Notes for

Analysis on Manifolds via the Laplacian

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Abstract

These are the notes of a course I taught on Fall 2013 at Harvard University. Any comments and suggestions are welcome. I plan on improving these notes next time I teach the course. The notes may have mistakes, so use them at your own risk. Also, many citations are missing. The following references were important sources for these notes:

- *Eigenvalues in Riemannian geometry.* By I. Chavel.
- *Old and new aspects in Spectral Geometry.* By M. Craiveanu, M. Puta and T. Rassias.
- *The Laplacian on a Riemannian manifold.* By S. Rosenberg.
- *Local and global analysis of eigenfunctions on Riemannian manifolds.* By S. Zelditch.

I would like to thank Evans Harrell and Richard Laugesen for sharing with me their thoughts and experiences on teaching courses like this one.

Enjoy!!

Contents

CHAPTER 1

What makes the Laplacian special?

In this Chapter we motivate the study of the Laplace operator. To simplify exposition, we do this by concentrating on planar domains.

1.1 Almost daily life problems

Let $\Omega \subset \mathbb{R}^n$ be a connected domain and consider the operator Δ acting on $C^\infty(\Omega)$ that simply differentiates a function $\varphi \in C^\infty(\Omega)$ two times with respect to each position variable:

$$\Delta\varphi = -\sum_{i=1}^{n} \frac{\partial^2 \varphi}{\partial x_i^2}.$$

This operator is called the *Laplacian* on Ω. You might also have seen it defined as $\Delta = -\mathrm{div}\nabla$. This is actually the definition of the *Laplacian* on a Riemannian manifold (M, g). Then the Riemannian Laplacian is defined as

$$\Delta_g = -\mathrm{div}_g \nabla_g$$

where div_g is the divergence operator and ∇_g is the gradient one. Here are some examples where the Laplacian plays a key role:

Steady-state Fluid Flow. Suppose you want to study the velocity $v(x_1, x_2, x_3, t)$ of a given fluid. If the flow is steady, then the velocity field should be independent of the time t. If the flow is irrotational, $\mathrm{curl}\, v = 0$, then $v = -\nabla u$ for some function u known as the velocity potential. If the flow is incompressible, then $\mathrm{div}\, v = 0$. It then follows that u must satisfy the equation

$$\Delta u = 0.$$

A function that satisfies such equation is called a *harmonic function*. Thus the velocity potential for an incompressible irrotational fluid is harmonic.

Static electric field. A static electric field E is governed by the following equations $\mathrm{curl}E = 0$ and $\mathrm{div}E = 4\pi\rho$ where ρ denotes the charge density. Since $\mathrm{curl}E = 0$ it follows that $E = -\nabla u$ for some function known as the electric potential. You then must have $\Delta u = 0$. In other words, the electric potential in a charge free region is harmonic.

Heat diffusion. If you are interested in understanding how would heat propagate along $\Omega \subset \mathbb{R}^n$ then you should solve the *Heat Equation*

$$\Delta u(x,t) = -\frac{1}{c}\frac{\partial}{\partial t}u(x,t)$$

where c is the conductivity of the material of which Ω is made of, and $u(x,t)$ is the temperature at the point $x \in \Omega$ at time t.

You could also think you have an insulated region Ω (it could be a wire, a ball, etc.) and apply certain given temperatures on the edge $\partial\Omega$. If you want to know what the temperature will be after a long enough period of time (that is, the steady state temperature distribution), then you need to find a solution of the heat equation that be independent of time. The steady state temperature solution will be a function $u(x_1, \ldots, x_n, t)$ such that

$$\Delta u = 0.$$

Wave propagation. Now, instead of applying heat to the surface suppose you cover it with a thin layer of some fluid and you wish to describe the motion of the surface of the fluid. Then you will need to solve the *Wave equation*

$$\Delta u(x,t) = -\frac{1}{c}\frac{\partial^2}{\partial t^2}u(x,t)$$

where $\sqrt{c}$ is the speed of sound in your fluid, and $u(x,t)$ denotes the height of the wave above the point x at time t.

You could also think of your domain Ω as the membrane of a drum, in which case its boundary $\partial\Omega$ would be attached to the rim of the drum. Suppose you want to study what will happen with the vibration you would generate if you hit it. Then, you should also solve the wave equation $\Delta u(x,t) = -\frac{\partial^2}{\partial^2 t}u(x,t)$ for your drum, but this time you want to make sure that you take into account that the border of the membrane is fixed. Thus, you should also ask your solution to satisfy $u(x,t) = 0$ for all points $x \in \partial\Omega$.

Quantum particles. If you are a bit more eccentric and wish to see how a quantum particle moves inside Ω (under the assumption that there are no external forces) then you need to solve the *Schrödinger Equation*

$$\frac{\hbar^2}{2m}\Delta u(x,t) = i\hbar\frac{\partial}{\partial t}u(x,t)$$

where $\hbar$ is Planck's constant and m is the mass of the free particle. Normalizing u so that $\|u(\cdot,t)\|_{L^2(\Omega)} = 1$ one interprets $u(x,t)$ as a probability density. That is, if $A \subset \Omega$ then the probability that your quantum particle be inside A at time t is given by $\int_A |u(x,t)|^2 dx$.

1.2 Why not another operator?

The Laplacian on $\mathbb{R}^n$ commutes with translations and rotations. That is, if T is a translation or rotation then $\Delta(\varphi \circ T) = (\Delta\varphi) \circ T$. Something more striking occurs, if S is any operator that commutes with translations and rotations then there exist coefficients $a_1, \dots, a_m$ making $S = \sum_{j=1}^{m} a_j \Delta^j$. Therefore, it is not surprising that the Laplacian will be a main star in any process whose underlying physics are independent of position and direction such as heat diffusion and wave propagation in $\mathbb{R}^n$. We will show that on general Riemannian manifolds the Laplacian commutes with isometries.

1.3 You need to solve $\Delta\varphi = \lambda\varphi$!

There are of course many more problems involving the Laplacian, but we will focus on these ones to stress the importance of solving the eigenvalue problem (also known as Helmholtz equation)

$$\Delta\varphi = \lambda\varphi.$$

It is clear that if one wants to study harmonic functions then one needs to solve the equation

$$\Delta\varphi = \lambda\varphi \qquad \text{with} \quad \lambda = 0.$$

So the need for understanding solutions of the Helmholtz equation for problems such as the static electric field or the steady-state fluid flow is straightforward. In order to attack the heat diffusion, wave propagation and Schrödinger problems described above, a standard method (inspired by Stone-Weierstrass Theorem) is to look for solutions $u(x,t)$ of the form $u(x,t) = \alpha(t)\varphi(x)$. For instance if you do this and look at the Heat equation then you must have

$$\frac{\Delta\varphi(x)}{\varphi(x)} = -\frac{\alpha'(t)}{\alpha(t)} \qquad x \in \Omega, \quad t > 0.$$

This shows that there must exist a $\lambda \in \mathbb{R}$ such that

$$\alpha' = -\lambda\alpha \quad \text{and} \quad \Delta\varphi = \lambda\varphi.$$

Therefore φ must be an eigenfunction of the Laplacian with eigenvalue λ and $\alpha(t) = e^{-\lambda t}$. Once you have these particular solutions $u_k = e^{-\lambda_k t}\varphi_k$ you use the superposition principle to write a general solution

$$u(x,t) = \sum_k a_k e^{-\lambda_k t}\varphi_k(x)$$

where the coefficients a_k are chosen depending on the initial conditions. You could do the same with the wave equation (we do it in detail for a guitar string in Section 2.1) or with the Schrödingier equation and you will also find particular solutions of the form $u_k(x,t) = \alpha_k(t)\varphi_k(x)$ with

$$\Delta\varphi_k = \lambda_k\varphi_k \qquad \text{and} \qquad \alpha_k(t) = \begin{cases} e^{-\lambda_k t} & \text{Heat eqn,} \\ e^{i\sqrt{\lambda_k}t} & \text{Wave eqn,} \\ e^{i\lambda_k t} & \text{Schrödinger eqn.} \end{cases}$$

1.4 A hard problem: understanding the eigenvalues

Section 1.3 shows why it is so important to understand the eigenvalues λ_k together with the eigenfunctions φ_k of the Laplacian. The truth is that doing so is a very hard task. Indeed one can only explicitly compute the eigenvalues for very specific choices of regions Ω such as rectangles, discs, ellipses and a few types of triangles (see Section **??**). Understanding the eigenvalues is so hard that for hexagons not even the first eigenvalues is known! This brings us to the following question:

Question 1 (Direct problem)**.** If I know (more or less) the shape of a domain, what can I deduce of its Laplace eigenvalues?

On the other side of the road, it wouldn't be weird to expect the eigenvalues of the Laplacian on Ω to carry some information of the geometry of Ω.

Question 2 (Inverse problem)**.** If I know (more or less) the Laplace eigenvalues of a domain, what can I deduce of its geometry?

1.4.1 Direct problems

The first eigenvalue: Rayleigh Conjecture

The first eigenvalue λ_1 of the Laplacian on an interval or a region of the plane is called the *fundamental tone*. This is because either on a vibrating guitar string or drum membrane the first eigenvalue corresponds to the leading frequency of oscillation and it is therefore the leading tone you hear when you play one of these instruments. Seen from a heat-diffusion point of view, since the solutions of the heat equation are of the form $u(x,y,t) = \sum_n a_n e^{-\lambda_n t}\varphi_n(x,y)$, it is clear that (λ_1, φ_1) give the dominant information because $e^{-\lambda_1 t}\varphi_1(x,y)$ is the mode that decays with slowest rate as time passes by. From this last point of view it is natural to expect that the geometry of Ω should be reflected on λ_1 to some extent. For instance the largest the boundary $\partial\Omega$ is, the more quickly the heat should wear off. That is, if we consider a domain Ω and a ball B of same area as Ω, then we expect the heat on Ω to diffuse more quickly than that of B. Therefore, we should have

Faber-Krahn Inequality:

$$\lambda_1(\Omega) \geq \lambda_1(B).$$

This result was proved by Faber and Krahn in 1923. As expected, it extends to any dimension.

Counting function: Lorentz conjecture

Jeans asked once what is the energy corresponding to an infinitesimal frequency interval. In 1966 Mark Kac told this story in a very illustrating manner:

> ...At the end of October of 1910 the great Dutch physicist H. A. Lorentz was invited to Götingen to deliver a Wolfskehl lecture... Lorentz gave five lectures under the overall title “Old and new problems of physics” and at

the end of the fourth lecture he spoke as follows (in free translation from the original German):

In conclusion, there is a mathematical problem which perhaps will arouse the interest of mathematicians who are present. It originates in the radiation theory of Jeans.

In an enclosure with a perfectly reflecting surface, there can form standing electromagnetic waves analogous to tones over an organ pipe: we shall confine our attention to very high overtones. Jeans asks for the energy in the frequency interval $d\nu$. To this end, he calculates the number of overtones which lie between frequencies ν and $\nu + d\nu$, and multiplies this number by the energy which belongs to the frequency ν, and which according to a theorem of statistical mechanics, is the same for all frequencies.

It is here that there arises the mathematical problem to prove that the number of sufficiently high overtones which lie between ν and $\nu + d\nu$ is independent of the shape of the enclosure, and is simply proportional to its volume. For many shapes for which calculations can be carried out, this theorem has been verified in a Leiden dissertation. There is no doubt that it holds in general even for multiply connected regions. Similar theorems for other vibrating structures, like membranes, air masses, etc., should also hold.

If we express the Lorentz conjecture in a vibrating membrane Ω, it becomes of the following form: Let $\lambda_1 \leq \lambda_2 \leq \dots$ be the Laplace eigenvalues corresponding to the problem

$$\Delta\varphi_k = \lambda_k \varphi_k \qquad \varphi_k|_{\partial\Omega} = 0.$$

Then

$$N(\lambda) = \#\{\lambda_k : \ \lambda_k < \lambda\} \sim \frac{\text{area}(\Omega)}{2\pi}\lambda \qquad \text{as } \lambda \to \infty.$$

D. Hilbert was attending these lectures and predicted as follows: “This theorem would not be proved in my life time.” But, in fact, Hermann Weyl, a graduate student at that time, was also attending these lectures. Weyl proved this conjecture four months later in February of 1911.

We will prove this in specific examples such as rectangles and the torus. Later on we will prove the analogue result for compact Riemannian manifolds (M, g). Let $\lambda_0 \leq \lambda_1 \leq \dots$ be the Laplace eigenvalues repeated according to its multiplicity. Then

$$N(\lambda) \sim \frac{\omega_n}{(2\pi)^n} Vol(M)\lambda^{n/2}, \qquad \lambda \to \infty$$

where ω_n is the volume of the unit ball in $\mathbb{R}^n$.
In particular,

$$\lambda_j \sim \frac{(2\pi)^2}{(\omega_n Vol(M))^{2/n}} j^{2/n}, \qquad j \to \infty.$$

1.4.2 Inverse problem: Can you hear the shape of a drum?

Suppose you have perfect pitch. Could you derive the shape of a drum from the music you hear from it? More generally, can you determine the structural soundness of an object by listening to its vibrations? This question was first posed by Schuster in 1882. As Berger says in his book *A panoramic view of Riemannian Geometry*,

> "Already in the middle ages bell makers knew how to detect invisible cracks by sounding a bell on the ground before lifting it up to the belfry. How can one test the resistance to vibrations of large modern structures by non-destructive essays?... A small crack will not only change the boundary shape of our domain, one side of the crack will strike the other during vibrations invalidating our use of the simple linear wave equation. On the other hand, heat will presumably not leak out of a thin crack very quickly, so perhaps the heat equation will still provide a reasonable approximation for a short time..."

Infinite sequences of numbers determine via Fourier analysis an integrable function. It wouldn't be that crazy if an infinite sequences of eigenvalues would determine the shape of the domain. Unfortunately, the answer to the question *can you hear the shape of a drum?* is no. This was proved in 1992 by Gordon, Web and Wolpert [**?**]. Nowadays many planar domains are known to have different shapes but exactly the same spectrum.

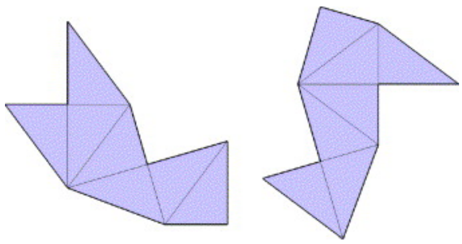


Figure: Two domains with the same eigenvalues
Picture from the paper *LaplaceBeltrami spectra as Shape-DNA of surfaces and solids*

Not all is lost. One can still derive a lot of information of a domain by knowing its eigenvalues. Using the heat kernel, in 1966 Mark Kac proved the formula

$$\sum_{k=1}^{\infty} e^{-\lambda_k t} \sim \frac{1}{4\pi t}\left(\operatorname{area}(\Omega) - \sqrt{4\pi t}\,\operatorname{length}(\partial\Omega) + \frac{2\pi t}{3}(1-\gamma(\Omega))\right)$$

where $\gamma(\Omega)$ is the genus of Ω and Ω is a polygon. The eigenvalues λ_n are the ones corresponding to the Laplacian on Ω enforcing $\varphi_k|_{\partial\Omega} = 0$. A year later McKean and Singer (1967) proved the same result in the context of Riemannian manifolds with boundary.

This means that if you know the full sequence of eigenvalues of your favorite domain Ω then you can deduce its area, its perimeter and the number of holes in it!!

On a compact Riemmanian manifold without boundary Minakshisundaram (1953) proved the analog weaker result

$$\sum_{k=1}^{\infty} e^{-\lambda_k t} \sim \frac{1}{(4\pi t)^{\frac{n}{2}}}\left(\operatorname{vol}(M) + \frac{t}{6}\int_M R_g(x)\,\omega_g + O(t^2)\right),$$

where R_g denotes the scalar curvature. So you can hear the dimension, the volume and the total scalar curvature of a compact Riemannian manifold.

1.5 An extremely hard problem: understanding the eigenfunctions

Eigenfunctions of the Laplacian play a key role whenever it comes to do analysis on Riemannian manifolds. One of the main reasons is that they are the key ingredient to carry an analog of Fourier series on manifolds. Indeed, as we shall prove later, we have

Sturm-Liouville's decomposition. Give a compact Riemannian manifold (M, g) there is an orthonormal basis $\varphi_1, \ldots, \varphi_j, \ldots$ of eigenfunctions of the Laplacian Δ_g with respective eigenvalues $0 \leq \lambda_1 \leq \cdots \leq \lambda_j \leq \ldots$ such that any function $\phi \in L^2(M)$ can be written as a convergent series in $L^2(M)$

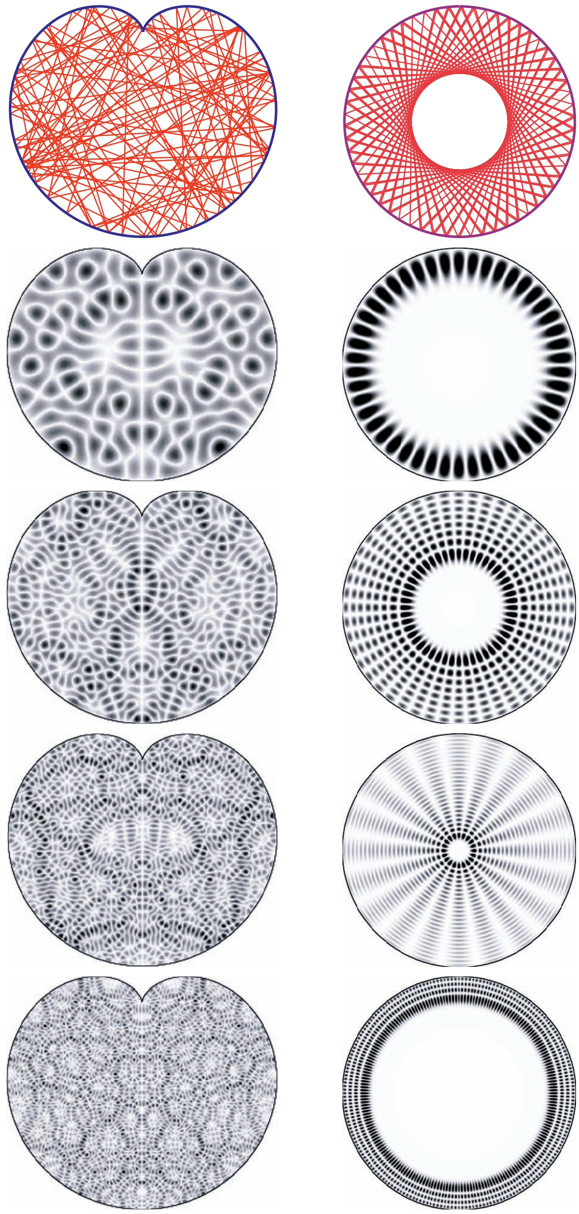
$$\phi = \sum_{j=1}^{\infty} a_j \varphi_j$$

for some coefficients $a_j \in \mathbb{R}$.

On the other hand, as already mentioned, we may interpret the eigenfunction φ_j as the probability density of a quantum particle in the energy state λ_j. That is, the probability that a quantum particle in the state φ_j belongs to the set $A \subset M$ if given by

$$\int_A |\varphi_j|^2 \, \omega_g.$$

High energy eigenfunctions are expected to reflect the dynamics of the geodesic flow. In the energy limit $\lambda \to \infty$ one should be able to recover classical mechanics from quantum mechanics. In the following picture (taken from *Many-body quantum chaos: Recent developments and applications to nuclei*) you can see how the dynamics of the geodesic flow for two different systems is reflected on the eigenfunctions. In the left column a cardioid billiard is represented. In the right column a ring billiard is shown. In the first line the trajectories of the geodesic flow for each system is shown. Then, from the second line to the fifth one, the graph of the functions $|\varphi_j|^2$ is shown for $\lambda_j = 100, 1000, 1500, 2000$. The darker the color, the higher the value of the modulus. One can see how a very chaotic system, such as the cardioid, yields a uniform distribution (chaotic) of the eigenfunctions. On the other hand, a very geometric dynamical system, such as the ring, yields geometric distributions of the eigenfunctions.



A beautiful result about the behavior of eigenfunctions takes place on manifolds with ergodic geodesic flow (like the cardioid above), including all manifolds with negative constant sectional curvature. This result says that in the high energy limit eigenfunctions are equidistributed.

Quantum ergodicity. If (M, g) is a compact manifold with ergodic geodesic flow then there exists a density one subsequence of eigenfunctions $\{\varphi_{j_k}\}_k$ such that for any $A \subset M$

$$\lim_{k\to\infty} \int_M \varphi_{j_k}^2\, \omega_g = \frac{\mathrm{vol}(A)}{\mathrm{vol\ (M)}}.$$

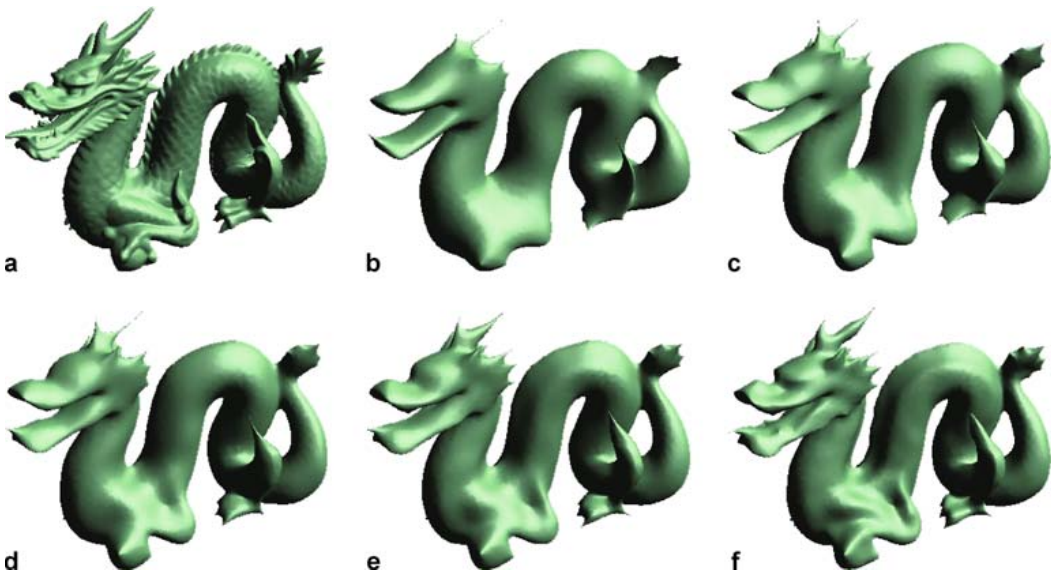
By density one subsequence it is meant that $\inf_m \frac{\#\{k:\, j_k \leq m\}}{m} = 1$. This result is due to Schnirelman (1973) finished by Colin de Verdiere (1975).

Image processing. One may describe a give surface $M \subset \mathbb{R}^3$ by a function such as the normal vector. That is, to every point $x \in M$ you associate the normal vector at x

$$x \mapsto (n_1(x), n_2(x), n_3(x)).$$

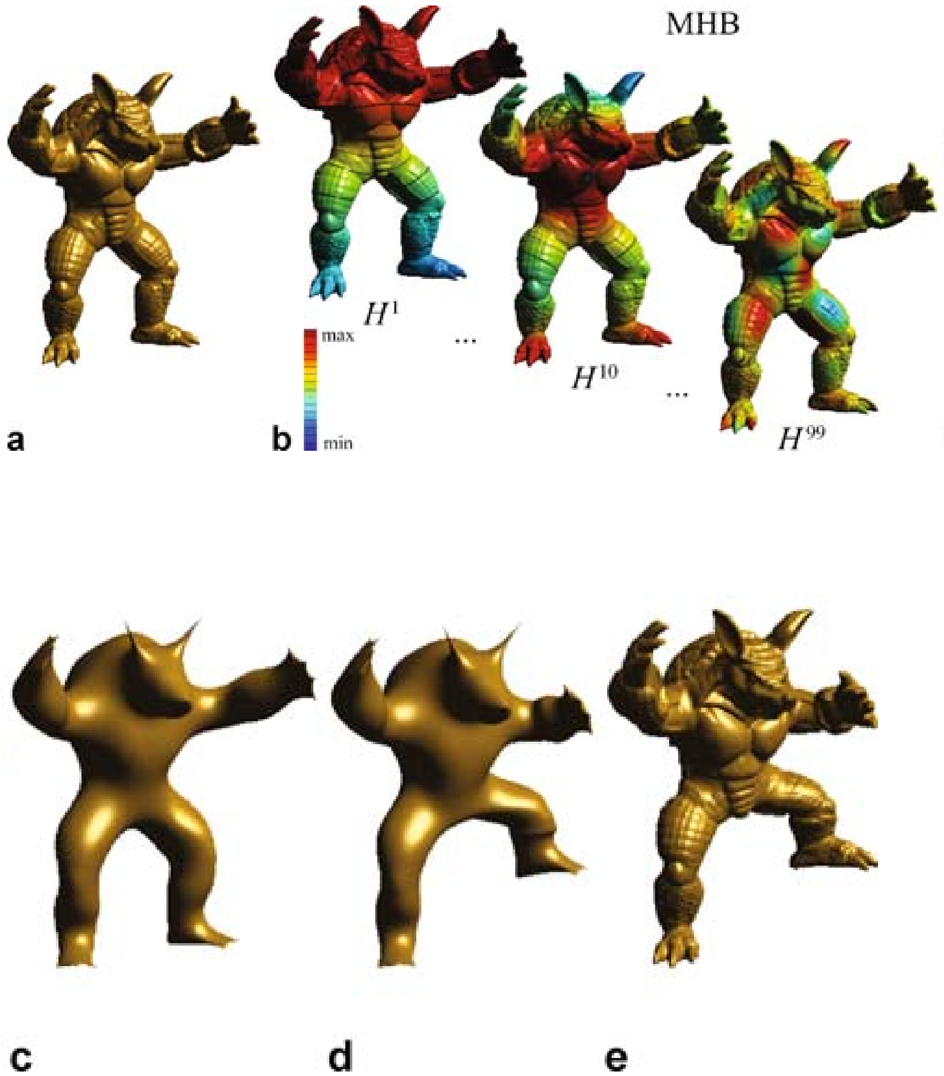
Each function $n_j : M \to \mathbb{R}$ can be rewritten as an infinite series $n_j = \sum_{i=1}^{\infty} a_i^{(j)} \varphi_i$ by the Sturm-Liouville decomposition Theorem. You may then truncate the series and work with the approximates manifold described by the function

$$x \mapsto \left(\sum_{i=1}^{N} a_i^{(1)} \varphi_i(x), \sum_{i=1}^{N} a_i^{(2)} \varphi_i(x), \sum_{i=1}^{N} a_i^{(3)} \varphi_i(x) \right).$$



a. M =Dragon. **b-f.** reconstructed manifold using $N = 100, 200, 300, 500$ and 900 eigenfunctions respectively. Picture from paper *Spectral mesh deformation.*

This is a way of encoding the geometry of a surface to some extent using as little information as you want (at the risk of having a worse approximation). In practice the the way people have of computing eigenfunctions on the dragon is to discretize it and work with a discretized version of the Laplacian and its corresponding eigenfunctions. For instance, the method of approximating a surface by a finite number of eigenfunctions is used to perform a change of the position of some part of the surface. Suppose you have an armadillo standing on two legs (figure **a**) and you wish to lift one of its legs (figure **e**) reducing as much as possible the amount of computations that need to be carried to get the final result. What people are doing is to compute the first 99 eigenfunctions on the (discretized) armadillo (figure **b**) and approximate the armadillo by them (figure **c**). Then, you apply the transformation to the approximate armadillo (figure **d**). Doing this is much cheaper -computation wise- than applying the transformation to the original armadillo. You may then add all the details to the transformed armadillo by an algorithm called.

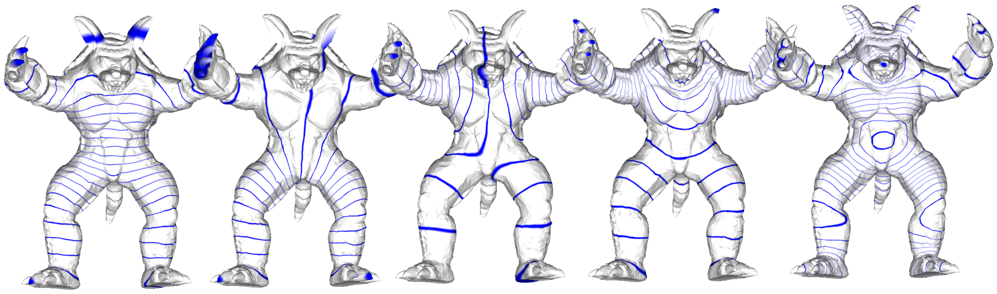


Picture from the article *Spectral mesh deformation.*

Another way of understanding the behavior of eigenfunctions is to study their nodal sets. The set of points where an eigenfunction vanishes is like the skeleton of the eigenfunction. Let us write

$$\mathcal{N}_{\varphi_j} := \{x \in M : \ \varphi_j(x) = 0\}.$$

Nodal sets play the role of the skeleton of your manifold. They encode several aspects of the geometry of the manifold. This is illustrated in the following figure where the nodal sets of some eigenfunctions on the armadillo are colored in blue.



Picture from the article *Laplace-Beltrami eigenfunctions towards an algorithm that "understands" the geometry.*

From a quantum mechanics point of view, nodal sets can be interpreted as the least likely place for a quantum particle in the state φ_j to be. This is because $\int_{\mathcal{N}_{\varphi_j}} \varphi_j^2 \, \omega_g = 0$.

One of the main problems about nodal sets is estimating their size. A famous conjecture on this matter is known as **Yau's conjecture on nodal set's sizes**. It says that there exist positive constants c, C such that

$$c\sqrt{\lambda_j} \leq \mathrm{vol}(\mathcal{N}_{\varphi_j}) \leq C\sqrt{\lambda_j} \qquad \text{as } j \to \infty.$$

Yau's conjecture has only been prove on compact manifolds with analytic Riemannian metric. This result is due to Donelly and Fefferman (1988).

CHAPTER 2

Laplacian in Euclidean spaces

Spectrum of the Laplacian

The aim of this chapter is to compute explicitly in some special easy cases the eigenvalues and eigenfunctions of the Laplacian operator subject to different boundary conditions. We first do this in one dimensional spaces (segments and circles). We then solve the wave equation of an interval in detail and explain the importance of the first eigenvalue and how it relates to the length of the interval. We then study two dimensional domains such as rectangles and discs. We then prove Weyl asymptotics for the rectangle and show how they encode the area of it.

Given a domain Ω with boundary (the reader might think of it as an interval, or a membrane, or an arbitrary manifold) there are two important boundary conditions one may impose on the solutions φ of the eigenvalue problem $\Delta\varphi = \lambda\varphi$:

Dirichlet boundary conditions: $\quad \varphi|_{\partial\Omega} = 0.$

This is used for instance when your domain $\Omega \subset \mathbb{R}^2$ is a membrane and you fix its boundary as if Ω was a drum. Since you don't have any vibrations on the rim of a drum you must have $\varphi|_{\partial\Omega} = 0$.

Neuman boundary conditions: $\quad \partial_\nu\varphi|_{\partial\Omega} = 0.$

Here ν is the unit outward normal vector to the boundary $\partial\Omega$. This condition is used for example when a surface has a prescribed heat flux, such as a perfect insulator (the heat doesn't go through the boundary, when it hits it it stays in).

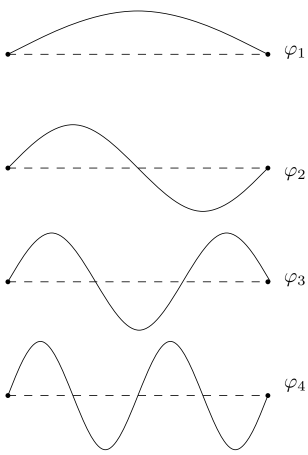
2.1 Interval

Consider an interval $[0, \ell]$.
– *Dirichlet boundary conditions:* $\varphi(0) = \varphi(\ell) = 0$.

The eigenfunctions are

$$\varphi_k(x) = \sin\big(\frac{k\pi}{\ell} x\big) \qquad \text{for } k \geq 1$$

with eigenvalues $\lambda_k = \big(\frac{k\pi}{\ell}\big)^2$ for $k \geq 1$.



– *Neumann boundary conditions:* $\varphi'(0) = \varphi'(\ell) = 0$.

The eigenfunctions are

$$\varphi_k(x) = \cos\big(\frac{k\pi}{\ell} x\big) \qquad \text{for } k \geq 1$$

with eigenvalues $\lambda_k = \big(\frac{k\pi}{\ell}\big)^2$ for $k \geq 0$.

