Region Counting Graphs

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Abstract

A new family of proximity graphs, called *region counting graphs (RCG)* is presented. The RCG for a finite set of points in the plane uses the notion of region counting distance introduced by Demaine et al. to characterize the proximity between two points p and q: the edge pq is in the RCG if and only if there is less than or exactly k vertices in a given geometric neighborhood defined by a region. These graphs generalize many common proximity graphs, such as k-nearest neighbor graphs, β-skeletons or Θ-graphs. This paper concentrates on RCGs that are invariant under translations, rotations and uniform scaling. For $k = 0$, we give conditions on regions R that define an RCG to ensure a number of properties including planarity, connectivity, triangle freeness, cycle freeness, bipartiteness, and bounded degree. These conditions take form of what we call *tight regions*: maximal or minimal regions that a region R must contain or be contained in to satisfy a given monotone property.

1 Introduction

We consider here *proximity graphs* [12], also called *neighborhood graphs*. Those graphs are defined on a finite set V of vertices in the plane and there exists an edge between any two vertices if they are "close" in some sense. The proximity can be measured for instance by the Euclidean distance between those vertices, the distance to other vertices of the graph, or the number of other vertices in a given neighborhood. Those graphs are well-studied and have numerous applications in computer graphics and classification; a survey of Jaromczyk and Toussaint [7] discusses many of them, such as Relative Neighborhood Graphs [7, 11], Gabriel Graphs [5, 10], β-skeletons [9], Rectangular Influence graph [6], and Θ-graphs [8, 14].

Previous work on proximity graphs traditionally consisted in the introduction of one or more graphs, followed by different contributions analyzing their properties. Surprisingly, the natural opposite approach does not seem to have been considered: to start from a set of desired graph properties to construct the definition of the proximity graph. For this, we have to define a class of proximity graphs general enough to encompass many useful graphs, but simple enough to be analyzed.

The simplest form of proximity graph is a *distance graph* that connects a point $p \in V$ to every point in V whose distance to p is at most some specified value D. Our class of graphs is a variant on this definition using the discrete *region counting distances* defined by Demaine, Iacono and Langerman [2]. These distance functions are parameterized by the finite point set V and the distance between two points is the count of items of V inside a region surrounding those points. In a *k-region counting graph* or *k-RCG* (respectively ($\leq k$)-RCG), two vertices are adjacent if and only if the region counting distance between them is equal to k (respectively at most k).

One of the motivations of our work was to design proximity graphs that are invariant under translations, rotations and uniform scaling. It can be shown that this property is satisfied if and only if the region defining the region counting distance between two points is obtained by translating, rotating and uniformly scaling a template region. In this paper, we concentrate on the case $k = 0$, where two vertices are adjacent if the region does not contain any other point of the set. We further focus on symmetric and convex regions and symmetric distances (undirected graphs).

The properties of those graphs are determined by the choice of the template region. More specifically, we say that a given template region satisfies a property when for all point sets V the graph generated using that region satisfies the property.

Graph properties that are monotone with respect to either edge removal or addition are good candidates for investigation, because monotone properties that are satisfied by a template region are satisfied by all template regions included in it in the case of edge addition, or containing it in the case of edge removal. This naturally raises the issue of template regions that are extremal with respect to the inclusion partial order. We show for instance that the lune, defined as the intersection of two disks of radius $||pq||$ and respective centers p and q, is the unique maximal region ensuring the connectivity of the graph. We call these extremal regions *tight regions*. However, because the inclusion relation is not a total order, tight regions need not be unique. Tight regions can somehow be seen as a deterministic geometric analogue to thresholds for monotone properties studied in random graph theory [4]. Table 1 summarizes our findings related to tight regions and their uniqueness for various graph properties.

