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Wavelet bases of Hermite cubic splines on the interval ⋆

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In this paper a pair of wavelets are constructed on the basis of Hermite cubic splines. These wavelets are in C^1 and supported on $[-1, 1]$. Moreover, one wavelet is symmetric, and the other is antisymmetric. These spline wavelets are then adapted to the interval $[0, 1]$. The construction of boundary wavelets is remarkably simple. Furthermore, global stability of the wavelet basis is established. The wavelet basis is used to solve the Sturm–Liouville equation with the Dirichlet boundary condition. Numerical examples are provided. The computational results demonstrate the advantage of the wavelet basis.

Keywords: wavelets on the interval, Hermite cubic splines, numerical solutions of differential equations.

Mathematics subject classifications (2000): 42C40, 41A15, 65L60.

1. Introduction

In this paper we shall construct wavelet bases of Hermite cubic splines on the interval. These wavelet bases are suitable for numerical solutions of differential equations.

By $L_2(\mathbb{R})$ we denote the linear space of all square-integrable real-valued functions on $\mathbb{R}$. The inner product in $L_2(\mathbb{R})$ is defined as

$$\langle u, v\rangle := \int_{\mathbb{R}} u(x)v(x)\,\mathrm{d}x, \quad u, v \in L_2(\mathbb{R}).$$

If $\langle u, v\rangle = 0$, then we say that u and v are orthogonal. The norm of a function f in $L_2(\mathbb{R})$ is given by $\|f\|_2 := \sqrt{\langle f, f\rangle}$.

Smooth orthogonal wavelets with compact support were constructed by Daubechies (see [9]). The Daubechies orthogonal wavelets were adapted to the interval [0, 1] by Cohen et al. [7]. Semi-orthogonal spline wavelets were constructed by Chui and Wang [5].

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These spline wavelets were adapted to the interval [0, 1] by Chui and Quak [4]. In [14] Wang constructed cubic spline wavelet bases for Sobolev spaces.

Orthogonal multi-wavelets were constructed by Donovan et al. [10]. In [11], Heil et al. considered the possibility of construction of wavelets on the basis of Hermite cubic splines.

Let ϕ_1 and ϕ_2 be the cubic splines given by

$$\phi_1(x) := \begin{cases} (x+1)^2(1-2x) & \text{for } x \in [-1, 0], \\ (1-x)^2(2x+1) & \text{for } x \in [0, 1], \\ 0 & \text{for } x \in \mathbb{R} \setminus [-1, 1], \end{cases}$$

and

$$\phi_2(x) := \begin{cases} x(x+1)^2 & \text{for } x \in [-1, 0], \\ x(x-1)^2 & \text{for } x \in [0, 1], \\ 0 & \text{for } x \in \mathbb{R} \setminus [-1, 1]. \end{cases}$$

In [8], Dahmen et al. constructed biorthogonal multi-wavelets on the basis of the Hermite cubic splines ϕ_1 and ϕ_2. These wavelets were adapted to the interval [0, 1]. However, their construction for the wavelet basis on the interval [0, 1] was quite complicated.

In this paper we take a new approach to the construction of wavelet bases of Hermite cubic splines. In contrast to the semi-orthogonal wavelets of Chui and Wang, the wavelets at different levels are orthogonal with respect to the inner product $\langle u', v'\rangle$, rather than $\langle u, v\rangle$. This requirement of orthogonality is more pertinent to applications of wavelets to numerical solutions of differential equations.

In section 2 we will give two wavelets ψ_1 and ψ_2 as follows:

$$\begin{aligned} \psi_1(x) &= -2\phi_1(2x+1) + 4\phi_1(2x) - 2\phi_1(2x-1) \\ &\quad - 21\phi_2(2x+1) + 21\phi_2(2x-1), \\ \psi_2(x) &= \phi_1(2x+1) - \phi_1(2x-1) + 9\phi_2(2x+1) + 12\phi_2(2x) + 9\phi_2(2x-1). \end{aligned}$$

Clearly, ψ_1 and ψ_2 are supported on $[-1, 1]$; ψ_1 is symmetric and ψ_2 is antisymmetric. Moreover,

$$\langle \psi_1', \phi_m'(\cdot - j)\rangle = \langle \psi_2', \phi_m'(\cdot - j)\rangle = 0, \quad m = 1, 2, \ \forall j \in \mathbb{Z}.$$

These wavelets can be easily adapted to the interval [0, 1].

By $L_2(0, 1)$ we denote the space of all square-integrable real-valued functions on $(0, 1)$. The inner product in $L_2(0, 1)$ is defined as

$$\langle u, v\rangle := \int_0^1 u(x)v(x)\,\mathrm{d}x, \quad u, v \in L_2(0, 1).$$

Let $H^1(0, 1)$ be the space of all functions u in $L_2(0, 1)$ for which (the distributional derivative) $u' \in L_2(0, 1)$. Let $H_0^1(0, 1)$ be the closure of the set

$$\{u \in C[0, 1] \cap C^1(0, 1):\ u(0) = u(1) = 0\}$$

in the space $H^1(0, 1)$, where $C[0, 1]$ denotes the space of all continuous functions on $[0, 1]$, and $C^1(0, 1)$ denotes the space of those continuous functions on $(0, 1)$ whose derivatives are also continuous.

For a nonnegative integer k, we denote by Π_k the set of all polynomials of degree at most k. For $n \geqslant 1$, let V_n be the space of those cubic splines $v \in C^1(0, 1) \cap C[0, 1]$ for which $v(0) = v(1) = 0$ and

$$v|_{(j/2^n,(j+1)/2^n)} \in \Pi_3|_{(j/2^n,(j+1)/2^n)} \quad \text{for } j = 0, \dots, 2^n - 1.$$

The dimension of V_n is 2^{n+1}. It is easily seen that the set

$$\Phi_n := \{\phi_1(2^n \cdot -j)\colon\ j = 1, \dots, 2^n - 1\} \cup \{\phi_2(2^n \cdot -j)|_{(0,1)}\colon\ j = 0, \dots, 2^n\} \quad (1.1)$$

is a basis for V_n. We label the elements in Φ_n as $\{v_1, v_2, \dots, v_{2^{n+1}}\}$.