Observations.
Note that if we scale our domain by a factor $a > 0$ we get $\lambda_k\big[(0, a\ell)\big] = \frac{1}{a^2}\,\lambda_k\big[(0,\ell)\big]$. Intuitively, the eigenvalue λ must balance $\frac{d^2}{dx^2}$, and so $\lambda \sim (\text{length scale})^{-2}$. We also note that we have the asymptotics

$$\lambda_k \sim C\, k^2$$

where C is a constant independent of k.
For $\lambda > 0$ consider the eigenvalue counting function

$$N(\lambda) = \#\{\text{eigenvalues} \leq \lambda\}.$$

Proposition 1. *(Weyl's law for intervals) Write λ_j for the Dirichlet or Neumann eigenvalues of the Laplacian on the interval $\Omega = [0, \ell]$. Then,*

$$N(\lambda) \sim \frac{\mathit{length}(\Omega)}{\pi}\sqrt{\lambda}.$$

Proof.
$$N(\lambda) = \max\{k : \lambda_k < \lambda\} = \max\left\{k : \frac{k^2\pi^2}{\ell^2} < \lambda\right\} \sim \frac{\ell}{\pi}\sqrt{\lambda}.$$
□

Hear the length of a guitar string (solving the wave equation)

After the first half of the 18th century mathematicians such as d'Alembert and Bernoulli developed the theory of a vibrational string. As one should expect, the vibrations of a string will depend on many factors such us its length, mass and tension. To simplify our exposition consider a guitar string of length ℓ which we model as the interval $[0, \ell]$. Assume further that the density mass and the tension are constant and equal to 1. Today it comes as no surprise that the behavior of a vibrating string is described by the wave equation. That is, if we write x for a point in the string $[0, \ell]$ and t for the time variable, then the height $u(x, t)$ of the string above the point x after a time t should satisfy the wave equation
$$\Delta u(x,t) = -\frac{\partial^2}{\partial t^2}u(x,t).$$
There are infinitely many solutions to this problem. But we already know that there are constraints to this problem that we should take into account since the endpoints of the string are fixed and so $u(x, t)$ must satisfy $u(0, t) = 0 = u(\ell, t)$ for all time t. In addition having a unique solution to our problem depends upon specifying the initial shape of the string $f(x) = u(x, 0)$ and its initial velocity $g(x) = \partial_t u(x, 0)$. All in all, we are solving the system

$$\begin{cases} -\frac{\partial^2}{\partial x^2}u(x,t) = -\frac{\partial^2}{\partial t^2}u(x,t) & x \in [0,\ell],\ t > 0, \\ u(0,t) = 0 = u(\ell,t) & t > 0, \\ u(x,0) = f(x) & x \in [0,\ell], \\ \partial_t u(x,0) = g(x) & x \in [0,\ell]. \end{cases}$$

A general sulution of this problem has the form

$$u(x,t) = \sum_{k=1}^{\infty} \alpha_k(t)\varphi_k(x), \tag{2.1}$$

where
$$\alpha_k(t) = a_k \cos\left(\frac{k\pi}{\ell}t\right) + b_k \sin\left(\frac{k\pi}{\ell}t\right).$$
The coefficients are $a_k = \langle f, \varphi_k\rangle$ and $b_k = \langle g, \varphi_k\rangle$.

The functions φ_k are called harmonic modes for the string $[0, \ell]$. Since $f_k = k\pi$ is the frequency of the wave $\varphi_k(x) = \sin\left(\frac{k\pi}{\ell}x\right)$ the connection between the eigenvalues λ_k and the frequencies f_k of the harmonic modes of the string is obvious:
$$f_k = \frac{1}{2\pi}\sqrt{\lambda_k}.$$

Therefore, the higher the eigenvalue, the higher the frequency is.

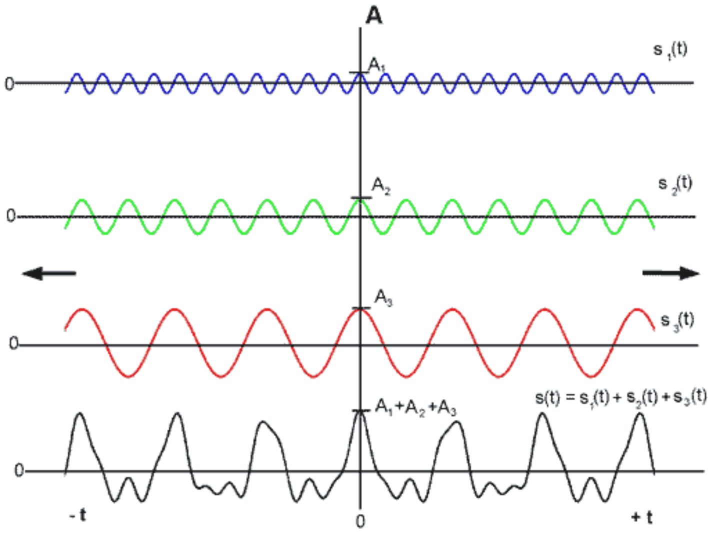
Consider the Fourier transform of a function φ as

$$\varphi(\xi) = \int_{-\infty}^{\infty} \varphi(x) e^{-2\pi i x \xi}.$$

The Fourier transform of the function $\varphi_k(x) = \sin\left(\frac{k\pi}{\ell} x\right)$ is

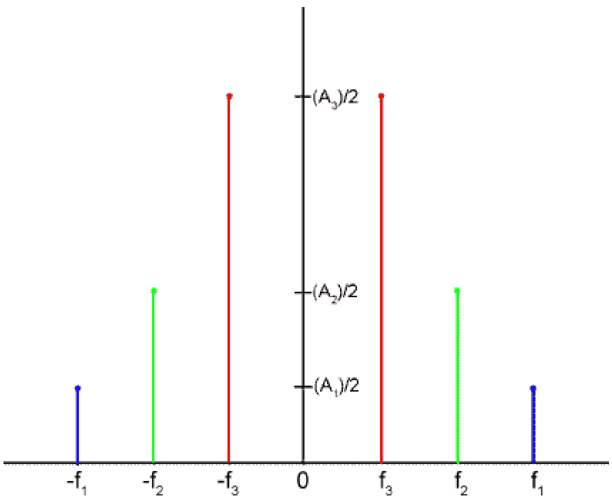
$$\mathcal{F}(\varphi_k)(\xi) = i\sqrt{\frac{\pi}{2}}\left(\delta\left(\xi - \frac{k\pi}{\ell}\right) - \delta\left(\xi + \frac{k\pi}{\ell}\right)\right).$$

In th following picture the graphs of $s_k(x) = A_k \cos\left(\frac{k\pi}{\ell} x\right)$ for $k = 1, 2, 3$ and in the last line we put the graph of $s_1 + s_2 + s_3$.



Picture from *www-rohan.sdsu.edu/ jiracek/DAGSAW/3.4.html.*

In the following picture the Fourier transform of the function $s_1 + s_2 + s_3$ is shown.



Picture from *www-rohan.sdsu.edu/ jiracek/DAGSAW/3.4.html.*

If you pluck a guitar string then you obtain a wave of the form $u(x,t) = \sum_{k=1}^{\infty} \alpha_k(t)\varphi_k(x)$, and by applying the Fourier transform to it you get

$$\mathcal{F}(u(\cdot,t))(\xi) = i\sqrt{\frac{\pi}{2}} \sum_{k=1}^{\infty} \alpha_k(t)\Big(\delta\Big(\xi - \frac{k\pi}{\ell}\Big) - \delta\Big(\xi + \frac{k\pi}{\ell}\Big)\Big).$$

So you recover all the relevant frequencies and hence all the eigenvalues.

2.2 Rectangle

Consider a rectangle $\Omega = [0,\ell] \times [0,m]$. Separate variables using carthesian coordinates x and y. That is, look for solutions of the form $\varphi(x,y) = f(x)g(y)$.

– *Dirichlet boundary conditions:* $\varphi|_{\partial\Omega} = 0$.

The eigenfunctions are

$$\varphi_{jk}(x,y) = \sin\left(\frac{j\pi}{\ell}x\right)\sin\left(\frac{k\pi}{m}y\right) \qquad \text{for } j,k \geq 1,$$

and have eigenvalues

$$\lambda_{jk} = \left(\frac{j\pi}{\ell}\right)^2 + \left(\frac{k\pi}{m}\right)^2 \qquad \text{for} j,k \geq 1.$$

– *Neumann boundary conditions:* $\partial_\nu\varphi|_{\partial\Omega} = 0$.

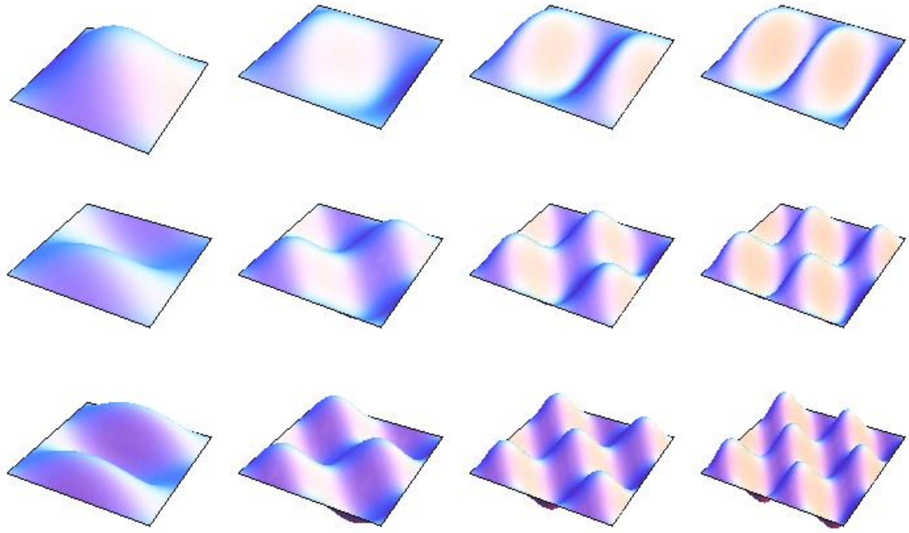
The eigenfunctions are

$$\varphi_{jk}(x,y) = \cos\left(\frac{j\pi}{\ell}x\right)\cos\left(\frac{k\pi}{m}y\right) \qquad \text{for } j,k \geq 0,$$

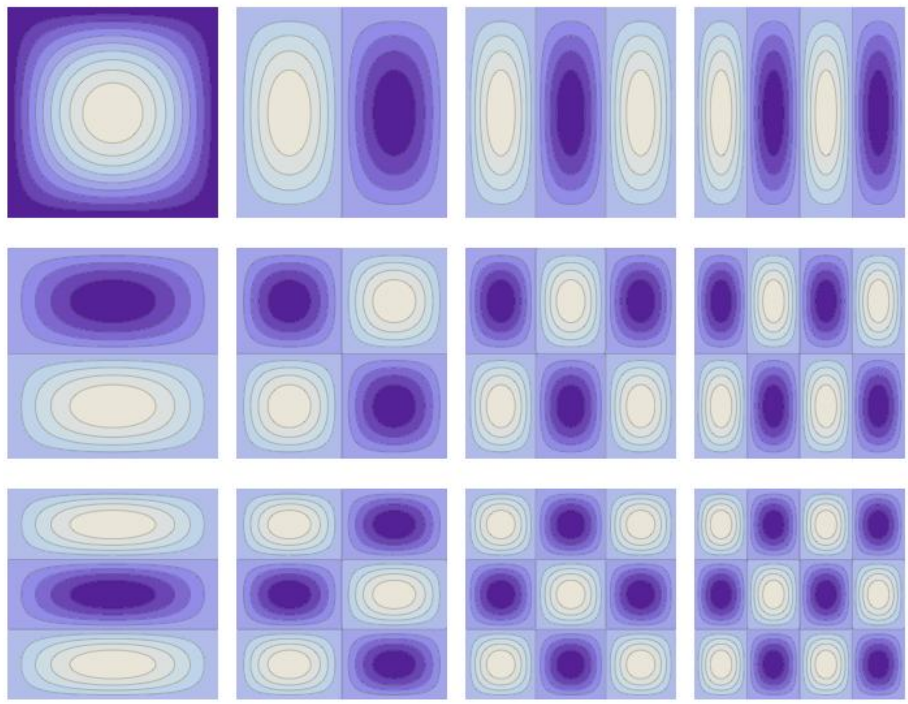
and have eigenvalues

$$\lambda_{jk} = \left(\frac{j\pi}{\ell}\right)^2 + \left(\frac{k\pi}{m}\right)^2 \qquad \text{for} j,k \geq 0.$$

In the following picture the eigenfunctions on a square are shown.



In the following picture the eigenfunctions on a square are shown as if the were seen from above.



Observation

Note that if we scale our domain by a factor $a > 0$ we get $\lambda_k\big(a\Omega\big) = \frac{1}{a^2}\,\lambda_k\big(\Omega\big)$ and so the eigenvalue λ must balance Δ. Again, $\lambda \sim (\text{length scale})^{-2}$.

Hearing the area of a rectangle (computing Weyl asymptotics)

For $\lambda > 0$ consider the eigenvalue counting function

$$N(\lambda) = \#\{\text{eigenvalues} \le \lambda\}.$$

Proposition 2. *(Weyl's law for rectangles) Write λ_j for the Dirichlet eigenvalues of the Laplacian on the rectangle $\Omega = [0,\ell] \times [0,m]$. Then,*

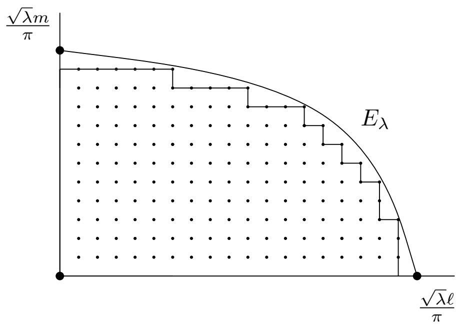
$$N(\lambda) \sim \frac{\mathit{area}(\Omega)}{4\pi}\lambda.$$

Proof.

$$N(\lambda) = \#\Big\{(j,k) \in \mathbb{N}\times\mathbb{N} : \Big(\frac{j\pi}{\ell}\Big)^2 + \Big(\frac{k\pi}{m}\Big)^2 \le \lambda\Big\} = \#\big\{(j,k) \in \mathbb{N}\times\mathbb{N} : (j,k) \in E_\lambda\big\}$$

where E_λ is the first quadrant of the ellipse $\Big(\frac{x}{\sqrt{\lambda}\ell/\pi}\Big)^2 + \Big(\frac{y}{\sqrt{\lambda}m/\pi}\Big)^2 \le 1$. To each point $(j,k) \in E_\lambda$ with the square

$$R_{j,k} = [j-1,j] \times [k-1,k].$$

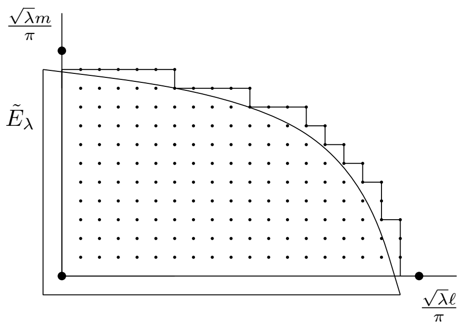


Since all these squares lie inside E_λ we get

$$N(\lambda) \le \text{area}(E_\lambda) = \frac{1}{4}\pi(\sqrt{\lambda}\ell/\pi)(\sqrt{\lambda}m/\pi) = \frac{\text{area}(\Omega)}{4\pi}\lambda.$$

Also, the union of the squares covers a copy, $\tilde{E}_\lambda$, of E_λ translated by $(-1,-1)$:

$$\Big(\tilde{E}_\lambda \cap \{(x,y) :\ x \ge 0, y \ge 0\}\Big) \subset \bigcup_{(j,k)\in E_\lambda} R_{j,k}.$$



Comparing areas shows that

$$\begin{aligned}
N(\lambda) &\geq \frac{1}{4}\pi(\sqrt{\lambda}\ell/\pi)(\sqrt{\lambda}m/\pi) - \sqrt{\lambda}\ell/\pi - \sqrt{\lambda}m/\pi \\
&= \frac{\ell m}{4\pi}\lambda - \frac{\ell + m}{\pi}\sqrt{\lambda} \\
&= \frac{\text{area}(\Omega)}{4\pi}\lambda - \frac{\text{perimeter}(\Omega)}{2\pi}\sqrt{\lambda}.
\end{aligned}$$

□

Since $N(\lambda_j) = j$, Proposition 2 yields $j \sim \frac{\text{area}(\Omega)}{4\pi}\lambda_j$ and so we obtain

Corollary 3. *Write $\lambda_1 \leq \lambda_2 \leq \ldots$ for the Dirichlet eigenvalues of the Laplacian on the rectangle $\Omega = [0, \ell] \times [0, m]$. Then,*

$$\lambda_j \sim \frac{4\pi j}{\mathit{area}(\Omega)} \qquad \text{as } j \to \infty.$$

2.3 Disc

Consider a disc $\Omega = \{x \in \mathbb{R}^2 : |x| \leq 1\}$.

In order to compute the eigenfunctions on the disc we need to separate variables using polar coordinates r and θ. We first derive a formula for the Laplacian in polar coordinates. Since $r = \sqrt{x^2 + y^2}$ and $\tan\theta = y/x$, one has

$$\begin{aligned}
\frac{\partial \varphi}{\partial x} &= \frac{\partial \varphi}{\partial r}\frac{x}{r} - \frac{\partial \varphi}{\partial \theta}\frac{y}{r^2}, \\
\frac{\partial \varphi}{\partial y} &= \frac{\partial \varphi}{\partial r}\frac{y}{r} + \frac{\partial \varphi}{\partial \theta}\frac{x}{r^2}.
\end{aligned}$$

Therefore,

$$\frac{\partial^2 \varphi}{\partial x^2} = \frac{x^2}{r^2}\frac{\partial^2 \varphi}{\partial r^2} - 2\frac{xy}{r^3}\frac{\partial^2 \varphi}{\partial r \partial \theta} + \frac{y^2}{r^4}\frac{\partial^2 \varphi}{\partial \theta^2} + \frac{y^2}{r^3}\frac{\partial \varphi}{\partial r} + 2\frac{xy}{r^4}\frac{\partial \varphi}{\partial \theta}$$

and

$$\frac{\partial^2 \varphi}{\partial y^2} = \frac{y^2}{r^2}\frac{\partial^2 \varphi}{\partial r^2} + 2\frac{xy}{r^3}\frac{\partial^2 \varphi}{\partial r \partial \theta} + \frac{x^2}{r^4}\frac{\partial^2 \varphi}{\partial \theta^2} + \frac{x^2}{r^3}\frac{\partial \varphi}{\partial r} - 2\frac{xy}{r^4}\frac{\partial \varphi}{\partial \theta}.$$

It follows that the Laplacian applied to φ has the form

$$\Delta\varphi = -\frac{\partial^2 \varphi}{\partial x^2} - \frac{\partial^2 \varphi}{\partial y^2} = -\frac{\partial^2 \varphi}{\partial r^2} - \frac{1}{r}\frac{\partial \varphi}{\partial r} - \frac{1}{r^2}\frac{\partial^2 \varphi}{\partial \theta^2}.$$

Therefore, the **Laplacian in polar coordinates** takes the form

$$\Delta = -\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right).$$

We now look for solutions of the form $\varphi(r,\theta) = R(r)\Phi(\theta)$.
From $\Delta\varphi = \lambda\varphi$ we get

$$-\Big(R''(r)\Phi(\theta) + \frac{1}{r}R'(r)\Phi(\theta) + \frac{1}{r^2}R(r)\Phi''(\theta)\Big) = \lambda R(r)\Phi(\theta)$$

and therefore

$$\frac{r^2}{R(r)}\Big(R''(r)\Phi(\theta) + \frac{1}{r}R'(r)\Phi(\theta) + \frac{1}{r^2}R(r)\Phi''(\theta)\Big) = -\frac{\Phi''(\theta)}{\Phi(\theta)}.$$

This means that there exists k such that

$$-\Phi''(\theta) = k^2\Phi(\theta)$$

and

$$R''(r)\Phi(\theta) + \frac{1}{r}R'(r)\Phi(\theta) + \Big(\lambda - \frac{k^2}{r^2}\Big)R(r) = 0.$$

Set $x = \sqrt{\lambda}r$ and $J(x) = R(x/\sqrt{\lambda})$. Then,

$$x^2J''(x) + xJ'(x) + (x^2 - k^2)J(x) = 0$$

which is known as Bessel's equation. The solution for it is the k-th Bessel function

$$J_k(x) = \sum_{\ell=0}^{\infty}\frac{(-1)^\ell}{\ell!\,(k+\ell)!}\Big(\frac{x}{2}\Big)^{k+2\ell}.$$

Since $R(r) = J_k(\sqrt{\lambda}\,r)$, we get that

$$\varphi_k^\lambda(r,\theta) = \Phi_k(\theta)J_k(\sqrt{\lambda}\,r)$$

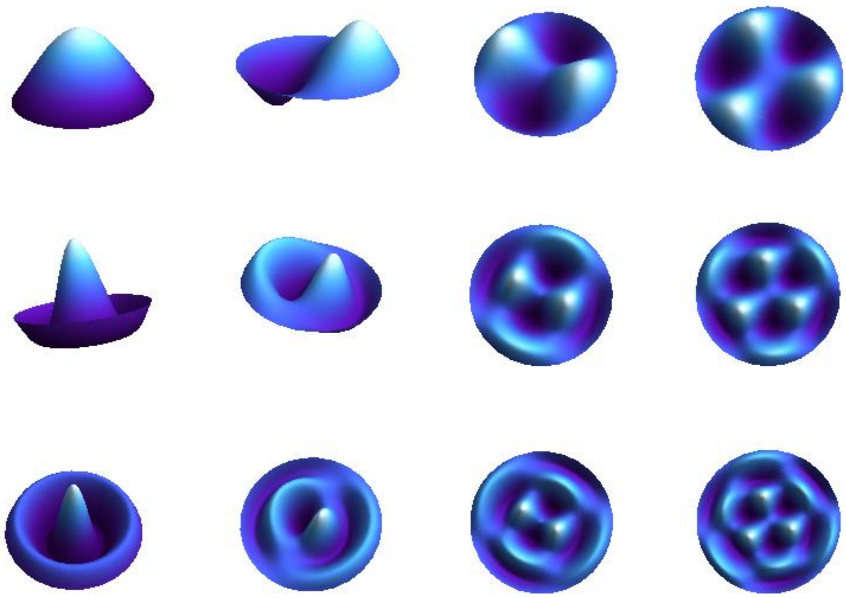
are eigenfunctions of Δ where $\Phi_k(\theta) = a_k\cos(k\theta) + b_k\sin(k\theta)$. The eigenvalue of φ_k^λ has eigenvalue λ.

Let us now impose boundary conditions:

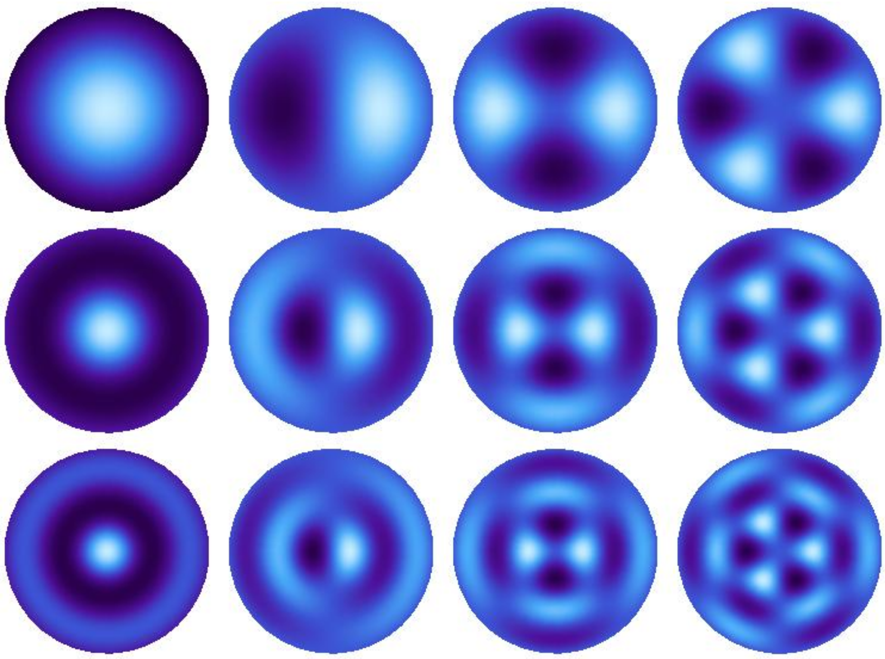
Dirichlet boundary conditions: We ask $\varphi_k^\lambda(1,\theta) = 0$ for all $\theta \in [0,2\pi]$. This implies $J_k(\sqrt{\lambda}) = 0$ and so $\sqrt{\lambda}$ must be a zero of the k-Bessel function.

Neumann boundary conditions: We ask $\partial_r\varphi_k^\lambda(1,\theta) = 0$ for all $\theta \in [0,2\pi]$. This implies $J_k'(\sqrt{\lambda}) = 0$ and so $\sqrt{\lambda}$ must be a zero of the derivative of the k-Bessel function.

In the figure below, the first eigenfunctions on the disk are shown.



The following are the eigenfunctions shown as if they where seen from above.



2.4 Harmonic functions

Let $\Omega \subset \mathbb{R}^n$ be an open connected region. A real valued function $\varphi \in C^2(\Omega)$ is said to be **harmonic** if

$$\Delta\varphi = 0.$$

The theory of harmonic functions is the same as the theory of conservative vector fields with zero divergence. Indeed, for any vector field F in a connected region $\Omega \subset \mathbb{R}^n$ one has that one has $\text{curl}F = 0$ and $\text{div}F = 0$ if and only if there exists a harmonic potential φ making $F = \nabla\varphi$. An example that we already mentioned in the introduction is that of an insulated region. Imagine a thin uniform metal plate that is insulated so no heat can enter or escape. Some time after a given temperature distribution is maintained along the edge of the plate, the temperature distribution inside the plate will reach a steady-state, that will be given by a harmonic function φ.

Throughout this section we write $B_r(x) \subset \mathbb{R}^n$ for the ball of radios r centred at $x \in \Omega$ and $S_r(x) \subset \mathbb{R}^n$ for the corresponding sphere. We set

$$c_n := \text{vol}(S_1(0))$$

Theorem 4 (Mean value Theorem). *Let $\Omega \subset \mathbb{R}^n$ and let $\varphi \in C^2(\Omega)$ be a harmonic function. Then, for each $x \in \Omega$ and $r > 0$ such that $\overline{B}_r(x) \subset \Omega$ one has*

$$\varphi(x) = \frac{1}{r^{n-1}c_n} \int_{S_r(x)} \varphi(y)\, d\sigma(y). \tag{2.2}$$

Proof. Without loss of generality assume that $x = 0$. By Green's identities, for any $U \subset \mathbb{R}^n$ open and $\phi, \psi \in C^1(\overline{U})$,

$$\int_U (\phi\Delta\psi - \psi\Delta\phi)\, dx = \int_{\partial U} (\psi\partial_\nu\phi - \phi\partial_\nu\psi)\, d\sigma. \tag{2.3}$$

For $0 < \varepsilon < r$ set $U := B_r(0)\backslash\overline{B_\varepsilon(0)}$. Define $\phi := \varphi$ and $\psi(y) := |y|^{2-n}$ whenever $n \neq 2$ and $\psi(y) = \log(|y|)$ when $n = 2$. Let us treat the case $n \neq 2$ (the other is done similarly). It turns out that

$$\Delta\psi = 0 \text{ on } \Omega$$

and

$$\frac{\partial\psi}{\partial\nu}|_{S_r(0)} = (2-n)r^{1-n} \quad \text{and} \quad \frac{\partial\psi}{\partial\nu}|_{S_\varepsilon(0)} = -(2-n)\varepsilon^{1-n}.$$

Then,

$$\begin{aligned}
0 &= \int_\Omega (\varphi\Delta\psi - \psi\Delta\varphi)\, dx \\
&= \int_{S_r(0)} (\psi\partial_\nu\varphi - \varphi\partial_\nu\psi)\, d\sigma - \int_{S_\varepsilon(0)} (\psi\partial_\nu\varphi - \varphi\partial_\nu\psi)\, d\sigma \\
&= r^{2-n} \int_{S_r(0)} \partial_\nu\varphi\, d\sigma - \varepsilon^{2-n} \int_{S_\varepsilon(0)} \partial_\nu\varphi\, d\sigma \\
&\qquad\qquad - (2-n)r^{1-n} \int_{S_r(0)} \varphi\, d\sigma + (2-n)\varepsilon^{1-n} \int_{S_\varepsilon(0)} \varphi\, d\sigma
\end{aligned}$$

It is also clear from (2.3) that if we pick $U := B_r(0)$ or $U := B_\varepsilon(0)$, $\psi := 1$ and $\phi := \varphi$, then

$$\int_{S_r(0)} \partial_\nu\varphi\, d\sigma = 0 \qquad \text{and} \qquad \int_{S_\varepsilon(0)} \partial_\nu\varphi\, d\sigma = 0.$$

Therefore,

$$0 = -r^{1-n} \int_{S_r(0)} \varphi \, d\sigma + \varepsilon^{1-n} \int_{S_\varepsilon(0)} \varphi \, d\sigma.$$

By continuity of φ,

$$\frac{1}{r^{n-1} c_n} \int_{S_r(0)} \varphi(y) \, d\sigma(y) = \frac{1}{\varepsilon^{n-1} c_n} \int_{S_\varepsilon(0)} \varphi(y) \, d\sigma(y) \quad \underset{\varepsilon \to 0}{\longrightarrow} \quad \varphi(0).$$

□

A function φ that satisfies (2.2) is said to have the **mean value property**. Satisfying the mean value property is equivalent to

$$\varphi(x) = \frac{n}{r^n c_n} \int_{B_r(x)} \varphi(y) \, dy. \tag{2.4}$$

Indeed, equation (2.4) follows from integrating $\varphi(x) r^{n-1}$ with respect to r. Equation (2.2) follows from differentiating $\varphi(x) r^n$ with respect to r.

In addition, the mean value property is also equivalent to satisfying

$$\varphi(x) = \frac{1}{c_n} \int_{S_1(0)} \varphi(x + rw) \, dS(w). \tag{2.5}$$

where dS is the area measure on the unit sphere. This follows easily from performing the change of variables $y = x + r\omega$.
Next we prove a converse to the Mean value Theorem.

Theorem 5. *Let $\Omega \subset \mathbb{R}^n$ and $\varphi \in C(\Omega)$ satisfy the mean value property. Then φ is smooth and harmonic in Ω.*

Corollary 6. *Harmonic functions are smooth.*

Proof of Theorem 5. Let u be a Friederich's mollifier. That is, $u \in C_0^\infty(B_1(0))$ is a radial function satisfying $\int_{B_1(0)} u(x) dx = 1$. For $\varepsilon > 0$ define

$$u_\varepsilon(y) := \frac{1}{\varepsilon^n} u\left(\frac{y}{\varepsilon}\right).$$

We will prove that $\varphi(x) = u_\varepsilon * \varphi(x)$ for $x \in \Omega$ with with $0 < \varepsilon < \operatorname{dist}(x, \partial\Omega)$. Since u_ε is smooth, it will follow that φ is smooth. In what follows, since u is radial, we introduce

$v(r) := u(rw)$ for $r \in [0, \infty)$ and $w \in S_1(0)$.

$$\begin{aligned}
u_\varepsilon * \varphi(x) &= \int_\Omega \varphi(y) u_\varepsilon(y-x) dy \\
&= \int_\Omega \varphi(x+y) u_\varepsilon(y) dy \\
&= \frac{1}{\varepsilon^n} \int_{B_\varepsilon(0)} \varphi(x+y) u\left(\frac{y}{\varepsilon}\right) dy \\
&= \int_{B_1(0)} \varphi(x+\varepsilon y) u(y) dy \\
&= \int_0^1 \int_{S_1(0)} \varphi(x+\varepsilon rw) u(rw) r^{n-1} \, dS(w) \, dr \\
&= \int_0^1 v(r) r^{n-1} \int_{S_1(0)} \varphi(x+\varepsilon rw) \, dS(w) \, dr \\
&= \varphi(x) c_n \int_0^1 v(r) r^{n-1} \, dr \\
&= u(x).
\end{aligned}$$

It remains to show that φ is harmonic. Since $\Delta\varphi$ is continuous, we deduce that $\Delta\varphi = 0$ from the fact that for all $r > 0$

$$\begin{aligned}
\int_{B_r(x)} \Delta\varphi(y) \, dy = &= -\int_{S_r(x)} \partial_\nu \varphi(y) dS(y) \\
&= -r^{n-1} \frac{\partial}{\partial r} \int_{S_1(0)} \varphi(x+rw) dS(w) \\
&= -r^{n-1} \frac{\partial}{\partial r} (c_n \varphi(x)) \\
&= 0.
\end{aligned}$$

□

Harmonic functions are analytic (Exercise).

1. Fix $x \in \mathbb{R}^n$ and let $R > 0$. Show that if $\varphi \in C^2(B_R(x)) \cap C(\overline{B_R(x)})$ is harmonic, then
$$|\partial_{x_i} \varphi(x)| \leq \frac{n}{R} \|\varphi\|_{L^\infty(\overline{B_R(x)})}.$$
Hint: Prove and use that $\partial_{x_i}\varphi$ is harmonic.

