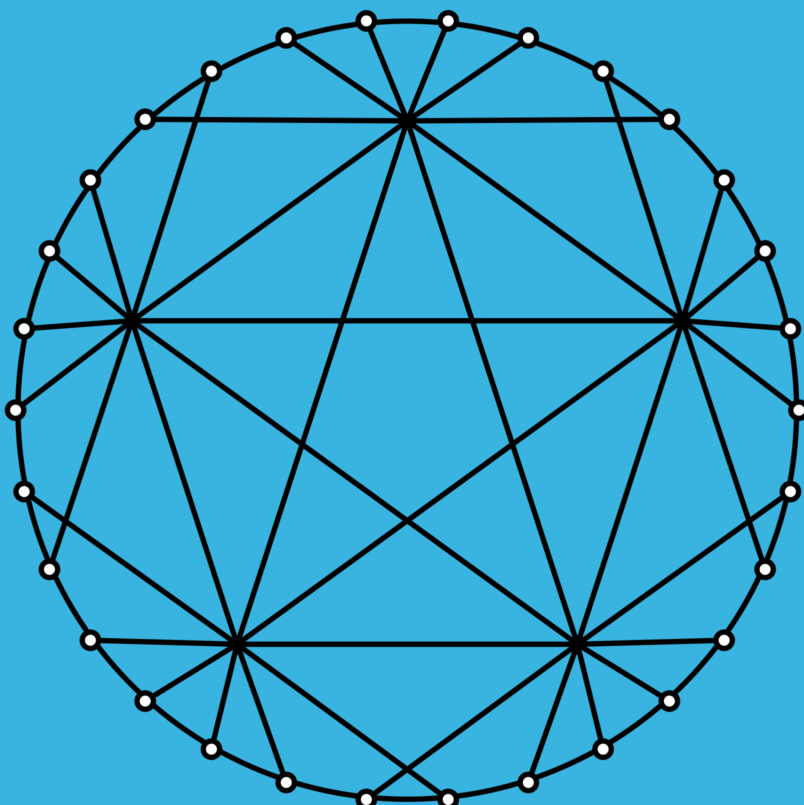
BULLETIN of The INSTITUTE of COMBINATORICS and its APPLICATIONS

Volume 92
June 2021



Boca Raton, FL, U.S.A.

ISSN: 2689-0674 (Online)
ISSN: 1183-1278 (Print)

BULLETIN OF THE ICA
Volume **92** (2021), 62–77

List colorings count rokudoku-pair squares

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Abstract

A rokudoku-pair square is an order-6 sudoku Latin square for both 2×3 and 3×2 tiling regions simultaneously. We count the distinct rokudoku-pair squares as well as orbits under the action of a suitable group. Our arguments employ group actions and list colorings of graphs. As an application we determine which rokudoku-pair squares are based on groups.

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Key words and phrases: sudoku-pair latin square, rokudoku, graph, list coloring, group action

AMS (MOS) Subject Classifications: 05B15 Primary; 05C15 and 05E16 Secondary

Received: 17 December 2020
Accepted: 23 March 2021

1 Introduction

We count rokudoku-pair squares, give a complete list of essentially different rokudoku-pair squares, and use this list to classify group-based rokudoku-pair squares. The computations employ group actions and list colorings of graphs. We hope this work sheds light on the counting and construction of sudoku-pair Latin squares.

A **rokudoku-pair square** is an order-6 Latin square that additionally has no repetition of symbols in any 2×3 tiling region, nor in any 3×2 tiling region when tiled in the natural way. An example is shown in Figure 1, where the gridlines illustrate the subdivision into 2×3 and 3×2 tiling regions. We let $\mathcal{R}$ denote the collection of rokudoku-pair squares.

2	5	1	3	0	4
4	3	0	5	2	1
0	1	2	4	5	3
3	4	5	2	1	0
1	2	4	0	3	5
5	0	3	1	4	2

Figure 1: A rokudoku-pair square

Given two rokudoku-pair squares, if one cannot be obtained from the other from some combination of relabeling and reasonable physical symmetries (described in Section 3), then the two squares are **essentially different**. In terms of groups, two squares are essentially different if they lie in different orbits under the action of a relabeling/symmetry group.

The counting of squares and essentially different squares has been performed in other settings. Felgenhauer and Jarvis [9] counted the number of distinct "classical" sudoku squares (order 9, with 3×3 regions). Jarvis and Russell [12] then used Burnside's orbit counting theorem to count the essentially different classical sudoku squares. Similar results [13] have been achieved for rokudoku squares (order 6, with no repetition in 2×3 regions): Pettersen counts 28200960 distinct rokudoku squares, and, according to Jarvis and Russell, there are 49 essentially different rokudoku squares.