Let Ψ_n be the set of wavelets given by

$$\Psi_n := \{\psi_1(2^n \cdot -j)\colon\ j = 1, \dots, 2^n - 1\} \cup \{\psi_2(2^n \cdot -j)|_{(0,1)}\colon\ j = 0, \dots, 2^n\}. \quad (1.2)$$

Let W_n be the linear span of Ψ_n. It is easily seen that Ψ_n is a basis for W_n. Consequently, the dimension of W_n is 2^{n+1}. In section 3 we shall show that

$$\int_0^1 w'(x)v'(x)\,\mathrm{d}x = 0 \quad \forall w \in \Psi_n \text{ and } v \in \Phi_n.$$

It follows that $V_n \cap W_n = \{0\}$. Moreover, we have $V_{n+1} \supseteq V_n + W_n$ and

$$\dim(V_{n+1}) = \dim(V_n) + \dim(W_n).$$

This shows that V_{n+1} is the direct sum of V_n and W_n. Therefore, we have the following decomposition of $H_0^1(0, 1)$:

$$H_0^1(0, 1) = V_1 + W_1 + W_2 + \cdots.$$

Recall that $\Phi_1 = \{v_1, v_2, v_3, v_4\}$. For $n = 1, 2, \dots$, we label the elements in Ψ_n as follows:

$$\Psi_n = \{w_{2^{n+1}+1}, \dots, w_{2^{n+2}}\}.$$

Let $g_k := v_k/\|v_k'\|_2$ for $k = 1, 2, 3, 4$ and $g_k := w_k/\|w_k'\|_2$ for $k > 4$. Thus, $\|g_k'\|_2 = 1$ for $k = 1, 2, \dots$. In section 3 we will show that $(g_k')_{k=1,2\dots}$ is a Riesz sequence in $L_2(0, 1)$.

In section 4 we shall apply the wavelets constructed in section 3 to numerical solutions of the Sturm–Liouville equation of the form

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left(p(x)\frac{\mathrm{d}u}{\mathrm{d}x}\right) + q(x)u(x) = f(x), \quad x \in (0, 1), \quad (1.3)$$

with the Dirichlet boundary condition $u(0) = u(1) = 0$. We assume that p and q are continuous functions on $[0, 1]$ and $p(x) > 0$, $q(x) \geqslant 0$ for all $x \in [0, 1]$. Let

$$a(u, v) := \int_0^1 p(x)u'(x)v'(x)\,\mathrm{d}x + \int_0^1 q(x)u(x)v(x)\,\mathrm{d}x, \quad u, v \in H_0^1(0, 1). \quad (1.4)$$

Then the variational form of the above equation with the Dirichlet boundary condition is

$$a(u, v) = \langle f, v\rangle \quad \forall v \in H_0^1(0, 1).$$

Wavelets have been used to discretize differential equations. In particular, Xu and Shann [15] successfully applied the wavelet method to numerical solutions of the Sturm–Liouville equation (1.3). The wavelet bases in their paper are anti-derivatives of the Daubechies orthogonal wavelets. Consequently, their basis functions are not locally supportd and, in general, the corresponding stiffness matrix is full (not sparse). Furthermore, the condition number of the stiffness matrix is not uniformly bounded.

In application of the wavelet method one often encounters the difficuly that the boundary conditions are hard to impose on wavelets. In our construction, only two wavelets in Ψ_n, $\psi_2(2^n\cdot)$ and $\psi_2(2^n\cdot -2^n)$, needed to be adapted to the interval (0, 1) by means of restriction. This is in sharp contrast to the complexity of the construction of boundary wavelets given in [8].

Recall that $\{g_k : k = 1, \ldots, 2^{n+1}\}$ is a wavelet basis for V_n. Let A_n denote the stiffness matrix $(a(g_j, g_k))_{j,k=1,\ldots,2^{n+1}}$. In section 4 we will prove that the condition number of A_n is uniformly bounded (independent of n). In particular, for the case $p = 1$ and $q = 1$, numerical computation suggests that the condition number of A_n be less than 3.75 for all n. By comparison, the condition number of the stiffness matrix with respect to the wavelet basis constructed in [8] is very large.

At the end of this paper, we shall provide two numerical examples using the above wavelet basis. The computational results demonstrate the advantage of our wavelet basis.

2. Spline wavelets

In this section we construct wavelets on the basis of Hermite cubic splines.

Let ϕ_1 and ϕ_2 be the cubic splines given in section 1. The graphs of ϕ_1 and ϕ_2 are depicted in figure 1. Clearly, both ϕ_1 and ϕ_2 belong to $C^1(\mathbb{R})$. Moreover, we have

$$\phi_1(0) = 1, \qquad \phi_1'(0) = 0, \qquad \phi_2(0) = 0, \qquad \phi_2'(0) = 1.$$

Hence, for a function $f \in C^1(\mathbb{R})$,

$$u = \sum_{j\in\mathbb{Z}} f(j)\phi_1(\cdot - j) + \sum_{j\in\mathbb{Z}} f'(j)\phi_2(\cdot - j)$$

is a Hermite interpolant to f on $\mathbb{Z}$, that is, $u(j) = f(j)$ and $u'(j) = f'(j)$ for all $j \in \mathbb{Z}$.

Let $\Phi := (\phi_1, \phi_2)^{\mathrm{T}}$, the transpose of the 1×2 vector (ϕ_1, ϕ_2). Then Φ satisfies the following vector refinement equation (see [11]):

$$\Phi(x) = \sum_{j=-1}^{1} a(j)\Phi(2x - j), \quad x \in \mathbb{R},$$

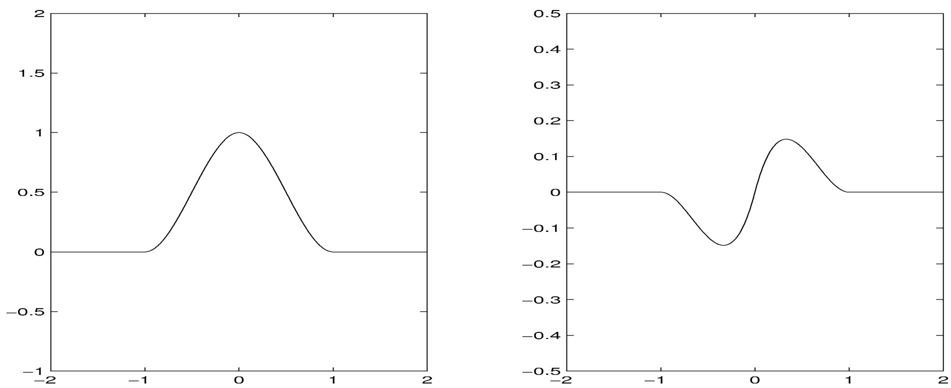

Figure 1. Hermit cubic splines on $\mathbb{R}$.