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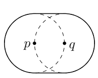
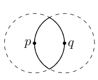
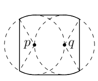
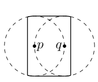
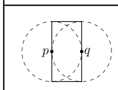
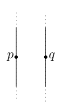
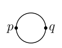
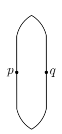
Region	Name	Property
$\mathbb{R}^2$	Plane	no edge $\star$ (unless $\|V\| < 3$)
	Mastercard	no chain $\star$, no cycle (Thm 10)
	Lune	no 3-cycle $\star$ (Thm 9), connected $\star$ (Thm 12)
	Pacman $P_{4\pi/3}$	no 3-star $\star$ (Thm 8: $P_{4\pi/k}$ for no k-star $\star$)
	Pacman $P_{6\pi/5}$	no 5-cycle (Thm 9)
	Pacman P_{π}	no 4-star $\star$ (Thm 8), no 4-cycle $\star$ (Thm 9)
	Slab	no cycle (Thm 10), bipartite (Thm 13)
	Ball	planar (Thm 11)
	Truncated Slab	no 5-cycle (Thm 9)

Table 1: Regions which are tight for various properties. Unique tight regions are marked with a $\star$.

In Section 2, we define the k- and $(\leq k)$-RCG and prove several facts, including how to combine tight regions for conjunction of properties. Section 3 is about geometric properties, which depend on the position of the vertices. We consider the planarity of the embedding and prove that no region counting graph invariant under translation, rotation and uniform scaling can guarantee a constant spanning ratio. This is interesting in light of known bounds on the spanning ratio of Θ-graphs [8], which are not rotation invariant. In section 4 we study the property of not having a given graph as subgraph, and how sets of tight regions can be constructed for forbidden combinations of graphs. Then we specifically consider the properties of not having a k-star or a k-cycle as subgraph. Finally, section 5 presents tightness results for planarity, cycle-freeness, connectivity and bipartiteness.

2 Region Counting Graphs

Definition 1 *An* influence region R *is a function mapping a pair* (p, q) *of points in* $\mathbb{R}^2$ *to a subset of* $\mathbb{R}^2$ *such that inclusion in* $R(p, q)$ *can be computed in* $O(1)$ *time.*

Definition 2 *An* anchored region R *is an influence region parameterized by a triple* (a, b, D)*, where* a *and* b *are points in* $\mathbb{R}^2$ *and* D *is a subset of* $\mathbb{R}^2$ *such that inclusion in* D *can be computed in* $O(1)$ *time. The set* $R(p, q)$ *is the subset of* $\mathbb{R}^2$ *obtained by translating, rotating and uniformly scaling* D *so that* a *maps to* p *and* b *maps to* q*.*

Definition 3 *A* region counting distance *[2]* $\mathrm{d}_R = \mathrm{d}_R^S(p, q)$ *parameterized by a finite point set* $S \subseteq \mathbb{R}^2$ *and an influence region* R*, is defined by* $\mathrm{d}_R(p, q) = |(S \setminus \{p, q\}) \cap R(p, q)|$*.*

Definition 4 *A* symmetric region counting distance *is a region counting distance satisfying* $\mathrm{d}_R(p, q) = \mathrm{d}_R(q, p)$*.*

For the region counting distances using an anchored region as influence region, the symmetry of the region counting distance implies that the region $R(p, q)$ is symmetric with respect to the center of the line segment pq.

Definition 5 *A* k-region counting graph $RCG_R^k(V) = (V, E)$ *(respectively* $(\leq k)$-region counting graph $RCG_R^{\leq k}(V)$*) parameterized by an influence region* R *and an integer* k *is a graph where* V *is a finite subset of* $\mathbb{R}^2$ *and*

$$\forall p, q \in V : pq \in E \Leftrightarrow \mathrm{d}_R(p, q) = k \text{ (respectively } \leq k).$$

If the region counting distance is not symmetric, then the graph is defined as a directed graph, and as an undirected graph otherwise.

We denote by $RCG_R(V)$ the 0-RCG using the influence region R, which is the region counting graph where the edge pq exists if no other point is included in the region $R(p, q)$. Many previously known proximity graphs such as nearest neighbor graphs [3], β-skeletons [9] and Θ-graphs [8] can be defined as 0-RCG.