2. Let φ, x and R as in the previous part. For $m \in \mathbb{N}$, prove by induction that there exists a constant $C > 0$ independent of m, n and R such that
$$|\partial^\alpha \varphi(x)| \leq \frac{n^m C^{m-1} m!}{R^m}$$
for any multi-index α with $|\alpha| = m$.

3. Prove by Taylor expansion that any harmonic function φ on $\Omega \subset \mathbb{R}^n$ is analytic.

Theorem 7 (The Maximum Principle)**.** *Assume $\Omega \subset \mathbb{R}^n$ is connected and open. If φ is harmonic and real-valued on Ω, then*

$$\text{either} \qquad \varphi(x) < \sup_\Omega \varphi \qquad \forall x \in \Omega, \qquad\qquad \text{or} \qquad \varphi = \sup_\Omega \varphi.$$

Proof. Consider the set

$$\mathcal{A} := \{x \in \Omega : \ \varphi(x) = \sup_\Omega \varphi\}.$$

The set $\mathcal{A}$ is clearly closed in $\mathcal{A}$. The set $\mathcal{A}$ is also open. Indeed, if $\varphi(x) = \sup_\Omega \varphi$ then $\varphi(y) = \sup_\Omega \varphi$ for all y in a ball centred at x for otherwise the Mean Value Theorem would lead to a contradiction. Since $\mathcal{A}$ is both open and closed in Ω we conclude that $\mathcal{A} = \Omega$ or $\mathcal{A} = \emptyset$. □

Theorem 8. *Suppose $\Omega \subset \mathbb{R}^n$ is open, and $\overline{\Omega}$ is compact. If φ is harmonic and real-valued on Ω, and continuous on $\overline{\Omega}$, then the maximum value of φ is achieved on $\partial\Omega$.*

Proof. If the maximum is achieved at an interior point, the φ must be constant on the connected component of Ω that contains such point, and therefore the maximum is also achieved at the border. □

Theorem 9. *Suppose $\overline{\Omega}$ is compact and that φ, ψ are harmonic on Ω and continuous up to $\partial\Omega$. If $\varphi|_{\partial\Omega} = \psi|_{\partial\Omega}$, then*

$$\varphi = \psi \qquad \text{on } \Omega.$$

Proof. Consider the functions $\phi_1 = \varphi - \psi$ and $\phi_2 = \psi - \varphi$. Both of them are harmonic and equal to zero when restricted to the boundary. The result follows from the fact that their maximums are achieved a the boundary. □

Dirichlet energy method. Fix $f \in C(\partial\Omega)$ and consider the space

$$\mathcal{B} = \{\phi \in C^2(\Omega) : \ \phi|_{\partial\Omega} = f\}$$

and define th *Dirichlet's energy of ϕ*

$$E(\phi) = \frac{1}{2} \int_\Omega |\nabla \phi(x)|^2 dx.$$

Theorem 10 (Dirichlet's Principle)**.** *Let Ω be open and bounded. Consider the problem*

$$(*) \begin{cases} \Delta\phi = 0 & \text{in } \Omega \\ \phi = f & \text{on } \partial\Omega. \end{cases}$$

The function $\varphi \in \mathcal{B}$ is a solution of $(*)$ if and only if

$$E(\varphi) = \min_{\phi \in \mathcal{B}} E(\phi).$$

Proof. Suppose φ solves $(*)$ and let $\psi \in \mathcal{B}$. Then,

$$\begin{aligned}
0 &= \int_\Omega \Delta\varphi(\varphi - \psi)\, dx \\
&= \int_\Omega \nabla\varphi \cdot \nabla(\varphi - \psi)\, dx - \int_{\partial\Omega} \partial_\nu \varphi\, (\varphi - \psi) \\
&= \int_\Omega |\nabla\varphi|^2 dx - \int_\Omega \nabla\varphi \cdot \nabla\psi dx \\
&= \int_\Omega |\nabla\varphi|^2 dx + \frac{1}{2}\int_\Omega (|\nabla(\psi - \varphi)|^2 - |\nabla\varphi|^2 - |\nabla\psi|^2) dx \\
&\geq \frac{1}{2}\int_\Omega |\nabla\varphi|^2 dx - \frac{1}{2}\int_\Omega |\nabla\psi|^2 dx.
\end{aligned}$$

Therefore,

$$\int_\Omega |\nabla\varphi|^2 dx \leq \int_\Omega |\nabla\psi|^2 dx.$$

Since ψ is arbitrary, it follows that φ minimizes the energy.

Suppose now that φ minimizes the energy. Then, for any $\psi \in C_0^2(\Omega)$ we must have that $\frac{d}{d\varepsilon}E(\varphi + \varepsilon\psi)|_{\varepsilon=0} = 0$. Observe that

$$E(\varphi + \varepsilon\psi) = \frac{1}{2}\int_\Omega (|\nabla\varphi|^2 + 2\varepsilon\nabla\varphi\nabla\psi + \varepsilon^2|\nabla\psi|^2) dx,$$

and therefore,

$$\frac{d}{d\varepsilon}E(\varphi + \varepsilon\psi) = \frac{1}{2}\int_\Omega (2\nabla\varphi\nabla\psi + 2\varepsilon|\nabla\psi|^2) dx.$$

It follows that

$$\begin{aligned}
\frac{d}{d\varepsilon}E(\varphi + \varepsilon\psi)\Big|_{\varepsilon=0} &= \int_\Omega \nabla\varphi\nabla\psi dx \\
&= \int_\Omega \Delta\varphi\, \psi\, dx + \int_{\partial\Omega} \partial_\nu\varphi\, \psi ds \\
&= \int_\Omega \Delta\varphi\, \psi\, dx.
\end{aligned}$$

We then must have $\int_\Omega \Delta\varphi\, \psi\, dx = 0$ for all $\psi \in C_0^2(\Omega)$, and so $\Delta\varphi = 0$ which implies that φ is harmonic. $\square$

CHAPTER 3

A very brief review of differential and Riemannian geometry

3.1 Differentiable Manifolds

A **topological manifold** is a topological space (E, τ) so that

1. It is Hausdorff.
2. $\forall x \in E$ there exists (U, φ) with U open and $x \in U$, such that $\varphi : U \to \varphi(U)$ is a homeomorphism. The pair (U, φ) is called **chart** and the real numbers $(x_1, \ldots, x_n) = \varphi(x)$ are called **local coordinates**.
3. (E, τ) has a countable basis of open sets.

A C^k**-differentiable structure** on a topological manifold M is a family of charts $\mathcal{U} = \{(U_\alpha, \varphi_\alpha\}$ so that

1. $\bigcup U_\alpha = M$
2. If $U_\alpha \cap U_\beta \neq \emptyset$ then $\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$ are C^k with C^k inverse. In this case we say that $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β) are compatible.
3. Completness property: If (V, ψ) is a chart which is compatible with every $(U_\alpha, \varphi_\alpha) \in \mathcal{U}$ then $(V, \psi) \in \mathcal{U}$.

Examples of differential manifolds include $\mathbb{R}^n$, the sphere S^n, the torus $\mathbb{T}^n$. Products of manifolds are manifolds as well.

Let M be a manifold and $W \subset M$ open. We say that $f : W \to \mathbb{R}$ is a C^k**-differentiable map** if for all $x \in W$ there exists (U, φ) coordinate chart (with $x \in U$) so that $f \circ \varphi^{-1} : \varphi(W \cap U) \to \mathbb{R}$ is C^k.

Let $A \subset M$. We say that $f : A \to \mathbb{R}$ is C^∞ if it has a C^∞ extension to an open set $U \subset M$ such that $A \subset U$.

Let M and N be differentiable C^k- manifolds. We will say that $f : M \to N$ is a **C^ℓ-differentiable map**, $\ell \leq k$, if forall $x \in M$ there exists (U, φ) coordinate chart with $x \in U$ and (V, ψ) coordinate with $f(x) \in V$ so that $f(U) \subset V$ and $\psi \circ f \circ \varphi^{-1} : \varphi(W \cap U) \to \mathbb{R}$ is C^ℓ-differentiable as a map of euclidean spaces.

A **manifold with boundary** is a Hausdorff space with a countable basis of open sets and a differentiable structure $\{(U_\alpha, \varphi_\alpha) : \alpha \in A\}$ such that it has compatibility on overlaps and $\varphi_\alpha(U_\alpha)$ is open in $\mathbb{H}^n = \{(x_1, \ldots, x_n) :\ x_1 > 0\}$. We denote the boundary of M by ∂M.

Examples of a manifold with boundary include intervals (a, b), $(a, b]$, $[a, b]$, balls, open subsets of $\mathbb{R}^n$ with some (or none) pieces of its boundary attached, open subsets of a manifold.
Let f, g be C^∞ in a neighborhood of $x \in M$. We say that $f \sim g$ if there exists $U \subset M$ open so that $f(y) = g(y)$ for all $y \in U$. The class $[f]_x$ is called **germ of C^∞ function at** x. The set of germs at x is denoted by $C^\infty(x, \mathbb{R})$.

Let $x \in M$ and $X_x : C^\infty(x, \mathbb{R}) \to \mathbb{R}$. If for every chart (U, φ) about x we have that there exist $a_1, \ldots, a_n \in \mathbb{R}$ so that

$$X_x([f]_x) = \sum_{i=1}^{n} a_i \, \frac{\partial (f \circ \varphi^{-1})}{\partial x_i}(\varphi(x)),$$

we will say that X_x **is a tangent vector at** x . If the equation holds for some chart (U, φ) about x, then it holds for every C^∞-compatible chart overlapping at x.

The **tangent space to** M at x is the vector space of tangent vectors based at x. We will denote it by $T_x M$. If $n = \dim M$ then $\dim T_x M = n$.

Given a coordinate chart (U, φ) about $x \in M$ the basis $\left\{ \left(\frac{\partial}{\partial x_i} \right)_x , \ \ 1 \leq i \leq n \right\}$ is called the **natural basis** of $T_x M$ associated to this chart, where

$$\left(\frac{\partial}{\partial x_i} \right)_x [f]_x := \frac{\partial f \circ \varphi^{-1}}{\partial x_i}(\varphi(x)).$$

Given $f : M \to N$ a C^∞-map and $x \in M$ we define the **push-forward** of f at x as follows:

$$(f_*)_x : T_x M \to T_{f(x)} N \qquad (f_*)_x X_x([g]_{f(x)}) := X_x([g \circ f]_x).$$

The push-forward is what we know as the differential of a function. For this reason we sometimes write

$$(f_*)_x = d_x f.$$

With this definition we have

$$\left(\frac{\partial}{\partial x_i}\right)_x = \varphi_*^{-1}\left(\left(\frac{\partial}{\partial x_i}\right)_{\varphi(x)}\right).$$

Let φ and ψ be charts about x. Let

$$\psi \circ \varphi^{-1}(x_1, \ldots, x_n) = (y_1(x_1, \ldots, x_n), \ldots, y_n(x_1, \ldots, x_n)).$$

Then

$$\left(\frac{\partial}{\partial x_i}\right)_x = \sum_k \frac{\partial y_k}{\partial x_i}(\varphi(x))\left(\frac{\partial}{\partial y_k}\right)_x.$$

We define the **cotangent space** to x at M to be the dual space of T_xM. We denote it by T_x^*M. The dual basis of $\left\{(\frac{\partial}{\partial x_i})_x,\ \ 1 \leq i \leq n\right\}$ is denoted by $\{(dx_i)_x,\ \ 1 \leq i \leq n\}$ and it is known as the **natural basis** of T_x^*M.

Let φ and ψ be charts about x. Let

$$\varphi \circ \psi^{-1}(y_1, \ldots, y_n) = (x_1(y_1, \ldots, y_n), \ldots, x_n(y_1, \ldots, y_n)).$$

Then

$$(dx_i)_x = \sum_k \frac{\partial x_i}{\partial y_k}(\psi(x))(dy_k)_x.$$

The **tangent bundle** of M, denoted by TM is defined as follows:

$$TM := \coprod_{x \in M} T_xM.$$

The tangent bundle TM is equipped with a projection map $\pi : TM \to M$ defined as $\pi(X_x) = x$ for $X_x \in T_xM$.

A **vector field** is a section of the tangent bundle. That is, a map

$$X : M \to TM$$

that satisfies that

$$\pi \circ X(x) = X(x).$$

To each point $x \in M$ the vector field X assigns a tangent vector at x, $X(x) \in T_xM$. The space of C^k-vector fields will be denoted by $\Gamma_{C^k}(TM)$.

Properties of vector fields:

1. $X(f + g) = X(f) + X(g)$
2. $X(\lambda f) = \lambda X(f) \quad \forall \lambda \in \mathbb{R}$
3. $X(fg) = gX(f) + fX(g)$

The **cotangent bundle** of M, denoted by T^*M is defined as follows:

$$T^*M := \coprod_{x \in M} T_x^*M.$$

3.2 Differential forms

Let V be a real n-dimensional space and let V^* be its dual space. We define the **space of alternating k-forms** as follows:

$$\Lambda^k(V^*) = \{\omega : V \oplus \cdots \oplus V_{\text{(k times)}} \to \mathbb{R} : \omega \text{ is linear and alternating}\}.$$

Observe that $\dim \Lambda^k(V^*) = \frac{n!k!}{(n-k)!}$. The form ω is linear and alternating if $\omega(v_1, \ldots, v_n)$ is linear in each argument and

$$\omega(v_{\pi(1)}, \ldots, v_{\pi(k)}) = (-1)^\pi \omega(v_1, \ldots, v_n).$$

$\Lambda^k(T^*M) := \coprod_{x \in M} \Lambda^k(T_x^*M)$. Choose a chart (U, φ) about x with local coordinates $(x_1, \ldots, x_n)$. An element $\omega_x \in \Lambda^k(T_x^*M)$ is called **k-form at** x and can be written as

$$\omega_x = \sum_{1 \leq i_1 < \cdots < i_k \leq n} a_{i_1 \ldots i_k} (dx_{i_1})_x \wedge \cdots \wedge (dx_{i_k})_x.$$

$\Lambda^k(T^*M)$ is a manifold of dimension $n + \frac{n!k!}{(n-k)!}$.

We define a **k-form on** M as a section of the bundle $\pi : \Lambda^k(T^*M) \to M$. That is, a C^∞ map $\omega : M \to \Lambda^k(T^*M)$ so that $\pi \circ \omega = id_M$. We denote the space of k-forms on M by $\Omega^k(M)$. We write $\Omega^*(M) := \bigoplus_{k=0}^n \Omega^k(M)$ and $\Omega^0(M) = C^\infty(M, \mathbb{R})$. Let $f : M \to N$ be a C^∞ map. We define the **pull back of** f as the map $f^* : \Omega^*(N) \to \Omega^*(M)$ so that

1. $f^*(g) = g \circ f$ for $f \in \Omega^0(N) = C^\infty(N, \mathbb{R})$.
2. $(f^*\omega)_x(X_1, \ldots, X_k) = \omega_{f(x)}(f_*X_1, \ldots, f_*X_k)$ for $\omega \in \Omega^k(N)$ with $k \geq 1$.

Properties of the pull-back map.

1. $f^*(\omega \wedge \tau) = f^*\omega \wedge f^*\tau$
2. $f^*(g\omega + h\tau) = f^*(g)\, f^*\omega \wedge f^*(h)\, f^*\tau$
3. $(f \circ g)^* = g^* \circ f^*$

Proposition. Pull-backs and d commute:

$$d(f^*\omega) = f^*(d\omega).$$

Integral of n-forms. Let M be an orientable manifold of dimension n.

1. If $\omega \in \Omega^n(\mathbb{R}^n)$ has compact support, and $\omega = f dx^1 \wedge \cdots \wedge dx^n$ then
$$\int_{\mathbb{R}^n} \omega := \int_{\mathbb{R}^n} f dx^1 \ldots dx^n.$$
2. If $\omega \in \Omega^n(M)$ we define
$$\int_{[M]} \omega = \sum_{\alpha \in a} \int_{U_\alpha} \rho_\alpha \omega := \sum_{\alpha \in a} \int_{\varphi(U_\alpha)} (\varphi_\alpha^*)^{-1}(\rho_\alpha \omega)$$
where $\{(U_\alpha, \varphi_\alpha) : \alpha \in A\}$ is a positively oriented atlas and $\{\rho_\alpha : \alpha \in A\}$ is a partition of unity subordinate to $\{(U_\alpha, \varphi_\alpha) : \alpha \in A\}$.

Theorem 11 (Stokes Theorem)**.** *Let M be a compact differentiable manifold of dimension n with boundary ∂M. Let $\omega \in \Omega^{n-1}(M)$. Then,*

$$\int_M d\,\omega = \int_{\partial M} \omega.$$

3.3 Riemannian Geometry

A **Riemannian Manifold** is a pair (M, g) where M is a C^∞ manifold and g is a map that assigns to any $x \in M$ a non-degenerate symmetric positive definite bilinear form $\langle \cdot, \cdot \rangle_{g(x)} : T_xM \times T_xM \to \mathbb{R}$ such that for all $X, Y \in \Gamma_{C^\infty}(TM)$, the map

$$x \mapsto \langle X(x), Y(x) \rangle_{g(x)}$$

is smooth.

Notation. Let (U, φ) be a chart with local coordinates $(x_1, \ldots, x_n)$, and consider the corresponding natural basis $\{(\frac{\partial}{\partial x^1})_x, \ldots, (\frac{\partial}{\partial x^n})_x\}$ of T_xM. We adopt the following notation:

$$g_{ij}(x) := \left\langle \left(\frac{\partial}{\partial x^i}\right)_x, \left(\frac{\partial}{\partial x^j}\right)_x \right\rangle_{g(x)}.$$

We will also denote by g^{ij} the entries of the inverse matrix of $(g_{ij})_{ij}$.

Proposition. Let M and N be manifolds and let g be a Riemannian metric on N. Let $f : M \to N$ be a C^∞ inmersion. Then the map f^* defined below defines a Riemannian metric f^*g on M:

$$\langle X(x), Y(x) \rangle_{(f^*g)(x)} := \langle f_*X(x), f_*Y(x) \rangle_{g(x)}.$$

Theorem. Every manifold carries a Riemannian metric.

Examples of Riemannian manifolds are $\Omega \subset \mathbb{R}^n$, $\mathbb{H}^n$, S^n and $\mathbb{T}^n$.

Given two Riemannian manifolds (M, g_M) and (N, g_N) we say that a map $\Phi : (M, g_M) \to (N, g_N)$ is a **local isometry** provided

$$\Phi^*(g_N) = g_M.$$

If Φ is a local isometry and a diffeomorphism at the same time, then we say that Φ is an **isometry**.

L^2- Integrable vector fields. Given any two vector fields $X, Y \in \Gamma_{C^\infty}(TM)$ the function on M $x \mapsto \langle X(x), Y(x) \rangle_{g(x)}$ is smooth and real valued. Therefore, we may define an inner product on $\Gamma_{C^\infty}(TM)$ by

$$\langle X, Y \rangle_g := \int_M \langle X(x), Y(x) \rangle_{g(x)}\, \omega_g(x).$$

The completion of $\Gamma_{C^\infty}(TM)$ with respect to $(\cdot,\cdot)_g$ is a Hilbert space denoted by

$$\Gamma_{L^2}(TM).$$

Geodesic normal coordinates. For $(x, v) \in TM$ write $\gamma_{x,v}$ for the geodesic on M starting at x with velocity v.
If the geodesic $\gamma_{x,v}(t)$ is defined on the interval $(-\delta, \delta)$, then the geodesic $\gamma_{x,av}$, $a \in \mathbb{R}$, $a > 0$, is defined on the interval $(-\delta/a, \delta/a)$ and $\gamma_{x,av}(t) = \gamma_{x,v}(at)$.
In addition, given $x \in M$, there exist a neighborhood V of x in M, a number $\varepsilon > 0$ and a C^∞ mapping $\gamma : (-2, 2) \times \mathcal{U} \to M$,

$$\mathcal{U} = \{(y, w) \in TM;\ y \in V,\ w \in T_M,\ |w| < \varepsilon\},$$

such that $t \to \gamma_{y,w}(t)$, $t \in (-2, 2)$, is the unique geodesic of M which, at the instant $t = 0$, passes through y with velocity w, for every $y \in V$ and for every $w \in T_qM$, with $|w| < \varepsilon$.
The **exponential map** $\exp : \mathcal{U} \to M$ is defined by

$$\exp(y, v) = \gamma(1, y, v) = \gamma\left(|v|, y, \frac{v}{|v|}\right), \quad (y, v) \in \mathcal{U}.$$

In most applications we shall utilize the restriction of exp to an open subset of the tangent space T_xM, that is,

$$\exp_y : B_\varepsilon(0) \subset T_yM \to M,$$

$$\exp_y(v) = \exp(y, v).$$

Given $x \in M$ there exists $\varepsilon > 0$ such that the exponential map $\exp_x : B_\varepsilon(0) \subset T_xM \to M$ is a diffeomorphism. Taking an orthonormal basis $v_1 \ldots, v_n$ of T_xM one can define a diffeomorphism

$$\begin{aligned} B_\varepsilon(0) \subset \mathbb{R}^n &\longrightarrow B_\varepsilon(0) \subset M \\ (x_1, \ldots, x_n) &\mapsto \exp_x\Big(\sum_{i=1}^n x_i\, v_i\Big). \end{aligned}$$

The coordinate map $y \mapsto (x_1(y), \ldots, x_n(y))$ is called **geodesic normal coordinates**.

CHAPTER 4

The Laplacian on a Riemannian manifold

4.1 Definition

Gradient. Given a Riemannian manifold (M^n, g) and a function $\varphi \in C^\infty(M)$, the differential map $d_x\varphi : T_xM \to \mathbb{R}$ is linear for all $x \in M$. Thus, there exists a vector field on TM named gradient of f and denoted by $\nabla_g\varphi$ so that

$$\langle \nabla_g\varphi(x), X_x\rangle_{g(x)} = d_x\varphi(X_x) \qquad \text{for all } \; X_x \in T_xM.$$

Formally speaking, here is the definition.

Definition 12. The **gradient** is the operator

$$\nabla_g : C^\infty(M) \to \Gamma_{C^\infty}(TM)$$

making

$$\langle \nabla_g\varphi, X\rangle_g = d\varphi(X) \qquad \text{for all } \; X \in \Gamma_{C^\infty}(TM).$$

Let's find the expression for the gradient vector field in local coordinates $(x_1, \dots, x_n)$. It has to be a linear combination of the form $\nabla_g\varphi = \sum_{i=1}^n a_i \frac{\partial}{\partial x_i}$ for some coefficients $a_i \in C^\infty(M)$. Since $d\varphi(\frac{\partial}{\partial x_i}) = \frac{\partial\varphi}{\partial x_i}$ we get $\sum_{m=1}^n a_m g_{mi} = \frac{\partial\varphi}{\partial x_i}$ and so $a_j = \sum_{i=1}^n g^{ij} \frac{\partial\varphi}{\partial x_i}$ which implies

$$\nabla_g\varphi = \sum_{i,j=1}^n g^{ij} \frac{\partial\varphi}{\partial x_i} \frac{\partial}{\partial x_j}.$$

Since $d(\varphi + \psi) = d\varphi + d\psi$ for all $\varphi, \psi \in C^1(M)$ we get

$$\nabla_g(\varphi + \psi) = \nabla_g\varphi + \nabla_g\psi.$$

Furthermore, since $d(\varphi \cdot \psi) = \varphi \cdot d\psi + \psi \cdot d\varphi$ we have

$$\nabla_g(\varphi \cdot \psi) = \varphi \cdot \nabla_g\psi + \psi \cdot \nabla_g\varphi.$$

Divergence. Given a n-form $\omega \in \Omega^n(M)$ where $n = \dim(M)$, and any vector field X on M one can define the $(n-1)$-form $\iota_X\omega \in \Omega^{n-1}$ by

$$\iota_X\omega(X_1, \dots, X_{n-1}) = \omega(X, X_1, \dots, X_{n-1})$$

where $X_1, \dots, X_{n-1}$ are any vector fields on M. Since $d(\iota_X\omega)$ is an n-form we know there must exist a number $\mathrm{div}_\omega X$ making

$$d(\iota_X\omega) = \mathrm{div}_\omega X \cdot \omega.$$

If ω_g is the volume form of (M, g), then the number $\mathrm{div}_g X := \mathrm{div}_{\omega_g} X$ is known as the divergence of X. Formally speaking we have the following definition.

Definition 13. The **divergence** is the operator

$$\mathrm{div}_g : \Gamma_{C^\infty}(TM) \to C^\infty(M)$$

making

$$d(\iota_X\omega_g) = \mathrm{div}_g X \cdot \omega_g \qquad \text{for all } X \in \Gamma_{C^\infty}(TM).$$

We will now find the expression for div_g in local coordinates $(x_1, \dots, x_n)$. For $X = \sum_{j=1}^n b_j \frac{\partial}{\partial x_j} \in \Gamma_{C^\infty}(TM)$ we get

$$\begin{aligned}
\iota_X\omega_g\left(\frac{\partial}{\partial x_1}, \dots, \frac{\hat{\partial}}{\partial x_i}, \dots, \frac{\partial}{\partial x_n}\right) &= \omega_g\left(X, \frac{\partial}{\partial x_1}, \dots, \frac{\hat{\partial}}{\partial x_i}, \dots, \frac{\partial}{\partial x_n}\right)\\
&= (-1)^{i-1}\omega_g\left(\frac{\partial}{\partial x_1}, \dots, X, \dots, \frac{\partial}{\partial x_n}\right)\\
&= (-1)^{i-1}\sqrt{|\det g|}\, dx_1 \wedge \dots \wedge dx_n\left(\frac{\partial}{\partial x_1}, \dots, X, \dots, \frac{\partial}{\partial x_n}\right)\\
&= b_i(-1)^{i-1}\sqrt{|\det g|}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
d(\iota_X\omega_g) &= d\left(\sum_{i=1}^n b_i(-1)^{i-1}\sqrt{|\det g|}\, dx_1 \wedge \dots \wedge \hat{dx_i} \wedge \dots \wedge dx_n\right)\\
&= \sum_{i=1}^n (-1)^{i-1}\frac{\partial}{\partial x_i}(b_i\sqrt{|\det g|})\, dx_i \wedge dx_1 \wedge \dots \wedge \hat{dx_i} \wedge \dots \wedge dx_n\\
&= \sum_{i=1}^n \frac{\partial}{\partial x_i}(b_i\sqrt{|\det g|}) dx_1 \wedge \dots \wedge dx_n\\
&= \frac{1}{\sqrt{|\det g|}}\sum_{i=1}^n \frac{\partial}{\partial x_i}(b_i\sqrt{|\det g|}) \ \cdot \omega_g,
\end{aligned}$$

and so, for $X = \sum_{j=1}^n b_j \frac{\partial}{\partial x_j} \in \Gamma_{C^\infty}(TM)$,

$$\mathrm{div}_g X = \frac{1}{\sqrt{|\det g|}}\sum_{i=1}^n \frac{\partial}{\partial x_i}(b_i\sqrt{|\det g|}). \tag{4.1}$$

Since for any $X, Y \in \Gamma(TM)$ and $\omega \in \Omega^n(M)$ we have $\iota_{X+Y}\omega = \iota_X\omega + \iota_Y\omega$ it follows that

$$\mathrm{div}_g(X+Y) = \mathrm{div}_g X + \mathrm{div}_g Y.$$

Furthermore, from (4.1), we get that for any $\varphi \in C^\infty(M)$

$$\mathrm{div}_g(\varphi X) = \varphi\,\mathrm{div}_g X + \langle \nabla_g \varphi, X\rangle_g\,. \tag{4.2}$$

We are now in conditions to define our star operator known as the Laplacian, or Laplace operator, or Laplace-Beltrami operator.

Definition 14. The **Laplacian** on (M, g) is the operator

$$\Delta_g : C^\infty(M) \to C^\infty(M)$$

defined as

$$\Delta_g = -\mathrm{div}_g \circ \nabla_g.$$

Since both ∇_g and div_g are linear operators it follows that for any $\varphi, \psi \in C^\infty(M)$

$$\Delta_g(\varphi + \psi) = \Delta_g\varphi + \Delta_g\psi.$$

In addition we have

$$\Delta_g(\varphi \cdot \psi) = \psi\Delta_g\varphi + \varphi\Delta_g\psi - 2\,\langle \nabla_g\varphi, \nabla_g\psi\rangle_g\,.$$

In order to prove the last equality we observe that from (4.2) we have

$$\mathrm{div}_g(\psi\nabla_g\varphi) = -\psi\,\Delta_g\varphi + \langle \nabla_g\varphi, \nabla_g\psi\rangle_g$$

and so

$$\begin{aligned}\Delta_g(\varphi \cdot \psi) &= -\mathrm{div}_g\nabla_g(\varphi\cdot\psi)\\ &= -\mathrm{div}_g(\psi\nabla_g\varphi) - \mathrm{div}(\varphi\nabla_g\psi)\\ &= \psi\Delta_g\varphi + \varphi\Delta_g\psi - 2\,\langle\nabla_g\varphi, \nabla_g\psi\rangle_g\,.\end{aligned}$$

Laplacian in local coordinates. From the expression of ∇_g and div_g in local coordinates $(x_1, \dots, x_n)$ it is straightforward to see that

$$\Delta_g = -\frac{1}{\sqrt{|\det g|}}\sum_{ij=1}^{n}\frac{\partial}{\partial x_i}\left(g^{ij}\sqrt{|\det g|}\frac{\partial}{\partial x_j}\right).$$

It is worth mentioning that from all the local expressions it follows that if $\varphi \in C^k(M)$ then $\nabla_g\varphi \in C^{k-1}(M)$ and so $\Delta_g\varphi = -\mathrm{div}_g\nabla_g\varphi \in C^{k-2}(M)$.

Average over orthogonal geodesics (local definition). Given $x \in M$ there exists $\varepsilon > 0$ such that the exponential map $\exp_x : B_\varepsilon(0) \subset T_xM \to M$ is a diffeomorphism. Taking an orthonormal basis $v_1 \dots, v_n$ of T_xM one can define a diffeomorphism

$$\begin{aligned}B_\varepsilon(0) \subset \mathbb{R}^n &\longrightarrow B_\varepsilon(0) \subset M\\ (x_1, \dots, x_n) &\mapsto \exp_x\Big(\sum_{i=1}^n x_i\, v_i\Big).\end{aligned}$$

The coordinate map $y \mapsto (x_1(y), \dots, x_n(y))$ gives rise to the so called **geodesic normal coordinates**.

For $x \in M$ consider an orthonormal basis $v_1, \dots v_n$ of T_xM and write $(x_1, \dots, x_n)$ for the geodesic normal coordinates system around x determined by such basis. Since in these coordinates $g_{ij}(x) = \delta_{ij}$ and $\frac{\partial g_{ij}}{\partial x_k}(x) = 0$ for all $i, j, k = 1, \dots, n$, it follows that

$$\Delta_g \varphi(x) = -\sum_{i=1}^{n} \frac{\partial^2 \varphi}{\partial x_i^2}(x).$$

For each $i = 1, \dots, n$ let γ_i be the geodesic satisfying the conditions $\gamma_i(0) = x$ and $\dot{\gamma}_i(0) = v_1$. Since for any $i, j = 1, \dots, n$ we have $x_i(\gamma_j(t)) = \delta_{ij}\, t$, then

$$\frac{\partial^2 \varphi}{\partial x_i^2}(x) = \frac{d^2}{dt^2}\varphi(\gamma_i(t))\Big|_{t=0}.$$

It follows that

$$\Delta_g \varphi(x) = -\sum_{i=1}^{n} \frac{d^2}{dt^2}\varphi(\gamma_i(t))\Big|_{t=0}.$$

Alternative definitions.

- The Laplacian is sometimes defined as $\Delta_g = -\mathrm{trace}_g(\mathrm{Hess}_g)$ where Hess_g is the Hessian operator on (M, g).

- If we write δ_g for the adjoint operator of d, then $\Delta_g = \delta_g\, d$. This definition can be generalized to define a Laplace operator acting on forms. Indeed, $\Delta_g : \Omega^*(M) \to \Omega^*(M)$ is defined as $\Delta_g := \delta_g d + d\delta_g$.

- Since $\Delta_g = -\sum_{ij} g^{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \mathit{lower\ order\ terms}$, the Laplacian is also characterized as the only symmetric linear partial differential operator whose principal symbol is $-\|\cdot\|_g^2$.

4.2 Examples

Laplacian on $\mathbb{R}^n$.

Let $g_{\mathbb{R}^n}$ be the euclidean metric on $\mathbb{R}^n$. Since $g_{ij}(x) = \delta_{ij}$ for all $x \in \mathbb{R}^n$ and $i, j = 1, \dots, n$ it follows that

$$\Delta_{g_{\mathbb{R}^n}} = -\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}.$$

Laplacian on $\mathbb{H}$.

Consider the upper half plane $\mathbb{H} = \{(x, y) \in \mathbb{R}^2 : \ y > 0\}$ endowed with the hyperbolic metric

$$g_{\mathbb{H}}(x, y) = \begin{pmatrix} \frac{1}{y^2} & 0 \\ 0 & \frac{1}{y^2} \end{pmatrix}.$$

It is straightforward that
$$\Delta_{g_{\mathbb{H}}} = -y^2 \frac{\partial^2}{\partial x^2} - y^2 \frac{\partial^2}{\partial y^2}.$$

Laplacian on S^2.