where

$$a(-1) = \begin{bmatrix} \frac{1}{2} & \frac{3}{4} \\ -\frac{1}{8} & -\frac{1}{8} \end{bmatrix}, \qquad a(0) = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}, \quad \text{and} \quad a(1) = \begin{bmatrix} \frac{1}{2} & -\frac{3}{4} \\ \frac{1}{8} & -\frac{1}{8} \end{bmatrix}.$$

Let S be the shift-invariant space generated by ϕ_1 and ϕ_2. A function g belongs to S if and only if there are two sequences b_1 and b_2 on $\mathbb{Z}$ such that

$$g = \sum_{j\in\mathbb{Z}} \bigl[b_1(j)\phi_1(\cdot - j) + b_2(j)\phi_2(\cdot - j)\bigr].$$

Let $S_1 := \{g(2\cdot)\colon\ g \in S\}$. Then $S \subset S_1$, since Φ is refinable. We look for a wavelet space W such that S_1 is the direct sum of S and W. We wish to find two wavelets ψ_1 and ψ_2 such that their shifts generate W. Moreover, we require

$$\bigl\langle \psi_1', \phi_m'(\cdot - j)\bigr\rangle = \bigl\langle \psi_2', \phi_m'(\cdot - j)\bigr\rangle = 0, \quad m = 1, 2,\ \forall j \in \mathbb{Z}. \tag{2.1}$$

For this purpose we need to calculate the inner product of the derivatives of shifts of ϕ_1 and ϕ_2. Note that

$$\phi_1'(x) := \begin{cases} -6x^2 - 6x & \text{for } x \in [-1, 0], \\ 6x^2 - 6x & \text{for } x \in [0, 1], \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\phi_2'(x) := \begin{cases} 3x^2 + 4x + 1 & \text{for } x \in [-1, 0], \\ 3x^2 - 4x + 1 & \text{for } x \in [0, 1], \\ 0 & \text{otherwise.} \end{cases}$$

Suppose

$$\psi(x) = \sum_{k\in\mathbb{Z}} \bigl[b_1(k)\phi_1(2x - k) + b_2(k)\phi_2(2x - k)\bigr], \quad x \in \mathbb{R}.$$

Then for $j \in \mathbb{Z}$ we have

$$\begin{aligned}\langle \psi', \phi_1'(\cdot - j)\rangle = \frac{1}{20}\bigl[&-21b_1(2j-2) + 42b_1(2j) - 21b_1(2j+2)\\ &- 3b_2(2j-2) + 4b_2(2j-1) - 4b_2(2j+1) + 3b_2(2j+2)\bigr]\end{aligned}$$

and

$$\begin{aligned}\langle \psi', \phi_2'(\cdot - j)\rangle = \frac{1}{120}\bigl[&33b_1(2j-2) - 60b_1(2j-1) + 60b_1(2j+1)\\ &- 33b_1(2j+2) + 4b_2(2j-2) - 12b_2(2j-1)\\ &+ 28b_2(2j) - 12b_2(2j+1) + 4b_2(2j+2)\bigr].\end{aligned}$$

For $z \in \mathbb{C} \setminus \{0\}$, let

$$\begin{aligned} q_{11}(z) &:= \sum_{j\in\mathbb{Z}} b_1(2j+1)z^{2j+1}, & q_{12}(z) &:= \sum_{j\in\mathbb{Z}} b_1(2j)z^{2j},\\ q_{21}(z) &:= \sum_{j\in\mathbb{Z}} b_2(2j+1)z^{2j+1}, & q_{22}(z) &:= \sum_{j\in\mathbb{Z}} b_2(2j)z^{2j}.\end{aligned}$$

Then $\langle \psi', \phi_m'(\cdot - j)\rangle = 0$ for $m = 1, 2$ and all $j \in \mathbb{Z}$ if and only if

$$B(z)\bigl(q_{11}(z), q_{12}(z), q_{21}(z), q_{22}(z)\bigr)^{\mathrm{T}} = 0 \quad \forall z \in \mathbb{C} \setminus \{0\},$$

where

$$B(z) := \begin{bmatrix} 0 & -21z^2 + 42 - 21z^{-2} & 4z - 4z^{-1} & -3z^2 + 3z^{-2} \\ -60z + 60z^{-1} & 33z^2 - 33z^{-2} & -12z - 12z^{-1} & 4z^2 + 28 + 4z^{-2} \end{bmatrix}.$$

We find two independent solutions as follows:

$$\begin{bmatrix} q_{11}(z) \\ q_{12}(z) \\ q_{21}(z) \\ q_{22}(z) \end{bmatrix} = \begin{bmatrix} -2z^{-1} - 2z \\ 4 \\ -21z^{-1} + 21z \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} q_{11}(z) \\ q_{12}(z) \\ q_{21}(z) \\ q_{22}(z) \end{bmatrix} = \begin{bmatrix} z^{-1} - z \\ 0 \\ 9z^{-1} + 9z \\ 12 \end{bmatrix}.$$

These two solutions induce two wavelets ψ_1 and ψ_2 given by

$$\begin{aligned}\psi_1(x) &= -2\phi_1(2x+1) + 4\phi_1(2x) - 2\phi_1(2x-1) - 21\phi_2(2x+1) + 21\phi_2(2x-1),\\ \psi_2(x) &= \phi_1(2x+1) - \phi_1(2x-1) + 9\phi_2(2x+1) + 12\phi_2(2x) + 9\phi_2(2x-1).\end{aligned}$$

By our construction, ψ_1 and ψ_2 are supported on $[-1, 1]$, they satisfy the conditions in (2.1), and their shifts generate the wavelet space W such that S_1 is the direct sum of S and W. Moreover, ψ_1 is symmetric and ψ_2 is antisymmetric (see figure 2).