2.1 Assumptions

In what follows, we are mainly concerned with 0-RCG, and refer to them as region counting graphs or simply RCG. We restrict ourselves to using anchored regions parameterized by triples (a, b, D) where D is closed, convex and symmetric with respect to segment ab. Using only anchored regions is necessary and sufficient to guarantee the invariance of the graph structure under translation, rotation and uniform scaling of the set of points. We further restrict ourselves to symmetric region counting distances, hence undirected graphs.

The regions presented in Table 1 are of particular interest. The pacman $P_\Theta(p, q)$ is bounded by the convex hull of two pie-wedges of angle Θ, with apex in p and in q facing each other, such that $P_0(p, q)$ is the segment pq. The lune $L(p, q)$ is defined as $P_{2\pi/3}(p, q)$, while the mastercard $M(p, q)$ is $P_{2\pi}(p, q)$. The slab $S(p, q)$ is the infinite strip perpendicular to the line segment pq.

2.2 Properties

Lemma 1 $\forall k, G = RCG_R^{\leq k}(V), G' = RCG_{R'}^{\leq k}(V) : R(p,q) \subseteq R'(p,q) \Rightarrow G' \subseteq G$.

Definition 6 *A* graph property $\mathcal{P}$ *on a family of graphs* $\mathcal{G}$ *is a subset* $\mathcal{P} \subseteq \mathcal{G}$. *A* graph G has property $\mathcal{P}$ *if* $G \in \mathcal{P}$.

Definition 7 *A graph property* $\mathcal{P}$ *is* monotone with respect to edge addition (respectively to edge removal) *if and only if* $\forall G = (V, E), G' = (V, E'), E \subseteq E'$ *(respectively* $E \supseteq E'$*):* $G \in \mathcal{P} \Rightarrow G' \in \mathcal{P}$.

Our definition of monotonicity is slightly different from the one commonly used in graph theory. Usually, this is stated as follows: a property is monotone if and only if it is closed upon taking subgraphs. We add the symmetrical definition with properties monotone upon taking supergraphs.

Definition 8 *An anchored region* R satisfies *a graph property* $\mathcal{P}$ *if and only if for all* $V \in \mathbb{R}^2$ *finite,* $RCG_R(V) \in \mathcal{P}$.

Definition 9 *An anchored region* R is tight *for a graph property* $\mathcal{P}$ *monotone with respect to edge addition (respectively to edge removal) if and only if* R *satisfies* $\mathcal{P}$ *and for all anchored region* $R' \supset R$ *(respectively* $R' \subset R$*),* R' *does not satisfy* $\mathcal{P}$.

Note that the monotonicity of the property with respect to edge removal implies that any region containing a tight region as subset satisfies the property as well. On the other hand, for regions that are strictly contained in a tight region, one can always find a set of points generating a graph that does not have the property. A similar observation holds the other way around for properties that are monotone with respect to edge addition. Another definition of a tight region for a monotone property is a region that satisfies the property and is extremal for the inclusion partial order.

Knowing tight regions for useful properties is important in practice, because it allows to check quickly if the properties we wish to obtain are satisfied or not. The uniqueness of a tight region is even more important, because knowing a single tight region R does not, in general, give any information on the properties guaranteed by regions that simultaneously do not contain R and are not contained in R. If the tight region R is unique, any region that does not contain R does not satisfy the property, even if it is not strictly included in R.

Lemma 2 *Let* $\mathcal{P}$ *be a monotone property with respect to edge removal and* R *be the unique tight region satisfying that property. Every region* $R' \not\supseteq R$ *does not satisfy* $\mathcal{P}$.

The same lemma holds for edge addition, where every region $R' \not\subseteq R$ does not satisfy $\mathcal{P}$.

