11}a_{22}a_{33}^2] = \boldsymbol{E}[a_{11}a_{22}]\boldsymbol{E}[a_{33}^2] + 2\boldsymbol{E}[a_{11}a_{22}]^2 = 5v^2,$$

Hence

$$\boldsymbol{E}[S_4] = 18m(m-1)v^2 + 5m(m-1)(m-2)v^2 = m(m-1)(5m+8)v^2.$$

We have

$$q(A)^2 = \left(\underbrace{\sum_i a_{ii}^2}_{X} + \underbrace{2\sum_{k<\ell} a_{k\ell}^2}_{Y} \right)^2 = X^2 + Y^2 + 2XY.$$

The random variables X and Y are independent and thus

$$\boldsymbol{E}\big[\, q(A)^2 \,\big] = \boldsymbol{E}[X^2] \; + \; \boldsymbol{E}[Y^2] + 2\boldsymbol{E}[X]\boldsymbol{E}[Y].$$

We have

$$\boldsymbol{E}[X] = 3mv, \;\; \boldsymbol{E}[Y] = m(m-1)v, \;\; 2\boldsymbol{E}[XY] = 6m^2(m-1)v^2.$$

Next,

$$X^2 = \sum_i a_{ii}^4 + 2\sum_{I<j} a_{ii}^2 a_{jj}^2,$$

$$\boldsymbol{E}[X^2] = m\boldsymbol{E}[a_{11}^4] + m(m-1)\boldsymbol{E}[a_{11}^2 a_{22}^2] \overset{(A.15)}{=} 27mv^2 + 11m(m-1)v^2 = 11m^2v^2 + 16mv^2,$$

$$Y^2 = 4\left(\sum_{k<\ell} a_{k\ell}^2\right)^2 = 4\sum_{k<\ell} a_{k\ell}^4 + 8 \sum_{\substack{i<j,\; k<\ell \\ (i,j)\neq(k,\ell)}} a_{ij}^2 a_{kl}^2$$

$$\boldsymbol{E}[Y^2] = 4\binom{m}{2}\boldsymbol{E}[a_{12}^4] + 8\binom{\binom{m}{2}}{2}\boldsymbol{E}[a_{12}^2]^2.$$

$$= 6m(m-1)v^2 + 8\binom{m}{2}\left(\binom{m}{2} - 1\right)v^2 = m(m-1)v^2\big(\,6 + 2(m+1)(m-2)\,\big).$$

We summarize the results we have obtained so far. Below we denote by $o(1)$ a function of m, independent of v that goes to 0 as $m \to \infty$.

$$\boldsymbol{E}\big[\, p(A) \,\big] = m(m+2)v,$$

$$\boldsymbol{E}\big[\, p(A)^2 \,\big] = 3m^2(m+2)^2v^2 = 3m^4v^1(1 + o(1)),$$

$$\boldsymbol{E}[q(A)] = m(m+2)v,$$

$$\boldsymbol{E}\big[\, q(A)^2 \,\big] = mv^2(2\,m^3 + 2\,m^2 + 9\,m + 14) = 2m^4v^2(1 + o(1)),$$

$$E[p(A)q(A)] = \big(m^3 + 3m^2 + 12m + 11\big)mv^2 = m^4v^2(1 + o(1)).$$

We have

$$\boldsymbol{E}[\bar{p}(A)^2] = \boldsymbol{E}[p(A)^2] - 2m(m+2)\boldsymbol{E}[p(A)] + m(m+2)v = 2m^4v^2(1 + o(1)).$$

$$\boldsymbol{E}[\,\bar{p}(A)\bar{q}(A)\,] = E[p(A)q(A)] - m^2(m+2)^2v^2 = -m^3v^2(1 + o(1)),$$

$$\boldsymbol{E}[\bar{q}(A)^2] = mv^4v^2(1 + o(1)).$$

Thus, in the basis $\bar{p}(A), \bar{q}(A)$ of $L^2(\mathcal{S}_m^v)_2^{inv}$ the inner product is given by the symmetric matrix

$$Q_m = m^4v^2 \begin{bmatrix} 2 & o(1) \\ o(1) & 1 \end{bmatrix}.$$

This proves that the component of $f(A)$ in $L^2(\mathcal{S}_m^v)_2^{inv}$ has a decomposition