Let us take a look at ψ_1' and ψ_2'. For $0 \leqslant x \leqslant 1/2$ we have

$$\begin{aligned} \psi_1'(x) &= 792x^2 - 312x, & \psi_1'(x-1) &= -408x^2 + 120x,\\ \psi_2'(x) &= 552x^2 - 288x + 24, & \psi_2'(x-1) &= 168x^2 - 48x.\end{aligned}$$

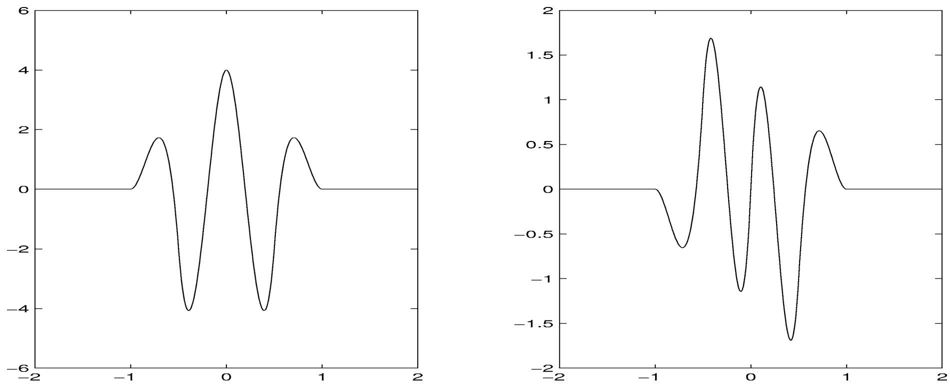

Figure 2. Wavelets ψ_1 and ψ_2.

For $1/2 \leqslant x \leqslant 1$ we have

$$\psi_1'(x) = 408x^2 - 696x + 288, \qquad \psi_1'(x-1) = -792x^2 + 1272x - 480,$$
$$\psi_2'(x) = 168x^2 - 288x + 120, \qquad \psi_2'(x-1) = 552x^2 - 816x + 288.$$

Hence, the shifts of ψ_1' and ψ_2' are linearly independent on the interval (0, 1). Because of shift invariance, the shifts of ψ_1' and ψ_2' are linear independent on the interval $(k, k+1)$ for every $k \in \mathbb{Z}$. Suppose b_1 and b_2 are two square summable sequences on $\mathbb{Z}$. Let

$$u := \sum_{j\in\mathbb{Z}} \bigl[b_1(j)\psi_1'(\cdot - j) + b_2(j)\psi_2'(\cdot - j)\bigr].$$

For $j < k$ or $j > k+1$, $\psi_1'(\cdot - j)$ and $\psi_2'(\cdot - j)$ vanish on $(k, k+1)$. Since the shifts of ψ_1' and ψ_2' are linearly independent on $(k, k+1)$, there exist two positive constants C_1 and C_2 independent of k, b_1, and b_2 such that

$$C_1^2 \sum_{j=k}^{k+1} \bigl[\lvert b_1(j)\rvert^2 + \lvert b_2(j)\rvert^2\bigr] \leqslant \int_k^{k+1} \lvert u(x)\rvert^2 \,\mathrm{d}x \leqslant C_2^2 \sum_{j=k}^{k+1} \bigl[\lvert b_1(j)\rvert^2 + \lvert b_2(j)\rvert^2\bigr].$$

It follows that

$$2C_1^2 \sum_{j\in\mathbb{Z}} \bigl[\lvert b_1(j)\rvert^2 + \lvert b_2(j)\rvert^2\bigr] \leqslant \int_{\mathbb{R}} \lvert u(x)\rvert^2 \,\mathrm{d}x \leqslant 2C_2^2 \sum_{j\in\mathbb{Z}} \bigl[\lvert b_1(j)\rvert^2 + \lvert b_2(j)\rvert^2\bigr].$$

In other words, the shifts of ψ_1' and ψ_2' are stable. See [12] for a study of stability of shifts of several functions.

3. Wavelets on the interval

In this section we use the spline wavelets in the previous section to construct a wavelet basis for the space $H_0^1(0, 1)$.

Recall that V_n is the linear space of those cubic splines $v \in C^1(0,1) \cap C[0,1]$ for which $v(0) = v(1) = 0$ and

$$v|_{(j/2^n,(j+1)/2^n)} \in \Pi_3|_{(j/2^n,(j+1)/2^n)} \quad \text{for } j = 0, \ldots, 2^n - 1.$$

The dimension of V_n is 2^{n+1}. Moreover,

(a) $V_1 \subset V_2 \subset \cdots \subset H_0^1(0,1)$;

(b) $\bigcup_{n=1}^{\infty} V_n$ is dense in $H_0^1(0,1)$.

Let Φ_n and Ψ_n be the sets defined in (1.1) and (1.2), respectively. Then Φ_n is a basis for V_n. Let W_n be the linear span of Ψ_n. Clearly, Ψ_n is a basis for W_n. Consequently, the dimension of W_n is 2^{n+1}.

We claim that

$$\int_0^1 w'(x)v'(x)\,\mathrm{d}x = 0 \quad \forall w \in \Psi_n \text{ and } v \in \Phi_n. \tag{3.1}$$