One aim of this project is to deepen our knowledge of **sudoku-pair Latin squares**. A sudoku-pair Latin square with parameters $a, b \in \mathbb{Z}^+$ is a Latin square of order ab that additionally has no repetition simultaneously in

$a \times b$ and $b \times a$ tiling regions. Much less is known about sudoku-pair Latin squares than about sudoku squares. For instance, it is not known whether sudoku-pair Latin squares exist for all parameters a, b, though the existence question boils down to existence for relatively prime parameters a, b (e.g., see [4]). The set $\mathcal{R}$ is the simplest possible "nontrivial" subset of sudoku-pair Latin squares, so it is important to have a firm understanding of $\mathcal{R}$. Also, one possible method for constructing new sudoku-pair Latin squares is to use (isotopisms of) Cayley tables for finite groups. Squares constructed in this way are said to be **based on groups**. This method is widely applicable for sudoku squares (e.g., [3]). As an application, we show that some members of $\mathcal{R}$ are based on the symmetric group S_3 (about 0.8%), but that none are based on $\mathbb{Z}_6$. Therefore, unlike the situation for sudoku squares, not every group of order n with subgroup of order k will induce a sudoku-pair Latin square with parameters $n/k, k$. More about group-based sudoku-pair Latin squares is forthcoming in [5].

While the group based approach has widely been used to count squares of various types, we combine this wealth of literature with list coloring techniques on the graph structure underlying the rokudoku-pair Latin square parameters. Hence in Section 2 we establish a list coloring scheme on the rokudoku-pair Latin square graph along with some combinatorial techniques related to the problem. Capitalizing on the counts in Section 2, Section 3 proceeds to utilize Burnside's Lemma to count the number of essentially different rokudoku-pair squares. Finally, in Section 4, we will determine precisely which orbit arises from a Cayley table of a group.

2 Counting rokudoku-pair squares

Theorem 2.1 $|\mathcal{R}| = 1393920$

We sketch a proof of Theorem 2.1 that uses list colorings of graphs. We begin with terminology. A **central figure** is an arranged collection of symbols $\{0, 1, 2, 3, 4, 5\}$ in a 6×6 (partially filled) array consisting of the third and fourth rows and columns so that:

(i) no symbol appears more than once in any row, column, 2×3 tiling region and 3×2 tiling region of the array, and

(ii) each 3×3 quadrant of the array has exactly three distinct symbols.

We mention that items (i) and (ii) must be satisfied by the 6×6 array formed from the middle two rows and columns in any legitimate rokudoku-pair square, so central figures make a good starting point for constructing rokudoku-pair squares.

An example of a central figure is given in Figure 2. Note that the three distinct symbols lying in the upper-left 3×3 quadrant of this central figure are $0, 1, 2$. Also note that when attempting to fill the remaining locations of the array, the only ambiguity in placing symbols involves 2 and 5, which are precisely the symbols in the central 2×2 subsquare.

		1	3		
		0	4		
0	1	2	5	3	4
3	4	5	2	0	1
		3	0		
		4	1		

Figure 2: Central figure

A central figure need not extend to a member of $\mathcal{R}$. For example, the central figure shown in Figure 2 does not extend to a member of $\mathcal{R}$. However, as we mentioned above, a central figure determines possible values, sometimes uniquely, in the remaining cells of the array. A **corner figure** is a 4×4 array consisting of the corner 2×2 regions in the complement of a central figure containing the possible values for extending the central figure to a member of $\mathcal{R}$. An example is given in Figure 3.

4	$2,5$	1	3	$2,5$	0
$2,5$	3	0	4	1	$2,5$
0	1	2	5	3	4
3	4	5	2	0	1
1	$2,5$	3	0	4	$2,5$
$2,5$	0	4	1	$2,5$	3

4	$2,5$		$2,5$	0
$2,5$	3		1	$2,5$
1	$2,5$		4	$2,5$
$2,5$	0		$2,5$	3

Figure 3: A central figure with possible corner region symbols (left) and corresponding corner figure (right)

A choice of specific values for these undetermined entries yielding a member of $\mathcal{R}$ is called a **valid corner choice**. Our count of $\mathcal{R}$ hinges upon enumerating the valid corner choices for each central figure. For this purpose it is convenient to write $\mathcal{R} = T_2 \cup T_3 \cup T_4$ where T_i is the collection of rokudoku-pair squares with exactly i distinct symbols in the central 2×2 subsquare. To count $\mathcal{R}$ we count each T_i separately. We let $\mathcal{C}$ denote the set of central figures and let $\mathcal{C}_i$ denote the members of $\mathcal{C}$ with exactly i distinct symbols in the central 2×2 subsquare. For example, the central figure shown in Figure 2 is a member of $\mathcal{C}_2$. Generally speaking, counting each T_i is done by counting, for each n, the members of $\mathcal{C}_i$ that can be completed to rokudoku-pair squares in n ways. As described below, this will be accomplished via list colorings of graphs. Elementary counting arguments give that

$$|\mathcal{C}_2| = 6!2^6 \text{ while } |\mathcal{C}_3| = |\mathcal{C}_4| = 6!2^8, \tag{1}$$

where the 6! counts the number of ways to place symbols in, say, the left-central 2×3 region, and the powers of 2 count the number of ways to fill the remaining locations in the central figure.

