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COMPARATIVE STUDY OF WAVELETS OF THE FIRST AND SECOND GENERATION

1 Abstract

Wavelet analysis is, among others, one of the most effective method of experimental data processing. Unlike traditional Fourier transform it is more informative in various applications because it provides one extra degree of freedom for analysis.

There are known a lot of methods of data analysis developed on the basis of continuous and discrete wavelets, such as wavelets suggested by Daubechies, Mallat, Meyer etc. (see for example [1]). Besides a whole family of fast-computed wavelets named as wavelets of the second generation appeared recently [2]. Therefore experimentalists are inevitable faced with the problem of choosing the type of wavelets and the algorithm of their implementation that have to be most suitable for a concrete application. It imposes the need in developing a basic set of test problems as a benchmark, on which one could compare capabilities of different types of wavelets.

In this paper in order to compare efficiency a comprehensive set of benchmarking tests is developped, which is used to compare abilities of continuous wavelets of the vanishing momenta type [3] as well as the second generation wavelets constructed on the basis of the lifting scheme [2]. A proposed set of tests is based on processing of various types of pure and contaminated harmonic signals, delta-function, study of the signal phase dependence and the gain-frequency characteristics. Results of a comparative multiscale analysis allow to reveal advantages and flaws of considered types of wavelets.

2 Filtering Data with Wavelets

Wavelet analysis provides a variety of tools for signal processing, which help to extract components out of source signal and to see what the signal looks like at different scales. We use the term filtering to refer something more than just denoising. Wavelet filters are so multifunctional that can be used not only for the purpose of noise suppression but for searching of different components of the signal as well. Being distinguishable by wavelets, some of those components might not be clearly observed before wavelet analysis is applied.

The idea of wavelet filtering is simple. First we build the set of wavelet coefficients for the given signal. This set is called a wavelet spectrum. Shift and scale parameters of analyzing wavelet may vary in wide ranges and are defined by the task of processing. Having a spectrum, it is possible to estimate, visually or automatically, what is the structure of the signal at different scales at each moment.

After the stage of processing mentioned above is passed there are two ways to continue. Both of them require a procedure of inverse wavelet transform. Before the inverse transform we set to zero some of wavelet coefficients, in other words, ignore them. Thus the inverse transform will use only the part (or several disjoint parts) of the spectrum of the signal.

2.1 Scale Selection

The appearance of wavelet spectrum of the signal allows to view the scales of components which present in the source. If it is known a priori which scales are to be extracted it is possible to run the inverse transform with the desired scales only. And it is also possible to ignore some components of prescribed scales before the inverse transform. As an example we can analyze electrocardiographical data. Besides the heart pulse signals it includes the signal induced by the power supply (50 or 60 Hz) and the low frequency additive background due to variations of the resistance of electrical contact body-sensor. So we can select the scales which correspond to frequencies within these limits to extract the desired heart pulses.

2.2 Thresholding

Obtained wavelet coefficients represent the weight of parts of the signal with different scales. The greater amplitude of the component the higher corresponding wavelet coefficient. Thus, before running the inverse transform, coefficients which absolute values are under some predefined threshold might be deleted from the array. For example, noise component could bring small wavelet coefficients into the spectrum, and by ignoring them the inverse transform removes the noise from the signal.

3 Types of Wavelets

There is a vast variety of wavelets with their own unique properties. We focus ourselves here on wavelets of so-called Vanishing Momenta Wavelet Family (VMWF), sometimes referred to as Gaussian Wavelets. VMWF wavelets represent the first generation of wavelets. Recently developed methods use so-called lifting scheme wavelets (LSW), which are often called as the second generation wavelets.