Suppose $w = \psi_r(2^n \cdot - j)$ for some $r \in \{1,2\}$ and $j \in \{1, \ldots, 2^n - 1\}$. Then $\psi_r'(2^n \cdot - j)$ is supported in the interval $[0,1]$. Hence, for $s = 1, 2$ and $k \in \mathbb{Z}$, we have

$$\int_0^1 \psi_r'\big(2^n x - j\big)\phi_s'\big(2^n x - k\big)\,\mathrm{d}x = \int_{\mathbb{R}} \psi_r'\big(2^n x - j\big)\phi_s'\big(2^n x - k\big)\,\mathrm{d}x = 0,$$

where (2.1) has been used to derive the second equality. For the same reason, (3.1) is valid if $v = \phi_s(2^n \cdot - k)$ for some $s \in \{1,2\}$ and $k \in \{1, \ldots, 2^n - 1\}$. Thus, in order to complete the proof of (3.1), it remains to deal with the case $w = \psi_2(2^n \cdot - j)|_{(0,1)}$ and $v = \phi_2(2^n \cdot - k)|_{(0,1)}$ for $j, k \in \{0, 2^n\}$. We have $v'(x)w'(x) = 0$ for $x \in (0,1)$ if $j = 0$ and $k = 2^n$, or if $j = 2^n$ and $k = 0$. Hence (3.1) is valid in this case. Suppose $j = k = 0$. Since ψ_2 and ϕ_2 are antisymmetric, ψ_2' and ϕ_2' are symmetric. Hence, $\psi_2'\phi_2'$ is symmetric. It follows that

$$\int_{-1}^0 \psi_2'(x)\phi_2'(x)\,\mathrm{d}x = \int_0^1 \psi_2'(x)\phi_2'(x)\,\mathrm{d}x.$$

But (2.1) tells us that

$$\int_{-1}^1 \psi_2'(x)\phi_2'(x)\,\mathrm{d}x = 0.$$

Therefore,

$$\int_0^1 \psi_2'(x)\phi_2'(x)\,\mathrm{d}x = 0.$$

Consequently,

$$\int_0^1 \psi_2'\big(2^n x\big)\phi_2'\big(2^n x\big)\,\mathrm{d}x = 2^{-n}\int_0^{2^n} \psi_2'(x)\phi_2'(x)\,\mathrm{d}x = 0.$$

This verifies (3.1) for $w = \psi_2(2^n\cdot)|_{(0,1)}$ and $v = \phi_2(2^n\cdot)|_{(0,1)}$. An analogous argument shows that (3.1) is valid for $w = \psi_2(2^n\cdot - 2^n)|_{(0,1)}$ and $v = \phi_2(2^n\cdot - 2^n)|_{(0,1)}$. The proof of (3.1) is complete.

It follows from (3.1) that

$$\int_0^1 w'(x)v'(x)\,\mathrm{d}x = 0 \quad \forall w \in W_n \text{ and } v \in V_n.$$

In particular, $V_n \cap W_n = \{0\}$. We have $V_{n+1} \supseteq V_n + W_n$ and

$$\dim(V_n + W_n) = \dim(V_n) + \dim(W_n) = 2^{n+1} + 2^{n+1} = \dim(V_{n+1}).$$

This shows that V_{n+1} is the direct sum of V_n and W_n. Consequently,

$$V_{n+1} = V_1 + W_1 + \cdots + W_n.$$

Therefore, we have the following decomposition of $H_0^1(0,1)$:

$$H_0^1(0,1) = V_1 + W_1 + W_2 + \cdots.$$

Suppose $v \in V_1$ and $w_n \in W_n$ for $n = 1, 2, \ldots$. The preceding discussion tells us that $\langle v', w_n'\rangle = 0$ for all n and $\langle w_m', w_n'\rangle = 0$ for $m \neq n$. Hence,

$$\left\| v' + \sum_{n=1}^{\infty} w_n' \right\|_{L_2(0,1)}^2 = \|v'\|_{L_2(0,1)}^2 + \sum_{n=1}^{\infty} \|w_n'\|_{L_2(0,1)}^2. \tag{3.2}$$

For $n = 1, 2, \ldots$ and $x \in (0,1)$, let

$$\psi_{n,j}(x) := \frac{2^{-n/2}}{\sqrt{729.6}}\psi_1\left(2^n x - \frac{j}{2}\right) \quad \text{for } j = 2, 4, \ldots, 2^{n+1} - 2,$$

$$\psi_{n,j}(x) := \frac{2^{-n/2}}{\sqrt{153.6}}\psi_2\left(2^n x - \frac{j-1}{2}\right) \quad \text{for } j = 3, 5, \ldots, 2^{n+1} - 1,$$

and

$$\psi_{n,1}(x) := \frac{2^{-n/2}}{\sqrt{76.8}}\psi_2\big(2^n x\big), \qquad \psi_{n,2^{n+1}}(x) := \frac{2^{-n/2}}{\sqrt{76.8}}\psi_2\big(2^n x - 2^n\big).$$

Note that $\psi_{n,j}$ are so normalized that $\|\psi_{n,j}'\|_{L_2(0,1)} = 1$ for $j = 1, \ldots, 2^{n+1}$.

Theorem 1. The sequence $(\psi_{n,j}')_{n=1,2,\ldots,1\leqslant j\leqslant 2^{n+1}}$ is a Riesz sequence in $L_2(0,1)$. In other words, there exist two positive constants A and B such that

$$A\left(\sum_{n=1}^{\infty}\sum_{j=1}^{2^{n+1}} |b_{n,j}|^2\right)^{1/2} \leqslant \left\|\sum_{n=1}^{\infty}\sum_{j=1}^{2^{n+1}} b_{n,j}\psi_{n,j}'\right\|_{L_2(0,1)} \leqslant B\left(\sum_{n=1}^{\infty}\sum_{j=1}^{2^{n+1}} |b_{n,j}|^2\right)^{1/2}$$

for every sequence $(b_{n,j})_{n=1,2,\ldots,1\leqslant j\leqslant 2^{n+1}}$.