The problem of counting valid corner choices can be translated to a graph coloring problem. In a corner figure, the collection of undetermined entries forms a set of vertices, with an edge connecting a pair of vertices whenever the corresponding entries lie in the same row, column, or 2×2 region. We refer to this graph as the **corner graph**. In the **list labeled corner graph**, each vertex in the corner graph will be labeled with the list of possible values the undetermined entry could take on in the corner figure to extend the central figure to a member of $\mathcal{R}$. A **list coloring** of the list labeled corner graph is a proper vertex coloring of the graph such that the colors assigned to each vertex come from the list of colors available to that vertex. List colorings were introduced by Erdős in [7] and a survey on list colorings has been compiled by Woodall in [14]. The number of valid corner choices for a particular corner figure is the number of list colorings of the list labeled corner graph.

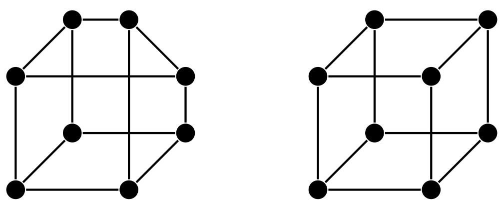

Figure 4: Corner graphs arising from members of $\mathcal{C}_2$

Half of the central figures in $\mathcal{C}_2$ have corner graph as shown at left in Figure 4, while the other half have corner graph as shown at right in Figure 4. Let us denote those halves by $\mathcal{C}_2^{(1)}$ and $\mathcal{C}_2^{(2)}$, respectively. If each vertex in Figure 4 is assigned two colors $\{a, b\}$, then the graph at left has no list colorings because it contains a 5-cycle, while the graph at right has two list colorings. Therefore, since $|\mathcal{C}_2| = 2^6 \cdot 6!$,

$$|T_2| = 0 \cdot |\mathcal{C}_2^{(1)}| + 2 \cdot |\mathcal{C}_2^{(2)}| = 0 \cdot \frac{2^6 \cdot 6!}{2} + 2 \cdot \frac{2^6 \cdot 6!}{2} = 2^6 \cdot 6!. \tag{2}$$

		*	*		
		*	*		
*	*	a	b	*	*
*	*	c	a	*	*
		*	*		
		*	*		

2, 4, 5	2, 5	1	3	0	2, 4
2, 4	3	0	5	2, 4	1
0	1	2	4	5	3
3	4	5	2	1	0
1	2, 5	4	0	3	2, 5
2, 5	0	3	1	2, 4	2, 4, 5

2, 4, 5	2, 5		0	2, 4
2, 4	3		2, 4	1
1	2, 5		3	2, 5
2, 5	0		2, 4	2, 4, 5

Figure 5: Central figures for T_3 (showing the repeated symbol on the main diagonal) together with an example and its corner figure

Next we sketch the count of T_3. An example is given in Figure 5. We can partition $\mathcal{C}_3$ into three sets $\mathcal{C}_3^{(1)}$, $\mathcal{C}_3^{(2)}$, and $\mathcal{C}_3^{(3)}$, where the central figures in these sets have corner graphs as shown left to right in Figure 6, respectively.

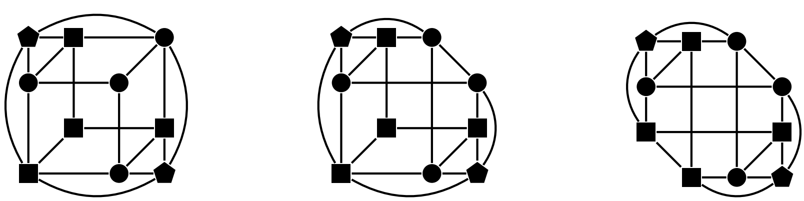

Figure 6: Corner graphs arising from members of $\mathcal{C}_3$

In the figure, three colors $\{a, b, c\}$ are assigned to the pentagonal vertices, while $\{a, b\}$ is the list for the circular vertices, and $\{a, c\}$ is the list for the square vertices. Therefore, from left to right in Figure 6 the graphs have three list colorings, two list colorings, and two list colorings, respectively. This yields

$$|T_3| = 3\cdot|\mathcal{C}_3^{(1)}|+2\cdot|\mathcal{C}_3^{(2)}|+2\cdot|\mathcal{C}_3^{(3)}| = 3\cdot\frac{2^8 6!}{4}+2\cdot\frac{2^8 6!}{2}+2\cdot\frac{2^8 6!}{4} = 9\cdot 2^6\cdot 6!. \quad (3)$$

The left side of Figure 7 shows the list labeled corner graph of the leftmost corner graph of Figure 6, corresponding to the example in Figure 5. To the right are induced subgraphs of the list labeled corner graph. The induced subgraph in the center of Figure 7 is the graph induced by the set of vertices labeled with $\{2, 4\}$ and the induced subgraph on the right side of the figure is the graph induced by the set of vertices labeled with $\{2, 5\}$. Both induced graphs have two possible list colorings. However, the list colorings which color all vertices of degree one in both induced subgraphs with 2, is not allowed in the coloring of the list labeled corner graph, as those vertices are adjacent in the list labeled corner graph. Hence there are $2 \cdot 2 - 1 = 3$ list colorings of the list labeled corner graph.