3.1 Vanishing Momenta Wavelet Family

Gaussian wavelets are obtained by taking the derivatives of Gaussian exponent. The order of the derivative corresponds the order of wavelet itself

$$g_n(x) = (-1)^{n+1} \frac{d^n}{dx^n} e^{-x^2/2}.$$

First two VMWF wavelets are most popular:

$$g_1(x) = -xe^{-x^2/2}, \quad g_2(x) = (1 - x^2)e^{-x^2/2}.$$

Gaussian Wavelets of higher orders are able to bring some new features into the wavelet signal processing. Hereafter we restrict ourselves by using the second order wavelet only, also referred to as Mexican Hat wavelet.

There are two types of wavelet transforms, discrete and continuous. Each of them have its own advantages and shortcomings. For instance, the discrete wavelet transform (DWT) is faster and allows to reveal exact representation of the signal structure with more compact resulting set of coefficients. Continuous wavelet transform (CWT) needs more computational resources but gives a chance to see the structure in details.

3.2 Lifting Scheme

This type of wavelet transform is referred to as a second generation wavelet tool. Let us give a short description following Wim Sweldens [1] [2].Consider a signal s_j with 2^j samples which we want to transform into a coarser signal s_{j-1} and a detail signal d_{j-1}. A typical case of a wavelet transform built through lifting consists of three steps: split, predict, and update. Let us discuss each stage in more detail.

- **Split:** This stage does not do much except for splitting the signal into two disjoint sets of samples. In our case one group consists of the even indexed samples s_{2l} and the other group consists of the odd indexed samples s_{2l+1}. Each group contains half as many samples as the original signal. The splitting into even and odds is called the Lazy wavelet transform. We thus built an operator so that

$$(even_{j-1}, odd_{j-1}) = Split(s_j) .$$

[1] One can find a detailed description of the lifting scheme concept and algorithms for some of second generation wavelets in [2] and also on web-site http://www.cm.bell-labs.com/who/wim/papers/

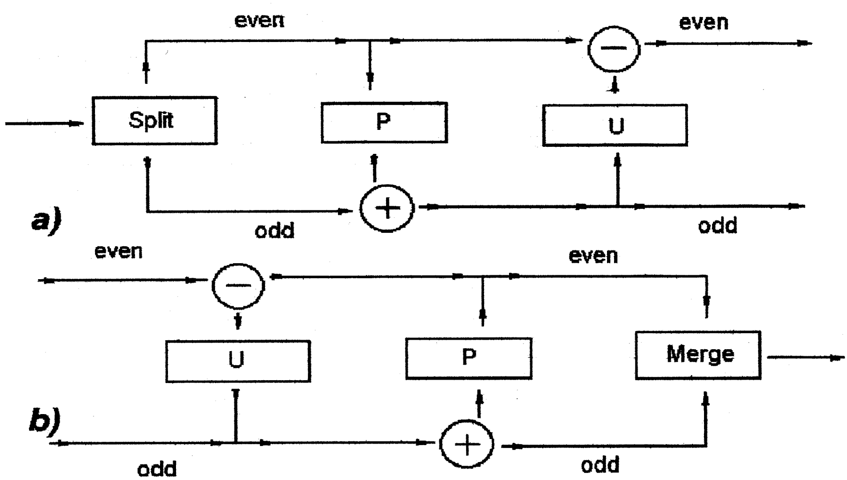


Figure 1: Scheme of one step of lifting decomposition a) and reconstruction b).

- **Predict:** The even and odd subsets are interspersed. If the signal has a local correlation structure, the even and odd subsets will be highly correlated. In other words given one of the two sets, it should be possible to predict the other one with reasonable accuracy. We always use the even set to predict the odd one. In the Haar case the prediction is particularly simple. An odd sample particularly simple. An odd sample $s_{j,2l+1}$ will use its left neighboring even sample $s_{j,2l}$ as its predictor. We then let the detail $d_{j-1,l}$ be the difference between the odd sample and its prediction, which defines an operator P such that

$$d_{j-1} = odd_{j-1} - P(even_{j-1}).$$

Our purpose is to represent the detail more efficiently. Note that if the original signal is a constant, then all details are exactly zero.