Proof. By (3.2) we have

$$\left\| \sum_{n=1}^{\infty} \sum_{j=1}^{2^{n+1}} b_{n,j} \psi'_{n,j} \right\|_{L_2(0,1)}^2 = \sum_{n=1}^{\infty} \left\| \sum_{j=1}^{2^{n+1}} b_{n,j} \psi'_{n,j} \right\|_{L_2(0,1)}^2 .$$

In light of the discussion at the end of section 3, we assert that the shifts of ψ'_1 and ψ'_2 are linearly independent on $(k, k+1)$ for every $k \in \mathbb{Z}$. Hence, there exist two positive constants A and B (independent of n) such that

$$A^2 \sum_{j=1}^{2^{n+1}} |b_{n,j}|^2 \leqslant \left\| \sum_{j=1}^{2^{n+1}} b_{n,j} \psi'_{n,j} \right\|_{L_2(0,1)}^2 \leqslant B^2 \sum_{j=1}^{2^{n+1}} |b_{n,j}|^2 .$$

This completes the proof of the theorem. □

For $x \in (0, 1)$, let

$$\phi_{1,1}(x) := \sqrt{\frac{5}{24}} \phi_1(2x-1), \qquad \phi_{1,2}(x) := \sqrt{\frac{15}{4}} \phi_2(2x),$$

$$\phi_{1,3}(x) := \sqrt{\frac{15}{8}} \phi_2(2x-1), \qquad \phi_{1,4}(x) := \sqrt{\frac{15}{4}} \phi_2(2x-2).$$

Note that each $\phi_{1,j}$ is so normalized that $\|\phi'_{1,j}\|_{L_2(0,1)} = 1$, $j = 1, \ldots, 4$. Clearly, V_1 is spanned by $\phi_{1,j}$, $j = 1, \ldots, 4$. Consequently, $H_0^1(0, 1)$ is spanned by $\phi_{1,j}$, $j = 1, \ldots, 4$, together with $\psi_{n,j}$, $n = 1, 2, \ldots$, $j = 1, \ldots, 2^{n+1}$. We relabel these functions as follows. Let $g_j := \phi_{1,j}$ for $j = 1, \ldots, 4$, and let $g_{2^{n+1}+j} := \psi_{n,j}$ for $n = 1, 2, \ldots$ and $j = 1, \ldots, 2^{n+1}$. With the same reasoning as in the proof of theorem 1, we see that the sequence $(g'_k)_{k=1,2,\ldots}$ is a Riesz sequence in $L_2(0, 1)$. In other words, there exist two positive constants A and B such that

$$A \left(\sum_{k=1}^{\infty} |b_k|^2 \right)^{1/2} \leqslant \left\| \sum_{k=1}^{\infty} b_k g'_k \right\|_{L_2(0,1)} \leqslant B \left(\sum_{k=1}^{\infty} |b_k|^2 \right)^{1/2} \tag{3.3}$$

for every square summable sequence $(b_k)_{k=1,2,\ldots}$.

4. Applications

In this section the wavelets constructed in the previous section are used to solve differential equations. We shall confine ourselves to the Sturm–Liouville equation of the form (1.3) with the Dirichlet boundary condition $u(0) = u(1) = 0$. We assume that p and q are continuous functions on $[0, 1]$ and $p(x) > 0$, $q(x) \geqslant 0$ for all $x \in [0, 1]$.

For $u, v \in H_0^1(0, 1)$, let $a(u, v)$ be the bilinear form given in (1.4). Then the variational form of equation (1.3) with the Dirichlet boundary condition is

$$a(u, v) = \langle f, v \rangle \quad \forall v \in H_0^1(0, 1). \tag{4.1}$$

The corresponding Galerkin approximation problem is the following: find $u_n \in V_n$ such that

$$a(u_n, v) = \langle f, v\rangle \quad \forall v \in V_n. \tag{4.2}$$

By the Lax–Milgram lemma (see, e.g., [2, p. 60]), the approximation problem (4.2) has a unique solution.

We propose to use the wavelet set $G_n := \{g_1, \ldots, g_{2^{n+1}}\}$ as a basis for V_n. Recall that $g_j := \phi_{1,j}$ for $j = 1, \ldots, 4$, and $g_{2^{n+1}+j} := \psi_{n,j}$ for $n = 1, 2, \ldots$ and $j = 1, \ldots, 2^{n+1}$, where $\phi_{1,j}$ ($j = 1, \ldots, 4$) and $\psi_{n,j}$ ($j = 1, \ldots, 2^{n+1}$) are the functions constructed in the previous section. With this basis for V_n, the Galerkin approximation problem (4.2) can be discretized as follows:

$$\sum_{k=1}^{2^{n+1}} a(g_j, g_k)\eta_k = \langle g_j, f\rangle, \quad j = 1, \ldots, 2^{n+1}.$$

The stiffness matrix

$$\big(a(g_j, g_k)\big)_{1\leqslant j,k\leqslant 2^{n+1}}$$

is denoted by A_n. We will prove that the condition number of A_n is uniformly bounded (independent of n). Therefore, the wavelet basis G_n is a good tool for preconditioning.

Let us recall that the condition number of an invertible square matrix A is defined by

$$\operatorname{cond}(A) := \|A\| \, \|A^{-1}\|,$$

where $\|\cdot\|$ is a matrix norm. The spectral condition number of A is defined as

$$\frac{\max_i |\lambda_i(A)|}{\min_i |\lambda_i(A)|},$$

where the numbers $\lambda_i(A)$ are eigenvalues of A. If A is a (real) symmetric matrix, then its condition number with respect to the 2-norm is equal to its spectral condition number (see [6, p. 51]).

Theorem 2. The condition number of the stiffness matrix A_n is uniformly bounded (independent of n).

Proof. It suffices to show that there exist two positive constants B and C independent of n such that

$$B\left(\sum_{j=1}^{4} |c_j|^2 + \sum_{m=1}^{n-1}\sum_{j=1}^{2^{m+1}} |b_{m,j}|^2\right) \leqslant a(u, u) \leqslant C\left(\sum_{j=1}^{4} |c_j|^2 + \sum_{m=1}^{n-1}\sum_{j=1}^{2^{m+1}} |b_{m,j}|^2\right) \tag{4.3}$$

for any

$$u = \sum_{j=1}^{4} c_j \phi_{1,j} + \sum_{m=1}^{n-1} \sum_{j=1}^{2^{m+1}} b_{m,j} \psi_{m,j}.$$

By (3.3) there exists a positive constant C_1 independent of n such that

$$\begin{aligned} \|u'\|_{L_2(0,1)} &= \left\| \sum_{j=1}^{4} c_j \phi'_{1,j} + \sum_{m=1}^{n-1} \sum_{j=1}^{2^{m+1}} b_{m,j} \psi'_{m,j} \right\|_{L_2(0,1)} \\ &\geqslant C_1 \left(\sum_{j=1}^{4} |c_j|^2 + \sum_{m=1}^{n-1} \sum_{j=1}^{2^{m+1}} |b_{m,j}|^2 \right)^{1/2}. \end{aligned}$$