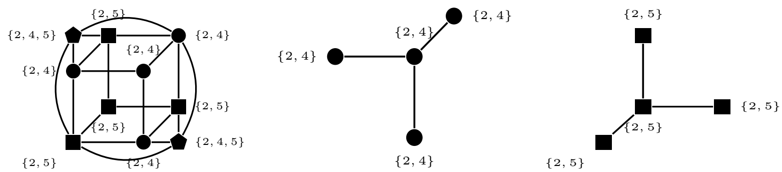

Figure 7: Example of corner graph corresponding to corner figure in Figure 5, and its subgraphs induced by the labelings $\{2, 4\}$ and $\{2, 5\}$, respectively

Finally, we sketch the count for T_4. An example of a $\mathcal{C}_4$ central figure, together with its corner figure and corner graph, are given in Figure 8.

Each corner figure arising from a member of $\mathcal{C}_4$ will have two undetermined pairs of symbols and one undetermined triple of symbols in each corner. Therefore, corresponding corner graphs will have twelve vertices: Eight have color lists of size two, and four have color lists of size three. It turns out that each corner graph Γ arising from a member of $\mathcal{C}_4$ is determined, up to isomorphism, by the induced subgraph $\hat{\Gamma}$ whose four vertices correspond to the four undetermined triples in the corner figure. There are six isomorphism classes of $\hat{\Gamma}$ (representatives shown in Figure 9), and hence there are six classes of corner graph Γ. The corner graph shown at right in

2, 5	2, 3, 5	0	4	2, 3	1
4	2, 3	1	5	0, 2, 3	0, 3
0	1	2	3	4	5
3	4	5	0	1	2
2, 5	0, 2, 5	4	1	0, 3, 5	0, 3
1	0, 5	3	2	0, 5	4

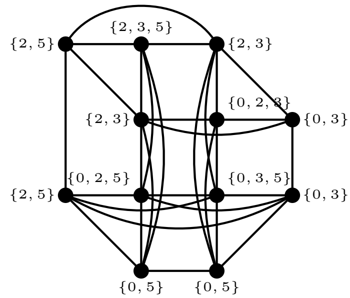

Figure 8: A member of $\mathcal{C}_4$ together with corner figure and list labeled corner graph

Figure 8 lies in the class determined by the graph $\hat{\Gamma}$ shown second in the second row of Figure 9.

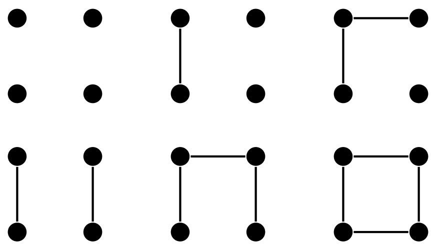

Figure 9: Subgraphs induced by undetermined triples, with color lists suppressed

We can partition $\mathcal{C}_4$ into six sets $\mathcal{C}_4^{(1)}, \ldots, \mathcal{C}_4^{(6)}$, where $\mathcal{C}_4^{(j)}$ is the collection of central figures that have corner graphs associated to the induced subgraph shown j-th from the left in Figure 9. The number of list colorings for corner graphs arising from $\mathcal{C}_4^{(1)}, \ldots, \mathcal{C}_4^{(6)}$ is $2, 4, 5, 6, 6, 7$, respectively. Therefore,

$$\begin{aligned} |T_4| &= 2 \cdot |\mathcal{C}_4^{(1)}| + 4 \cdot |\mathcal{C}_4^{(2)}| + 5 \cdot |\mathcal{C}_4^{(3)}| + 6 \cdot |\mathcal{C}_4^{(4)}| + 6 \cdot |\mathcal{C}_4^{(5)}| + 7 \cdot |\mathcal{C}_4^{(6)}| \\ &= 2 \cdot \frac{2^8 \cdot 6!}{16} + 4 \cdot \frac{2^8 \cdot 6!}{4} + 5 \cdot \frac{2^8 \cdot 6!}{4} + 6 \cdot \frac{2^8 \cdot 6!}{8} + 6 \cdot \frac{2^8 \cdot 6!}{4} + 7 \cdot \frac{2^8 \cdot 6!}{16} \\ &= 81 \cdot 2^4 \cdot 6!. \end{aligned} \tag{4}$$

From Equations (2), (3), and (4), we have

$$|\mathcal{R}| = 2^6 \cdot 6! + 9 \cdot 2^6 \cdot 6! + 81 \cdot 2^4 \cdot 6! = 1393920, \tag{5}$$

yielding the result of Theorem 2.1.