- **Update:** One of the key properties of the coarser signals is that they have the same average value as the original signal. The update stage ensures this by letting

$$s_{j-1} = even_{j-1} + U(d_{j-1})$$

All this can be computed in-place: the even locations can be overwritten with the averages and the odd ones with the details. These three stages are depicted in a wiring diagram in fig. 1, a). We can immediately build the inverse scheme, see the wiring diagram in fig. 1, b). Again we have three stages:

- **Undo update:** Given d_{j-1} and s_{j-1} we can recover the even samples by simply subtracting the update information:

$$even_{j-1} = s_{j-1} - U(d_{j-1})$$

- **Undo predict:** Given $even_{j-1}$ and d_{j-1} we can recover the odd samples by adding the prediction information

$$odd_{j-1} = d_{j-1} + P(even_{j-1})$$

- **Merge:** Now that we have the even and odd samples we simply have to zipper them together to recover the original signal. This is the inverse Lazy wavelet:

$$s_j = Merge(even_{j-1}, odd_{j-1})$$

he inverse transform is thus always found by reversing the order of the operations and ipping the signs.
The lifting scheme has a number of algorithmic advantages

- **In-place:** all calculations can be performed in-place which can be an important memory savings.

- **Efficiency:** in many cases the number of floating point operations needed to compute both smooth and detail parts is reduced since subexpressions are reused.

  But perhaps more importantly lifting has some structural advantages which are both theoretically and practically relevant:

- **Inverse Transform:** writing the wavelet transform as a sequence of elementary predict and update (lifting) steps, it is immediately obvious what the inverse transform is: simply run the code backwards. In the classical setting, the inverse transform can typically only be found with the help of Fourier techniques.

- **Generality:** this is the most important advantage. Since the design of the transform is performed without reference to Fourier techniques it is very easy to extend it to settings in which, for example, samples are not placed evenly or constraints such as boundaries need to be incorporated. It also carries over directly to curves, surfaces and volumes.

It is for these reasons that we built our exposition entirely around the lifting scheme.

The indisputable advantage of it is speed. Moreover, the set of coefficients obtained after the transformation is of same length as the source. The inverse transform restores the signal without any errors, what is not possible with the VMWF wavelets.

However there are some shortcomings of the lifting scheme technique. The most significant is that it is possible to choose the scale of the transformation from predefined set only. And it is not possible to use scales less than 1 in principle.

3.3 Software

Authors have developed an object-oriented C++ software package implementing both continuous VMWF transform and its very fast disctrete version. Besides a LSF package was developed (see [4] for detailes).

3.4 Problem

Particular tasks of signal processing require their own demands for the processing tool. Before applying wavelet transform the researcher have to select one from a variety of approaches. We are trying to test two types of wavelets more in details in order to understand when each of them is more desirable in a particular task of signal processing. We should mention that the results obtained are applicable to wavelet families described above.

4 Data Samples

To compare the different approaches of applying wavelets for data filtering we generate a number of test data. All the sample signals are of the length 512. There are several groups of signals, listed below.

1. Harmonical signals;

2. Harmonical signals with noises;

3. Dirac's delta function.

Using these classes of sample data the following characteristics are examined.

1. Ability of filter to recover signals;

2. Denoising properties;

3. Influence of the phase of the signal;

4. Gain-frequency characteristics (GFC).

4.1 Harmonical Signals

Samples of this class may be grouped into two subclasses.

1. The sum of two sine waves with different frequencies and amplitudes. The purpose of the analysis is to split the signal into two components, those are the different frequency harmonical signals.

   $10\sin(0.0368x) + 5\sin(0.1223x)$ (3 periods of low frequency component and about 10 periods of high frequency one);

   $10\sin(0.0368x) + 5\sin(0.2454x)$ (3 and approximately 20 periods);

   $10\sin(0.0368x) + 5\sin(0.3677x)$ (3 and approximately 30 periods).

   An example of this type of signal is presented on fig. 2, a).