But

$$a(u,u) \geqslant \langle pu', u' \rangle \geqslant \mu \langle u', u' \rangle = \mu \|u'\|^2_{L_2(0,1)},$$

where $\mu := \min_{x \in [0,1]} p(x) > 0$. This establishes the first inequality in (4.3). Furthermore, we observe that

$$a(u,u) \leqslant \nu \big(\|u\|^2_{L_2(0,1)} + \|u'\|^2_{L_2(0,1)} \big),$$

where $\nu := \max_{0 \leqslant x \leqslant 1} \{p(x), q(x)\} < \infty$. By (3.3) there exists a positive constant C_2 independent of n such that

$$\|u'\|_{L_2(0,1)} \leqslant C_2 \left(\sum_{j=1}^{4} |c_j|^2 + \sum_{m=1}^{n-1} \sum_{j=1}^{2^{m+1}} |b_{m,j}|^2 \right)^{1/2}.$$

Moreover,

$$\|u\|_{L_2(0,1)} \leqslant \left\| \sum_{j=1}^{4} c_j \phi_{1,j} \right\|_{L_2(0,1)} + \sum_{m=1}^{n-1} \left\| \sum_{j=1}^{2^{m+1}} b_{m,j} \psi_{m,j} \right\|_{L_2(0,1)}.$$

Note that $\|\psi_{m,j}\|_{L_2(0,1)} = \mathrm{O}(2^{-m})$ as $m \to \infty$. Hence, there exists a positive constant C_3 independent of n such that

$$\|u\|_{L_2(0,1)} \leqslant C_3 \left[\left(\sum_{j=1}^{4} |c_j|^2 \right)^{1/2} + \sum_{m=1}^{n-1} 2^{-m} \left(\sum_{j=1}^{2^{m+1}} |b_{m,j}|^2 \right)^{1/2} \right].$$

With the help of the Schwarz inequality we see that there exists a positive constant C_4 independent of n such that

$$\|u\|^2_{L_2(0,1)} \leqslant C_4 \left(\sum_{j=1}^{4} |c_j|^2 + \sum_{m=1}^{n-1} \sum_{j=1}^{2^{m+1}} |b_{m,j}|^2 \right).$$

The second inequality in (4.3) follows. The proof of the theorem is complete. □

In what follows we apply the wavelet basis G_n to two numerical examples.

Example 1. Consider the Dirichlet problem:

$$\begin{cases} -u'' = f & \text{on } (0,1), \\ u(0) = u(1) = 0, \end{cases}$$

where f is given by

$$f(x) = (53.7\pi)^2 \sin(53.7\pi x) + (2.3\pi)^2 \sin(2.3\pi x), \quad x \in (0,1).$$

The exact solution of the problem is

$$u(x) = \sin(53.7\pi x) + \sin(2.3\pi x), \quad x \in (0,1), \tag{4.4}$$

which could be regarded as the sum of a high-frequency component and a low-frequency component.

Let us use the wavelet basis $G_n := \{g_1, \ldots, g_{2^{n+1}}\}$ to solve the Dirichlet problem. With $u_n = \sum_{k=1}^{2^{n+1}} \eta_k g_k$, the Galerkin approximation problem (4.2) is discretized as

$$\sum_{k=1}^{2^{n+1}} \langle g_j', g_k' \rangle \eta_k = \langle g_j, f \rangle, \quad j = 1, \ldots, 2^{n+1}. \tag{4.5}$$

The stiffness matrix $A_n := (\langle g_j', g_k' \rangle)_{1 \leqslant j,k \leqslant 2^{n+1}}$ is block diagonal. Moreover, each block is a banded matrix. By theorem 2, the condition number of the matrix A_n is uniformly bounded (independent of n). This assertion is confirmed by numerical computation of the maximal eigenvalue $\lambda_{\max}$, the minimal eigenvalue $\lambda_{\min}$, and the condition number $\kappa = \lambda_{\max}/\lambda_{\min}$ of the matrix A_n for $n = 6, \ldots, 12$ (see table 1).

We use the CG (conjugate gradient) method to solve the system (4.5) of linear equations. The convergence of CG method will be judged by the threshold $\varepsilon = 10^{-10}$. More precisely, the process of iteration will terminate if the difference of two consecutive iterations is less than 10^{-10}. Since the stiffness matrix A_n is well conditioned, the CG method converges very fast. Up to $n = 12$, only 21 iterations are needed for convergence to the solution of the system of linear equations. For $n = 1, 2, \ldots,$ let $e_n := \|u_n - u\|_{L_2(0,1)}$, where u is the exact solution given in (4.4). For $n = 6, \ldots, 12$, table 2 lists the error e_n and the rate of convergence $\log_2 e_{n-1}/e_n$.

It is well known from approximation theory that the Hermite cubic splines provide approximation of order 4. The preceding computation confirms this assertion.

Table 1
Condition number of the matrix A_n.

n	6	7	8	9	10	11	12
$\lambda_{\max}$	1.5780	1.5787	1.5789	1.5789	1.5789	1.5789	1.5789
$\lambda_{\min}$	0.4220	0.4213	0.4211	0.4211	0.4211	0.4211	0.4211
κ	3.7397	3.7474	3.7494	3.7498	3.7498	3.7498	3.7498

Table 2
Error e_n and its convergence rate.

n	6	7	8	9	10	11	12
e_n	1.210 E−2	1.326 E−3	1.082 E−4	7.358 E−6	4.705 E−7	2.9584 E−8	1.852 E−9
$\log_2\left(\frac{e_{n-1}}{e_n}\right)$	4.10	3.19	3.62	3.88	3.97	3.99	4.00