Finally, we note that $\mathcal{R}$ can be counted manually without appealing to list colorings. However, there are advantages to the list coloring approach: The stratification of T_4 into six classes is more transparent, and the process of counting valid corner choices can be simplified by counting the list colorings of certain smaller subgraphs. An example of one of the more difficult list labeled corner graph reductions follows in Figure 10.

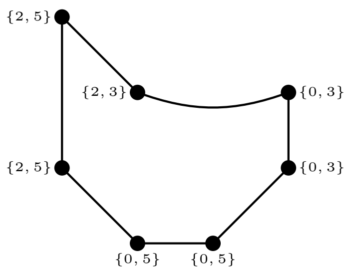

Figure 10: Reduced graph of list labeled corner graph in Figure 8

In Figure 10, we show a subgraph of the list labeled corner graph in Figure 8 to more easily determine the possible list colorings for the corner graph. We reduce the graph by removing all edges between vertices which have disjoint lists of possible colors. Since the colors between these vertices are disjoint, a choice to color one vertex will have no effect on the choice to color the other, regardless of the edge, so those edges are unnecessary. In our example, this means we can remove the edges between $\{2, 3\}$ and $\{0, 5\}$ and between $\{2, 5\}$ and $\{0, 3\}$. We also can remove one vertex from each pair of vertices which both share the same list of possible colors, and which are nonadjacent. In our example, this means we remove one of the vertices with possible colors $\{2, 3\}$. The vertices with possible colors $\{2, 3\}$ must always be colored the same in any list coloring for the following reason. The vertex with possible colors $\{2, 3, 5\}$ is adjacent to both vertices with possible colors $\{2, 3\}$ and is also adjacent to the vertex with possible colors $\{2, 5\}$. The vertex with possible colors $\{2, 5\}$ is also adjacent to both vertices with possible colors $\{2, 3\}$. This means that if the vertices with possible colors $\{2, 3\}$ are colored with different colors, one must be labeled a 2, and hence the vertex with list $\{2, 5\}$ must be labeled a 5. However, this means that the vertex with list $\{2, 3, 5\}$ has no possible color as its neighbors have used all possible colors. This same property is true for any vertices which are nonadjacent and have the same list of possible colors in a list labeled corner graph. So, this reduction process can be used to reduce any of the list labeled corner graphs.

3 Counting essentially different rokudoku-pair squares

There are various ways one can modify a rokudoku-pair square to obtain a new rokudoku-pair square. These modifications include relabelings, together with combinations of the following physical symmetries: 90° rotations, transpose, reflections across either the horizontal or vertical midline, swapping the leftmost two columns, swapping the rightmost two columns, swapping the top two rows, and swapping the bottom two rows. Two rokudoku-pair squares are **essentially different** if one cannot be obtained from the other by some combination of these modifications. We seek to know the number of essentially different rokudoku-pair squares.

This problem is naturally cast in terms of groups. Let $G = S_6 \times H$ be the group of modifications, with S_6 denoting the symmetric group of relabelings and H denoting the group of physical symmetries generated by those mentioned above. We call G the **rokudoku-pair group**. We observe that

$$H \cong (D_4 \times D_4) \rtimes \mathbb{Z}_2$$

where D_4 is the dihedral group of order 8: The left-most D_4 represents the allowable permutations of rows (combinations of a swap of the top-most two rows, a swap of the bottom-most two rows, and reflection across horizontal midline) and the other D_4 represents the analogous allowable permutations of columns (combinations of a swap of the left-most two columns, a swap of the right-most two columns, and reflection across vertical midline). The nontrivial element of $\mathbb{Z}_2$ represents transpose, and the semi-direct product reflects the fact that $tr = ct$, where t is transpose, r is a row permutation, and c is the corresponding column permutation. For example, applying a swap of the top two rows followed by transpose is the same as applying transpose first, followed by a swap of the left-most two columns. Rotations are achieved by transpose and row/column permutations. Altogether we have

$$G \cong S_6 \times [(D_4 \times D_4) \rtimes \mathbb{Z}_2],$$

a group of order $6! \cdot 128 = 92160$ acting on the set $\mathcal{R}$ of rokudoku-pair squares. Determining the number of essentially different rokudoku-pair squares translates to counting the orbits in $\mathcal{R}$ under the action of G.