2. The sum of two components. One of them is the sine wave with low frequency and is defined on the whole interval [0, 512). The second one is a high frequency sine wave which lives inside [160, 360] interval only (see fig. 2, b). There are three versions of this type of sample, with the same frequencies as above. The wavelet filter has to split those two parts of the signal.

4.2 Harmonical Signals with Noise

The source signals are the same as described in previous section. After the data set is built an additional random value is put to every point of samples. The distribution law of those random values is uniform with amplitude varying from 1 to 100. We use both non-correlated and correlated noise values. In the second case the correlation vector is (0.25, 0.5, 1, 0.5, 0.25). Figure 3 presents an example of noisy data – non-correlated noise is applied to samples from fig. 2.

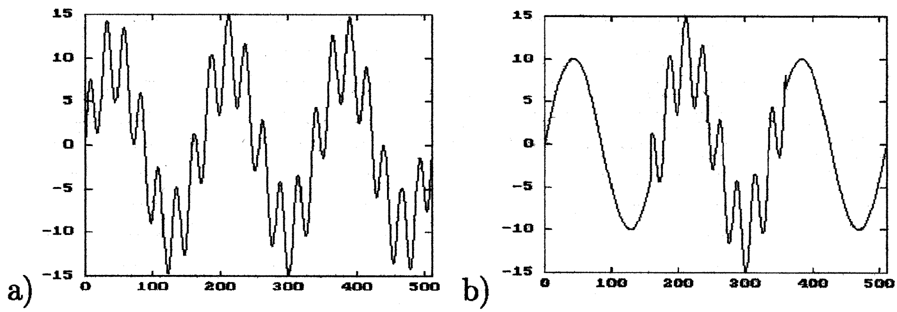


Figure 2: Sample signals, the sum of two harmonics.

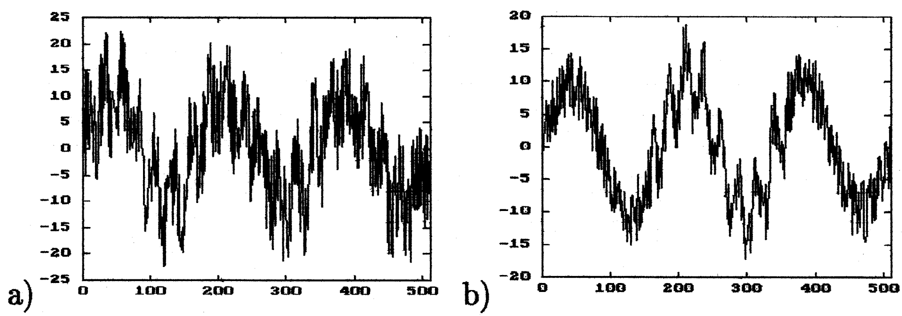


Figure 3: The same sample signals with added noise.

4.3 Gain-Frequency Characteristics

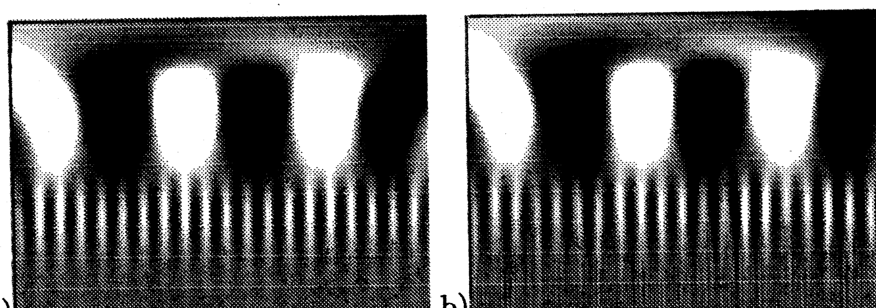
Let the source signal be a sine wave with fixed amplitude. Running it through the direct and then the inverse wavelet transforms we will obtain some new recovered signal. The ratio of their amplitudes is the gain of the filter at the frequency of the signal. Varying the frequency we obtain the gain-frequency characteristics (GFC) of the filter. Gain-frequency characteristics clearly shows the ability of the filter to pass different frequencies presented in the signal.