Let g_{S^2} be the round metric on the 2-sphere S^2. Endow the sphere with spherical coordinates
$$T : (0, \pi) \times (0, 2\pi) \to S^2 \subset \mathbb{R}^3$$
$$T(\theta, \phi) = (\sin\theta\, \cos\phi,\ \sin\theta\, \sin\phi,\ \cos\theta).$$

Since
$$g_{S^2}(\theta, \phi) = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix},$$
we get:

$$\begin{aligned}
\Delta_{g_{S^2}} &= -\frac{1}{\sqrt{|\det g_{S^2}|}} \left(\frac{\partial}{\partial\theta} \left(g^{\theta\theta} \sqrt{|\det g_{S^2}|} \frac{\partial}{\partial\theta} \right) + \frac{\partial}{\partial\phi} \left(g^{\phi\phi} \sqrt{|\det g_{S^2}|} \frac{\partial}{\partial\phi} \right) \right) \\
&= -\frac{1}{\sin\theta} \left(\frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial}{\partial\theta} \right) + \frac{\partial}{\partial\phi} \left(\frac{1}{\sin\theta} \frac{\partial}{\partial\phi} \right) \right) \\
&= -\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial}{\partial\theta} \right) - \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2}.
\end{aligned}$$

And so, in spherical coordinates,
$$\Delta_{g_{S^2}} = -\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial}{\partial\theta} \right) - \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2}. \tag{4.3}$$

4.3 The Laplacian under conformal deformations (Exercise)

Consider a conformal deformation $\tilde{g}$ of the metric g. Thas is, $\tilde{g} = e^f g$ with $f \in C^\infty(M)$. When you modify a metric conformally all you are doing is to change the distances between points while maintaining the angles between vectors. The aim of this exercise is to prove that
$$\Delta_{\tilde{g}} = e^{-f} \Delta_g + \left(1 - \frac{n}{2}\right) e^{-2f} \nabla_g f.$$

Notice that on surfaces this formula simplifies to $\Delta_{\tilde{g}} = e^{-f} \Delta_g$. Operators that satisfy such law are known as conformally covariant operators. We suggest you prove the assertion by showing the following:

1. $\nabla_{\tilde{g}} = e^{-f} \nabla_g$.
2. $\operatorname{div}_{\tilde{g}}(X) = \operatorname{div}_g(X) + \frac{n}{2} e^{-f} X(f)$.
3. $\Delta_{\tilde{g}} = e^{-f} \Delta_g + \left(1 - \frac{n}{2}\right) e^{-2f} \nabla_g f$.

4.4 The Laplacian commutes with isometries

The purpose of this exercise is to show that the Laplace operator commutes with isometries. That is, given any two manifolds (M, g_M), (N, g_N) and an isometry $\Phi : (M, g_M) \to (N, g_N)$ between them, show that

$$\Delta_{g_M} \Phi^* = \Phi^* \Delta_{g_N}.$$

To simplify your approach to this problem we suggest you break its solution into three steps. Using the definitions of ∇_g and div_g show:

1. $\Phi_* \nabla_{g_M} \Phi^* = \nabla_{g_N}$.
2. $\Phi_* \mathrm{div}_{g_N} \Phi^* = \mathrm{div}_{g_M}$.
3. $\Delta_{g_M} \Phi^* = \Phi^* \Delta_{g_N}$.

Solutions:

1. Let $\varphi \in C^1(N)$ and V a vector field on TN. Then

$$\begin{aligned}
\langle \Phi_* \nabla_{g_M} \Phi^*(\varphi), V \rangle_{g_N} &= \left\langle \Phi_* \nabla_{g_M} \Phi^*(\varphi), \Phi_* \Phi_*^{-1} V \right\rangle_{g_N} \\
&= \left\langle \nabla_{g_M} \Phi^*(\varphi), \Phi_*^{-1} V \right\rangle_{g_M} \\
&= d(\Phi^* \varphi)[\Phi_*^{-1} V] \\
&= \varphi_* \Phi_* [\Phi_*^{-1} V] \\
&= d\varphi(V) \\
&= \langle \nabla_{g_N} \varphi, V \rangle_{g_N}.
\end{aligned}$$

2. If we fix any $X \in \Gamma(TM)$, the assertion would follow if could prove $d(\iota_X \omega_{g_M}) = \Phi^* \mathrm{div}_{g_N} \Phi_*(X) \cdot \omega_{g_M}$. But this is equivalent to showing that $(\Phi^*)^{-1}(d\iota_X \omega_{g_M}) = \mathrm{div}_{g_N} \Phi_*(X) \cdot (\Phi^*)^{-1} \omega_{g_M}$, or what is the same, we need to show that

$$d((\Phi^*)^{-1}(\iota_X \omega_{g_M}) = \mathrm{div}_{g_N}(\Phi_* X) \cdot \omega_{g_N}.$$

   This last equality is true since $(\Phi^*)^{-1}(\iota_X \omega_{g_M}) = \iota_{\Phi_* X} (\Phi^*)^{-1}(\omega_{g_M})$.

3.

$$\begin{aligned}
\Delta_{g_M} \Phi^* &= \mathrm{div}_{g_M} \nabla_{g_M} \Phi^* = \mathrm{div}_{g_M} (\Phi *)^{-1} \Phi^* \nabla_{g_M} \Phi^* \\
&= \mathrm{div}_{g_M} (\Phi_*)^{-1} \nabla_{g_N} = \Phi^* (\Phi^*)^{-1} \mathrm{div}_{g_M} (\Phi_*)^{-1} \nabla_{g_N} \\
&= \Phi^* \mathrm{div}_{g_N} \nabla_{g_N} = \Phi^* \Delta_{g_N}
\end{aligned}$$

4.5 The Laplacian and Riemannian coverings

Given two Riemannian manifolds (M, g_M) and $(\tilde{M}, g_{\tilde{M}})$, a map $p : \tilde{M} \to M$ is a **Riemannian covering** if p is a differentiable covering (that is, a local surjective homeomorphism) that is a local isometry. In particular, $p^* g_M = g_{\tilde{M}}$. If $p : \tilde{M} \to M$ is a

Riemannian covering, the **deck transformation group** of the covering p is the group of homeomorphisms $\alpha : \tilde{M} \to \tilde{M}$ satisfying

$$p \circ \alpha = p.$$

When $p : \tilde{M} \to M$ is a Riemannian covering, the elements of the deck transformation group are isometries. Indeed, they are homeomorphisms and local isometries:

$$\alpha^* g_{\tilde{M}} = \alpha^* \circ p^* g_M = (p \circ \alpha)^* g_M = p^* g_M = g_{\tilde{M}}.$$

If $p : \tilde{M} \to M$ is a Riemannian covering, the functions on M can be identified with that of $\tilde{M}$ that are invariant under the desk transformation group. Indeed, if $\tilde{\varphi} : M \to \mathbb{C}$ satisfies $\tilde{\varphi} \circ \alpha = \tilde{\varphi}$ for every deck transformation α, then there exists a unique function $\varphi : M \to \mathbb{C}$ for which $\tilde{\varphi} = \varphi \circ p$. In particular, the eigenvalues of M are precisely those of $\tilde{M}$ in whose eigenspace there are eigenfunctions of $\tilde{M}$ invariant under the deck transformation group. Indeed, if $\tilde{\varphi}$ is an eigenfunction of $\Delta_{g_{\tilde{M}}}$ of eigenvalue λ invariant under the deck transformation group, then there exists a unique $\varphi : M \to \mathbb{R}$ such that $\lambda\varphi \circ p = \lambda\tilde{\varphi}$. On the other hand,

$$\lambda\tilde{\varphi} = \Delta_{g_{\tilde{M}}} \tilde{\varphi} = \Delta_{g_{\tilde{M}}} p^* \varphi = (p^* \Delta_{g_M})\varphi = \Delta_{g_M} \varphi \circ p$$

where we used that p is an isometry and that the Laplacian commutes with it. It follows that we must have

$$\Delta_g \varphi = \lambda\varphi.$$

In addition, if φ is an eigenfunction of Δ_{g_M}, then $\varphi \circ p$ is an eigenfunction of $\Delta_{\tilde{g}_M}$. Indeed,

$$\Delta_{g_{\tilde{M}}}(\varphi \circ p) = p^* \Delta_{g_M} \varphi = p^*(\lambda\varphi) = \lambda\, \varphi \circ p.$$

Quotients by discrete group of isometries. A discrete group Γ is said to act **properly** on a Riemannian manifold $(\tilde{M}, g_{\tilde{M}})$ if for any $\tilde{x}, \tilde{y} \in \tilde{M}$ there exist open neighborhoods U and V of $\tilde{x}$ and $\tilde{y}$ respectively such that $\#\{\gamma \in \Gamma :\ \gamma U \cap V \neq\} < \infty$. The group Γ is said to act **freely** if for any $\gamma_1, \gamma_2 \in \Gamma$ for which there exists a point $\tilde{x} \in \tilde{M}$ one has $\gamma_1 = \gamma_2$. The following theorem gives the existence of a canonical metric on a manifold obtained as a quotient by a discrete group of isometries.

Theorem 15. *Let Γ be a discrete group of isometries acting properly and freely on a Riemannian manifold $(\tilde{M}, g_{\tilde{M}})$. There exists a unique canonical Riemannian metric g_M on the quotient $M = \tilde{M}/\Gamma$ such that $p : (\tilde{M}, g_{\tilde{M}}) \to (M, g_M)$ is a Riemannian covering.*

Examples of manifolds where we may apply this result are the circle, the torus and the Klein bottle.

4.6 The Laplacian on product manifolds

If (M, g) and (N, h) are Riemannian manifolds, we can endow $M \times N$ with the product metric: If $\pi_M : M \times N \to M$ and $\pi_N : M \times N \to M$ are the projections then the product metric is defined as follows

$$\langle X, Y \rangle_{g \oplus h(x_1, x_2)} := \langle X, Y \rangle_{\pi_M^*(g)(x_1)} + \langle X, Y \rangle_{\pi_N^*(g)(x_2)}$$

where $(x_1, x_2) \in M \times N$.

Note that by the Stone-Weirstass Theorem the set of functions $(x_1, x_2) \mapsto \phi(x_1)\psi(x_2) \in C^\infty(M \times N)$ with $\phi \in C^\infty(M)$ and $\psi \in C^\infty(N)$ is dense in $L^2(M \times N)$. In addition, if $\Delta_{g_M}\phi = \lambda\phi$ and $\Delta_{g_N}\psi = \beta\psi$, then

$$\begin{aligned}\Delta_{g_M \oplus g_N}(\phi \cdot \psi) &= \psi \Delta_{g_M \oplus g_N}\phi + \phi \Delta_{g_M \oplus g_N}\psi - 2\langle \nabla_{g_M \oplus g_N}\phi, \nabla_{g_M \oplus g_N}\psi\rangle_{g_M \oplus g_N} \\ &= \psi \Delta_{g_M}\phi + \phi \Delta_{g_N}\psi \\ &= (\lambda + \beta)\phi \cdot \psi.\end{aligned}$$

As we will show later, if M and N are compact, one can find an orthonormal basis $\{\phi_j\}_j$ of $L^2(M, g_M)$ (reps. $\{\psi_j\}_j$ of $L^2(N, g_N)$) of eigenfunctions of Δ_{g_M} with eigenvalues λ_j (resp. of eigenfunctions of Δ_{g_N} with eigenvalues β_j). It then follows that

$$\{(x_1, x_2) \mapsto \phi_j(x_1) \cdot \psi_k(x_2)\}_{j,k} \subset C^\infty(M \times N)$$

is a basis of eigenfunctions of $L^2(M \times N)$ with eigenvalues

$$\{\lambda_j + \beta_k\}_{j,k}.$$

4.7 Green's theorem

Theorem 16 (Divergence theorem)**.** *Let M be a Riemannian manifold and let $X \in \Gamma_{C^1}(TM)$. Then,*

$$\int_M div_g X \, \omega_g = \int_{\partial M} \langle X, \nu\rangle \, \sigma_g$$

where ν is the unit vector normal to ∂M.

Proof. We claim that

$$\iota_X \omega_g = \langle X, \nu\rangle \, \sigma_g. \tag{4.4}$$

Assuming this holds, since $\mathrm{div}_g X \, \omega_g = d(\iota_X \omega_g)$, we get

$$\int_M \mathrm{div}_g X \, \omega_g = \int_M d(\iota_X \omega_g) = \int_{\partial M} \iota_X \omega_g = \int_{\partial M} \langle X, \nu\rangle \, \sigma_g$$

as desired. To prove (4.4) let us first show that

$$\sigma_g = \iota_\nu \omega_g.$$

Indeed, the metric h on ∂M is defined as follows. Let $x \in \partial M$ and let $v_1, \dots, v_n$ be an orthonormal basis of $T_x M$ with $v_1 = \nu \in (T_x \partial M)^\perp$. Let $x_1, \dots, x_n$ be the corresponding system of coordinates on M. Writing h for the metric on ∂M we have $\left\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right\rangle_h = \left\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right\rangle_g$ for all $i, j = 2, \dots, n$. Note that

$$\iota_\nu \omega_g\Big(\frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n}\Big) = \omega_g\Big(\nu, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n}\Big) = \sqrt{|\det h|} = \sigma_g\Big(\frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n}\Big).$$

To prove equation (4.4) we also need to show that $\iota_X \omega_g = \langle X, \nu \rangle_g \, \iota_\nu \omega_g$. But this follows from the following chain of equalities:

$$\begin{aligned}
\iota_X \omega_g \Big(\frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n} \Big) &= \omega_g \Big(X, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n} \Big) \\
&= \omega_g \Big(\langle X, \nu \rangle \, \nu, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n} \Big) \\
&= \langle X, \nu \rangle \, \iota_\nu \omega_g \Big(\frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n} \Big).
\end{aligned}$$

□

Theorem 17 (Green's Theorem)**.** *Let (M, g) be a compact Riemannian manifold with boundary ∂M. Let $\psi \in C^1(\overline{M})$ and $\phi \in C^2(\overline{M})$. Then,*

$$\int_M \psi \cdot \Delta_g \phi \;\, \omega_g = \int_M \langle \nabla_g \psi, \nabla_g \phi \rangle \, \omega_g - \int_{\partial M} \psi \cdot \frac{\partial \phi}{\partial \nu} \, \sigma_g \quad .$$

Proof. From equation (4.2) and Green's Theorem we have

$$\begin{aligned}
\int_M \psi \cdot \Delta_g \phi - \int_M \langle \nabla_g \psi, \nabla_g \phi \rangle \, \omega_g &= - \int_M \operatorname{div}_g (\psi \nabla_g \phi) \, \omega_g \\
&= - \int_{\partial M} \langle \psi \nabla_g \phi, \nu \rangle_g \, \sigma_g \\
&= - \int_{\partial M} \psi \cdot \frac{\partial \phi}{\partial \nu} \, \sigma_g.
\end{aligned}$$

□

Corollary 18. *If (M, g) is a compact Riemannian manifold without a boundary, then*

$$\langle \psi, \Delta_g \phi \rangle_g = \langle \nabla_g \psi, \nabla_g \phi \rangle_g.$$

Corollary 19 (Formal self-adjointness)**.** *If (M, g) is a compact Riemannian manifold without a boundary, then*

$$\langle \psi, \Delta_g \phi \rangle_g = \langle \phi, \Delta_g \psi \rangle_g \, .$$

Corollary 20 (Positivity)**.** *If (M, g) is a compact Riemannian manifold without a boundary, then*

$$\langle \Delta_g \phi, \phi \rangle_g \geq 0.$$

Remark 21. All the previous corollaries are valid for a manifold (M, g) with a boundary where Dirichlet or Neumann boundary conditions are imposed.

CHAPTER 5

Examples on manifolds

5.1 Circle

Consider the circle $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$. The deck transformation group consists of the translations $\alpha_j(t) = t + 2\pi j$ with $j \in \mathbb{Z}$. Then, the eigenfunctions of $\mathbb{T}$ are eigenfunctions of $-\frac{d^2}{dt^2}$ on $\mathbb{R}$ that satisfy $\varphi(t) = \varphi(t + 2\pi j)$ for all $j \in \mathbb{Z}$ and all $t \in \mathbb{R}$. The eigenfunctions are therefore

$$t \mapsto e^{ikt} \qquad \text{with } k \in \mathbb{Z}.$$

Taking real and imaginary parts we get the functions

$$1, \quad \cos(t), \quad \sin(t), \quad \cos(2t), \quad \sin(2t), \quad \ldots, \quad \cos(kt), \quad \sin(kt), \ldots$$

with eigenvalues $0, 1, 1, 4, 4, \ldots, k^2, k^2, \ldots$ respectively.

5.2 Torus

Let Γ be an n-dimensional lattice in $\mathbb{R}^n$. That is, there exist $\gamma_1, \ldots, \gamma_n$ so that Γ is generated by $\{\gamma_1 e_1, \ldots, \gamma_n e_n\}$ over $\mathbb{Z}$ where $\{e_1, \ldots, e_n\}$ is a basis of $\mathbb{R}^n$. Consider the torus

$$\mathbb{T} := \mathbb{R}^n/\Gamma.$$

We define the dual lattice

$$\Gamma^* := \{x \in \mathbb{R}^n : \ \langle x, \gamma \rangle \in \mathbb{Z} \ \text{ for all } \gamma \in \Gamma\}.$$

Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be so that $A(\mathbb{Z}^n) = \Gamma$. Then, $\mathrm{vol}(\mathbb{T}) = \det A$. Consider $A^* : \mathbb{R}^n \to \mathbb{R}^n$. Then $(A^*)^{-1}(\mathbb{Z}^n) = \Gamma^*$. Therefore,

$$\mathrm{vol}(\mathbb{T}^*) = \det((A^*)^{-1}) = \frac{1}{\det A}.$$

Since the eigenfunctions of $\mathbb{T}$ are those of $\mathbb{R}^n$ which are invariant under the deck transformation group $\{\alpha_\gamma(x) = x + \gamma : \gamma \in \Gamma\}$ we consider the family of functions

$$\varphi_y(x) := e^{2\pi i \langle x,y\rangle} \qquad \text{for } y \in \Gamma^*.$$

Clearly, φ_y is invariant under the deck transformation group. If you want, you may take their real and imaginary part to get the eigenfunctions

$$1, \qquad \sin(2\pi i\langle x,y\rangle), \qquad \cos(2\pi i\langle x,y\rangle), \qquad y \in \Gamma^*$$

with eigenvalues

$$0, \qquad 4\pi^2|y|^2, \qquad 4\pi^2|y|^2, \qquad y \in \Gamma^*.$$

Let us see that the

$$\{\varphi_y : y \in \Gamma^*\}$$

is a basis of $L^2(\mathbb{T})$.

We first check that these functions are linearly independent by induction: suppose $\varphi_{y_1}, \ldots, \varphi_{y_k}$ are linearly independent and suppose also that $\sum_{i=1}^{k+1} a_i\varphi_{y_i} = 0$. Since $\varphi_{y_j} \circ \varphi_{y_i} = \varphi_{y_j+y_i}$ we know that $0 = \sum_{i=1}^{k+1} a_i\varphi_{y_i-y_{k+1}} = a_{k+1} + \sum_{i=1}^{k} a_i\varphi_{y_i-y_{k+1}}$, after applying the Laplacian we get $\sum_{i=1}^{k} a_i 4\pi^2|y_i - y_{k+1}|^2\varphi_{y_i-y_{k+1}} = 0$. It then follows that $a_1 = \cdots = a_k = 0$. Therefore $a_{k+1} = 0$.

To see that $\mathcal{B} = \mathrm{span}\{\varphi_y : y \in \Gamma^*\}$ is dense in $L^2(\mathbb{T})$ we use Stone Weirstarss Theorem. Clearly $\mathcal{B}$ is a subalgebra. Let us see that it separates points. Fix two points $x_1, x_2 \in \mathbb{T}$ and assume $x_1 \neq x_2$ (then $x_1 - x_2 \notin \Gamma$). Suppose now that $\varphi_y(x_1) = \varphi_y(x_2)$ for all $y \in \Gamma^*$. It then follows that $e^{2\pi i\langle x_1-x_2,y\rangle} = 1$ for all $y \in \Gamma^*$ and so $\langle x_1 - x_2, y\rangle \in \mathbb{Z}$ for all $y \in \Gamma^*$ which implies that $x_1 - x_2 \in (\Gamma^*)^* = \Gamma$, and this is a contradiction.

Weyl's law on the Torus

We continue to write $\omega_n := \mathrm{vol}(B_1(0))$. Denote by $N(\lambda)$ the counting function $N(\lambda) := \#\{j : \lambda_j < \lambda\}$.

Theorem 22 (Weyl's law on the torus)**.**

$$N(\lambda) \sim \frac{\omega_n \; vol(\mathbb{T})}{(2\pi)^n}\lambda^{\frac{n}{2}} \qquad \lambda \to \infty.$$

Note that

$$\begin{aligned} N(\lambda) &= \#\{j : \lambda_j < \lambda\} \\ &= \#\{y \in \Gamma^* : 4\pi^2|y|^2 < \lambda\} \\ &= \#\{y \in \Gamma^* : |y| < \sqrt{\lambda}/2\pi\} \\ &= \#\{\Gamma^* \cap B_{\frac{\sqrt{\lambda}}{2\pi}}(0)\}. \end{aligned}$$

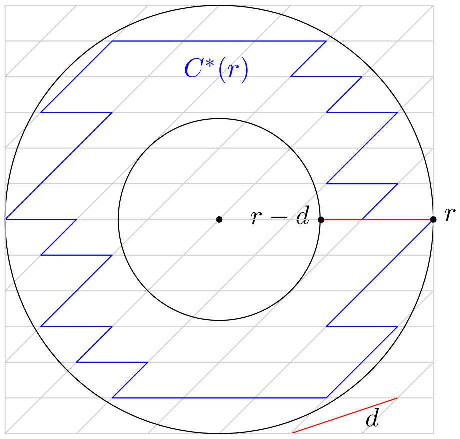
We therefore define

$$N^*(r) := \#\{\Gamma^* \cap B_r(0)\},$$

and so we have $N(\lambda) = N^*\left(\frac{\sqrt{\lambda}}{2\pi}\right)$.
We have reduced our problem to show

$$N^*(r) \sim \omega_n \mathrm{vol}(\mathbb{T}) r^n = \frac{\omega_n}{\mathrm{vol}(\mathbb{T}^*)} r^n \qquad r \to \infty.$$

Let $P^*(r)$ denote the number of polygons inside $B_r(0)$ and write $C^*(r)$ for the polygon formed by all the parallelepipeds inside $B_r(0)$.



Observe that

$$P^*(r) \le N^*(r) \le P^*(r+d).$$

On the one hand,

$$P^*(r)\mathrm{vol}(\mathbb{T}^*) = \mathrm{vol}(C^*(r)) \le \mathrm{vol}(B_r(0)) = \omega_n r^n.$$

On the other hand, if we write d for the diameter of each parallelogram, we get

$$(r-d)^n \omega_n \le P^*(r)\mathrm{vol}(\mathbb{T}^*)$$

for $B_{r-d}(0) \subset C^*(r)$. We conclude

$$\frac{\omega_n}{\mathrm{vol}(\mathbb{T}^*)}(r-d)^n \le P^*(r) \le N^*(r) \le P^*(r+d) \le \frac{\omega_n}{\mathrm{vol}(\mathbb{T}^*)}(r+d)^n.$$

5.3 Rectangular prisms

We work with rectangles $\Omega = [0,\gamma_1] \times \cdots \times [0,\gamma_n]$ with Dirichlet boundary conditions. Since we know that the eigenvalues for $[0,\gamma_j]$ are $\frac{\pi^2 j^2}{\gamma_j^2}$ with $j \in \mathbb{Z}_+$ we know that the eigenvalues of Ω are

$$\pi^2\left(\frac{j_1^2}{\gamma_1^2} + \cdots + \frac{j_n^2}{\gamma_n^2}\right) \qquad j_1, \dots, j_n \in \mathbb{Z}_+.$$

Weyl's law on rectangular prisms

Consider the prism $\Omega = [0,\gamma_1]\times\cdots\times[0,\gamma_n]$ and the lattice Γ generated by $\{\gamma_1 e_1, \ldots, \gamma_n e_n\}$ where $\{e_1, \ldots, e_n\}$ is the standard basis of $\mathbb{R}^n$. Then Γ^* is generated by $\{\frac{e_1}{\gamma_1}, \ldots, \frac{e_n}{\gamma_n}\}$. It follows that

$$\begin{aligned}
N(\lambda) &= \#\left\{(j_1, \ldots, j_n) \in \mathbb{Z}_+^n : \pi^2\left(\frac{j_1^2}{\gamma_1^2} + \cdots + \frac{j_n^2}{\gamma_n^2}\right) < \lambda\right\} \\
&= \left\{y \in \Gamma^*,\ y > 0,\ |y| < \frac{\sqrt{\lambda}}{\pi}\right\} \\
&\sim \frac{\#\left\{\Gamma^* \cap B_{\frac{\sqrt{\lambda}}{\pi}}(0)\right\}}{2^n} \\
&= \frac{N^*\left(\frac{\sqrt{\lambda}}{\pi}\right)}{2^n}
\end{aligned}$$

Since $N^*\left(\frac{\sqrt{\lambda}}{\pi}\right) \sim \frac{\omega_n}{\mathrm{vol}(\mathbb{T}^*)\pi^n}\lambda^{n/2}$ we get the desired result. Observe that $\mathrm{vol}(\mathbb{T}^*) = \frac{1}{\gamma_1}\cdots\frac{1}{\gamma_n} = \frac{1}{\mathrm{vol}(\Omega)}$.

Theorem 23 (Weyl's law on prisms)**.**

$$N(\lambda) \sim \frac{\omega_n\ vol(\Omega)}{(2\pi)^n}\lambda^{\frac{n}{2}} \qquad \lambda \to \infty.$$

5.4 Sphere

Let $f(\xi_1, \ldots, \xi_n)$ represent the spherical coordinate system for the sphere $S^n \subset \mathbb{R}^{n+1}$. Then any $x \in \mathbb{R}^{n+1}$ can be written as $rf(\xi_1, \ldots \xi_n)$ for $r > 0$ and is represented by the coordinates $(r, \xi_1, \ldots, \xi_n)$. We proceed to compute the euclidean metric of $\mathbb{R}^{n+1}$ in terms of the round metric on S^n. Notice that as vector fields in $\mathbb{R}^n$

$$\begin{aligned}
\frac{\partial}{\partial r} &= \sum_{i=1}^{n+1}\frac{\partial(rf_i)}{\partial r}\frac{\partial}{\partial x_i} = \sum_{i=1}^{n+1} f_i\frac{\partial}{\partial x_i}, \qquad \text{and} \\
\frac{\partial}{\partial \xi_j} &= \sum_{i=1}^{n+1}\frac{\partial(rf_i)}{\partial \xi_j}\frac{\partial}{\partial x_i} = r\sum_{i=1}^{n+1}\frac{\partial f_i}{\partial \xi_j}\frac{\partial}{\partial x_i}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\left\langle \frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right\rangle_{g_{\mathbb{R}^{n+1}}} &= \sum_{k=1}^{n+1} f_k^2 = 1, \\
\left\langle \frac{\partial}{\partial r}, \frac{\partial}{\partial \xi_j}\right\rangle_{g_{\mathbb{R}^{n+1}}} &= r\sum_{k=1}^{n+1} f_k\frac{\partial f_k}{\partial \xi_j} = 0, \\
\left\langle \frac{\partial}{\partial \xi_i}, \frac{\partial}{\partial \xi_j}\right\rangle_{g_{\mathbb{R}^{n+1}}} &= r^2\sum_{k=1}^{n+1}\frac{\partial f_k}{\partial \xi_i}\frac{\partial f_k}{\partial \xi_j} = r^2\left\langle \frac{\partial}{\partial \xi_i}, \frac{\partial}{\partial \xi_j}\right\rangle_{g_{S^n}}.
\end{aligned}$$

It then follows that

$$g_{\mathbb{R}^{n+1}}(r,\xi) = \begin{pmatrix} 1 & 0 \\ 0 & r^2\, g_{S^n}(\xi) \end{pmatrix}.$$

It is straightforward to check that in these coordinates then

$$\Delta_{g_{\mathbb{R}^{n+1}}} = -\frac{1}{r^n}\frac{\partial}{\partial r}\left(r^n \frac{\partial}{\partial r}\right) + \frac{1}{r^2}\Delta_{g_{S^n}}.$$

Define

$$\begin{aligned} \mathcal{P}_k &= \{\text{homogeneous polynomials of degree } k\}, \\ \mathcal{H}_k &= \{P \in \mathcal{P}_k :\ \Delta_{g_{\mathbb{R}^{n+1}}} P = 0\}, \\ H_k &= \{P|_{S^n} :\ P \in \mathcal{H}_k\}. \end{aligned}$$

Proposition 24. *The spaces $\mathcal{H}_k$ and H_k are isomorphic. In addition,*

$$H_k \subset \{Y \in C^\infty(S^n) : \quad \Delta_{g_{S^n}} Y = k(n+k-1)Y\}.$$

Proof. The idea is to prove that the restriction map

$$\begin{aligned} \mathcal{H}_k &\to H_k \\ P &\mapsto P|_{S^n} \end{aligned}$$

has inverse

$$\begin{aligned} H_k &\to \mathcal{H}_k \\ Y &\mapsto r^k Y. \end{aligned}$$

Indeed, if $Y \in H_k$, then $Y = P|_{S^n}$ for some $P \in \mathcal{H}_k$. Since P is a homogeneous polynomial of degree k we have

$$P(x) = P\Big(\|x\|\frac{x}{\|x\|}\Big) = \|x\|^k P\Big(\frac{x}{\|x\|}\Big) = \|x\|^k Y\Big(\frac{x}{\|x\|}\Big).$$

Let us see that if $Y \in H_k$ then Y is an eigenfunction of $\Delta_{g_{S^n}}$. Indeed, $Y = P|_{S^n}$ for some $P \in \mathcal{H}_k$, and since P is homogenous of degree k, we must have $P(r,\xi) = r^k Y(\xi)$. Then

$$\begin{aligned} 0 &= \Delta_{g_{\mathbb{R}^{n+1}}} P \\ &= -Y\frac{1}{r^n}\frac{\partial}{\partial r}(kr^{n+k-1}) + r^{k-2}\Delta_{g_{S^n}} Y \\ &= -k(n+k-1)r^{k-2}Y + r^{k-2}\Delta_{g_{S^n}} Y \\ &= r^{k-2}(\Delta_{g_{S^n}} Y - k(n+k-1)Y). \end{aligned}$$

□

Since any $Y \in H_k$ are restrictions to the sphere of a harmonic polynomial, the space H_k is known as the space of **spherical harmonics of degree** k.

Proposition 25. *Write* $r^2\mathcal{P}_{k-2} := \{r^2 P : \; P \in \mathcal{P}_{k-2}\}$. *Then,*

$$\mathcal{P}_k = \mathcal{H}_k \oplus r^2\mathcal{P}_{k-2}.$$

Proof. We start defining an inner product on $\mathcal{P}_k$. For $Q \in \mathcal{P}_k$ and $P = \sum_\alpha a_\alpha x^\alpha \in \mathcal{P}_k$ we set

$$(P, Q) := \sum_\alpha a_\alpha \partial^\alpha \overline{Q}$$

which we may rewrite as $(P, Q) = P(\partial)\overline{Q}$. Note that if both $|\alpha| = |\beta| = k$, then

$$(x^\alpha, x^\beta) = \begin{cases} \alpha! & \text{if } \alpha = \beta \\ 0 & \text{else.} \end{cases}$$

In particular

$$\Big(\sum_\alpha a_\alpha x^\alpha, \sum_\beta a_\beta x^\beta\Big) = \sum_\alpha \alpha!\, a_\alpha\, \overline{b_\alpha}.$$

It follows that (,) is a scalar product on $\mathcal{P}_k$.

Note that if $P \in \mathcal{P}_{k-2}$ and $Q \in \mathcal{P}_k$, and since $r^2 = x_1^2 + \dots + x_{n+1}^2$,

$$(r^2 P,\, Q) = P(\partial) r^2(\partial)\, \overline{Q} = P(\partial)\, \overline{\Delta_{g_{\mathbb{R}^{n+1}}} Q} = (P, \Delta_{g_{\mathbb{R}^{n+1}}} Q) = 0.$$

This shows that $\mathcal{H}_k$ is the orthogonal complement of $r^2\mathcal{P}_{k-2}$ with respect to (,).
♣The proof that $\mathcal{P}_k = \mathcal{H}_k + r^2\mathcal{P}_{k-2}$ needs to be added ♣. □

By induction one proves the following corollary.

Corollary 26.

$$\mathcal{P}_{2k} = \mathcal{H}_{2k} \oplus r^2\mathcal{H}_{2k-2} \oplus r^4\mathcal{H}_{2k-4} \oplus \dots r^{2k}\mathcal{H}_0,$$
$$\mathcal{P}_{2k+1} = \mathcal{H}_{2k+1} \oplus r^2\mathcal{H}_{2k-1} \oplus r^4\mathcal{H}_{2k-3} \oplus \dots r^{2k}\mathcal{H}_1.$$

Corollary 27. *If* $P \in \mathcal{P}_k$, *then* $P|_{S^n}$ *is a sum of spherical harmonics of degree* $\leq k$.