Now given a set of compatible properties we wish to have on the graph, we can easily construct a region guaranteeing these properties. In the case of convex and symmetric regions and properties that are monotone with respect to edge removal, this is achieved by taking the convex hull of the union of the tight regions for each property. In some cases, this region can be proved to be extremal with respect to the inclusion ordering among the considered type of regions.

Lemma 3 *Let* $\mathcal{P}$ *and* $\mathcal{P}'$ *be two monotone properties with respect to edge removal, and* R *and* R' *two regions satisfying* $\mathcal{P}$ *and* $\mathcal{P}'$ *respectively. Then* $R \cup R'$ *satisfies* $\mathcal{P} \cap \mathcal{P}'$. *Furthermore, if* R *and* R' *are the unique tight regions for* $\mathcal{P}$ *and* $\mathcal{P}'$*, then the convex hull of* $R \cup R'$ *is tight and unique for* $\mathcal{P} \cap \mathcal{P}'$.

A similar Lemma holds for properties that are monotone with respect to edge addition as well.

In the class of all possible properties, we can identify various families. For instance the class of properties corresponding to graphs not containing any of the graphs in a given set as subgraph. Another way used to describe properties is to express them as a set of forbidden minors.

Our study showed that there is not always a unique tight region satisfying a property expressed as a set of forbidden subgraphs, or as a set of forbidden minors. However, we do not know whether every monotone property can be expressed as a *finite* set of symmetric, convex and closed tight regions.

3 Geometric Properties

Here we consider geometric properties, which are properties depending on the position of the vertices.

The following theorem shows that the graph embedding obtained by linking adjacent points by straight line segments is planar if and only if the region contains the ball of diameter pq.

Theorem 4 *The ball* B *is the unique tight region ensuring a planar embedding.*

In the following, $\mathrm{d}_G(u, v)$ is the minimum Euclidean length of a path between u and v.

Definition 10 *A graph* G *in the plane is a* t-spanner *for* $t \in [1, \infty)$*, if and only if* $\forall u, v \in V : \mathrm{d}_G(u,v)/||uv|| \leq t$*, where* t *is called the spanning ratio.*

The spanning ratio is also called the dilation. We say that the spanning ratio is unbounded whenever it cannot be bounded by a constant independent of n, i.e. it can be made arbitrarily large for sufficiently large n.

Theorem 5 *For every convex anchored region R with non-empty interior, there exists a real number $\alpha > 0$ such that we can find a set S of n vertices for any n for which the spanning ratio of $RCG_R(S)$ is $\Omega(n^{\alpha})$.*

The proof uses recent results on the spanning ratio of β-skeletons [1, 13]. When we introduced the region counting distances, one of our motivations was to find an anchored region such that the corresponding graph would be a t-spanner not affected by rotations of the set of points. The well-known Θ-graph, which is not invariant to rotations, exhibits a constant spanning ratio. The theorem above shows that it is not possible to find an anchored region corresponding to a constant spanning ratio other than the segment or the empty region if the anchored region is convex.

4 Forbidden Subgraphs

A property $\mathcal{P}$ defined by a forbidden subgraph F is a set of graphs not having any subgraph isomorphic to F. We denote by $F \subseteq F'$ the fact that F' has a subgraph isomorphic to F. The union $F \cup F'$ of two graphs $F = (V, E)$ and $F' = (V', E')$ with $V \cap V' = \emptyset$ is $(V \cup V', E \cup E')$.

Lemma 6 *If the region R is tight for a forbidden subgraph F and for a forbidden subgraph $F' \supseteq F$, then it is tight for all forbidden subgraph G which satisfies $F \subseteq G \subseteq F'$.*

Theorem 7 *If $\mathcal{R}$ is the set of tight regions for a forbidden subgraph F and if $\mathcal{R}'$ is the set of tight regions for a forbidden subgraph F', then the set of tight regions for the forbidden subgraph $F \cup F'$ is $\{R \in \mathcal{R} \cup \mathcal{R}' | \forall R' \in \mathcal{R} \cup \mathcal{R}' : R \not\supseteq R'\}$.*

We will now study properties which can be explained as forbidden subgraphs.