5 Results of Applying Filters

Data samples mentioned in previous sections were passed through wavelet filters built on both VMWF and LSW wavelet sets. In the following subsections we will present the results of applying these filters to sample data.

5.1 Wavelet Spectrum and Skeleton

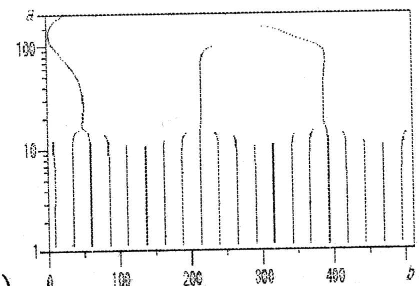
Wavelet analysis provides the powerful tool which helps to view the structure of the signal. The set of wavelet coefficients can be presented as a projection of 3-dimensional surface onto the plane in a-b axes. Coefficients with higher values are depicted in more light color while lower ones are darker. The image obtained after such a projection is called wavelet spectrum. Two examples of wavelet spectrum of signals from figs. 2 - 3 are shown in fig. 4. One can see the fine structure of spectrum in the lower part of vertical scalig axis.



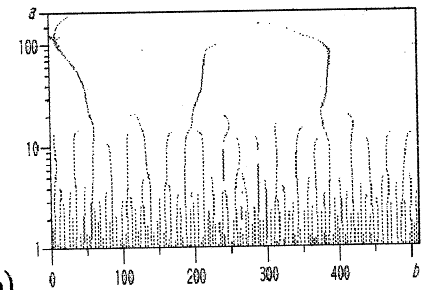
a) b)

Figure 4: Two examples of wavelet spectrum of signals from figs. 2 - 3.

Being highly informative wavelet spectrum often brings too much redundant information. To avoid it gray-scaled image is transformed to so-called wavelet skeleton. Lines on the skeleton correspond local maxima in the spectrum.



a)



b)

Figure 5: Wavelet skeletons of signals, presented in figs. 2, a) and 3, a)

Figures 5, a) and 5, b) present wavelet skeletons of data samples depicted in figs. 2, a) and 3, a), respectively. Note on the first spectrum that it clearly shows the positions of maxima of both sine waves presented in the source. A grid of vertical lines in the bottom part of the skeleton corresponds to high frequency component, while lines going higher show the presence of low frequency one.

When the noise is added to the signal, additional curved lines appear in the small-scale region of the skeleton (see fig. 5).

5.2 VMWF Filter

Here we present some results of applying VMWF filter to data samples. First we take the sample signal depicted in fig. 2, b) and pass it through the filter which allows to select components with scales corresponding gaussian wavelet g_2 at scales 32, 64, 128 and 256. The result is the signal in fig. 5, a). After the sample was processed at scales 1, 2, 4, 8 and 16 we obtain data as in fig. 6, b).

It is seen that the filter allows to extract the initial parts of the signal. Thus selecting the scales of wavelet transformation it is possible to highlight the components with desired scales. Note that in case of signal consisted of two sine waves (fig.2, a) wavelet-based extraction works like traditional Fourier filters.

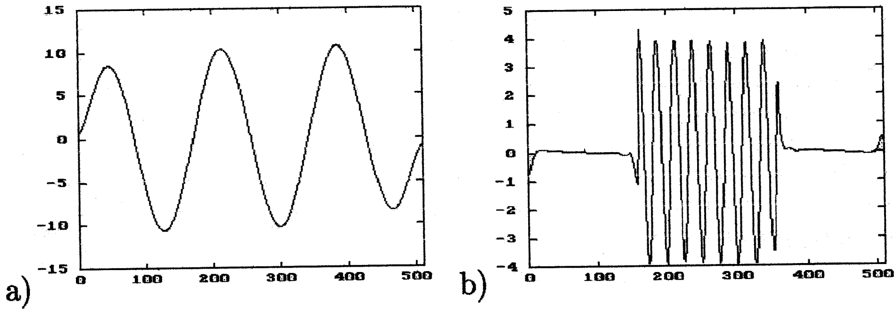


Figure 6: The result of extracting data from the signals using VMWF filter.