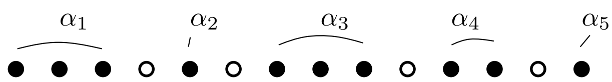
Corollary 28.

$$dim\, H_k = (2k + n - 1)\frac{(k + n - 2)!}{k!(n-1)!}.$$

Proof. Since H_k and $\mathcal{H}_k$ are isomorphic, $\dim H_k = \dim\mathcal{H}_k$. And from Proposition 25,

$$\dim H_k = \dim\mathcal{P}_k - \dim\mathcal{P}_{k-2}.$$

It only remains to compute the dimension of $\mathcal{P}_k$. In order to do that note that the monomials x^α with $|\alpha| = k$ are a basis for $\mathcal{P}_k$. Therefore, $\dim \mathcal{P}_k$ is the number of ways in which you can form an $n+1$-tuple $\alpha = (\alpha_1, \dots, \alpha_{n+1})$ such that $\alpha_1 + \dots + \alpha_{n+1} = k$. Imagine you want to know in how many way you can arrange k black balls into $n+1$ subsets, and that you separate any two subsets using n white balls.



It follows that the dimension of $\mathcal{P}_k$ is the number of ways in which you can choose from a total of $n+k$ black balls a subset of n balls and paint them in white. That is,

$$\dim \mathcal{P}_k = \binom{n+k}{n}.$$

$\square$

Theorem 29.

$$L^2(S^n) = \bigoplus_{k=1}^{\infty} H_k.$$

Proof. Taking $r=1$ in the above corollary we get that

$$\oplus_{k=1}^{\infty} H_k = \oplus_{k=1}^{\infty} \mathcal{H}_k|_{S^n} = \oplus_{k=1}^{\infty} \mathcal{P}_k|_{S^n}.$$

Since $\mathcal{P}_k \cdot \mathcal{P}_\ell \subset \mathcal{P}_{k+\ell}$ we get that $\oplus_{k=1}^{\infty} \mathcal{P}_k|_{S^n}$ is a subalgebra of $C^\infty(S^n)$. Note that it separates points. Indeed, if $y, z \in S^n$ are different points, $y = (y_1, \ldots, y_{n+1})$ and $x = (z_1, \ldots, z_{n+1})$, then there must exist a coordinate for which $y_j \neq z_j$. Therefore we may choose $P(x_1, \ldots, x_{n+1}) := x_j \in \mathcal{P}_1$ which makes $P(y) \neq P(z)$. By Stone Weirstrass Theorem we obtain that $\oplus_{k=1}^{\infty} H_k$ is dense in $L^2(S^n)$. $\square$

We proved that the eigenfunctions of the Laplacian on the sphere $S^n \subset \mathbb{R}^{n+1}$ are restrictions of harmonic polynomials to the sphere. The eigenspaces are H_k with corresponding eigenvalues $k(k+n-1)$ of multiplicities $(2k+n-1)\frac{(k+n-2)!}{k!(n-1)!}$.

The 2-sphere.

The eigenvalues are $k(k+1)$ with multiplicities $2k+1$ for $k \in \mathbb{N}$.

Let us parametrize S^2 with spherical coordinates

$$(\theta, \phi) \mapsto (\sin\theta\, \cos\phi,\ \sin\theta\, \sin\phi,\ \cos\theta).$$

Let us first find the first degree spherical harmonics. First observe that $\mathcal{P}_1 = \operatorname{span}\{x, y, z\}$ and $\dim \mathcal{P}_1 = \binom{2+1}{2} = 3$. Also $\mathcal{H}_1 = \operatorname{span}\{x, y, z\}$ and therefore

$$H_1 = \operatorname{span}\{\sin\theta\, \cos\phi, \sin\theta\, \sin\phi, \cos\theta\}$$

with corresponding eigenvalue 2 of multiplicity 3.

We now find the second degree spherical harmonics. First observe that the second degree harmonic polynomials are $\mathcal{P}_2 = \operatorname{span}\{x^2, y^2, z^2, xy, xz, yz\}$ and $\dim \mathcal{P}_2 = \binom{2+2}{2} = 6$. Also $\mathcal{H}_2 = \operatorname{span}\{z^2 - x^2, z^2 - y^2, xy, xz, yz\}$ and therefore

$$\begin{aligned} H_2 = \operatorname{span}\{&\cos^2\theta - \sin^2\theta\cos^2\phi,\ \cos^2\theta - \sin^2\theta\sin^2\phi, \\ &\sin^2\theta\cos\phi\sin\phi,\ \sin\theta\cos\phi\cos\theta, \sin\theta\sin\phi\cos\theta\} \end{aligned}$$

with corresponding eigenvalue 6 of multiplicity 5.

To find a basis of H_k for general k, let us try to solve Laplace's equation by separation of variables. Suppose $Y_k(\theta,\phi) = P_k(\theta)\Phi_k(\phi)$ solves

$$\Delta_{g_{S^2}} Y_k = k(k+1)Y_k.$$

Then, by the formula for the Laplacian (4.3) in spherical coordinates we have

$$\left(-\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) - \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right) Y_k(\theta,\phi) = k(k+1)Y_k(\theta,\phi)$$

which reduces our problem to the following two systems:

$$-\frac{\Phi_k''(\phi)}{\Phi_k(\phi)} = m^2,$$

$$k(k+1)\sin^2\theta + \frac{\sin\theta}{P_k(\theta)}\frac{d}{d\theta}\big(\sin\theta\, P_k'(\theta)\big) = m^2,$$

where m is a priori any complex number. Then $\Phi_k(\phi) = e^{im\phi}$, and, since Φ_k is 2π-periodic m must be an integer. Changing variables $\theta \mapsto t = \cos\theta$ we may rewrite the second equation as

$$(1-t^2)P_k'' - 2tP_k' + \left(k(k+1) - \frac{m^2}{1-t^2}\right)P_k = 0$$

which is known as the generalized Legendre's equation and has $2k+1 = \dim H_k$ solutions known as the **associated Legendre polynomials** $P_k^m(t)$ where $-k \le m \le k$. It then follows that

$$H_k = \operatorname{span}\{Y_k^m : \ -k \le m \le k\}$$

where

$$Y_k^m(\theta,\phi) = e^{im\phi}\, P_k^m(\cos\theta).$$

If we restrict ourselves to real eigenfunctions we then have that ♣Indices reverted because of the picture!♣

$$H_k = \operatorname{span}\{y_m^k : \ -k \le m \le k\}$$

where

$$y_m^k(\theta,\phi) = \begin{cases} \sqrt{2}\, C_k^m\, \cos(m\phi)\, P_k^m(\cos\theta) & \text{if } m > 0, \\ C_k^0\, P_k^0(\cos\theta) & \text{if } m = 0, \\ \sqrt{2}\, C_k^m\, \sin(-m\phi)\, P_k^{-m}(\cos\theta) & \text{if } m < 0, \end{cases}$$

where C_k^m are so that $\|y_k^m\|_2 = 1$

$$C_k^m := \sqrt{\frac{(2k+1)(k-|m|)!}{4\pi(k+|m|)!}}.$$

Zonal harmonics. When $m = 0$ the eigenfunctions y_k^0 are known as **zonal harmonics**. They are invariant under rotations that fix the south and north poles.

$$y_k^0(\theta,\phi) = C_k^0\, P_k^0(\cos\theta).$$

Highest weight harmonics. This is the case $k = m$. Here,

$$y_k^k(\theta,\phi) = \sqrt{2}\, C_k^k\, \cos(k\phi)\, P_k^k(\cos\theta) = \sqrt{2}\, C_k^k\, (-1)^k(2k-1)!!\, \cos(k\phi)\sin(\theta)^k.$$

$$
\begin{array}{rll}
 & \textbf{Spherical} & \textbf{Cartesian} \\
y_0^0(\theta,\phi) = & \sqrt{\frac{1}{4\pi}} & \sqrt{\frac{1}{4\pi}}, \\
\left\{\begin{array}{r} y_1^{-1}(\theta,\phi) = \\ y_1^{0}(\theta,\phi) = \\ y_1^{1}(\theta,\phi) = \end{array}\right. &
\begin{array}{l} \sqrt{\frac{3}{4\pi}}\sin\phi\sin\theta \\ \sqrt{\frac{3}{4\pi}}\cos\theta \\ \sqrt{\frac{3}{4\pi}}\cos\phi\sin\theta \end{array} &
\begin{array}{l} \sqrt{\frac{3}{4\pi}}x, \\ \sqrt{\frac{3}{4\pi}}z, \\ \sqrt{\frac{3}{4\pi}}y, \end{array} \\
\left\{\begin{array}{r} y_2^{-2}(\theta,\phi) = \\ y_2^{-1}(\theta,\phi) = \\ y_2^{0}(\theta,\phi) = \\ y_2^{1}(\theta,\phi) = \\ y_2^{2}(\theta,\phi) = \end{array}\right. &
\begin{array}{l} \sqrt{\frac{15}{4\pi}}\sin\phi\cos\phi\sin^2\theta \\ \sqrt{\frac{15}{4\pi}}\sin\phi\sin\theta\cos\theta \\ \sqrt{\frac{5}{16\pi}}(3\cos^2\theta-1) \\ \sqrt{\frac{15}{4\pi}}\cos\phi\sin\theta\cos\theta \\ \sqrt{\frac{15}{16\pi}}(\cos^2\phi-\sin^2\phi)\sin^2\theta \end{array} &
\begin{array}{l} \sqrt{\frac{15}{4\pi}}xy, \\ \sqrt{\frac{15}{4\pi}}yz, \\ \sqrt{\frac{5}{16\pi}}(3z^2-1), \\ \sqrt{\frac{15}{8\pi}}xz, \\ \sqrt{\frac{15}{32\pi}}(x^2-y^2). \end{array}
\end{array}
$$

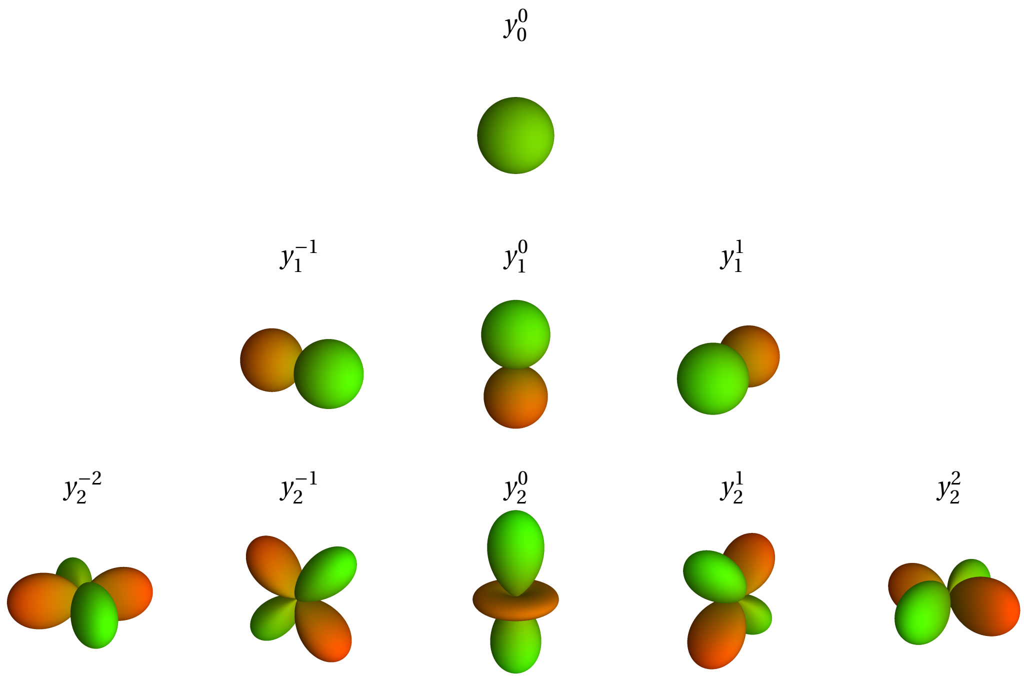


Figure B.1: Plots of the real-valued spherical harmonic basis functions. Green indicates positive values and red indicates negative values.

5.5 Klein Bottle (Exercise)

For $a > 0$ and $b > 0$ consider the discrete lattice $\Gamma_{a,b}$ generated by $\{ae_1, be_2\}$ where $e_1 = (1, 0)$ and $e_2 = (0, 1)$. We write $\hat{\Gamma}_{a,b}$ for the group generated by the translations induced by $\Gamma_{a,b}$ and the transformation

$$\alpha : \mathbb{R}^2 \to \mathbb{R}^2, \qquad \alpha(x, y) = \big(x + \frac{a}{2}, b - y\big).$$

1. Show that $\hat{\Gamma}_{a,b}$ is a discrete group of isometries that acts properly and freely on $\mathbb{R}^2$.
2. By the previous part we may endow the **Klein bottle** $K_{a,b} := \mathbb{R}^2/\Gamma_{a,b}$ with a canonical Riemannian structure $g_{K_{a,b}}$. Find the eigenvalues of the Laplacian on $(K_{a,b}, g_{K_{a,b}})$.
3. Deduce that the flat torus $\mathbb{T} = R^n/\mathbb{Z}^n$ and $K_{a,b}$ are never isospectral (that is, they cannot have the same eigenvalues).

5.6 Projective space (Exercise)

Consider the sphere S^n endowed with the round metric g_{S^n}. Consider the isometry

$$\alpha : S^n \to S^n \qquad \alpha(x) = -x$$

and the group of isometries $\Gamma = \{Id, \alpha\}$. The **projective space** is defined as the quotient $\mathbb{P}^n(\mathbb{R}) := S^n/\Gamma$.

The projective space inherites a canonical Riemanian metric $g_{\mathbb{P}^n(\mathbb{R})}$. Find the eigenvalues and eigenfunctions of $(\mathbb{P}^n(\mathbb{R}), g_{\mathbb{P}^n(\mathbb{R})})$.

CHAPTER 6

Heat Operator

6.1 Sobolev Spaces

We start this chapter reviewing some basics on the Fourier transform. For $\phi \in C_0^\infty(\mathbb{R}^n)$ its Fourier transform is

$$\hat{\phi}(\xi) := \int_{\mathbb{R}^n} e^{-ix\cdot\xi}\phi(x)dx$$

where we write (throughout this section)

$$dx := \frac{1}{(2\pi)^{n/2}} dx_1 \dots dx_n.$$

We need some notation. For $\phi, \psi \in L^2(\mathbb{R}^n)$ their convolution is

$$\phi * \psi(x) = \int_{\mathbb{R}^n} \phi(x-y)\psi(y)dy.$$

Also, given an n-tuple $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n$ we write

$$D^\alpha := (-i)^{|\alpha|}\partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n}.$$

The following collection of results is well known:

Lemma 30.

- *The Fourier transform is an isometry on $C_0^\infty(\mathbb{R}^n)$ in the L^2-norm, and so it extends to an isometry of $L^2(\mathbb{R}^n)$.*
- $(\phi * \psi)\hat{} = \hat{\phi} \cdot \hat{\psi}$.
- $(\phi\psi)\hat{} = \hat{\phi} * \hat{\psi}$.
- $\phi(x) = \int_{\mathbb{R}^n} e^{ix\cdot\xi}\hat{\phi}(\xi)\, d\xi$.

- $(D^\alpha \phi)\hat{}(\xi) = \xi^\alpha \, \hat{\phi}(\xi)$.
- $(x^\alpha \phi)\hat{}(\xi) = D^\alpha \hat{\phi}(\xi)$.

For $\Omega \subset \mathbb{R}^n$ open with $\overline{\Omega}$ compact, $\phi \in C_0^\infty(\Omega)$ and $s = 0, 1, 2, \dots$ set

$$\|\phi\|_{H_s} := \Big(\sum_{|\alpha| \le s} \|D^\alpha \phi\|_2^2 \Big)^{\frac{1}{2}}.$$

If $\|\phi\|_{H_s} < \infty$ we say the ϕ belongs to the s- Sobolev Space $H_s(\Omega)$. Note that $H_0(\Omega) = L^2(\Omega)$. The Sobolev s-norm can be generalized for $s \in \mathbb{R}$. In order to do this note that (setting $\phi \equiv 0$ on Ω^c)

$$\begin{aligned}
\|\phi\|_{H_s}^2 &= \sum_{|\alpha| \le s} \|D^\alpha \phi\|_2^2 = \sum_{|\alpha| \le s} \|(D^\alpha \phi)\hat{}\|_2^2 \\
&= \sum_{|\alpha| \le s} \int_{\mathbb{R}^n} |\xi^\alpha \hat{\phi}(\xi)|^2 d\xi = \int_{\mathbb{R}^n} |\hat{\phi}(\xi)|^2 \big(\sum_{|\alpha| \le s} |\xi^\alpha|^2 \big) \, d\xi.
\end{aligned}$$

Since both $\sum_{|\alpha| \le s} |\xi^\alpha|^2$ and $(1 + |\xi|^2)^s$ are polynomials on ξ of the same degree, then there exist two positive constants C_1, C_2 for which

$$C_1 \|\phi\|_{H_s}^2 \le \int_{\mathbb{R}^n} |\hat{\phi}(\xi)|^2 (1 + |\xi|^2)^s d\xi \le C_2 \|\phi\|_{H_s}^2.$$

Which means that for $s = 0, 1, 2, \dots$ the norms $\| \cdot \|_{H_s}$ and $\big(\int_{\mathbb{R}^n} |(\,\cdot\,)\hat{}(\xi)|^2 (1 + |\xi|^2)^s d\xi \big)^{1/2}$ are equivalent. Since the second norm is meaningful for any $s \in \mathbb{R}$, we define the **Sobolev s-norm** of a function $\phi \in C_0^\infty(\Omega)$ as

$$\|\phi\|_{H_s} := \left(\int_{\mathbb{R}^n} |\hat{\phi}(\xi)|^2 (1 + |\xi|^2)^s \right)^{\frac{1}{2}}$$

where ϕ is defined to vanish on Ω^c ($\phi(x) = 0$ if $x \notin \Omega$). The completion of $C_0^\infty(\Omega)$ with respect to the Sobolev s-norm is called the **Sobolev s-space** and is denoted by $H_s(\Omega)$. Note that for $0 < t < s$ one has continuous inclusions

$$H_s(\Omega) \subset H_t(\Omega) \subset H_0(\Omega) = L^2(\Omega).$$

It is clear that $C^k(\overline{\Omega}) \subset H^k(\Omega)$. This property has a partial converse:

Theorem 31 (Sobolev embedding). *If $\phi \in H_k(\Omega)$, then for all $s < k - \frac{n}{2}$ we have $f \in C^s(\overline{\Omega})$.*

Proof. Fix k and s such that $k > s + \frac{n}{2}$. We first show that the map $D^\alpha : H_k(\Omega) \to H_{k-|\alpha|}(\Omega)$ is continuous as long as $|\alpha| \le s$. Indeed, for $\phi \in H_k(\Omega)$,

$$\begin{aligned}
\|D^\alpha \phi\|_{H_{k-|\alpha|}}^2 &= \int_{\mathbb{R}^n} |(D^\alpha \phi)\hat{}|^2 (1 + |\xi|^2)^{k-|\alpha|} d\xi \\
&= \int_{\mathbb{R}^n} |\xi^\alpha \hat{\phi}(\xi)|^2 (1 + |\xi|^2)^{k-|\alpha|} d\xi.
\end{aligned}$$

Since both $(1+|\xi|^2)^k$ and $|\xi^\alpha|^2(1+|\xi|^2)^{k-|\alpha|}$ are polynomials of degree $2k$ there exists $C>0$ making

$$\|D^\alpha\phi\|_{H_{k-|\alpha|}} \leq C\|\phi\|_{H_k}.$$

We now want to show that if $D^\alpha\phi \in H_{k-|\alpha|}(\Omega)$ then $D^\alpha\phi \in C^0(\Omega)$ as long as $|\alpha| \leq s$. If we prove this we would then have that $\phi \in C^s(\Omega)$ as desired. Let $x \in \Omega$,

$$\begin{aligned}
|D^\alpha\phi(x)| &= \left|\int_{\mathbb{R}^n} e^{ix\cdot\xi}\,(D^\alpha\phi)\hat{}(\xi)d\xi\right| \\
&= \left|\int_{\mathbb{R}^n} e^{ix\cdot\xi}\,(1+|\xi|^2)^{-(k-|\alpha|)/2}(1+|\xi|^2)^{(k-|\alpha|)/2}\,(D^\alpha\phi)\hat{}(\xi)d\xi\right| \\
&\leq \left(\int_{\mathbb{R}^n}(1+|\xi|^2)^{-(k-|\alpha|)}d\xi\right)^{1/2}\left(\int_{\mathbb{R}^n}|(D^\alpha\phi)\hat{}(\xi)|^2(1+|\xi|^2)^{k-|\alpha|}d\xi\right)^{1/2}
\end{aligned}$$

and since $k-\|\alpha| > \frac{n}{2}$, there exists some $C>0$ for which

$$|D^\alpha\phi(x)| \leq C\|D^\alpha\phi\|_{H_{k-|\alpha|}}. \tag{6.1}$$

Since $D^\alpha\phi \in H_{k-|\alpha|}(\Omega)$, there exists a sequence $\{\psi_j\}_j \subset C_0^\infty(\Omega)$ such that $\psi_j \to D^\alpha\phi$ in $H_{k-|\alpha|}(\Omega)$. Equation (6.1) implies that $\psi_j \to D^\alpha\phi$ uniformly and so $D^\alpha\phi$ is continuous. □

Theorem 32 (Compactness)**.** *If $s<t$, then the inclusion $H_t(\Omega) \subset H_s(\Omega)$ is compact.*

Proof. Saying that the inclusion is compact means that any bounded sequence $\{\phi_j\}_j \subset H_t(\Omega)$ has a subsequence $\{\phi_{j_k}\}_k$ which is convergent in $H_s(\Omega)$. So let us start fixing the sequence $\{\phi_j\}_j \subset H_t(\Omega)$. The argument is as follows. We first show that if $\{\phi_j\}_j$ is bounded, $\|\phi_j\|_{H_t} \leq 1$, then $\{\hat{\phi}_j\}_j$ and $\{\partial_{\xi_i}\hat{\phi}_j\}_j$ are uniformly bounded on compact subsets of $\mathbb{R}^n$. By Arzela-Ascoli's Theorem this implies that there is a subsequence $\{\hat{\phi}_{j_k}\}_k$ that converges uniformly on compact subsets of $\mathbb{R}^n$. Using this, we then prove that $\{\phi_{j_k}\}_k$ is a Cauchy sequence in $H_s(\Omega)$. Since by definition $H_s(\Omega)$ is complete, we get that $\{\phi_{j_k}\}_k$ is convergent in $H_s(\Omega)$.

Let $\xi \in \mathbb{R}^n$ and let $\chi \in C_0^\infty(\mathbb{R}^n)$ be so that $\chi(x)=1$ for all $x \in \Omega$.

$$\begin{aligned}
|\hat{\phi}_j(\xi)| &= |(\chi\cdot\phi_j)\hat{}(\xi)| \\
&= |\hat{\chi} * \hat{\phi}_j(\xi)| \\
&= \left|\int_{\mathbb{R}^n}\hat{\chi}(\xi-\eta)\hat{\phi}_j(\eta)d\eta\right| \\
&\leq \left(\int_{\mathbb{R}^n}|\hat{\chi}(\xi-\eta)|^2(1+|\eta|^2)^{-t}d\eta\right)^{1/2}\left(\int_{\mathbb{R}^n}|\hat{\phi}_j(\eta)|^2(1+|\eta|^2)^{t}d\eta\right)^{1/2} \\
&= f(\xi)\|\phi_j\|_{H_t}
\end{aligned}$$

where $f(\xi)$ is a continuous function on $\mathbb{R}^n$. Since $\|\phi_j\|_{H_t} \leq 1$ this shows that $\{\hat{\phi}_j\}_j$ is uniformly bounded on compact subsets of $\mathbb{R}^n$. Same argument works for $\{\partial_{\xi_i}\hat{\phi}_j\}_j$. Let

$\{\phi_{j_k}\}_k$ be the subsequence given by Arzela-Ascoli's Theorem. It only remains to show that $\{\phi_{j_k}\}_k$ is a Cauchy sequence in $H_s(\Omega)$. Fix $\varepsilon > 0$. For $r > 0$,

$$\begin{aligned}\|\phi_{j_k} - \phi_{j_\ell}\|_{H_s}^2 &= \int_{\mathbb{R}^n} |\hat{\phi}_{j_k}(\xi) - \hat{\phi}_{j_\ell}(\xi)|^2 (1+|\xi|^2)^s d\xi \\ &= \int_{|\xi|\le r} |\hat{\phi}_{j_k}(\xi) - \hat{\phi}_{j_\ell}(\xi)|^2 (1+|\xi|^2)^s d\xi + \int_{|\xi|>r} |\hat{\phi}_{j_k}(\xi) - \hat{\phi}_{j_\ell}(\xi)|^2 (1+|\xi|^2)^s d\xi \end{aligned} \tag{6.2}$$

The idea is to pick r sufficiently large so that the second term on the RHS of (6.2) is smaller than $\varepsilon/2$. Since $\{\xi \in \mathbb{R}^n : |\xi| \le r\}$ is compact and $\{\hat{\phi}_{j_k}\}_k$ converges uniformly on compact subsets we can make the first therm on the RHS of (6.2) be smaller than $\varepsilon/2$.

To choose r simply note that if $|\xi| > r$ then

$$(1+|\xi|^2)^t = (1+|\xi|^2)^s(1+|\xi|^2)^{t-s} \ge (1+|\xi|^2)^s(1+r^2)^{t-s}.$$

Therefore,

$$\begin{aligned}&\int_{|\xi|>r} |\hat{\phi}_{j_k}(\xi) - \hat{\phi}_{j_\ell}(\xi)|^2 (1+|\xi|^2)^s d\xi \\ &\qquad \le \frac{1}{(1+r^2)^{t-s}} \int_{|\xi|>r} |\hat{\phi}_{j_k}(\xi) - \hat{\phi}_{j_\ell}(\xi)|^2 (1+|\xi|^2)^t d\xi \\ &\qquad \le \frac{1}{(1+r^2)^{t-s}} \|\hat{\phi}_{j_k} - \hat{\phi}_{j_\ell}\|_{H_t}^2 \\ &\qquad \le \frac{4}{(1+r^2)^{t-s}},\end{aligned}$$

and so we may pick r as large as we want to make $\frac{4}{(1+r^2)^{t-s}} \le \varepsilon/2$. □

Sobolev spaces on compact manifolds. We now extend the definition of the Sobolev spaces to a compact Riemannian manifolds (M, g). Let $\{u_i : U_i \subset \mathbb{R}^n \to M\}$ be an atlas of coordinate maps on M with $\overline{U}_i$ compact, and write $\{\rho_i : M \to [0,1]\}$ for a partition of unity associated to $\{U_i\}_i$. We define $H_s(M)$ as the completion of $C_0^\infty(M)$ with respect to the norm

$$\|\phi\|_{H_s} := \left(\sum_i \|\rho_i \phi \circ u_i\|_{H_s}^2\right)^{\frac{1}{2}}.$$

It is not difficult to show that this definition is invariant under changes of coordinate charts and partitions of unity. All the results in this section extend to Sobolev spaces on compact manifolds in a natural manner.

6.2 Heat propagator

Throughout this section we assume that (M, g) is a **compact Riemannian manifold without boundary**. The heat operator $L := \Delta + \partial_t$ acts on functions in $C(M \times$

$(0,+\infty))$ that are C^2 on M and C^1 on $(0,\infty)$. The *Heat equation* is given by

$$\begin{cases} Lu(x,t) = F(x,t) & (x,t) \in M \times (0,+\infty) \\ u(x,0) = f(x) & x \in M \end{cases}.$$

The *homogeneous Heat equation* is

$$\begin{cases} Lu(x,t) = 0 & (x,t) \in M \times (0,+\infty) \\ u(x,0) = f(x) & x \in M \end{cases}.$$

Lemma 33. *If $u(x,t)$ solves the homogeneous heat equation, then the function $t \mapsto \|u(\cdot,t)\|_{L^2}$ is decreasing with t.*

Proof. The proof reduces to show that the t-derivative of the map is negative:

$$\begin{aligned} \frac{d}{dt}\|u(\cdot,t)\|_{L^2}^2 &= 2\int_M \partial_t u(x,t)\, u(x,t)\omega_g(x) \\ &= -2\int_M \Delta_g u(x,t)\, u(x,t)\omega_g(x) \\ &= -2\|\nabla_g u(\cdot,t)\|^2 \\ &\leq 0. \end{aligned}$$

□

Lemma 34. *The solution to the inhomogenous problem is unique.*

Proof. Suppose that both u_1 and u_2 are solutions to the homogeneous problem. Then $u = u_1 - u_2$ solves

$$\begin{cases} Lu(x,t) = 0 & (x,t) \in M \times (0,+\infty) \\ u(x,0) = 0 & x \in M \end{cases}.$$

The proof follows from the fact that $\int_M u(x,t)^2 dx$ is a decreasing function of t while $u(x,0) = 0$ for $x \in M$. It follows that $u(x,0) = 0$ for all $x \in M$. □

Proposition 35 (Duhamel's principle)**.** *Let $u, v : M \times (0,+\infty) \to \mathbb{R}$ be C^2 on M and C^1 on $(0,+\infty)$. Then, for any $t > 0$ and α,β such that $[\alpha,\beta] \subset (0,t)$, we have*

$$\begin{aligned} &\int_M u(y,t-\alpha)v(y,\alpha) - u(y,t-\beta)v(y,\beta)\,\omega_g(y) = \\ &\qquad = \int_\alpha^\beta \int_M Lu(y,t-s)\,v(y,s) - u(y,t-s)\,Lv(y,s)\,\omega_g(y)ds. \end{aligned}$$

Proof.

$$\begin{aligned} &Lu(y,t-s)\,v(y,s) - u(y,t-s)\,Lv(y,s) = \\ &= \Delta_g u(y,t-s)v(y,s) - \Delta_g(y,t-s)Lv(y,s) + \partial_s u(y,t-s)v(y,s) - u(y,t-s)\partial_s v(y,s) \\ &= \Delta_g u(y,t-s)v(y,s) - \Delta_g(y,t-s)Lv(y,s) - \partial_s(u(y,t-s)v(y,s)). \end{aligned}$$

The result follows from integrating first with respect to x (this makes the first two terms cancel out) and then with respect to t. □

We say that a **fundamental solution** of the heat equation is a continuous function $p : M \times M \times (0, \infty) \to \mathbb{R}$ which is C^2 with respect to x, C^1 with respect to t and such that

$$L_y p = 0 \qquad \text{and} \qquad \lim_{t\to 0} p(\cdot, y, t) = \delta_y.$$

Proposition 36. *The fundamental solution to the heat equation on M is* unique *and symmetric in the two space variables.*

Proof. Let p_1 and p_2 be two fundamental solutions. Fix $x, z \in M$. In Duhamel's principle set $u(y, t) = p_1(x, y, t)$ and $v(y, t) = p_2(z, y, t)$. Using that $L_y p_1 = L_y p_2 = 0$ we get

$$\int_M p_1(x, y, t-\alpha)p_2(z, y, \alpha) - p_1(x, y, t-\beta)p_2(z, y, \beta)\, \omega_g(y) = 0.$$

Let $\alpha \to 0$ and $\beta \to t$. We then get

$$p_1(x, z, t) = p_2(z, x, t).$$

In particular, choosing $p_1 = p_2$ we get that the fundamental solution is symmetric. Since the fundamental solution is symmetric we deduce that

$$p_1(x, z, t) = p_2(z, x, t) = p_2(x, z, t)$$

and so the fundamental solution is unique. □

Proposition 37. *Let $f \in C(M)$ and $F \in C(M \times (0, +\infty))$. Then*

$$u(x, t) = \int_M p(x, y, t)f(y)\, d\omega_g(y) + \int_0^t \int_M p(x, y, s)F(y, t-s)d\omega_g(y)$$

solves the problem

$$\begin{cases} Lu = F, \\ u(\cdot, 0) = f. \end{cases}$$

Proof. Fix $x \in M$. Apply Duhamel's principle to u and $v(y, t) = p(x, y, t)$ and get

$$\int_M u(y, t-\alpha)p(x, y, \alpha) - u(y, t-\beta)p(x, y, \beta)\, \omega_g(y) = \int_\alpha^\beta \int_M F(y, t-s)\, p(x, y, s)ds.$$

Let $\alpha \to 0$ and $\beta \to t$. □

Remark 38. We stop to observe that $\int_M p(x, y, t)\, d\omega_g(y) = 1 \qquad \forall x \in M, t \in (0, \infty)$.
This is because $u(x, t) \equiv 1$ solves $\begin{cases} Lu = F, \\ u(\cdot, 0) = f. \end{cases}$

Remark 39. For any $t, s > 0$ we get

$$p(x, z, t+s) = \int_M p(x, y, t)p(y, z, s)\omega_g(y).$$

This is because for fixed $y \in M$ we have $u(z, s) = p(z, y, s+t)$ solves $\begin{cases} Lu = 0 \\ u(\cdot, 0) = p(\cdot, y, t) \end{cases}$.