We proceed to count these orbits. For $x \in \mathcal{R}$ let $G_x = \{g \in G \mid g.x = x\}$. Observe that if $x, y \in \mathcal{R}$ and $g \in G$ with $g.x = y$, then the mapping $\gamma \mapsto g\gamma g^{-1}$ is an isomorphism $G_x \to G_y$. In particular, $|G_x| = |G_y|$ if $x, y \in \mathcal{R}$ lie in the same G-orbit. Now suppose there are N orbits $\mathcal{R}_1, \cdots, \mathcal{R}_N$ in $\mathcal{R}$ under the action of G, with base-points $x_1, \cdots, x_N$. Then $|\mathcal{R}_i| = |G|/|G_{x_i}|$ for $1 \le i \le N$, and

$$N \cdot |G| = \sum_{i=1}^{N} \frac{|G|}{|G_{x_i}|}|G_{x_i}| = \sum_{i=1}^{N} |\mathcal{R}_i||G_{x_i}| = \sum_{i=1}^{N} \sum_{x \in \mathcal{R}_i} |G_x| = \sum_{x \in \mathcal{R}} |G_x|. \quad (6)$$

The series of equations (6) falls one equation short of being a proof Burnside's classical orbit counting theorem (e.g., see [10], p. 205). However, we take (6) in another direction: For $x \in \mathcal{R}$, let

$$\hat{H}_x = \{h \in H \mid h.x = \sigma.x \text{ for some } \sigma \in S_6\}.$$

Because $G = S_6 \times H$, we have $G_x = \{\sigma h \in G \mid \sigma h.x = x \text{ with } \sigma \in S_6,\ h \in H\}$. Further, if $\sigma_1 h, \sigma_2 h \in G_x$, then $\sigma_1 = \sigma_2$. These facts together imply that $|G_x| = |\hat{H}_x|$, and we have

$$\sum_{x \in \mathcal{R}} |G_x| = \sum_{x \in \hat{\mathcal{R}}} 6!|G_x| = \sum_{x \in \hat{\mathcal{R}}} 6!|\hat{H}_x|, \quad (7)$$

where $\hat{\mathcal{R}}$ denotes the set of rokudoku-pair squares possessing a left-central 2×3 subrectangle[1] of the form

$$\boxed{\begin{array}{ccc} 0 & 1 & 2 \\ 3 & 4 & 5 \end{array}}\ .$$

Combining (6) and (7) tells us that

$$N = \frac{1}{128} \sum_{x \in \hat{\mathcal{R}}} |\hat{H}_x|. \quad (8)$$

Equation (8) is effective for counting orbits because both H and $\hat{\mathcal{R}}$ are relatively small. A MATLAB program [11] employing (8) gives:

[1] The location of the fixed subrectangle is irrelevant. It could just as easily be in the upper-left.

Theorem 3.1 *There are* 26 *G-orbits in $\mathcal{R}$.*

Representatives $x_1, \ldots, x_{26}$ of these G-orbits are given in Figure 11. Henceforth the G-orbits in $\mathcal{R}$ will be listed in this particular order, with these particular representatives.

The size of each orbit can be computed via

$$|\mathcal{R}_i| = |G|/|G_{x_i}| = |G|/|\hat{H}_{x_i}| \qquad 1 \le i \le 26.$$

The size of $\hat{H}_{x_i}$ is crucial for this computation. Again using MATLAB in [11], we have:

Theorem 3.2 *(a)* $|\hat{H}_{x_i}| = 1$ *for* $i \in \{3, 4, 6, 9, 11, 13, 14, 17, 19\}$

(b) $|\hat{H}_{x_i}| = 2$ *for* $i \in \{1, 5, 7, 10, 12, 15, 18, 20, 23\}$

(c) $|\hat{H}_{x_i}| = 4$ *for* $i \in \{2, 8, 16, 21, 25\}$

(d) $|\hat{H}_{x_i}| = 8$ *for* $i \in \{22, 24, 26\}$.

We finish this subsection with a discussion of the minimality of G. We say that G is **minimal** for $\mathcal{R}$ if there is no proper subgroup of G that acts on $\mathcal{R}$ with the same orbits. It is known [2] that the obvious group of symmetries/relabelings for ordinary order-9 sudoku squares is minimal, while the similar groups for shidoku (2×2 regions) and certain magic sudoku variants are not minimal (see [1] and [2]). In our present situation, Theorem 3.2 tells us that $|\hat{H}_{x_i}| = 1$ for at least one value of i, so there are G-orbits in $\mathcal{R}$ that are of size $|G|$. This forces G to be minimal.

Corollary 3.3 *G is minimal for $\mathcal{R}$.*

4 An application to group-based squares

Here we discuss the existence of group-based rokudoku-pair squares. Let $K = \{k_1, \ldots, k_n\}$ be a finite group. The **Cayley table** L_K for K is the body of the multiplication table for K with row/column headers $k_1, \ldots, k_n$