However with signal in fig. 2, b) simple Fourier methods would fail. If you need to select high frequency short-lived wave you should first extract low-frequency wave and then subtract it from the signal. Wavelet filter allows direct extraction of the requested component.

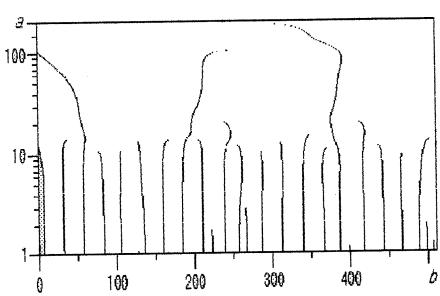


Figure 7: Skeleton of the resulted signal after VMWF denoising.

Another application of wavelet filtering is denoising. The procedure of denoising consists of deleting wavelet coefficients with small amplitude from the set before inverse transform. In fig. 5, a) the skeleton of the signal corrupted with noise was presented. Comparing this image with the skeleton in fig. 7 we may say, that the noise is efficiently suppressed. There are only few short lines at small scales region, which appear at the locations of most significant noise.

5.3 LSW Filter

An alternative to VMWF filter is a filter built on lifting scheme wavelets. In this section we present the results of applying this type of filter to the sample data.

In figs. 8, a) and 8, b) signals obtained after passing the signal in fig. 2, a) through the LSW filter are presented. To extract low frequency component we use scales larger than 32, lower scales are used to select high frequency feature. Unlike the VMWF filter LSW one do not work properly at the edge of the signal, note the downwards initial part in the fig. 8, b).

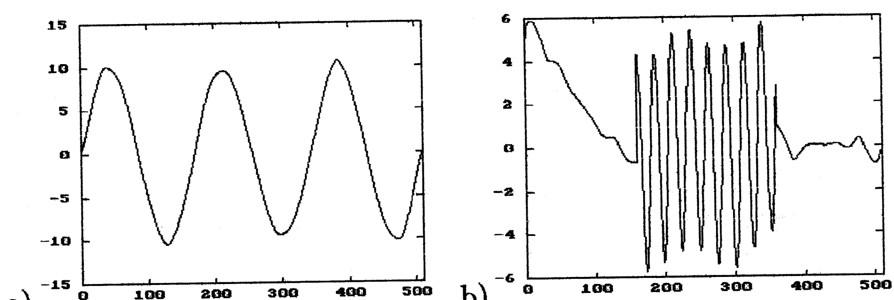


Figure 8: The result of extracting data from the signals using LSW filter.

Denoising abilities of LSW filter are worser of those of VMWF filter. As it is seen from the fig. 9 where the skeleton of the denoised data is presented there are much more noise traces left.

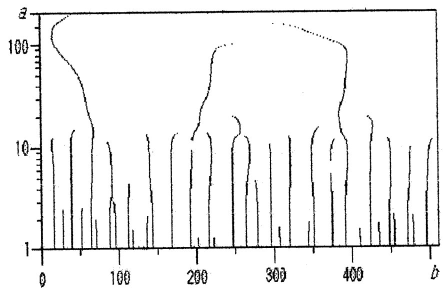


Figure 9: Here presented skeleton of the signal after LSW denoising.

5.4 Gain-Frequency Characteristics

Gain-frequency characteristics of filters based on both VMWF and LSW wavelets are shown in figs. 10 and 11, respectively. Note that VMWF filter provides more flat GFC than LSW filter does. Higher order g_n wavelets give better frequency properties of the filter.