Remark 40. All these results can be proved for compact manifolds with boundary using an adaptation of Duhamel's principle. One should use that both in the Dirichlet and Neumann boundary conditions setting one has $\int_{\partial M} \phi \partial_\nu \psi\, d\sigma = 0$ for all $\phi, \psi \in C^\infty(M)$ satisfying the appropriate boundary conditions.

6.3 Basis of eigenfunctions on compact manifolds

Given $t > 0$, let us define the **heat propagator** $e^{-t\Delta_g} : L^2(M) \to L^2(M)$ by

$$e^{-t\Delta_g} f(x) := \int_M p(x,y,t) f(y)\, \omega_g(y).$$

Lemma 41.

1. *$e^{-t\Delta_g} \circ e^{-s\Delta_g} = e^{-(t+s)\Delta_g}$*
2. *$e^{-t\Delta_g}$ is self-adjoint and positive.*
3. *$e^{-t\Delta_g}$ is a compact operator.*

Proof.

1. Follows from the fact that $p(x,z,t+s) = \int_M p(x,y,t)p(y,z,s)\omega_g(y)$.
2. For $f, g \in L^2(M)$,

$$\begin{aligned}
\langle e^{-t\Delta_g} f, g\rangle &= \int_M \left(\int_M p(x,y,t) f(y)\, \omega_g(y) \right) \overline{g(x)}\, \omega_g(x) \\
&= \int_M \int_M p(y,x,t) f(y) g(x)\, \overline{\omega_g(x)}\, \omega_g(y) \\
&= \int_M \left(\int_M p(y,x,t) \overline{g(x)}\, \omega_g(x) \right) f(y)\, \omega_g(y) \\
&= \langle f, e^{-t\Delta_g}(g)\rangle.
\end{aligned}$$

   This shows that $e^{-t\Delta_g}$ is self-adjoint. To show that it's positive it suffices to notice that $\langle e^{-t\Delta_g} f, f\rangle = \|e^{-\frac{t}{2}\Delta_g} f\|^2 \geq 0$.

3. The operator $e^{-t\Delta_g} : L^2(M) \to H_1(M)$ is continuous and the inclusion $H_1(M) \subset L^2(M)$ is compact. Their composition $e^{-t\Delta_g} : L^2(M) \to L^2(M)$ is therefore compact. To see that $e^{-t\Delta_g} : L^2(M) \to H_1(M)$ is continuous simply note that if $\{f_j\}_j \subset L^2(M)$ converges to 0 in L^2 then $\|e^{-t\Delta_g} f_j\|_{L^2} \to_j 0$ and similarly $\|\partial_{x_k} e^{-t\Delta_g} f_j\|_{L^2} \to_j 0$. The last two statements can be shown by splitting M in coordinate charts and pasting them by partition of unity.

□

Lemma 42. *As $t \to 0$ we have $e^{-t\Delta_g} f \to f$ in $L^2(M)$.*

Proof.

$$\begin{aligned}
\left\| f - \int_M p(x,y,t) f(y) \omega_g(y) \right\|_{L^2}^2 &= \int_M \left| f(x) - \int_M p(x,y,t) f(y) d\omega_g(y) \right|^2 \omega_g(x) \\
&= \int_M \left| \int_M p(x,y,t)(f(x) - f(y))\, \omega_g(y) \right|^2 \omega_g(x) \\
&\leq \int_M \int_M p(x,y,t) \left| f(x) - f(y) \right|^2 \omega_g(y) \omega_g(x).
\end{aligned}$$

Since $p(x,y,t)$ is uniformly bounded and $p(x,y,t)|f(x)-f(y)|^2 \to 0$ as $t \to 0$, we conclude our result from the Dominated convergence theorem. □

Theorem 43. *For any $f \in L^2(M)$ the function $e^{-\Delta_g} f$ converges uniformly as $t \to \infty$ to a harmonic function. In particular, it converges to a constant function when $\partial M = \emptyset$.*

Proof. The strategy is to show first that $e^{-t\Delta_g} f$ converges in $L^2(M)$. Then show that $e^{-t\Delta_g} f$ converges uniformly to a continuos function $\tilde{f}$. Lastly, prove that $e^{-t\Delta_g}\tilde{f} = \tilde{f}$. If we do this, then $0 = L(e^{-t\Delta_g}\tilde{f}) = L\tilde{f} = \Delta_g \tilde{f}$. This would show that $\tilde{f}$ is harmonic and since M is compact we must have that $\tilde{f}$ is the constant function.

We know that $\|e^{-t\Delta_g} f\|_{L^2}$ is a decreasing function of t, so it must converge to something. Now,

$$\|e^{-t\Delta_g} f - e^{-s\Delta_g} f\|^2 = \|e^{-t\Delta_g} f\|^2 + \|e^{-s\Delta_g} f\|^2 - 2\|e^{-\frac{t+s}{2}\Delta_g} f\|^2.$$

This shows that $\|e^{-t\Delta_g} f - e^{-s\Delta_g} f\| \to 0$ as $s, t \to \infty$.

Let $x \in M$.

$$\begin{aligned} |(e^{-(T+t)\Delta_g} f - e^{-(T+s)\Delta_g} f)(x)|^2 &= |e^{-T\Delta_g}(e^{-t\Delta_g} f - e^{-s\Delta_g} f)(x)|^2 \\ &= \left| \int_M p(x,y,T)(e^{-t\Delta_g} f - e^{-s\Delta_g} f)(y)\omega_g(y) \right|^2 \\ &\le \text{constant} \cdot \|e^{-t\Delta_g} f - e^{-s\Delta_g} f\|_{L^2}^2. \end{aligned}$$

We now show that $e^{-t\Delta_g}\tilde{f} = \tilde{f}$:

$$\begin{aligned} |(e^{-(t+s)\Delta_g} f - e^{-t\Delta_g}\tilde{f})|^2(x) &= \left| \int_M p(x,y,t)(e^{-s\Delta_g} f - \tilde{f})(y)\omega_g(y) \right|^2 \\ &\le \text{constant} \cdot \|e^{-s\Delta_g} f - \tilde{f}\|_{L^2}^2. \end{aligned}$$

□

Theorem 44 (Sturm-Liouville decomposition)**.** *For M compact, there exists a complete orthonormal basis $\{\varphi_0, \varphi_1, \dots\}$ of $L^2(M)$ consisting of eigenfunctions of Δ_g with φ_j having eigenvalue λ_j satisfying*

$$\lambda_0 \le \lambda_1 \le \lambda_2 \le \dots \quad \to \infty.$$

For every j we have $\varphi_j \in C^\infty(M)$ and

$$p(x,y,t) = \sum_{j=0}^{\infty} e^{-\lambda_j t}\varphi_j(x)\varphi_j(y).$$

Proof. We recall that so far we haven't proved the existence of the fundamental solution p. Once we do so we will show that p is C^∞ in the spatial variables. The proof that follows will yield that the smoothness of φ_j is that of p in the spatial variables and so it will follow that $\varphi_j \in C^\infty(M)$.

Our first claim is that $e^{-t\Delta_g} = (e^{-\Delta_g})^t$ for $t > 0$. If $k \in \mathbb{Z}$, then $e^{-k\Delta_g} = (e^{-\Delta_g})^k$. Fix $p, q \in \mathbb{Z}$. Since $(e^{-\frac{1}{q}\Delta_g})^q = e^{-\Delta_g}$, then $e^{-\frac{1}{q}\Delta_g} = (e^{-\Delta_g})^{\frac{1}{q}}$. We then need to generalize this for $t \in \mathbb{R}$. Note that

$$\|e^{-t\Delta_g} - (e^{-\Delta_g})^t\| \leq \|e^{-t\Delta_g} - e^{-\frac{p}{q}\Delta_g}\| + \underbrace{\|e^{-\frac{p}{q}\Delta_g} - (e^{-\Delta_g})^{\frac{p}{q}}\|}_{=0} + \|(e^{-\Delta_g})^{\frac{p}{q}} - (e^{-\Delta_g})^t\|.$$

Observe that

$$\|(e^{-\Delta_g})^{\frac{p}{q}} - (e^{-\Delta_g})^t\| = \sup_{\beta \in \operatorname{spec}(e^{-\Delta_g})} |\beta^{\frac{p}{q}} - \beta^t| \to 0 \qquad \text{as } \frac{p}{q} \to t.$$

On the other hand,

$$\|(e^{-t\Delta_g} - e^{-\frac{p}{q}\Delta_g})f\|_{L^2}^2 \leq \int_M \int_M |p(x,y,t) - p(x,y,p/q)|^2 |f(y)|^2 \omega_g(y)\omega_g(x)$$

also converges to 0 by dominated convergence.

Since $e^{-\Delta_g}$ is a compact self-adjoint operator we know that it has eigenvalues $\beta_0 \geq \beta_1 \geq \cdots \geq \beta_k \to 0$ as $n \to \infty$ with respective eigenfunctions φ_0, $\varphi_1, \ldots$ forming a complete orthonormal basis of $L^2(M)$. Since $e^{-t\Delta_g} = (e^{-\Delta_g})^t$, we must have $e^{-t\Delta_g}\varphi_0 = \beta_0^t \varphi_0$. From the fact that $u(x,t) = \int_M p(x,y,t)\varphi_0(y)\omega_g(y)$ is a solution to the homogeneous equation

$$\begin{cases} Lu = 0 \\ u(\cdot, 0) = \varphi_0 \end{cases}$$

and that $\int_M u(x,t)^2 \omega_g(x)$ decreases with t we have $\beta_0 \leq 1$.

Set $\lambda_k := -\ln \beta_k$. Then

$$e^{-t\Delta_g}\varphi_k = e^{-t\lambda_k}\varphi_k.$$

Since $e^{-t\Delta_g}\varphi_k$ is as solution of the heat equation for all k we get

$$0 = L(e^{-t\Delta_g}\varphi_k) = e^{-\lambda_k t}(\Delta\varphi_k - \lambda_k \varphi_k),$$

which implies that φ_k is an eigenfunction of the Laplacian with eigenvalue λ_k.
Since $p(x,y,t) = \sum_{k=0}^{\infty} \langle p(x,\cdot,t), \varphi_k \rangle \varphi_k(y)$, and

$$\langle p(x,\cdot,t), \varphi_k \rangle = e^{-t\Delta_g}\varphi_k(x) = e^{-\lambda_k t}\varphi_k(x)$$

we finally obtain that

$$p(x,y,t) = \sum_{k=0}^{\infty} e^{-\lambda_k t}\varphi_k(x)\varphi_k(y).$$

$\square$

Since $\int_M \varphi_k^2 \omega_g = 1$ for all k we conclude the following

Corollary 45. *For every $t > 0$*

$$\int_M p(x,x,t)\omega_g(x) = \sum_{k=0}^{\infty} e^{-\lambda_k t}.$$

6.4 Fundamental solution in $\mathbb{R}^n$

Eventhough we have been working on compact manifolds throughout this chapter, we digress briefly to inspire the form of the fundamental solution on compact manifolds. Suppose $p(x,y,t)$ is a fundamental solution for Heat equation in $\mathbb{R}^n$. That is, $p : \mathbb{R}^n \times \mathbb{R}^n \times (0,+\infty)$ is continuous, C^2 on $\mathbb{R}^n$, C^1 on $(0,+\infty)$

$$\begin{aligned}
&\Delta_{g_{\mathbb{R}^n}} p + \partial_t p = 0,\\
&p(x,y,t) \to \delta_x(y) \qquad t \to 0.
\end{aligned}$$

Then $u(x,t) = \int_{\mathbb{R}^n} p(x,y,t) f(y) dy$ solves the homogeneous heat equation

$$\begin{cases} (\Delta_{g_{\mathbb{R}^n}} + \partial_t) u = 0 \\ u(\cdot, 0) = f \end{cases}$$

where f is a continuous and bounded function on $\mathbb{R}^n$. Let us use the notation " ˆ " for the Fourier tranform with respect to the spatial variables:

$$\begin{aligned}
\|y\|^2 \hat{u}(y,t) &= \sum_{j=1}^n y_j^2 \hat{u}(y,t) = (\sum_{j=1}^n D_{x_j}^2 u\widehat{)}(y,t)\\
&= (-\sum_{j=1}^n \partial_{x_j}^2 u\widehat{)}(y,t) = (\Delta_{g_{\mathbb{R}^n}} u\widehat{)}(y,t)\\
&= -\widehat{\partial_t u}(y,t) = -\partial_t \hat{u}(y,t).
\end{aligned}$$

It follows that there exists a function $h : \mathbb{R}^n \to \mathbb{R}$ such that

$$\hat{u}(y,t) = h(y) e^{-t\|y\|^2}$$

so therefore

$$h(y) = \hat{u}(y,0) = \hat{f}(y).$$

We deduce that

$$\hat{u}(y,t) = \hat{f}(y) e^{-t\|y\|^2} = \hat{f}(y) e^{-\|\sqrt{2t}y\|^2/2} = \hat{f}(y) \cdot \widehat{\left((2t)^{-\frac{n}{2}} e^{-\|\cdot\|^2/4t}\right)}(y)$$

and so

$$\hat{u}(\cdot,t) = \widehat{\left(f * \left[(2t)^{-\frac{n}{2}} e^{-\|\cdot\|^2/4t}\right]\right)}.$$

It follows that formally

$$u(y,t) = f * \left[(2t)^{-\frac{n}{2}} e^{-\|\cdot\|^2/4t}\right](y) = \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} f(x) e^{-\|y-x\|^2/4t} dx.$$

We then should have

$$p(x,y,t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\|y-x\|^2/4t}.$$

Proposition 46. *The function* $p : \mathbb{R}^n \times \mathbb{R}^n \times (0, +\infty) \to [0, +\infty)$

$$p(x, y, t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\|y-x\|^2/4t}$$

is a fundamental solution for the heat equation on $\mathbb{R}^n$.

Proof. It is easy to check that $\Delta_{g_{\mathbb{R}^n}} p + \partial_t p = 0$. Let us prove that $p(x, y, t) \to \delta_x(y)$ $t \to 0$. We first need to show that $\int_{\mathbb{R}^n} p(x, y, t)dy = 1$ for all $(x, t) \in \mathbb{R}^n \times (0, +\infty)$. Indeed,

$$\begin{aligned}
\int_{\mathbb{R}^n} p(x, y, t)dy &= \int_0^\infty \int_{S^{n-1}(x)} p(x, x + r\xi, t) r^{n-1} d\xi dr \\
&= \int_0^\infty \int_{S^{n-1}(x)} \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{r^2}{4t}} r^{n-1} d\xi dr \\
&= \int_0^\infty \int_{S^{n-1}(x)} \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-s^2} s^{n-1} (4t)^{\frac{n-1}{2}} (4t)^{\frac{1}{2}} d\xi dr \\
&= \frac{1}{\pi^{\frac{n}{2}}} \int_0^\infty \int_{S^{n-1}(x)} e^{-s^2} s^{n-1} d\xi ds \\
&= \frac{1}{\pi^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\|y\|^2} dy \\
&= 1.
\end{aligned}$$

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a bounded and continuous function.

$$\begin{aligned}
&\left| f(x) - \int_{\mathbb{R}^n} p(x, y, t) f(y) dy \right| = \\
&= \left| \int_{\mathbb{R}^n} p(x, y, t)(f(x) - f(y)) dy \right| \\
&\leq \int_{B_{2\sqrt{t}R}(x)} p(x, y, t)|f(x) - f(y)| dy + \left| \int_{\mathbb{R}^n \setminus B_{2\sqrt{t}R}(x)} p(x, y, t)(f(x) - f(y)) dy \right| \\
&\leq \sup_{y \in B_{2\sqrt{t}R}(x)} |f(x) - f(y)| + \int_{2\sqrt{t}R}^\infty \int_{S^{n-1}(x)} p(x, x + r\xi, t)|f(x) - f(x + r\xi)| r^{n-1}\, d\xi dr.
\end{aligned}$$

We now need to choose R sufficiently large so that the second term is as small as we wish. Once R is fixed the first term can be chosen to be small since $t \to 0$.

$$\begin{aligned}
&\int_{2\sqrt{t}R}^\infty \int_{S^{n-1}(x)} p(x, x + r\xi, t)|f(x) - f(x + r\xi)| r^{n-1}\, d\xi dr = \\
&\leq \int_R^\infty \int_{S^{n-1}(x)} p(x, x + 2\sqrt{t}s, t) 2\|f\|_\infty (2\sqrt{t}s)^{n-1} 2\sqrt{t}\, d\xi ds \\
&= \int_R^\infty \int_{S^{n-1}(x)} e^{-s^2} s^{n-1} 2\|f\|_\infty d\xi ds \\
&= 2\mathrm{vol}(S^{n-1})\|f\|_\infty \int_R^\infty e^{-s^2} s^{n-1} ds.
\end{aligned}$$

$\square$

6.5 Existence of the fundamental solution on compact manifolds

Let (M, g) be a compact manifold and choose $\varepsilon > 0$ such that the exponential map $\exp_x$ is a well defined diffeomorphism on $B_\varepsilon(0) \subset T_xM$. We then identify a point $y \in B_\varepsilon(x) := \exp_x(B_\varepsilon(0))$ with its polar coordinates (r, ξ), $y = \exp_x(r\xi)$ where $r \in (0, \epsilon)$ and $\xi \in S^{n-1}(0) \subset T_xM$. By performing a computation very similar to the one we carried when we expressed the Laplacian on $\mathbb{R}^n$ in terms of that of a radial operator and the Laplacian of S^{n-1}, once can show that using the geodesic polar coordinates $y = (r, \xi)$ one gets

$$\Delta_g = -\partial_r^2 - \frac{\partial_r(\sqrt{\det g}\partial_r)}{\sqrt{\det g}} + \Delta_{g_{S_r^{n-1}(x)}},$$

where $g_{S^{n-1}(x)}$ is the metric induced on the geodesic sphere $S_r^{n-1}(x) \subset M$ of radius $r \in (0, \varepsilon)$. Remember that the fundamental solution on $\mathbb{R}^n$ has the form $(x, y, t) \mapsto (4\pi t)^{-n/2} e^{\|x-y\|^2/4t}$. Inspired on this formula we wish to find the fundamental solution on M, but in order to do this we need to work with the Riemannian distance function on M. For $y = (r, \xi) \in B_\varepsilon(x)$ we set $d_g(x, y) := r$ which is the length of the radial geodesic joining x and y. Consider the set $\mathcal{V}_\varepsilon := \{(x, y) \in M : \ y \in B_\varepsilon(x), \ d_g(x, y) < \epsilon\}$ and the function

$$G : \mathcal{V}_\varepsilon \times (0, +\infty) \to \mathbb{R}$$
$$G(x, y, t) := \frac{1}{(4\pi t)^{n/2}} e^{d_g(x-y)^2/4t}.$$

Unfortunately, not only G is not defined on all M but also one may check that $(\Delta_g + \partial_t)G \neq 0$. First, note that in geodesic polar coordinates $y = (r, \xi)$ at $x \in M$ one has

$$G(x, y, t) := \frac{1}{(4\pi t)^{n/2}} e^{r^2/4t}$$

and so (writing Δ_g for the Laplacian on the spatial variable y)

$$\Delta_g G = -\partial_r^2 G - \frac{\partial_r(\sqrt{\det g}\partial_r G)}{\sqrt{\det g}} = -\partial_r^2 G - \partial_r G \, \frac{\partial_r \sqrt{\det g}}{\sqrt{\det g}}.$$

Defining the function $D : B_\varepsilon(x) \to \mathbb{R}$

$$D(y) := \frac{\sqrt{\det g(y)}}{r^{n-1}}$$

where $g(y)$ is thought of as a matrix in the $y = (r, \xi)$ geodesic polar coordinates at x we get

$$\begin{aligned} \Delta_g G &= -\partial_r^2 G - \partial_r G\Big(\frac{\partial_r D}{D} + \frac{n+1}{r}\Big) \\ &= \Big(\frac{n}{2t} - \frac{r^2}{4t^2}\Big)G + \frac{r}{2t}\frac{\partial_r D}{D} G. \end{aligned}$$

On the other hand,

$$\partial_t G = -\Big(\frac{n}{2t} - \frac{r^2}{4t^2}\Big)G.$$

This clearly shows that $(\Delta_g + \partial_t)G \neq 0$.

We then try to modify G and we do so by considering for each $k \in \mathbb{N}$ the function

$$S_k : \mathcal{V}_\varepsilon \times (0, +\infty) \to \mathbb{R}$$

$$S_k(x, y, t) := G(x, y, t)\big(u_0(x, y) + tu_1(x, y) + \cdots + t^k u_k(x, y)\big)$$

and we hope we can choose the functions $u_j \in \mathcal{V}_\varepsilon$ that make $(\Delta_g + \partial_t)S_k = 0$. Spoilers alert: we won't be able to get the 0, but we will be able to get an expression that decays to zero like $t^{k-n/2}$ as $t \to 0$. Let us compute $(\Delta_g + \partial_t)S_k$:

$$\Delta_g S_k = \Delta_g G(u_0 + \cdots + t^k u_k) + G(\Delta_g u_0 + \cdots + t^k \Delta_g u_k) - 2\langle \nabla_g G, \nabla_g(u_0 + \cdots + t^k u_k)\rangle_g.$$

Note that since G is not a function of the angular variables ξ, then $\nabla_g G$ involves no ∂_{ξ_j} terms. Since $\langle \partial_r, \partial_{\xi_j}\rangle_g = 0$ and $\langle \partial_r, \partial_r\rangle_g = 1$ we get $\langle \nabla_g G, \nabla_g(u_0 + \cdots + t^k u_k)\rangle_g = \partial_r G(\partial_r u_0 + \cdots + t^k \partial_r u_k) = -\frac{r}{2t}(\partial_r u_0 + \cdots + t^k \partial_r u_k)G$. It then follows that

$$(\Delta_g + \partial_t)S_k = G \cdot \Big(u_1 + \ldots kt^{k-1}u_k + \frac{r}{2t}\frac{\partial_r D}{D}(u_0 + \cdots + t^k u_k) \tag{6.3}$$

$$+ \frac{r}{t}(\partial_r u_0 + \cdots + t^k \partial_r u_k) + \Delta_g u_0 + \cdots + t^k \Delta_g u_k\Big). \tag{6.4}$$

Although we are not able to build the functions u_j so that $(\Delta_g + \partial_t)S_k = 0$ we will be able to build them so that

$$(\Delta_g + \partial_t)S_k = G \cdot t^k \cdot \Delta_g u_k \tag{6.5}$$

which makes $(\Delta_g + \partial_t)S_k$ vanish to order $t^{k-n/2}$. Rearranging the terms in (6.3) one gets the following system of equations that ensure (6.5). We get

$$r\partial_r u_0 + \frac{r}{2}\frac{\partial_r D}{D}u_0 = 0, \tag{6.6}$$

$$r\partial_r u_j + \frac{r}{2}\Big(\frac{\partial_r D}{D} + j\Big)u_j + \Delta_g u_{j-1} = 0, \qquad j = 1, \ldots, k. \tag{6.7}$$

Equation (6.6) gives $u_0 = fD^{-1/2}$ where f is a function of the angular variables ξ. We wish that $u_0(x, y)$ be defined at $x = y$, or equivalently $r = 0$. This means that f must be constant and we set it to be 1. Therefore,

$$u_0(x, y) = D^{-1/2}(y).$$

Since we chose $f \equiv 1$ we get

$$u_0(x, x) = 1.$$

We now need to prove the existence of the other $u_j's$. Instead of solving (6.7) we note that $v_j = f\, r^{-j}\, D^{-1/2}$ with $f = f(\xi)$ solves the simpler problem

$$r\partial_r v_j + \frac{r}{2}\Big(\frac{\partial_r D}{D} + j\Big)v_j = 0.$$

One can then check that

$$u_j := h\, r^{-j}\, D^{-1/2}$$

with $h = h(r)$ solves (6.7) as long as

$$\partial_r h = -D^{1/2}\, \Delta_g u_{j-1}\, r^{j-1}.$$

Let γ be the geodesic on M joining x and y. Since $\Delta_g u_{j-1}(\cdot, y)$ is a functions of r along γ, we may find h by integration in r

$$h(r) = \int_0^r D^{1/2}(\gamma(s))\Delta_g u_{j-1}(\gamma(s), y)\, s^{j-1} ds.$$

Finally,

$$u_j(x, y) = -(d(x, y))^{-j} D^{-1/2}(y) \int_0^r D^{1/2}(\gamma(s))\Delta_g u_{j-1}(\gamma(s), y)\, s^{j-1} ds.$$

It is easy to see that by induction $u_j \in C^\infty(\mathcal{V}_\epsilon)$.
Let us record what we proved so far:

Proposition 47. *The family of functions $u_k \in C^\infty(\mathcal{V}_\epsilon)$ defined by the recursion formulas*

$$\begin{aligned} u_0(x, y) &= D^{-1/2}(y) \\ u_k(x, y) &= -(d(x, y))^{-k} D^{-1/2}(y) \int_0^r D^{1/2}(\gamma(s))\Delta_g u_{j-1}(\gamma(s), y)\, s^{k-1} ds \end{aligned}$$

satisfies

$$L_y S_k = G \cdot t^k \cdot \Delta_g u_k \qquad \text{for all } k = 0, 1, \ldots$$

where $L_y = \Delta_{g,y} + \partial_t$. In particular, $u_0(x, x) = 1$ and $u_1(x, x) = \frac{1}{6} R_g(x)$.

We now wish to extend the definition of S_k to all of M. We then introduce a bump function $\alpha \in C^\infty(M \times M, [0, 1])$ with $\alpha \equiv 0$ on $\mathcal{V}_\varepsilon^c$ and $\alpha \equiv 1$ on $\mathcal{V}_{\varepsilon/2}$. We may now define

$$H_k : M \times M \times (0, +\infty) \to \mathbb{R}$$
$$H_k(x, y, t) := \alpha(x, y) S_k(x, y, t).$$

Lemma 48. *For $k > m/2$ the function $H_k \in C^\infty(M \times M \times (0, +\infty))$ satisfies*

a) $L_y H_k \in C^\ell(M \times M \times [0, +\infty)) \quad \text{for } 0 \le \ell < k - n/2.$

b) For every $x \in M$, $H_k(x, y, t) \to \delta_x(y) \quad \text{as } t \to 0 \text{ for all } y \in M.$

Proof.
a) We prove this statement for $\ell = 0$. The problem with establishing the continuity of H_k is to show we have it at $t = 0$. By the definition of the bump function we have $H_k \equiv 0$ on $\mathcal{V}_\varepsilon^c \times (0, +\infty)$. So we may extend the definition of H_k to $\mathcal{V}_\varepsilon^c \times [0, +\infty)$ to be 0 at $t = 0$. Since $\alpha \equiv 1$ on $\mathcal{V}_{\varepsilon/2}$ and

$$L_y H_k = \frac{1}{(4\pi t)^{n/2}} e^{-r^2/4t} t^k\, \Delta_g u_k \to 0 \quad \text{as } t \to 0,$$

we may also extend H_k on $\mathcal{V}_{\varepsilon/2} \times [0,+\infty)$ by setting it to be 0 at $t=0$. It only remains to deal with the domain $\mathcal{V}_\epsilon \cap \mathcal{V}^c_{\varepsilon/2} \times [0,+\infty)$. We then have

$$\begin{aligned} L_y H_k &= \alpha \cdot (\Delta_g + \partial_t) S_k + \Delta_g \alpha \cdot S_k - 2\langle \nabla_g \alpha, \nabla_g S_k \rangle_g \\ &= \frac{1}{(4\pi t)^{n/2}} e^{-r^2/4t} \beta(x,y,t) \end{aligned}$$

where $\beta(x,y,t) \in C^\infty(M \times M \times (0,+\infty))$ has at most a pole of order $\frac{1}{t}$ at $t=0$ coming from $\nabla_g S_k$. Since on $\mathcal{V}_\epsilon \cap \mathcal{V}^c_{\varepsilon/2} \times [0,+\infty)$ we have $r > \varepsilon/2$, we may also extend H_k to be 0 at $t=0$.

b) Let $f \in C(M)$. Since $H_k = \alpha\, G\, (u_0 + \ldots t^k u_k)$, we are interested in understanding the behavior of the integral

$$\begin{aligned} &\int_M \alpha(x,y) G(x,y,t) u_j(x,y) f(y)\, dv_g(y) \\ &= \int_{B_{\varepsilon/2}(x)} G(x,y,t) u_j(x,y) f(y)\, dv_g(y) + \int_{B^c_{\varepsilon/2}(x)} \alpha(x,y) G(x,y,t) u_j(x,y) f(y)\, dv_g(y) \end{aligned}$$

First note that since $G(x,y,t) = \frac{1}{(4\pi t)^{n/2}} e^{-r^2/4t}$ and $r > \varepsilon/2$ on $B^c_{\varepsilon/2}(x)$, we have that as $t \to 0$

$$\int_{B^c_{\varepsilon/2}(x)} \alpha(x,y) G(x,y,t) u_j(x,y) f(y)\, dv_g(y) \to 0.$$

We now deal with the other term

$$\begin{aligned} &\int_{B_{\varepsilon/2}(x)} \alpha(x,y) G(x,y,t) u_j(x,y) f(y)\, dv_g(y) = \\ &= \int_{B_{\varepsilon/2}(x)} \frac{1}{(4\pi t)^{n/2}} e^{-d^2(x,y)/4t} u_j(x,y) f(y) dv_g(y) \\ &= \int_{T_x M} \frac{1}{(4\pi t)^{n/2}} e^{-\|v\|^2/4t} u_j(x, \exp_x(v)) f(\exp_x(v)) J(v) dv \end{aligned}$$

to get the last equality we passed to normal coordinates $y \to v$ with $y = \exp_x(v)$, we wrote J is the jacobian for the change of variables, and we set $u_j \equiv 0$ on $B_{\varepsilon/2}(0)^c$. Since $\frac{1}{(4\pi t)^{n/2}} e^{-\|v\|^2/4t} = p(0,v,t)$ is the fundamental solution to the heat equation on $\mathbb{R}^n$, we get that as $t \to 0$

$$\int_{B_{\varepsilon/2}(x)} \alpha(x,y) G(x,y,t) u_j(x,y) f(y)\, dv_g(y) \to u_j(x,x) f(x).$$

Using that $u_0(x,x) = 1$ we get as $t \to 0$

$$\int_{B_{\varepsilon/2}(x)} \alpha(x,y) G(x,y,t) u_0(x,y) f(y)\, dv_g(y) \to f(x).$$

For any other values of j we get as $t \to 0$

$$\int_{B_{\varepsilon/2}(x)} \alpha(x,y) G(x,y,t) t^j u_j(x,y) f(y)\, dv_g(y) \to 0.$$

□

Before we find the actual expression for the fundamental solution it is convenient to understand how the Heat operator acts on convolutions. Let us define the convolution operation for functions $F, H \in C^0(M \times M \times [0,\infty))$ by

$$F * H\ (x,y,t) := \int_0^t \int_M F(x,z,s)H(z,y,t-s)\omega_g(z)ds.$$

We first note that if $F \in C^0(M \times M \times [0,\infty))$, then

$$F * H_k \in C^\ell(M \times M \times (0,\infty)) \qquad \text{for } \ell < k - n/2.$$

This is easy to check, the only problem being the discontinuity of H_k at $t = 0$. We now explore how the heat operator $L_y = \Delta_{g,y} + \partial_t$ acts on $F * H_k(x,y,t)$. First observe that

$$\begin{aligned}
&\partial_t(F * H_k)(x,y,t)\\
&\quad = \partial_t \int_0^t \int_M F(x,z,s)H_k(z,y,t-s)\omega_g(z)ds\\
&\quad = \lim_{s\to t}\int_M F(x,z,s)H_k(z,y,t-s)\omega_g(z) + \int_0^t \int_M F(x,z,s)\partial_t H_k(z,y,t-s)\omega_g(z)ds\\
&\quad = F(x,y,t) + \int_0^t \int_M F(x,z,s)\partial_t H_k(z,y,t-s)\omega_g(z)ds,
\end{aligned}$$

and so

$$\begin{aligned}
L_y(F * H_k)(x,y,t) = &= F(x,y,t) + \int_0^t \int_M F(x,z,s)\, L_y H_k(z,y,t-s)\omega_g(z)ds\\
&= F(x,y,t) + F * (L_y H_k)\,(x,y,t).
\end{aligned}$$

We then look for fundamental solutions of the form

$$p = H_k - F * H_k$$

for some suitable choice of F. Note that

$$\begin{aligned}
L_y p &= L_y(H_k - F * H_k)\\
&= L_y H_k - L_y(F * H_k)\\
&= L_y H_k - F - F * (L_y H_k).
\end{aligned} \tag{6.8}$$

This suggest that we consider, at least formally,

$$F_k := \sum_{j=1}^{\infty} (-1)^{j+1} (L_y H_k)^{*j}.$$

Indeed, if we had that this series is convergent then from (6.8), for $p = H_k - F_k * H_k$,

$$L_y p = L_y H_k - \sum_{j=1}^{\infty}(-1)^{j+1}(L_y H_k)^{*j} - \Big(\sum_{j=1}^{\infty}(-1)^{j+1}(L_y H_k)^{*j}\Big) * (L_y H_k) = 0. \tag{6.9}$$

Lemma 49. *For all $\ell < k - n/2$ the series $F_k \in C^\ell(M \times M \times [0, +\infty))$. In addition, for $t_0 > 0$ there exists a constant $C = C(t_0)$ such that*

$$\|F_k(\cdot, \cdot, t)\|_{L^\infty(M\times M)} \leq Ct^{k-n/2} \qquad \text{for all } t \in [0, t_0].$$

Proof. As we did in the proof of Lemma 48, using that $\nabla_g \alpha$ and $\Delta_g \alpha$ have their supper away from the diagonal in $M \times M$, it is easy to get the existence of a constant $C = A(t_0)$ such that

$$\|L_y H_k(\cdot, \cdot, t)\|_{L^\infty(M\times M)} \leq Ct^{k-n/2} \qquad \text{for all } t \in [0, t_0].$$

In particular,

$$\|L_y H_k\|_{L^\infty(M\times M\times[0,t_0])} \leq Ct_0^{k-n/2}$$

We claim that for $j = 1, 2, \ldots$ and $t \in [0, t_0]$

$$\|(L_y H_k)^{*j}(\cdot, \cdot, t)\|_{L^\infty(M\times M)} \leq \frac{C\ (Ct_0^{k-n/2})^{j-1}\ \mathrm{vol}_g(M)^{j-1}}{(k - n/2 + j - 1)\ldots(k - \frac{n}{2} + 2)(k - \frac{n}{2} + 1)} t^{k-\frac{n}{2}+j-1}. \tag{6.10}$$

Indeed, by induction, assuming that it is true for $j - 1$ we get for $x, y \in M$

$$\begin{aligned}
&|(L_y H_k)^{*j}(x, y, t)| \leq \\
&\leq \int_0^t \int_M |(L_y H_k)^{*(j-1)}(x, z, s)||L_y H_k(z, y, t - s)|\, \omega_g(z) ds \\
&\leq \int_0^t \int_M \frac{C\ (Ct_0^{k-n/2})^{j-2}\ \mathrm{vol}_g(M)^{j-2}}{(k - n/2 + j - 2)\ldots(k - \frac{n}{2} + 2)(k - \frac{n}{2} + 1)} t^{k-\frac{n}{2}+j-2} Ct_0^{k-n/2}\, \omega_g(z) ds \\
&= \frac{C\ (Ct_0^{k-n/2})^{j-1}\ \mathrm{vol}_g(M)^{j-2}}{(k - n/2 + j - 2)\ldots(k - \frac{n}{2} + 2)(k - \frac{n}{2} + 1)}\ \mathrm{vol}_g(M) \int_0^t t^{k-\frac{n}{2}+j-2} ds.
\end{aligned}$$

Using (6.10) the ratio test shows that $\sum_j \|(L_y H_k)^{*j}\|_{L^\infty}$ is convergent if $k > n/2$, and so in particular $\sum_j (-1)^{j+1}(L_y H_k)^{*j}$ converges to a continuous function.