$$f_2(A) = x_m \bar{p}(A) + y_m \bar{q}_m q(A),$$

where, as $m \to \infty$

$$\begin{aligned} x_m &\sim \frac{1}{2m^4v^2}\Big(\boldsymbol{E}_{\mathcal{S}_m^v}\big[\,p(A)f(A)\,\big] - m(m+2)v\boldsymbol{E}_{\mathcal{S}_m^v}\big[\,f(A)\,\big]\Big)\\ &\sim \frac{(2v)^{\frac{m+2}{2}}}{2m^4v^2}\Big(\boldsymbol{E}_{\mathcal{S}_m^{1/2}}\big[\,p(A)f(A)\,\big] - \frac{m(m+2)}{2}\boldsymbol{E}_{\mathcal{S}_m^v}\big[\,f(A)\,\big]\Big),\\ y_m &\sim \frac{1}{m^4v^2}\Big(\boldsymbol{E}_{\mathcal{S}_m^v}\big[\,p(A)f(A)\,\big] - m(m+2)v\boldsymbol{E}_{\mathcal{S}_m^v}\big[\,f(A)\,\big]\Big),\\ &\sim \frac{(2v)^{\frac{m+2}{2}}}{2m^4v^2}\Big(\boldsymbol{E}_{\mathcal{S}_m^{1/2}}\big[\,q(A)f(A)\,\big] - \frac{m(m+2)}{2}\boldsymbol{E}_{\mathcal{S}_m^v}\big[\,f(A)\,\big]\Big). \end{aligned}$$

Using (A.1),(A.2) and (A.3) we deduce that there exist two universal constant z_1, z_2,indepndent of m and v such that, as $m \to \infty$.

$$x_m = z_1 \boldsymbol{C}_m v^{\frac{m-2}{2}} m^{-5/2},\;\; y_m \sim z_2 \boldsymbol{C}_m v^{\frac{m-2}{2}} m^{-1/2}. \tag{A.16}$$

In the problem investigated in this paper the variance v also depends on m, $v = h_m$. Recall that the constant $\boldsymbol{C}_m$ grows really fast as $m \to \infty$

$$\log \boldsymbol{C}_m \sim \frac{1}{2} m \log m.$$

Proposition A.6. *The Gaussian vector*

$$J_2(X) := \big(\,X(0), \nabla X(0), \nabla^2 X(0)\,\big).$$

is nondegenerate.

Proof. We set $H := \nabla^2(0)$ and we denote by H_{ij} its entries. The equality (2.8d) shows that $H \in \mathcal{S}_m^{h_m}$ is a centered Gaussian random real symmetric matrix whose statistic is defined by the equalities

$$\boldsymbol{E}\big[\,H_{ii}^2\,\big] = 3h_m,\;\; \boldsymbol{E}\big[H_{ii}H_{jj}\,\big] = \boldsymbol{E}\big[\,H_{ij}^2\,\big] = h_m,\;\; \forall i \neq j,$$

while all the other covariances are trivial. This shows that the second jet $J_2(X)$ is the direct sum of mutually independent Gaussian vectors, $J_2(X) = A \oplus H_0 \oplus D$, where $D = \nabla X(0)$, H_0 is the vector with independent entries $(H_{ij})_{i<j}$ and A is the vector

$$A = \big(\,X(0), H_{11}, \dots, H_{mm}\,\big).$$

The components H_0 and D are obviously nondegenerate Gaussian vectors. Thus, the jet $J_2(X)$ is nondegenerate if and only if the component A is. The covariance matrix of A is $R_m(s_m, d_m, h_m)$ where for any $s, d, h > 0$ we denote by $R_m(s, d, h)$ symmetric $(m+1)\times(m+1)$ matrix with entries

$$r_{00} = s,\;\; r_{0i} = -d,\;\; \forall i = 1, \dots, m,\;\; r_{ii} = 3h,\;\; r_{ij} = h,\;\; \forall 1 \leq i < j \leq m.$$

Note that multiplying the first row by $s^{-1/2}$ and then the first column by $s^{-1/2}$ we deduce

$$\det R_m(s, h, d) = s \det R_m(1, \bar{d}, h),\;\; \bar{d} = ds^{-1/2}.$$

If we add the first column multiplied by $\bar{d}$ to the other columns we deduce that

$$\det R_m(1, \bar{d}, h) = \det G_m\big(\,3h - \bar{d}^2, h - \bar{d}^2\,\big),$$

where $G_m(a,b)$ denotes the symmetric $m \times m$ matrix whose diagonal entries are equal to a, and the off diagonal entries equal to b. As explained in [21, Appendix B], we have

$$\det G_m(a) = (a-b)^{m-1}\big(\, a + (m-1)b\,\big).$$

Thus

$$\det R_m(s,d,h) = s(2h)^{m-1}\big(\,(m+2)h - m\bar{d}^2\,\big) = (2h)^{m-1}\big(\,(m+2)hs - md^2\,\big).$$

Thus $J_2(X)$ is nondegenerate if and only if $\frac{h_m s_m}{d_m^2} \neq \frac{m}{m+2}$. Using (2.7) we deduce that

$$\frac{h_m s_m}{d_m^2} = \frac{m}{m+2}\frac{I_{m-1}(w)I_{m+3}(w)}{I_{m+1}(w)^2}.$$

From the Cauchy inequality we deduce that $I_{m+1}(w)^2 \leq I_{m-1}(w)I_{m+3}(w)$. We cannot have equality because the functions $\sqrt{w(r)}\, r^{\frac{m-1}{2}}$ and $\sqrt{w(r)}\, r^{\frac{m+3}{2}}$ are linearly independent. □

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