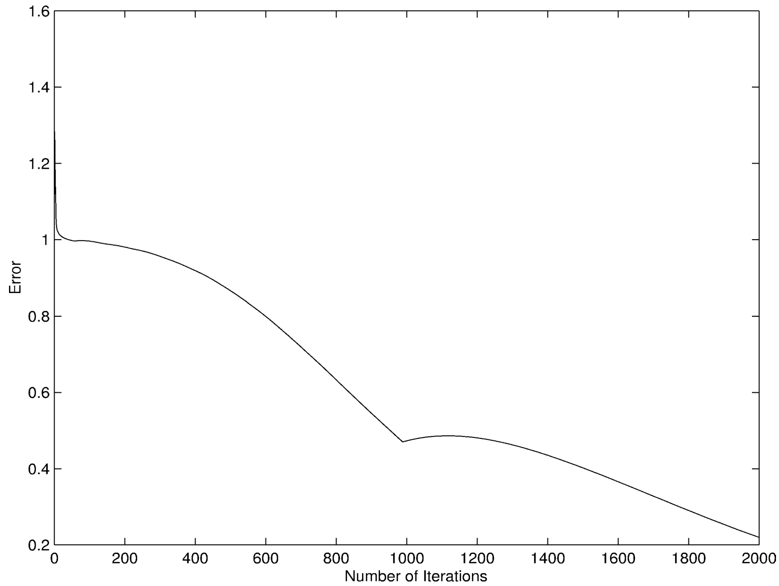

Figure 3. The error against the number of iterations without preconditioning.

If we use the finite elements in Φ_n given in (1.1) to discretize equation (4.2), then the resulting stiffness matrix is ill conditioned. For $n = 12$, the system of linear equations has 8192 unknowns. Without preconditioning, it takes more than 2000 iterations for the CG method to converge. The graph in figure 3 depicts the error against the number of iterations.

In [1], Bramble et al. proposed the so-called BPX method for preconditioning. This method was developed on the nodal basis (piecewise linear functions). We observe that piecewise linear functions only provide approximation of order 2. In order to achieve convergence of order 4, one may extend the BPX method to Hermite cubic splines. For $n = 6, \ldots, 12$, table 3 gives the maximal eigenvalue $\lambda_{\max}$, the minimal eigenvalue $\lambda_{\min}$, and the spectral condition number of the corresponding matrix after preconditioning.

We see that the condition number induced by our wavelet basis is smaller than that given by the BPX method. For $n = 12$, after preconditioning by the BPX method, it takes 26 iterations for the PCG (preconditioned conjugate gradient) method to converge. Hence, the preconditioning method induced by our wavelet basis is competitive.

Table 3
BPX preconditioning for Hermite cubic splines.

n	6	7	8	9	10	11	12
$\lambda_{\max}$	3.562	3.632	3.682	3.718	3.743	3.763	3.777
$\lambda_{\min}$	0.7693	0.7696	0.7696	0.7696	0.7696	0.7696	0.7696
κ	4.630	4.719	4.784	4.831	4.864	4.890	4.907

Table 4
Condition number of the matrix A_n.

n	6	7	8	9	10	11	12
$\lambda_{\max}$	1.5780	1.5787	1.5789	1.5789	1.5789	1.5789	1.5789
$\lambda_{\min}$	0.4220	0.4213	0.4211	0.4211	0.4211	0.4211	0.4211
κ	3.7396	3.7474	3.7494	3.7498	3.7498	3.7498	3.7498

Example 2. Consider the Dirichlet problem

$$\begin{cases} -u'' + u = f & \text{on } (0, 1), \\ u(0) = u(1) = 0, \end{cases}$$

where

$$f(x) = \left[(53.7\pi)^2 + 1\right]\sin(53.7\pi x) + \left[(2.3\pi)^2 + 1\right]\sin(2.3\pi x), \quad x \in (0, 1).$$

The function u given in (4.4) is the exact solution of the problem.

In this case, the bilinear form $a(u, v)$ is given by

$$a(u, v) = \langle u', v' \rangle + \langle u, v \rangle, \quad u, v \in H_0^1(0, 1).$$

With the wavelet basis G_n the Galerkin approximation problem (4.2) is discretized as

$$\sum_{k=1}^{2^{n+1}} \left(\langle g_j', g_k' \rangle + \langle g_j, g_k \rangle\right)\eta_k = \langle g_j, f \rangle, \quad j = 1, \ldots, 2^{n+1}. \tag{4.6}$$

The stiffness matrix

$$A_n := \left(\langle g_j', g_k' \rangle + \langle g_j, g_k \rangle\right)_{1 \leqslant j,k \leqslant 2^{n+1}}$$

is still a sparse matrix. By theorem 2, the condition number of the matrix A_n is uniformly bounded (independent of n). This assertion is confirmed by numerical computation of the maximal eigenvalue $\lambda_{\max}$, the minimal eigenvalue $\lambda_{\min}$, and the condition number κ of A_n for $n = 6, \ldots, 12$ (see table 4).

For $n = 6, \ldots, 12$, table 5 gives the maximal eigenvalue $\lambda_{\max}$, the minimal eigenvalue $\lambda_{\min}$, and the spectral condition number of the corresponding matrix after preconditioning by using the BPX method. By comparsion, the condition number of our wavelet basis is smaller than that of the BPX method.

Table 5
BPX preconditioning for Hermite cubic splines.

n	6	7	8	9	10	11	12
$\lambda_{\max}$	3.562	3.632	3.682	3.718	3.743	3.763	3.777
$\lambda_{\min}$	0.7696	0.7696	0.7696	0.7696	0.7696	0.7696	0.7696
κ	4.628	4.719	4.784	4.831	4.864	4.890	4.907

Table 6
Error e_n and its convergence rate.

n	6	7	8	9	10	11	12
e_n	1.210 E−2	1.326 E−3	1.082 E−4	7.359 E−6	4.706 E−7	2.967 E−8	1.918 E−9
$\log_2\left(\frac{e_{n-1}}{e_n}\right)$	4.10	3.19	3.62	3.88	3.97	3.99	3.95

We use the CG method to solve the system (4.6) of linear equations. The computational results are similar to those in example 1. Up to $n = 12$, only 19 iterations are needed for convergence to the solution of the system of linear equations. For $n = 6, \ldots, 12$, table 6 lists the error e_n and the rate of convergence $\log_2 e_{n-1}/e_n$.

Finally, we remark that our wavelet basis can also be used to solve integral equations numerically. A discrete wavelet Petrov–Galerkin method was developed by Chen et al. [3] for numerical solutions of integral equations of the second kind with weakly singular kernels. Recently, Shen and Lin [13] used the wavelet basis G_n constructed in this paper to find numerical solutions of integral equations on the upper half-plane.

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