The same kind of argument can be carried for the derivatives of $L_y H_k$. □

We have showed that F_k is well defined. We then prove

Proposition 50. *The function $p = H_k - F_k * H_k$ is a fundamental solution for the Heat equation for all $k > n/2 + 2$.*

Proof. For $k > n/2 + 2$ we know that $p \in C^2(M \times M \times (0, +\infty))$. Since $F_k := \sum_{j=1}^\infty (-1)^{j+1}(L_y H_k)^{*j}$,

$$L_y p = L_y(H_k - F * H_k) = L_y H_k - L_y(F * H_k) = L_y H_k - F - F * (L_y H_k) = 0.$$

Let us now prove that $p(x, y, t) \to \delta_x(y)$ as $t \to 0$:

$$\begin{aligned}
\lim_{t\to 0} \int_M p(x, y, t) f(y)\, \omega_g(y) &= \lim_{t\to 0} \int_M \big(H_k(x, y, t) - F_k * H_k(x, y, t)\big) f(y)\, \omega_g(y) \\
&= f(x) - \lim_{t\to 0} \int_M F_k * H_k(x, y, t)\ f(y)\, \omega_g(y)
\end{aligned}$$

By Lemma 49 we have $\|F_k(\cdot,\cdot,t)\|_{L^\infty(M\times M)} \le Ct^{k-n/2}$ for all $t \in [0,t_0]$ and so the kernel $R_k := t^{-(k-n/2)}F_k$ is uniformly bounded for $(x,y,t) \in M \times M \times [0,t_0]$. Therefore, since $H_k(x,y,t) \to \delta_x(y)$ as $t \to 0$ we get that for $k > n/2$

$$\lim_{t\to 0}\int_M F_k * H_k(x,y,t)\; f(y)\,\omega_g(y) = \lim_{t\to 0} t^{k-n/2}\int_M R_k * H_k(x,y,t)\; f(y)\,\omega_g(y) = 0.$$

□

Theorem 51. *There exists $\epsilon > 0$ such that the fundamental solution for the heat equation has expansion*

$$p(x,y,t) = \frac{e^{-d_g^2(x,y)/4t}}{(4\pi t)^{n/2}}\Big(\sum_{j=0}^{k} t^j u_j(x,y) + O(t^{k+1})\Big),$$

for all $x,y \in M$ with $d_g(x,y) \le \varepsilon/2$. In addition, $u_0(x,x) = 1$ and $u_1(x,x) = \frac{1}{6}R_g(x)$.

Proof. Since $p = H_k - F_k * H_k$ is a fundamental solution for the heat equation for any $k > n/2+2$ and such solution is unique, we have that the definition of p is independent of k. Note also that $H_k - F_k * H_k \in C^{k-n/2}(M \times M \times (0,+\infty))$ for all $k > n/2$ which shows that $p \in C^\infty(M \times M \times (0,+\infty))$.

Since $\|F_k(\cdot,\cdot,t)\|_{L^\infty(M\times M)} \le Ct^{k-n/2}$ and $H_k \in L^1(M \times M \times (0,t))$, we get $\|F_k * H_k(\cdot,\cdot,t)\|_{L^\infty(M\times M)} \le Dt^{k+1-n/2}$ for all $t \le t_0$ for some given small t_0. The asymptotics follow from the definition of $H_k = \frac{e^{-d_g^2(x,y)/4t}}{(4\pi t)^{n/2}}\alpha(x,y)\cdot(u_0 + \cdots + t^k u_k)$. □

6.6 Fundamental solution in Riemannian coverings

Let $(\tilde M, g_{\tilde M})$ be a Riemannian manifold and let Γ be a discrete group acting properly and freely on $\tilde M$. Write (M, g_M) for the compact quotient manifold $M = \tilde M/\Gamma$. Since M is compact, there exists a relatively compact open set $D \subset \tilde M$ such that $\tilde M = \cup_{\gamma\in\Gamma}\gamma D$. Also, the group Γ is finitely generated and so Γ is a countable set.

The construction of the fundamental solution for the heat equation on a compact manifold can also be carried on $\tilde M$ with almost no modification. Since $\tilde M = \cup_{\gamma\in\Gamma}\gamma D$ with D relatively compact we can choose $\varepsilon > 0$ so that $d(x,y) < \varepsilon$ implies that y lies in a normal coordinate neighborhood of x. On the sets $\mathcal{V}_\varepsilon := \{(x,y) \in \tilde M \times \tilde M :\ d_{g_{\tilde M}}(x,y) < \varepsilon\}$ and defines the parametrix H_k as in the previous section. Then

$$\tilde p \in C^\infty(\tilde M \times \tilde M \times (0,+\infty))$$

$$\tilde p := H_k - \Big(\sum_{j=1}^{\infty}(-1)^{j+1}(L_y H_k)^{*j}\Big) * H_k$$

is a fundamental solution for all $k > n/2+2$.

We need a few results before we explain the relation between the heat kernel on $\tilde M$ and that of M. First, by comparison with spaces of contant curvature one has the following upper bound for the volume of geodesic balls (which we won't prove):

Lemma 52. *Let $(\tilde{M}, g_{\tilde{M}})$ be a Riemannian manifold whose all sectional curvatures are bounded below by κ. Then, for each $x \in \tilde{M}$ and $r \leq inj(\tilde{M})$, one has*

$$vol_{g_{\tilde{M}}}(B_r(x)) \leq C_1 e^{C_2 r}$$

where $C_1, C_2 > 0$ depend on κ and $dim\tilde{M}$.

Lemma 53. *Set $N_D := \#\{\gamma \in \Gamma : \ D \cap \gamma D \neq \emptyset\}$. For each $x, y \in D$ and $r > diam(D)$*

$$\#\{\gamma \in \Gamma : \ B_r(x) \cap \gamma y \neq \emptyset\} \leq N_D \frac{vol_{g_{\tilde{M}}} B_{2r}(x)}{vol_{g_{\tilde{M}}}(D)}.$$

Proof. For $x, y \in D$ suppose that there is a number ℓ of Γ-translates of y inside $B_r(x)$. Since $r >$ diam D, then $B_{2r}(x)$ contains ℓ translates of D. However, any point of $\tilde{M}$ is contained in at most N_D translates of D. In consequence,

$$\frac{\ell \cdot \mathrm{vol}_{g_{\tilde{M}}}(D)}{N_D} \leq \mathrm{vol}_{g_{\tilde{M}}}(B_{2r}(x)).$$

$\square$

Write $\tilde{p} : \tilde{M} \times \tilde{M} \times (0, +\infty) \to \mathbb{R}$ for the fundamental solution of the heat equation in $(\tilde{M}, g_{\tilde{M}})$ and $p : M \times M \times (0, +\infty) \to \mathbb{R}$ for the fundamental solution of the heat equation in (M, g_M), where we continue to write $M = \tilde{M}/\Gamma$. We also write $\pi : \tilde{M} \to M$ for the covering map.

Proposition 54. *The fundamental solution for the heat equation on (M, g) is given by $p : M \times M \times (0, +\infty) \to \mathbb{R}$*

$$p(x, y, t) = \sum_{\gamma \in \Gamma} \tilde{p}(\tilde{x}, \gamma\, \tilde{y}, t)$$

where $\tilde{x}, \tilde{y}$ are such that $\pi(\tilde{x}) = x$, $\pi(\tilde{y}) = y$. The sum on the right hand side converges uniformly on $D \times D \times [t_0, t_1]$ with $0 < t_0 \leq t_1$.

Proof. We first show that the series converges:

$$\begin{aligned}
\sum_{\gamma \in \Gamma} \tilde{p}(\tilde{x}, \gamma\, \tilde{y}, t) &\leq C t^{-n/2} \sum_{\gamma \in \Gamma} e^{-d^2_{g_{\tilde{M}}}(\tilde{x}, \gamma \tilde{y})/4t} \\
&\leq C t^{-n/2} \sum_{j=1}^{\infty} \sum_{\{\gamma \in \Gamma :\ \gamma \tilde{y} \in B_{jr}(\tilde{x}) \setminus B_{(j-1)r}(x)\}} e^{-d^2_{g_{\tilde{M}}}(\tilde{x}, \gamma \tilde{y})/4t} \\
&\leq C t^{-n/2} \sum_{j=1}^{\infty} \frac{N_D}{\mathrm{vol}_{g_{\tilde{M}}}(D)} \mathrm{vol}_{g_{\tilde{M}}}(B_{2jr}(x)) e^{-(j-1)^2 r^2/4t} \\
&\leq C C_1 t^{-n/2} \sum_{j=1}^{\infty} \frac{N_D}{\mathrm{vol}_{g_{\tilde{M}}}(D)} e^{2C_2 jr} e^{-(j-1)^2 r^2/4t}
\end{aligned}$$

where to get the second inequality we decomposed M into rings whose boundary are geodesic spheres centred a t x of radious jr with $r > \mathrm{diam}(D)$ and $j = 1, 2 \ldots$. We then used that for $B_{jr}(x)\backslash B_{(j-1)r}(x)$ one has $d_{g_{\tilde{M}}}(x,y) > (j-1)r$, and we also used the previous Lemma to bound the number of Γ-translates of y inside $B_{jr}(x)$.

We now show that p is a fundamental solution:
Fix $(x,t) \in M \times (0,+\infty)$ and consider the function $\tilde{q} : \tilde{M} \to \mathbb{R}$ defined by $\tilde{q}(\tilde{y}) := \sum_{\gamma\in\Gamma} \tilde{p}(\tilde{x}, \gamma\, \tilde{y}, t)$ where $\tilde{x}$ is so that $\pi\tilde{x} = x$. It is clearly invariant under the deck transformation group Γ and so there exists a unique function $q : M \to \mathbb{R}$ making $q \circ \pi = \tilde{q}$. Since q is unique then $p(x,y,t) = q(y)$. We then have, for $\tilde{y}$ such that $\pi\tilde{y} = y$,

$$\begin{aligned}\Delta_{g_M,y} q(y) &= \Delta_{g_M} q(\pi\tilde{y}) = \pi^*\Delta_{g_M} q(\tilde{y}) \\ &= \Delta_{g_{\tilde{M}}} \pi^* q(\tilde{y}) = \Delta_{g_{\tilde{M}}} \tilde{q}(\tilde{y}) \\ &= \sum_{\gamma\in\Gamma} \Delta_{g_{\tilde{M}},\tilde{y}} \tilde{p}(\tilde{x}, \gamma\tilde{y}, t).\end{aligned}$$

Since $(\Delta_{g_{\tilde{M}},y} + \partial_t)\tilde{p} = 0$, it follows that $L_y q(y) = 0$. To see that $p(x,y,t) \to \delta_x(y)$, simply note that

$$p(x,y,t) \to \sum_{\gamma\in\Gamma} \delta_{\tilde{x}}(\gamma\tilde{y}).$$

It only remains to note that

$$\sum_{\gamma\in\Gamma} \delta_{\tilde{x}}(\gamma\tilde{y}) = \begin{cases} 1 & \exists\gamma :\ \tilde{x} = \gamma\tilde{y} \\ 0 & \text{else} \end{cases} \qquad \text{if and only if} \qquad \sum_{\gamma\in\Gamma} \delta_{\tilde{x}}(\gamma\tilde{y}) = \begin{cases} 1 & x = y \\ 0 & \text{else} \end{cases},$$

and so $\sum_{\gamma\in\Gamma} \delta_{\tilde{x}}(\gamma\tilde{y}) = \delta_x(y)$. □

Flat Torus. From the expression for the fundamental solution of the heat equation on $\mathbb{R}^n$ it follows that the fundamental solution for the heat equation on the flat torus $\mathbb{T} = \mathbb{R}^n/\Gamma$ is given by

$$p_{\mathbb{T}}(x,y,t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} \sum_{\gamma\in\Gamma} e^{-\|\tilde{y}+\gamma-\tilde{x}\|^2/4t}$$

where $\tilde{x}, \tilde{y}$ are such that $\pi(\tilde{x}) = x$, $\pi(\tilde{y}) = y$ for the corresponding covering map

$$\pi : (\mathbb{R}^n, g_{\mathbb{R}^n}) \to (\mathbb{R}^n/\Gamma, g_{\mathbb{R}^n}/\Gamma).$$

Hyperbolic surfaces. Let (M,g) be a hyperbolic compact surface which is realized as the quotient $\mathbb{H}/\Gamma$ where $\mathbb{H}$ is the hyperbolic plane and Γ is a discrete subgroup acting properly and freely. To get the fundamental solution on $M = \mathbb{H}/\Gamma$ we just need to know that of $\mathbb{H}$. We won't prove the next result but you may find it in the book by Buser. Let $d_{\mathbb{H}} : \mathbb{H} \times \mathbb{H} \to \mathbb{R}$ be the hyperbolic distance.

Proposition 55. *The function $p : \mathbb{H} \times \mathbb{H} \times (0,+\infty) \to \mathbb{R}$ given by*

$$p_{\mathbb{H}}(x,y,t) = \frac{\sqrt{2}}{(4\pi t)^{3/2}} e^{-t/4} \int_{d_{\mathbb{H}}(z,w)}^{\infty} \frac{re^{-r^2/4t}}{\sqrt{\cosh r - \cosh d_{\mathbb{H}}(z,w)}} dr$$

is a fundamental solution for the heat equation on $\mathbb{H}$.

We note that

$$\cosh d_{\mathbb{H}}(z,w) = 1 + \frac{|z-w|^2}{2\mathrm{Im}z\,\mathrm{Im}w}.$$

Negatively curved manifolds. If (M,g) is a compact manifold with negative sectional curvature everywhere, then by Hadamard's Theorem for every $x \in M$ the exponential map $\pi := \exp_x : T_xM \to M$ is a covering map. Identifying T_xM with $\mathbb{R}^n$, $n = \dim M$, we may think of (M,g) as the quotient of $\mathbb{R}^n$ by the deck transformation group Γ associated to π. The metric we use on $\mathbb{R}^n$ is exactly π^*g. One may define the Dirichlet domain

$$D_{Dir} := \{\tilde{y} \in \mathbb{R}^n : \; d_{\pi^*g}(0,\tilde{y}) < d_{\pi^*g}(0,\gamma\tilde{y}) \quad \forall \gamma \in \Gamma \backslash Id.\}$$

We can add to D_{Dir} a subset of $\partial D_{Dir} = \overline{D_{Dir}} \backslash \mathrm{int}(D_{Dir})$ to obtain a fundamental domain D which has the property that $\mathbb{R}^n$ is the disjoint union of the γD as γ ranges in Γ. One may then identify every point $x \in M$ with a unique point $\tilde{x} \in D$ and

$$p_{(M,g)}(x,y,t) = \sum_{\gamma \in \Gamma} p_{(\mathbb{R}^n,\pi^*g)}(\tilde{x},\gamma\,\tilde{y},t).$$

CHAPTER 7

Eigenvalues

Throughout this section we assume that M is compact. Further, if M has a boundary, then we impose either Dirichlet or Neumann boundary conditions. Let $\varphi_0, \varphi_1, \ldots$ be an orthonormal basis of eigenfunctions of the Laplacian with respective eigenvalues $\lambda_0 \leq \lambda_1 \leq \ldots$, satisfying the corresponding boundary conditions when needed.

7.1 Self-adjoint extension of the Laplacian

We know that the Laplacian $\Delta_g : C^\infty(M) \to C^\infty(M)$ is formally self-adjoint. The purpose of this section is to prove that the Laplacian admits a self-adjoint extension to $H^2(M)$ in the sense that the domains of Δ_g and Δ_g^* coincide. To show this, we first introduce a characterization of $H^1(M)$ and $H^2(M)$. Before we do this, we note that the Sobolev space $H_1(M)$ can be defined as the completion of $C^\infty(M)$ with respect to the inner product

$$\langle u, v\rangle_{H_1} := \langle u, v\rangle_g + \langle \nabla_g u, \nabla_g v\rangle_g$$

for $u, v \in C^\infty(M)$. Similarly, $H_k(M)$ is the completion of $C^\infty(M)$ with respect to

$$\langle u, v\rangle_{H_k} := \langle u, v\rangle_{H_{k-1}} + \langle \nabla_g^k u, \nabla_g^k v\rangle_g.$$

Proposition 56.

$$H_k(M) = \Big\{ f \in L^2(M) : \quad \sum_{j=0}^{\infty} \lambda_j^k \langle f, \varphi_j\rangle^2 < \infty \Big\}.$$

Proof. We give the proof of the characterization of $H_1(M)$. The one for $H_k(M)$ is analogous. Define $\mathcal{A} := \{ f \in L^2(M) : \quad \sum_{j=0}^{\infty} \lambda_j \langle f, \varphi_j\rangle^2 < \infty \Big\}$.

We first prove that $\mathcal{A} \subset H_1(M)$. For $f \in \mathcal{A}$ and $k \in \mathbb{N}$ consider the smooth approximation in $L^2(M)$

$$S_f^k := \sum_{j=0}^{k} a_j \varphi_j \qquad a_j := \langle f, \varphi_j \rangle.$$

Since $S_f^k \in C^\infty(M)$,

$$\|\nabla(S_f^k - S_f^l)\|_{L^2}^2 = \langle S_f^k - S_f^l, \Delta_g(S_f^k - S_f^l)\rangle_g = \sum_{j=l+1}^{k} \lambda_j a_j^2.$$

It follows that $\|S_f^k - S_f^l\|_{H_1}^2 = \sum_{j=l+1}^{k}(1+\lambda_j)a_j^2 \to 0$ as $l, k \to \infty$. Since $H_1(M)$ is complete $\lim_k S_f^k = f \in H_1(M)$ and so $\mathcal{A} \subset H_1(M)$.

Next we prove that $\mathcal{A}$ is closed in $H_1(M)$, that is $\overline{\mathcal{A}}^{\| \|_{H_1}} = \mathcal{A}$. Let $f = \sum_j a_j \varphi_j \in \overline{\mathcal{A}}^{\| \|_{H_1}}$. Since $C^\infty(M)$ is dense in $H_1(M)$, there exists a sequence $\{f_j\}_j \subset C^\infty(M)$ such that $f_j \to f$ in $H_1(M)$. We may also consider the Fourier expansion for each f_j. Say $f_j = \sum_{k=0}^\infty b_k^{(j)} \varphi_k$ in $L^2(M)$. Since $\|f_j - f\|_{H_1} \to_j 0$, we get $\|f_j - f\|_{L^2} \to_j 0$. Therefore $\| \sum_k (b_k^{(j)} - a_k)\varphi_k\|_{L^2} \to_j 0$, and this gives $b_k^{(j)} \to_j a_k$ for all k.
In addition,

$$\langle \Delta_g f_j, \varphi_k \rangle_g = \langle f_j, \Delta_g \varphi_k \rangle_g = \lambda_k b_k^{(j)}$$

and so $\Delta_g f_j = \sum_k \lambda_k b_k^{(j)} \varphi_k$. It follows that

$$\begin{aligned}
\|f_j\|_{H_1}^2 &= \|f_j\|_{L^2}^2 + \|\nabla_g f_j\|_{L^2}^2 \\
&= \|f_j\|_{L^2}^2 + \langle f_j, \Delta_g f_j \rangle_g \\
&= \sum_{k=0}^{\infty} (1+\lambda_k)(b_k^{(j)})^2 \\
&\geq \sum_{k=0}^{\ell} (1+\lambda_k)(b_k^{(j)})^2
\end{aligned}$$

for all $\ell \in \mathbb{N}$. Since $\|f_j - f\|_{H_1} \to_j 0$, we know $\|f_j\|_{H_1} \to_j \|f\|_{H_1}$. In particular, since $\|f_j\|_{H_1}^2 \geq \sum_{k=0}^{\ell}(1+\lambda_k)(b_k^{(j)})^2$ for all $\ell \in \mathbb{N}$ and $b_k^{(j)} \to_j a_k$ for all $k \in \mathbb{N}$, we deduce $\|f\|_{H_1}^2 \geq \sum_{k=0}^{\ell}(1+\lambda_k)a_k^2$ for all $\ell \in \mathbb{N}$. We then know that both $\sum_{k=0}^\infty (1+\lambda_k)a_k^2$ and $\sum_{k=0}^\infty a_k^2$ converge, which yields the convergence of $\sum_{k=0}^\infty \lambda_k a_k^2$. We have proved $\overline{\mathcal{A}}^{\| \|_{H_1}} \subset \mathcal{A}$.

Having shown that $\mathcal{A}$ is closed in $H_1(M)$, all that remains to be proved is that $\mathcal{A}^\perp = \{0\}$ in $H_1(M)$. Let $f \in H_1(M) \cap \mathcal{A}^\perp$. Since $C^\infty(M)$ is dense in $H_1(M)$ and $\mathcal{A}$ is closed in $H_1(M)$, there exists $\{f_j\}_j \subset C^\infty(M) \cap \mathcal{A}^\perp$ such that $\|f_j - f\|_{H_1} \to_j 0$, and so in particular, $\|f_j - f\|_{L^2} \to_j 0$. By Green's identities

$$\langle f_j, \varphi_k \rangle_{H_1} = \langle f_j, \varphi_k \rangle_g + \langle \nabla_g f_j, \nabla_g \varphi_k \rangle_g = \langle f_j, \varphi_k \rangle_g + \langle f_j, \Delta_g \varphi_k \rangle_g = (1+\lambda_k)\langle f_j, \varphi_k \rangle_g.$$

Since $\langle f_j, \varphi_k \rangle_{H_1} = 0$ for all k, it follows that $\langle f, \varphi_k \rangle_g = 0$ for all k and so $f_j = 0$. This shows that $f \equiv 0$. $\square$

Note that if $f = \sum_j a_j\varphi_j \in C^\infty(M)$, then $\Delta_g f = \sum_j a_j\lambda_j\varphi_j$ and so $\|\Delta_g f\|_{L^2} = \|f\|_{H_2(M)}$. This suggests we define an extension $\tilde{\Delta}_g : H_2(M) \to L^2(M)$ of the Laplacian. For $f \in H_2(M)$, $f = \sum_j a_j\varphi_j$, we set

$$\tilde{\Delta}_g f := \sum_{j=0}^{\infty} a_j\lambda_j\varphi_j.$$

Before we present the properties of the extension it is convenient to introduce a bilinear form in $H^1(M)$. Consider the bilinear form on $D_g : C^\infty(M) \times C^\infty(M) \to \mathbb{R}$

$$D_g(f,h) := \langle \nabla_g f, \nabla_g h\rangle_g.$$

Given $f, h \in H_1(M)$, there exist sequences $\{f_j\}_j, \{h_j\}_j \subset C^\infty(M)$ such that $f_j \to_j f$ and $h_j \to_j h$ in $H_1(M)$. We then define the bilinear form on $H_1(M) \times H_1(M)$ by

$$D_g(f,h) := \lim_{j\to\infty} D_g(f_j, h_j).$$

Theorem 57. *The extension $\tilde{\Delta}_g : H_2(M) \to L^2(M)$ of the Laplacian has the following properties.*

1. $\langle \tilde{\Delta}_g f, h\rangle_g = D_g(f,h)$ *for all $f \in H_2(M)$ and $h \in H_1(M)$.*
2. $\tilde{\Delta}_g$ *is self-adjoint.*

Proof.
1) Let $f = \sum_j a_j\varphi_j \in H_2(M)$ and $h = \sum_j b_j\varphi_j \in H_1(M)$. If we set $S_f^k = \sum_{j=0}^k a_j\varphi_j$ and $S_h^k = \sum_{j=0}^k b_j\varphi_j$, we get that $\{S_f^k\}_k$ and $\{S_h^k\}_k$ are Cauchy sequences in $H_1(M)$. Then,

$$D_g(f,h) = \lim_{k\to\infty} D(S_f^k, S_h^k) = \lim_{k\to\infty} \langle \nabla_g S_f^k, \nabla_g S_h^k\rangle_g = \lim_{k\to\infty} \langle \Delta_g S_f^k, S_h^k\rangle_g = \sum_{j=0}^{\infty} \lambda_j a_j b_j.$$

In particular, by definition,

$$\langle \tilde{\Delta}_g f, h\rangle_g = \Big\langle \sum_{j=0}^{\infty} a_j\lambda_j\varphi_j, \sum_{j=0}^{\infty} b_j\varphi_j \Big\rangle_g = \sum_{j=0}^{\infty} \lambda_j a_j b_j = D(f,h).$$

2) From the previous part we know that $\tilde{\Delta}_g$ is formally self-adjoint. Indeed, if $f, h \in H_2(M)$,

$$\langle \tilde{\Delta}_g f, h\rangle_g = D_g(f,h) = D_g(h,f) = \langle \tilde{\Delta}_g h, f\rangle_g.$$

Set

$$\mathrm{Dom}(\tilde{\Delta}_g^*) := \Big\{ u \in L^2(M) :\ \exists h_u \in L^2(H) \text{ such that } \langle f, h_u\rangle_g = \langle \tilde{\Delta}_g f, u\rangle_g \ \ \forall f \in H_2(M) \Big\}.$$

Note that if such h_u exists, then h_u is unique. Indeed, if there were two such choices $h_u^{(1)}$ and $h_u^{(2)}$ then $\langle f, h_u^{(1)}\rangle_g = \langle \tilde{\Delta}_g f, u\rangle_g = \langle f, h_u^{(2)}\rangle_g$ for all $f \in H_2(M)$. Since $H_2(M)$ is dense in $L^2(M)$ we must have $h_1 = h_2$. This shows that we may define the operator

$$\tilde{\Delta}_g^* : \mathrm{Dom}(\tilde{\Delta}_g^*) \to L^2(M)$$

$$\tilde{\Delta}_g^* \, u := h_u.$$

In order to show that $\tilde{\Delta}_g$ is self-adjoint we need to prove that $\mathrm{Dom}(\tilde{\Delta}_g^*) = \mathrm{Dom}(\tilde{\Delta}_g)$ where $\mathrm{Dom}(\tilde{\Delta}_g) = H_2(M)$.

Since $\tilde{\Delta}_g$ is formally self-adjoint we have $\mathrm{Dom}(\tilde{\Delta}_g) \subset \mathrm{Dom}(\tilde{\Delta}_g^*)$. Indeed, if $u \in \mathrm{Dom}(\tilde{\Delta}_g)$, then for all $f \in H_2(M)$ we have $\langle \tilde{\Delta}_g f, u\rangle_g = \langle f, \tilde{\Delta}_g u\rangle_g$ and so trivially we may set $h_u := \Delta_g u$.

Let us now prove the converse inclusion. Let $u \in \mathrm{Dom}(\tilde{\Delta}_g^*)$, say $u = \sum_j a_j \varphi_j$. Then there exists a unique $h_u = \sum_j b_j \varphi_j \in L^2(M)$ such that $\langle \tilde{\Delta}_g f, u\rangle_g = \langle f, h_u\rangle_g$ for all $f \in H_2(M)$. In particular, setting $f = \varphi_j$ we get

$$\lambda_j a_j = \lambda_j \langle \varphi_j, u\rangle_g = \langle \tilde{\Delta}_g \varphi_j, u\rangle_g = \langle \varphi_j, h_u\rangle_g = b_j$$

which gives $\lambda_j a_j = b_j$. Therefore $\sum_j \lambda_j^2 a_j^2 = \sum_j b_j^2 = \|h_u\|_{L^2}^2 < \infty$ and so by the characterization of $H_2(M)$ we got that $u \in H_2(M) = \mathrm{Dom}(\tilde{\Delta}_g)$ as we wanted. □

7.2 Characterization

Let (M, g) be a compact Riemannian manifold and write $\lambda_1 \leq \lambda_2 \leq \ldots$ for the eigenvalues of the Laplacian repeated according to multiplicity (for any initial problem). Write $\varphi_1, \varphi_2, \ldots$ for the corresponding L^2-normalized eigenfunctions.
Since we require $\varphi_i \neq 0$ and $\Delta_g(1) = 0.1$ we have the following easy remark:

$$\begin{cases} \lambda_1 = 0 & \text{if } \partial M = \emptyset, \\ \lambda_1 > 0 & \text{Dirichlet boundary conditions,} \\ \lambda_1 = 0 & \text{Neumann boundary conditions.} \end{cases}$$

Theorem 58. *For $k \in \mathbb{N}$ and $E_k(g) := \{\varphi_1, \varphi_s, \ldots, \varphi_{k-1}\}^{\perp}$,*

$$\lambda_k = \inf\Big\{ \frac{D_g(\phi,\phi)}{\|\phi\|_g^2} : \quad \phi \in H_1(M) \cap E_k(g) \Big\}.$$

The infimum is achieved if and only if ϕ is an eigenfunction of eigenvalue λ_k.

Proof. Fix $\phi \in H_1(M) \cap E_k(g)$ and assume it has expansion $\phi = \sum_{j=1}^{\infty} a_j \varphi_j$. Fix $\ell \in \mathbb{N}$,

$$\begin{aligned} 0 &\leq D_g\left(\phi - \sum_{j=1}^{\ell} a_j \varphi_j \ , \ \phi - \sum_{j=1}^{\ell} a_j \varphi_j \right) \\ &= D_g(\phi,\phi) - 2\sum_{j=1}^{\ell} a_j D_g(\phi, \varphi_j) + \sum_{i,j=1}^{\ell} a_j a_i D_g(\varphi_j, \varphi_i) \\ &= D_g(\phi,\phi) - 2\sum_{j=1}^{\ell} a_j \, \langle \phi, \Delta_g \varphi_j\rangle + \sum_{i,j=1}^{\ell} a_j a_i \langle \varphi_j, \Delta_g \varphi_l\rangle_g \\ &= D_g(\phi,\phi) - \sum_{j=1}^{\ell} \lambda_j a_j^2. \end{aligned}$$

Therefore, we get $D_g(\phi,\phi) \geq \sum_{j=1}^{\ell} \lambda_j \alpha_j^2$ for all $\ell \in \mathbb{N}$ and so

$$D_g(\phi,\phi) \geq \sum_{j=1}^{\infty} \lambda_j a_j^2 \geq \sum_{j=k}^{\infty} \lambda_j a_j^2 \geq \lambda_k \sum_{j=k}^{\infty} a_j^2 = \lambda_k \|\phi\|_g^2 \tag{7.1}$$

and so

$$\lambda_k \leq \frac{D_g(\phi,\phi)}{\|\phi\|_g^2}$$

for all $\phi \in E_k(g)$. If ϕ is an eigenfunction of eigenvalue λ_k, then according to Corollary 18

$$\frac{D_g(\phi,\phi)}{\|\phi\|_g^2} = \frac{\langle \phi, \Delta_g \phi \rangle_g}{\|\phi\|_g^2} = \lambda_k$$

and so the infimum is achieved.
On the other hand, if the infimum is achieved, from (7.1) follows that

$$\sum_{j=k}^{\infty} \lambda_j a_j^2 = \lambda_k \sum_{j=k}^{\infty} a_j^2$$

and therefore

$$\sum_{j=k}^{\infty} (\lambda_j - \lambda_k) a_j^2 = 0.$$

If $\lambda_j \neq \lambda_k$ then $a_j = 0$ and so ϕ must be a linear combination of eigenfunctions with eigenvalue λ_k. □

The previous theorem is a nice characterization of the eigenvalues, but in practise one need to know the eigenfunctions to use. Since finding the eigenfunctions is a much harder problem than determining the eigenvalues it is better to try to understand how to characterize the eigenvalues without using any eigenfunctions. We proceed to give a min-max characterization of the eigenvalues in terms of infimums and supremums over vectors spaces that are independent of the eigenfunctions.