GFC in fig. 9 is obtained for the filter with scales 1 through 256 with the base $a_0 = 2^{1/4}$ – that is scales are 1, a_0, a_0^2, etc. Higher base would damage the flat lookup of the characteristics.

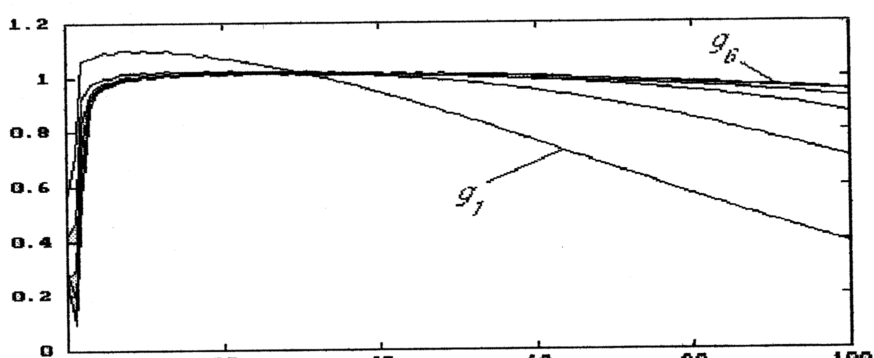


Figure 10: Gain-frequency characteristics of the VMWF filter.

Gain-frequency characteristics of the LSW filter at different

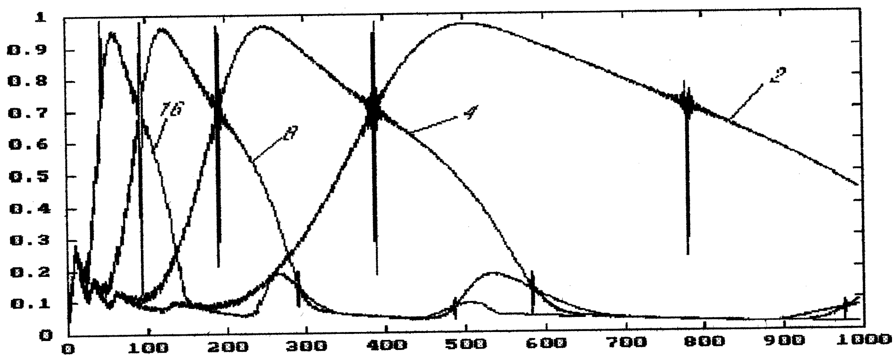


Figure 11: Gain-frequency characteristics of the LSW filter.

scales are shown in fig. 11. Distinctive peaks appear on it.

6 Conclusion

A comparison of two types of wavelets, namely vanishing momentum wavelets (VMWF) and wavelets based on the lifting scheme (LSF) is accomplished. It demands to elaborate a set of tests as a benchmark for data filtering problems and to develope object-oriented C++ software tools to implement both continuous and discrete versions of VMWF transform as well as LSF wavelets transform.

Results of applying these software to the proposed benchmark tests show that despite of obvious speed advantages of the LSF wavelets, the VMWF wavelets have better accuracy and gain-frequency characteristics. Therefore they could be preferable in problems where more delicate data handling is needed, for example to preserve a fine structure of a processed image.

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Ососков Г.А., Шитов А.Б., Стадник А.В. E11-2001-38
Сравнительный анализ вейвлетов первого и второго поколений

Предложен многосторонний набор тестовых сигналов для сравнения эффективности и возможностей непрерывного вейвлет-преобразования, основанного на вейвлетах с нулевыми моментами и вейвлетов второго поколения, построенных на основе лифтинг-схемы. Тесты заключаются в обработке различных типов неискаженных и зашумленных сигналов, дельта-функции, изучении зависимости фазы сигнала и амплитудно-частотных характеристик. Результаты сравнения позволяют определить преимущества и недостатки рассмотренных типов вейвлетов.

Работа выполнена в Лаборатории информационных технологий ОИЯИ.

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The investigation has been performed at the Laboratory of Information Technologies, JINR.

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