Theorem 8 *The pacman $P_{4\pi/k}$ is the unique tight region forbidding a k-Star, which is equivalent to bounding the maximum degree by $k - 1$.*

There are many regions corresponding forbidding a k-cycle, depending on the parameter k. The tight regions for 3-cycle and 4-cycle are unique, while there are at least two regions for $k \geq 5$.

Theorem 9
1. *The Lune L is the unique tight region forbidding a 3-cycle.*
2. *The pacman P_{π} is the unique tight region forbidding a 4-cycle.*
3. *There are at least two tight regions forbidding a 5-cycle: the truncated slab and $P_{6\pi/5}(p, q)$.*

We show in the next sections that cycle freeness, which corresponds to forbidding a k-cycle for every k, has also two tight regions, and that the tight regions for the 5-cycle are subregions of those for cycle freeness.

5 Other Properties

Properties discussed in theorems 10 and 11 correspond to forbidding graph minors: cycle freeness corresponds to forbidding a triangle as minor; Kuratowski's theorem says that planarity corresponds to forbidden minors $K_{3,3}$ and K_5.

Theorem 10 *There are at least two tight regions for cycle freeness: the slab S and the mastercard M.*

Theorem 11 *The ball B is a tight region for planarity.*

Theorem 12 *The lune L is the unique tight region for connectivity.*

Theorem 13 *There are at least two tight regions for bipartiteness: the slab S and the mastercard M.*

References

[1] P. Bose, L. Devroye, W. Evans, and D. Kirkpatrick. On the spanning ratio of gabriel graphs and beta-skeleton. *Siam Journal of Discrete Math*, 2004. accepted.

[2] E. D. Demaine, J. Iacono, and S. Langerman. Proximate point searching. In *Proceedings of the 14th Canadian Conference on Computational Geometry (CCCG)*, 2002.

[3] D. Eppstein, M. Paterson, and F. Yao. On nearest-neighbor graphs. *Discrete and Computational Geometry*, 17:263–282, 1997.

[4] E. Friedgut and G. Kalai. Every monotone graph property has a sharp threshold. In *Proceedings of the AMS*, pages 2993–3002, 1996.

[5] K. Gabriel and R. Sokal. A new statistical approach to geographic variation analysis. *Systematic Zoology*, 18:259–278, 1969.

[6] M. Ichino and J. Sklansky. The relative neighborhood graph for mixed feature variables. *Pattern Recognition*, 18:161–167, 1985.

[7] J. Jaromczyk and G. Toussaint. Relative neighborhood graphs and their relatives. *Proceedings of the IEEE*, 80(9):1502–1571, 1992.

[8] J. Keil and C. Gutwin. Classes of graphs which approximate the complete euclidean graph. *Discrete and Computational Geometry*, 7(1):13–28, 1992.

[9] D. Kirkpatrick and J. Radke. A framework for computational morphology. *Computational Geometry*, pages 217–248, 1985.

[10] T. Su and R. Chang. The k-gabriel graphs and their applications. In *Proceedings of the International Symposium SIGAL'90*, pages 66–75, 1990.

[11] T. Su and R. Chang. Computing the k-relative neighborhood graphs in euclidean plane. *Pattern Recognition*, 24:231–239, 1991.

[12] G. Toussaint. Some unsolved problems on proximity graphs. In *Proceedings of the First Workshop on Proximity Graphs*, 1991.

[13] W. Wang, X. Li, K. Moaveninejad, Y. Wang, and W. Song. The spanning ratio of beta-skeletons. In *Proceedings of the Canadian Conference on Computational Geometry (CCCG)*, 2003.

[14] A. Yao. On constructing minimum spanning trees in k-dimensional spaces and related problems. *SIAM Journal on Computing*, 11(4):721–736, 1982.