$$X_1 \begin{matrix} 2&3&0&4&1&5\\ 4&5&1&3&0&2\\ 0&1&2&5&4&3\\ 3&4&5&0&2&1\\ 1&0&3&2&5&4\\ 5&2&4&1&3&0 \end{matrix} \quad X_2 \begin{matrix} 2&3&0&4&1&5\\ 4&5&1&3&0&2\\ 0&1&2&5&4&3\\ 3&4&5&1&2&0\\ 1&0&3&2&5&4\\ 5&2&4&0&3&1 \end{matrix} \quad X_3 \begin{matrix} 2&3&0&4&1&5\\ 4&5&1&3&2&0\\ 0&1&2&5&4&3\\ 3&4&5&1&0&2\\ 1&0&3&2&5&4\\ 5&2&4&0&3&1 \end{matrix} \quad X_4 \begin{matrix} 2&3&0&5&1&4\\ 4&5&1&3&0&2\\ 0&1&2&4&5&3\\ 3&4&5&0&2&1\\ 1&0&3&2&4&5\\ 5&2&4&1&3&0 \end{matrix}$$

$$X_5 \begin{matrix} 2&3&0&5&1&4\\ 4&5&1&3&0&2\\ 0&1&2&4&5&3\\ 3&4&5&1&2&0\\ 1&0&3&2&4&5\\ 5&2&4&0&3&1 \end{matrix} \quad X_6 \begin{matrix} 2&3&0&5&1&4\\ 4&5&1&3&2&0\\ 0&1&2&4&5&3\\ 3&4&5&1&0&2\\ 1&0&3&2&4&5\\ 5&2&4&0&3&1 \end{matrix} \quad X_7 \begin{matrix} 2&3&0&4&1&5\\ 4&5&1&3&0&2\\ 0&1&2&5&3&4\\ 3&4&5&0&2&1\\ 1&0&4&2&5&3\\ 5&2&3&1&4&0 \end{matrix} \quad X_8 \begin{matrix} 2&3&0&4&1&5\\ 4&5&1&3&0&2\\ 0&1&2&5&3&4\\ 3&4&5&1&2&0\\ 1&0&4&2&5&3\\ 5&2&3&0&4&1 \end{matrix}$$

$$X_9 \begin{matrix} 2&3&0&4&1&5\\ 4&5&1&3&2&0\\ 0&1&2&5&3&4\\ 3&4&5&1&0&2\\ 1&0&4&2&5&3\\ 5&2&3&0&4&1 \end{matrix} \quad X_{10} \begin{matrix} 2&3&0&5&1&4\\ 4&5&1&3&0&2\\ 0&1&2&4&3&5\\ 3&4&5&0&2&1\\ 1&0&4&2&5&3\\ 5&2&3&1&4&0 \end{matrix} \quad X_{11} \begin{matrix} 2&3&0&5&1&4\\ 4&5&1&3&0&2\\ 0&1&2&4&5&3\\ 3&4&5&0&2&1\\ 1&0&4&2&3&5\\ 5&2&3&1&4&0 \end{matrix} \quad X_{12} \begin{matrix} 2&3&0&5&1&4\\ 4&5&1&3&0&2\\ 0&1&2&4&3&5\\ 3&4&5&1&2&0\\ 1&0&4&2&5&3\\ 5&2&3&0&4&1 \end{matrix}$$

$$X_{13} \begin{matrix} 2&3&0&5&1&4\\ 4&5&1&3&0&2\\ 0&1&2&4&5&3\\ 3&4&5&1&2&0\\ 1&0&4&2&3&5\\ 5&2&3&0&4&1 \end{matrix} \quad X_{14} \begin{matrix} 2&3&0&5&1&4\\ 4&5&1&3&2&0\\ 0&1&2&4&5&3\\ 3&4&5&1&0&2\\ 1&0&4&2&3&5\\ 5&2&3&0&4&1 \end{matrix} \quad X_{15} \begin{matrix} 2&3&1&4&0&5\\ 4&5&0&3&2&1\\ 0&1&2&5&4&3\\ 3&4&5&0&1&2\\ 1&0&3&2&5&4\\ 5&2&4&1&3&0 \end{matrix} \quad X_{16} \begin{matrix} 2&3&1&5&0&4\\ 4&5&0&3&2&1\\ 0&1&2&4&5&3\\ 3&4&5&0&1&2\\ 1&0&3&2&4&5\\ 5&2&4&1&3&0 \end{matrix}$$

$$X_{17} \begin{matrix} 2&3&1&5&0&4\\ 4&5&0&3&1&2\\ 0&1&2&4&5&3\\ 3&4&5&1&2&0\\ 1&0&3&2&4&5\\ 5&2&4&0&3&1 \end{matrix} \quad X_{18} \begin{matrix} 2&3&1&4&0&5\\ 4&5&0&3&2&1\\ 0&1&2&5&3&4\\ 3&4&5&0&1&2\\ 1&0&4&2&5&3\\ 5&2&3&1&4&0 \end{matrix} \quad X_{19} \begin{matrix} 2&3&1&5&0&4\\ 4&5&0&3&1&2\\ 0&1&2&4&3&5\\ 3&4&5&0&2&1\\ 1&0&4&2&5&3\\ 5&2&3&1&4&0 \end{matrix} \quad X_{20} \begin{matrix} 2&3&1&5&0&4\\ 4&5&0&3&1&2\\ 0&1&2&4&5&3\\ 3&4&5&1&2&0\\ 1&0&4&2&3&5\\ 5&2&3&0&4&1 \end{matrix}$$

$$X_{21} \begin{matrix} 2&3&0&4&1&5\\ 4&5&1&3&2&0\\ 0&1&2&5&4&3\\ 3&4&5&2&0&1\\ 1&2&3&0&5&4\\ 5&0&4&1&3&2 \end{matrix} \quad X_{22} \begin{matrix} 2&3&0&4&1&5\\ 4&5&1&3&2&0\\ 0&1&2&5&3&4\\ 3&4&5&2&0&1\\ 1&2&4&0&5&3\\ 5&0&3&1&4&2 \end{matrix} \quad X_{23} \begin{matrix} 2&3&0&5&1&4\\ 4&5&1&3&0&2\\ 0&1&2&4&5&3\\ 3&4&5&1&2&0\\ 1&2&4&0&3&5\\ 5&0&3&2&4&1 \end{matrix} \quad X_{24} \begin{matrix} 2&3&1&4&0&5\\ 4&5&0&3&2&1\\ 0&1&2&5&4&3\\ 3&4&5&2&1&0\\ 1&2&3&0&5&4\\ 5&0&4&1&3&2 \end{matrix}$$

$$X_{25} \begin{matrix} 2&3&1&5&0&4\\ 4&5&0&3&1&2\\ 0&1&2&4&5&3\\ 3&4&5&1&2&0\\ 1&2&3&0&4&5\\ 5&0&4&2&3&1 \end{matrix} \quad X_{26} \begin{matrix} 2&5&0&3&1&4\\ 4&3&1&5&0&2\\ 0&1&2&4&5&3\\ 3&4&5&1&2&0\\ 1&0&4&2&3&5\\ 5&2&3&0&4&1 \end{matrix}$$

Figure 11: Representatives of the 26 G-orbits in $\mathcal{R}$

in some fixed order. Observe that L_K is a latin square.[2] Note also that if we form a new square L from L_K by some combination of permuting rows, permuting columns, and relabeling, then L is again a Latin square. We say that L is **isotopic** to L_K and we say that L is **based on $\boldsymbol{K}$**. The combination of permutations and relabeling used to form L from L_K is called an **isotopism**. Group-based Latin squares have been used to construct large collections of pairwise mutually orthogonal Latin squares (see [8]). Regarding sudoku, in [3] it is shown, using basic group theory, that given any group K of order k possessing a subgroup J of order j, one can produce a K-based sudoku square of order k with $k/j \times k$ regions.

If we're presented with a Latin square, how do we know whether it is based on a group? From [6], for example, we have:

Proposition 4.1 (Quadrangle Criterion) *A Latin square $A = (a_{i,j})$ is based on a group if and only if whenever $a_{j,k} = a_{j_1,k_1}$, $a_{i,k} = a_{i_1,k_1}$, and $a_{i,\ell} = a_{i_1,\ell_1}$, we also have $a_{j,\ell} = a_{j_1,\ell_1}$.*

Upon inspection, each representative $x_1, x_2, \ldots, x_{26}$ from Figure 11 fails the Quadrangle Criterion except for x_{26}. So x_{26} is based on a group, either the symmetric group S_3 or the cyclic group $\mathbb{Z}_6$. We may reconstruct a group table whose body is x_{26} by selecting an arbitrary row and column to be the column headers and row headers, respectively. Selecting the top row and leftmost column yields the operation table

$*$	2	5	0	3	1	4
2	2	5	0	3	1	4
4	4	3	1	5	0	2
0	0	1	2	4	5	3
3	3	4	5	1	2	0
1	1	0	4	2	3	5
5	5	2	3	0	4	1

Since no element in this table is of order six, we conclude that x_{26} is based on S_3. Further, if $K \in \{S_3, \mathbb{Z}_6\}$, then the property of being a K-based square is invariant under the action of the rokudoku-pair group G: This is because elements G are isotopisms, modulo transpose, and transpose carries a K-based square to a K-based square. Therefore:

Theorem 4.2 *The only group-based rokudoku-pair squares are the members of $\mathcal{R}_{26}$; all of these squares are based on the symmetric group S_3.*

[2] Every Latin square is the body of a quasigroup operation table. A quasigroup is a set with a binary operation that satisfies left and right cancellation.

Recall our mention of the theorem from [3] asserting that given any group K of order k possessing a subgroup J of order j, one can produce a K-based sudoku square of order k with $k/j \times j$ regions. Because there are no $\mathbb{Z}_6$-based rokudoku-pair squares, this theorem does not extend to the setting of sudoku-pair Latin squares. More on group-based sudoku-pair Latin squares is forthcoming in [5]. Also, from Theorem 3.2 we know that $\mathcal{R}_{26}$ is among the smallest G-orbits in $\mathcal{R}$, with size 11520. Combining this with Theorem 2.1 we find that only about 0.8% of rokudoku-pair squares are based on groups.

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