Theorem 59 (Max-Min Theorem)**.** *For $k \in \mathbb{N}$ let $\mathcal{V}_{k-1}$ be the collection of all subspaces $V \subset C^\infty(M)$ of dimension $k-1$. Then*

$$\lambda_k \geq \sup_{V \in \mathcal{V}_{k-1}} \inf_{\phi \in (V^\perp \cap H_1(M)) \setminus \{0\}} \frac{D_g(\phi,\phi)}{\|\phi\|_g^2}.$$

Proof. Fix $V \in \mathcal{V}_{k-1}$ and let $\psi_1, \ldots, \psi_{k-1}$ be a basis of orthonormal smooth functions of V. We claim that there exists $\tilde{\phi} = \sum_{i=1}^{k} a_i \varphi_i \neq 0$ so that $\left\langle \tilde{\phi}, \psi_j \right\rangle = 0$ for all $j = 1, \ldots, n-k$. Indeed, the existence of such function is equivalent to finding $a_1, \ldots, a_k$ so that $\sum_{i=1}^{k} a_i \langle \varphi_i, \psi_j \rangle = 0$ for $j = 1, \ldots, k-1$. This forms a system of $k-1$ equations with k unknowns, so it must have a solution.

We then have

$$\begin{aligned}
\inf_{\phi\in(V^\perp\cap H_1(M))\setminus\{0\}} \frac{D_g(\phi,\phi)}{\|\phi\|_g^2} &\leq \frac{D_g(\tilde\phi,\tilde\phi)}{\|\tilde\phi\|_g^2}\\
&= \frac{1}{\|\tilde\phi\|_g^2}\sum_{ij=1}^k a_i a_j \langle \nabla_g\varphi_i, \nabla_g\varphi_j\rangle_g\\
&= \frac{1}{\|\tilde\phi\|_g^2}\sum_{ij=1}^k a_i a_j \langle \varphi_i, \Delta_g\varphi_j\rangle_g\\
&= \frac{1}{\|\tilde\phi\|_g^2}\sum_{i=1}^k a_i^2\lambda_i\\
&\leq \lambda_k.
\end{aligned}$$

□

Theorem 60 (Min-Max Theorem)**.** *For $k\in\mathbb{N}$ let $\mathcal{V}_k$ be the collection of all subspaces $V\subset C^\infty(M)$ of dimension k. Then*

$$\lambda_k \leq \inf_{V\in\mathcal{V}_k}\sup_{\phi\in V}\frac{D_g(\phi,\phi)}{\|\phi\|_g^2}.$$

Proof. Note that $V_\Delta := \operatorname{span}\{\varphi_1,\dots,\varphi_k\}\in\mathcal{V}_k$ and for $\phi=\sum_{j=1}^k a_j\varphi_j\in V_\Delta$ one has $D_g(\phi,\phi)=\sum_{j=1}^k\lambda_j a_j^2\leq\lambda_k\|\phi\|_{L^2}^2\leq\lambda_k$ and so $\sup_{\phi\in V_\Delta}\frac{D_g(\phi,\phi)}{\|\phi\|_g^2}\leq\lambda_k$. On the other hand, $D_g(\varphi_k,\varphi_k)=\lambda_k\|\varphi_k\|_{L^2}^2$ and so $\sup_{\phi\in V_\Delta}\frac{D_g(\phi,\phi)}{\|\phi\|_g^2}\geq\lambda_k$. It follows that

$$\sup_{\phi\in V_\Delta}\frac{D_g(\phi,\phi)}{\|\phi\|_g^2}=\lambda_k.$$

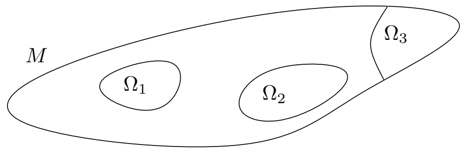
□

Remark 61. The statement of the Max-min (resp. Min-max) Theorem holds writing an "=" sign istead of $\geq$ (resp. $\leq$). The proof is not too hard but we skip it.

7.3 Domain monotonicity

Theorem 62 (Domain monotonicity for Dirichlet data)**.** *Let M be a compact Riemannian manifold with piece-wise smooth (or empty) boundary, with a given eigenvalue problem on ∂M. Let $\Omega_1,\dots,\Omega_\ell\subset M$ be pairwise disjoint open sets whose boundaries are piece-wise smooth and such that any intersection with ∂M is transversal. For each $i=1,\dots,\ell$ impose Dirichlet boundary conditions on Ω_i except on $\partial M\cap\Omega_i$ where the initial data remains unchanged. Write $\mu_1\leq\mu_2\leq\dots$ for the eigenvalues of all the Ω_i's. Then,*

$$\lambda_k\leq\mu_k.$$



Proof. Let ψ_i denote the eigenfunction of μ_i on the corresponding Ω_{ψ_i} with Dirichlet boundary conditions on $\partial\Omega_{\psi_i}$. Set $\psi_i = 0$ on $M\backslash\Omega_{\psi_i}$. Then $\psi_i \in H_1(M)$ for all i. Fix $k \in \mathbb{N}$. We may make $\psi_1, \dots, \psi_k$ orthonormal.

We claim that there exists $\phi = \sum_{i=1}^k a_i\psi_i \in H_1(M) \cap (\mathrm{span}\{\varphi_1, \dots, \varphi_{k-1}\})^\perp$. The argument is the same as in Theorem 59, it amounts to solve $k-1$ equations having k unknowns.

By Theorem 58

$$\lambda_k \|\phi\|_g^2 \leq D_g(\phi, \phi) = \sum_{i,j=1}^k a_i a_j D_g(\psi_i, \psi_j),$$

and

$$\begin{aligned} D_g(\psi_i, \psi_j) &= \int_M \tilde{\Delta}_g \psi_i \cdot \psi_j \, \omega_g \\ &= \int_{\Omega_{\psi_j}} \tilde{\Delta}_g \psi_i \cdot \psi_j \, \omega_g \\ &= \int_{\Omega_{\psi_j}} \psi_i \cdot \tilde{\Delta}_g \psi_j \, \omega_g + 0 \\ &= \mu_j \int_{\Omega_{\psi_j}} \psi_i \cdot \psi_j \, \omega_g \\ &= \mu_j \delta_{ij}. \end{aligned}$$

Therefore $\lambda_k \|\phi\|_g^2 \leq \sum_{i,j=1}^k a_i a_j D_g(\psi_i, \psi_j) = \sum_{i=1}^k a_i^2 \mu_i \leq \mu_k \|\phi\|_g^2$. □

Theorem 63 (Domain monotonicity for Neumann data)**.** *Let M be a compact Riemannian manifold with piece-wise smooth (or empty) boundary, with a given eigenvalue problem on ∂M. Let $\Omega_1, \dots, \Omega_\ell \subset M$ be pairwise disjoint open sets whose boundaries are piece-wise smooth and such that any intersection with ∂M is transversal. Assume further that*

$$\overline{M} = \cup_{i=1}^\ell \overline{\Omega_i}.$$

For each $i = 1, \dots, \ell$ impose Neumann boundary conditions on Ω_i except on $\partial M \cap \Omega_i$ where the initial data remains unchanged. Write $\nu_1 \leq \nu_2 \leq \dots$ for the eigenvalues of all the Ω_i's. Then,

$$\nu_k \leq \mu_k.$$

Proof. Let ψ_i be the eigenfunction corresponding to ν_i on the appropriate Ω_{ψ_i}, and set $\psi_i = 0$ on $M\backslash\Omega_{\psi_i}$. As before, we may assume the ψ_i are orthonormal and we have $\psi_i \in H_1(M)$. As we have shown before, there exists $\phi = \sum_{i=1}^k a_i\varphi_i$ so that $\phi \in H_1(M) \cap (span\{\psi_1, \dots, \psi_{k-1}\})^\perp$. Note that in particular $\phi \in H(\Omega_j)$ for all j.

Write $D_g^{\Omega_j} : H_1(\Omega_j) \times H_1(\Omega_j) \to \mathbb{R}$ for the Dirichlet form on Ω_j. Note that by Theorem 58

$$D_g^{\Omega_j}(\phi, \phi) \geq \nu_k \int_{\Omega_j} |\phi|^2 \, \omega_g.$$

It then follows that

$$D_g(\phi, \phi) = \sum_{j=1}^{\ell} D_g^{\Omega_j}(\phi, \phi) \geq \nu_k \sum_{j=1}^{\ell} \int_{\Omega_j} |\phi|^2 \, \omega_g = \nu_k \|\phi\|_g^2.$$

On the other hand,

$$D_g(\phi, \phi) = \sum_{i,j=1}^{k} a_i a_j D_g(\psi_i, \psi_j) = \sum_{i=1}^{k} a_i^2 \lambda_i \leq \lambda_k \|\phi\|_g^2.$$

□

7.4 Simple lower bound for λ_k

The operator $(\Delta_g + I)^m : H_{2m}(M) \to L^2(M)$ is elliptic for all $m \in \mathbb{N}$ and so by Gärding's inequality there exists $C > 0$ making

$$\|\phi\|_{H_{2m}} \leq C^m \|(\Delta_g + I)^m \phi\|_{L^2}.$$

Theorem 64. *Let $n = \dim M$. For any integer $m > n/4$ there exist constants $C_m > 0$ and $k_m > 0$ such that*

$$\lambda_k \geq C_m \, k^{\frac{1}{2m}}$$

for all $k \geq k_m$.

Proof. We start noticing that if $\Delta_g \varphi_j = \lambda_j \varphi_j$ then $(\Delta_g + I)^m \varphi_j = (\lambda_j + 1)^m \varphi_j$. Let

$$\mathcal{H}_\lambda := \bigoplus_{(\lambda_j+1)^m \leq \lambda} \ker\Big((\Delta_g + I)^m - (\lambda_j + 1)^m\Big).$$

For $\phi = \sum_j a_j \varphi_j \in \mathcal{H}_\lambda$ we have

$$\|(\Delta_g + I)^m \phi\|_{L^2}^2 = \|\sum_j (\lambda_j + 1)^m a_j \varphi_j\|_{L^2}^2 = \sum_j (\lambda_j + 1)^{2m} \|a_j \varphi_j\|_{L^2}^2 \leq \lambda^2 \|\phi\|_{L^2}^2$$

and so

$$\|(\Delta_g + I)^m \phi\|_{L^2} \leq \lambda \|\phi\|_{L^2}.$$

By the Sobolev embedding, if $2m > n/2$ then $H_{2m}(M) \subset C^0(M)$. In particular, there exists $C_1 > 0$ for which

$$\|\phi\|_\infty \leq C_1 \|\phi\|_{H_{2m}} \qquad \text{for all } \phi \in H_{2m}(M).$$

Putting all together, if $\phi \in \mathcal{H}_\lambda$, then there is $C_2 > 0$

$$\|\phi\|_\infty \leq C_1 \|\phi\|_{H_{2m}} \leq C_2 \|(\Delta_g + I)^m \phi\|_{L^2} \leq C_2 \lambda \|\phi\|_{L^2}.$$

Fix $x \in M$ and $\ell \leq \dim \mathcal{H}_\lambda$. Then, for any real numbers $a_1, \ldots, a_\ell$,

$$\Big|\sum_{j=1}^{\ell} a_j\, \varphi_j(x)\Big| \leq C_2\lambda \Big\|\sum_{j=1}^{\ell} a_j\, \varphi_j\Big\|_{L^2} = C_2\lambda\Big(\sum_{j=1}^{\ell} \|a_j\, \varphi_j\|_{L^2}^2\Big)^{1/2} = C_2\lambda\Big(\sum_{j=1}^{\ell} a_j^2\Big)^{1/2}.$$

Now pick $a_j = \varphi_j(x)$. We then get

$$\sum_{j=1}^{\ell} \varphi_j(x)^2 \leq C_2\lambda\Big(\sum_{j=1}^{\ell} \varphi_j(x)^2\Big)^{1/2}$$

and so for all $x \in M$

$$\sum_{j=1}^{\ell} \varphi_j(x)^2 \leq C_2^2\lambda^2.$$

Integrating over M

$$\ell \leq C_2^2\alpha^2 \mathrm{vol}_g(M)$$

and therefore in particular $\dim \mathcal{H}_\lambda \leq C_2^2\lambda^2\mathrm{vol}_g(M)$ for any λ. Picking $\lambda := (\lambda_k + 1)^m$ we get $k = \dim \mathcal{H}_{\lambda_k}$ and so

$$(\lambda_k + 1)^m \geq \frac{\sqrt{k}}{C_2\sqrt{\mathrm{vol}_g(M)}}.$$

□

7.5 First eigenvalue: Faber Krahn inequality

Given two regions with the same volume, the one with the largest boundary should be the one that looses heat the fastest. Since a solution for the heat equation is dominates by the term $e^{-\lambda_1 t}\varphi_1$, we should have that the first eigenvalue corresponding to the region with larger boundary should be greater. The results in this section are presented for subset of $\mathbb{R}^n$ but can be rewritten with almost no modification for a compact Riemannian manifolds (M, g).

Theorem 65 (Faber Krahn inequality)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let $B \subset \mathbb{R}^n$ denote a ball satisfying vol(Ω) = vol(B). Then,*

$$\lambda_1(\Omega) \geq \lambda_1(B)$$

Where $\lambda_1(\Omega)$ and $\lambda_1(B)$ are the first eigenvalues for the Dirichlet eigenvalues on Ω and B respectively.

To prove this Theorem we will use in several ocassions the co-area formula:

$$\int_{\{\phi>t\}} \psi\, dx = \int_t^{\max\phi}\int_{\varphi^{-1}(s)} \frac{\psi}{|\nabla\phi|}d\tau\, ds.$$

Proof. Let $\phi \in V$ and set

$$\Omega_t := \{x \in \mathbb{R}^n : \ \phi(x) > t\}.$$

We now define a *symmetrization* $\phi_* : B \to [0, +\infty)$ of ϕ. Let B_t be a ball centered at the origin satisfying $\mathrm{vol}(B_t) = \mathrm{vol}(\Omega_t)$. The symmetrization ϕ_* is defined as the radially symmetric function such that

$$\{x \in \mathbb{R}^2 : \ \phi_*(x) > t\} = B_t.$$

By the co-area formula

$$\int_t^{\max\phi} \int_{\phi^{-1}(s)} \frac{1}{|\nabla\phi|} d\tau\, ds = \mathrm{vol}(\Omega_t) = \mathrm{vol}(B_t) = \int_t^{\max\phi_*} \int_{\phi_*^{-1}(s)} \frac{1}{|\nabla\phi_*|} d\tau\, ds.$$

Differentiation with respect to t we get

$$\int_{\phi^{-1}(t)} \frac{1}{|\nabla\phi|} d\tau \ = \int_{\phi_*^{-1}(t)} \frac{1}{|\nabla\phi_*|} d\tau \qquad \text{for all } t. \tag{7.2}$$

Then,

$$\begin{aligned}
\int_\Omega \phi^2\, dx &= \int_0^{\max\phi} \int_{\phi^{-1}(s)} \frac{\phi^2}{|\nabla\phi|} d\tau\, ds \\
&= \int_0^{\max\phi} s^2 \int_{\phi^{-1}(s)} \frac{1}{|\nabla\phi|} d\tau\, ds \\
&= \int_0^{\max\phi} s^2 \int_{\phi_*^{-1}(s)} \frac{1}{|\nabla\phi_*|} d\tau\, ds \\
&= \int_B \phi_*^2\, dx
\end{aligned}$$

where the last equality follows from the fact that $\max\phi = \max\phi_*$.

For $t \in [0, \max\phi]$ define the functions

$$G(t) = \int_{D_t} |\nabla\phi|^2\, dx \qquad \text{and} \qquad G_*(t) = \int_{B_t} |\nabla\phi_*|^2\, dx.$$

By the co-area formula

$$G(t) = \int_t^{\max\phi} \int_{\phi^{-1}(s)} |\nabla\phi|\, d\tau\, ds$$

and so

$$G'(t) = -\int_{\phi^{-1}(t)} |\nabla\phi|\, d\tau.$$

Analogously,

$$G_*'(t) = -\int_{\phi_*^{-1}(t)} |\nabla\phi_*|\, d\tau.$$

By Cauchy-Schwartz inequality

$$(\mathrm{vol}(\phi^{-1}(t)))^2 = \left(\int_{\phi^{-1}(t)} 1\, d\tau\right)^2 \leq \left(\int_{\phi^{-1}(t)} |\nabla\phi|\, d\tau\right)\left(\int_{\phi^{-1}(t)} \frac{1}{|\nabla\phi|}\, d\tau\right).$$

On the other hand, since $|\nabla\phi_*|$ is constant on $\phi_*^{-1}(t)$,

$$(\mathrm{vol}(\phi_*^{-1}(t)))^2 = \left(\int_{\phi_*^{-1}(t)} 1\, d\tau\right)^2 = \left(\int_{\phi_*^{-1}(t)} |\nabla\phi_*|\, d\tau\right)\left(\int_{\phi_*^{-1}(t)} \frac{1}{|\nabla\phi_*|}\, d\tau\right).$$

□

The isoperimetric inequality says that

$$\mathrm{vol}(\phi^{-1}(t)) \geq \mathrm{vol}(\phi_*^{-1}(t)).$$

It follows from (7.2) that

$$G'(t) = -\int_{\phi^{-1}(t)} |\nabla\phi|\, d\tau \leq -\int_{\phi_*^{-1}(t)} |\nabla\phi_*|\, d\tau = G_*'(t).$$

Integrate with respect to t (using that $G(\max\phi) = 0 = G_*(\max\phi)$) and apply the co-area formula to get

$$\int_\Omega |\nabla\phi|^2\, dx = G(0) \geq G_*(0) = \int_B |\nabla\phi_*|^2\, dx.$$

It follows that

$$\lambda_1(\Omega) = \frac{\int_\Omega |\nabla\phi|^2\, dx}{\int_\Omega \phi^2\, dx} \leq \frac{\int_B |\nabla\phi_*|^2\, dx}{\int_B \phi_*^2\, dx} = \lambda_1(B).$$

7.6 Continuity of eigenvalues

The notes in this section where written by Dmitri Gekhtman.

Theorem 66 (Continuity in the C^0-topology of metrics)**.** *Let M be a compact manifold and let g and $\tilde{g}$ be two Riemannian metrics on M that are close in the sense that there exists $\varepsilon > 0$ small making*

$$(1-\varepsilon)\tilde{g} \leq g \leq (1+\varepsilon)\tilde{g}.$$

Then,

$$1-(n+1)\varepsilon + O(\varepsilon^2) \leq \frac{\lambda_k(g)}{\lambda_k(\tilde{g})} \leq 1+(n+1)\varepsilon + O(\varepsilon^2).$$

Proof. At the level of the volume measures we have that

$$(1-\varepsilon)^{\frac{n}{2}}\omega_{\tilde{g}} \leq \omega_g \leq (1+\varepsilon)^{\frac{n}{2}}\omega_{\tilde{g}}$$

and so $(1-\varepsilon)^{\frac{n}{2}}\|\varphi\|_{\tilde{g}}^2 \leq \|\varphi\|_g^2 \leq (1+\varepsilon)^{\frac{n}{2}}\|\varphi\|_{\tilde{g}}^2$.

Note that for any $\varphi \in C^\infty(M)$ we have $\|\nabla_g \varphi\|_g^2 = \int_M \sum_{i,j=1}^n g^{ij} \frac{\partial \varphi}{\partial x_i} \frac{\partial \varphi}{\partial x_j} \, \omega_g$ and therefore, since $(1-\varepsilon)\tilde{g}^{-1} \leq g^{-1} \leq (1+\varepsilon)\tilde{g}^{-1}$, we get

$$(1-\varepsilon)^{\frac{n}{2}+1} \|\nabla_{\tilde{g}} \varphi\|_{\tilde{g}}^2 \leq \|\nabla_g \varphi\|_g^2 \leq (1+\varepsilon)^{\frac{n}{2}+1} \|\nabla_{\tilde{g}} \varphi\|_{\tilde{g}}^2.$$

It then follows that

$$\frac{(1-\varepsilon)^{\frac{n}{2}+1}}{(1+\varepsilon)^{\frac{n}{2}}} \frac{\|\nabla_{\tilde{g}} \varphi\|_{\tilde{g}}^2}{\|\varphi\|_{\tilde{g}}^2} \leq \frac{\|\nabla_g \varphi\|_g^2}{\|\varphi\|_g^2} \leq \frac{(1+\varepsilon)^{\frac{n}{2}+1}}{(1-\varepsilon)^{\frac{n}{2}}} \frac{\|\nabla_{\tilde{g}} \varphi\|_{\tilde{g}}^2}{\|\varphi\|_{\tilde{g}}^2}.$$

□

We aim to show that the kth eigenvalue of the Laplacian on Riemannian manifold depends continuously on the metric. For this to make sense, we must first establish a topology on the space of metrics $\mathcal{M}$. Let M be a compact, connected, smooth manifold of dimension and let $\mathcal{T}^2(M)$ be the set of type $(2,0)$ tensor fields on M. Fix a finite cover $\{U_\sigma\}_{\sigma \in I}$ of M by open neighborhoods, each satisfying $\overline{U_\sigma} \subset V_\sigma$ for some open coordinate neighborhood V_σ. For any $h \in \mathcal{T}^2(M)$, let h_{ij} denote the components of h with respect to the coordinates on V_σ. For every nonnegative integer k and $\sigma \in I$ we define

$$\|h\|_{k,\sigma} = \sum_{|\alpha| \leq k} \sum_{ij} |\partial^\alpha h_{ij}|.$$

We define

$$\|h\|_k = \sum_{\sigma \in I} \|h\|_{k,\sigma}.$$

Finally, we define a norm on $\mathcal{T}^2(M)$ by

$$\|h\| = \sum_{k=0}^{\infty} 2^{-k} \|h\|_k (1 + \|h\|_k)^{-1}$$

and define $d' : \mathcal{T}^2(M) \times \mathcal{T}^2(M) \to \mathbb{R}$ to be the associated distance

$$d'(h_1, h_2) = \|h_1 - h_2\|.$$

Next, we define a distance d'' on $\mathcal{M}$ by

$$d''(g_1, g_2) = \sup_{x \in M} d''_x(g_1, g_2),$$

where

$$d''_x(g_1, g_2) = \inf\{\delta > 0 | e^{-\delta}(g_2)_x < (g_1)_x < e^{\delta}(g_2)_x\}.$$

If A, B are inner products $A < B$ means that $B - A$ is positive definite.
Now, we define $d = d' + d''$. The distance d on $\mathcal{M}$ is complete and defines the C^∞ topology on $\mathcal{M}$.

Theorem 67. *Let $\lambda_k(g)$ be the kth eigenvalue of the Laplacian associated to g. λ_k is a continuous function on $\mathcal{M}$ with respect to the C^∞ topology. More precisely, $d(g, g') < \delta$ implies*

$$\exp(-(n+1)\delta)\lambda_k(g') \leq \lambda_k(g) \leq \exp((n+1)\delta)\lambda_k(g').$$

Proof. Suppose $d(g, g') < \delta$. Then $d''(g, g') < \delta$, which implies that

$$e^{-\sigma} g' < g < e^{\delta} g'.$$

Let $\{U, (x^i)\}$ be a coordinate patch on M. Then

$$e^{-\sigma}(g'_{ij}) < (g_{ij}) < e^{\delta}(g'_{ij}),$$

where (g_{ij}) is the positive definite symmetric matrix defined by the components of g. If A, B are positive definite symmetric matrices satisfying $A < B$, then we have the estimates $|B| < |A|$ for the determinants and $B^{-1} < A^{-1}$ for the inverses. Hence, we have

$$\exp(-\frac{n}{2}\sigma)\sqrt{|g'_{ij}|} < \sqrt{|g_{ij}|} < \exp(\frac{n}{2}\delta)\sqrt{|g'_{ij}|}$$

and

$$e^{-\sigma}(g'^{ij}) < (g_{ij}) < e^{\delta}(g'^{ij}).$$

Now, suppose f is a smooth function compactly supported in U. Then we have

$$\|f\|_g^2 = \int_U f^2 \sqrt{|g_{ij}|} dx \leq \exp(\frac{n}{2}\delta) \int_U f^2 \sqrt{|g'_{ij}|} dx = \exp(\frac{n}{2}\delta)\|f\|_{g'}^2,$$

and similarly $\|f\|_g^2 \geq \exp(-\frac{n}{2}\delta)\|f\|'_g$.
If ω is a one-form compact supported in U, we have

$$\|\omega\|_g^2 = \int_U g^{ij}\omega_i\omega_j \sqrt{|g_{ij}|} dx \leq \exp((\frac{n}{2}+1)\delta) \int_U g^{ij}\omega_i\omega_j \sqrt{|g'_{ij}|} dx = \exp((\frac{n}{2}+1)\delta)\|\omega\|_{g'}^2,$$

and similarly $\|\omega\|_g^2 \geq \exp(-(\frac{n}{2}+1)\delta)\|\omega\|_{g'}^2$.
Hence, we have the inequalities

$$\exp(-(\frac{n}{2}+1)\delta)\|df\|_{g'}^2 \leq \|df\|_g^2 \leq \exp((\frac{n}{2}+1)\delta)\|df\|_{g'}^2$$

and

$$\exp(-\frac{n}{2}\delta)\|f\|_{g'}^2 \leq \|f\|_g^2 \leq \exp(\frac{n}{2}\delta)\|f\|_{g'}^2$$

for all smooth functions f compactly supported in a coordinate neighborhood. Using a partition of unity, we find that the inequalities hold for all f in $C^\infty(M)$. Combining the inequalities, we obtain

$$\exp(-(n+1)\delta)\frac{\|df\|_{g'}^2}{\|f\|_{g'}^2} \leq \frac{\|df\|_g^2}{\|f\|_g^2} \leq \exp((n+1)\delta)\frac{\|df\|_{g'}^2}{\|f\|_{g'}^2}.$$

Recalling

$$\lambda_k(g) = \inf_{V^k} \sup_{f\in V^k\setminus\{0\}} \frac{\|df\|_g^2}{\|f\|_g^2},$$

the last inequality yields

$$\exp(-(n+1)\delta)\lambda_k(g') \leq \lambda_k(g) \leq \exp((n+1)\delta)\lambda_k(g').$$

This implies

$$\lambda_k(g') - \lambda_k(g) \leq (\exp((n+1)\delta) - 1)\lambda_k(g),$$

which proves continuity of λ_k. ∎

In the proof of the last statement, we established the inequality

$$\exp(-(n+1)\delta) \leq \frac{\lambda_k(g)}{\lambda_k(g')} \leq \exp((n+1)\delta),$$

which shows that if g, g' are close, $\frac{\lambda_k(g)}{\lambda_k(g')}$ is close to 1 uniformly in k. As a consequence, we have the following

Corollary 68. *The multiplicity of the kth eigenvalue is an upper-semicontinuous function of the metric. That is, for any $g \in \mathcal{M}$, $k \in \mathbb{N}$, there is a δ so that $d(g, g') < \delta$ implies $\#\{j|\lambda_j(g') = \lambda_k(g')\} \leq \#\{j|\lambda_j(g) = \lambda_k(g)\}$.*

7.7 Multiplicity of eigenvalues

If (M, g) is a compact boundary-less Riemannian manifold of dimension $n \geq 3$, then Colin de Verdiere proved that every finite sequence $0 = \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_k$ is the sequence of the first k eigenvalues counted with multiplicity of the Laplacian. In particular, there are no restrictions on the multiplicities of eigenvalues on manifolds of dimension $n \geq 3$. In contrast, this picture is very different on surfaces. Indeed, the following holds:

- on (S^2, g) one has $m_j \leq 2j + 1$
- on $(P^2(\mathbb{R}), g)$ one has $m_j \leq 2j + 3$
- on (T^2, g) one has $m_j \leq 2j + 4$

where g in the above examples denotes a general Riemannian metric. Observe that if one chooses the standard metrics on these surfaces then one obtains the equalities on the multiplicities.

7.8 High energy eigenvalue asymptotics

Let (M, g) be a compact boundary-less Riemannian manifold. Write

$$0 = \lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots$$

for all the Laplace eigenvalues repeated according to their multiplicity. We begin this section by introducing the **Zeta function** $Z_g : (0, +\infty) \to \mathbb{R}$

$$Z_g(t) = \sum_{j=1}^{\infty} e^{-\lambda_j t}.$$

Since the series is uniformly convergent on intervals of the form $[t_0, +\infty)$ for all $t_0 > 0$ we know that Z_g is continuous. We also have that it is decreasing in t, that $\lim_{t\to 0^+} Z_g(t) = +\infty$, and $\lim_{t\to+\infty} Z_g(t) = 0$.

Proposition 69.

$$Z_g(t) \sim \frac{1}{(4\pi t)^{n/2}}(vol_g(M) + O(t)) \qquad as\ \ t \to 0^+.$$

Proof.

$$\begin{aligned} Z_g(t) &= \sum_{j=0}^{\infty} e^{-\lambda_j t} \\ &= \int_M p(x,x,t)\,\omega_g(x) \\ &= \frac{1}{(4\pi t)^{n/2}} \left(\sum_{j=0}^{k} t^j \int_M u_j(x,x)\,\omega_g(x) + O(t^{k+1}) \right) \\ &= \frac{1}{(4\pi t)^{n/2}} \left(\operatorname{vol}_g(M) + O(t)\right) \end{aligned}$$

□

Let us now write

$$0 = \nu_1 < \nu_2 < \nu_3 < \dots$$

for all the distinct eigenvalues. Then, setting m_j for the multiplicity of ν_j we can rewrite

$$Z_g(t) = \sum_{j=1}^{\infty} m_j e^{-\nu_j t}.$$

Theorem 70. *The function Z_g determines all the eigenvalues and their multiplicities.*

Proof. Note that for $\mu > 0$ with $\mu \neq 2$,

$$\lim_{t\to\infty} e^{\mu t}\big(Z_g(t) - 1\big) = \lim_{t\to\infty} \sum_{j=2}^{\infty} m_j e^{(\mu-\nu_j)t} = \begin{cases} 0 & \text{if } \mu < \nu_2, \\ +\infty & \text{if } \mu < \nu_2 \\ m_2, & \text{if } \mu = \nu_2. \end{cases}$$

It follows that ν_2 is the unique strictly positive real number μ such that the limit $\lim_{t\to\infty} e^{\mu t}\big(Z_g(t) - 1\big)$ is a natural number. By induction, ν_k is the unique strictly positive real number μ such that the limit

$$m_k := \lim_{t\to\infty} e^{\mu t}\Big(Z_g(t) - 1 - \sum_{j=2}^{k-1} m_j e^{-\nu_j t}\Big)$$

is a natural number.

□

Theorem 71 (Karamata)**.** *Suppose that μ is a positive measure on $[0,\infty)$ and that $\alpha \in (0,\infty)$. Then*

$$\int_0^{\infty} e^{-tx}\, d\mu(x) \sim at^{-\alpha} \qquad\qquad t \to 0$$

implies

$$\int_0^{\lambda} d\mu(x) \sim \frac{a}{\Gamma(\alpha+1)}\lambda^{\alpha} \qquad\qquad x \to \infty.$$

Proof. Define the measures on $\mathbb{R}^+$ by setting $\mu_t(A) := t^\alpha \mu(t^{-1}A)$ for $A \subset \mathbb{R}^+$. Observe that if χ_A is the indicator function for any set A, then by definition

$$\int \chi_A(\lambda) d\mu_t(\lambda) = t^\alpha \int \chi_A(t\lambda) d\mu(\lambda).$$

It follows that for any $f \in L^2(\mathbb{R}^+)$

$$\int f(\lambda) d\mu_t(\lambda) = t^\alpha \int f(t\lambda) d\mu(\lambda),$$

and so in particular,

$$\lim_{t\to\infty} \int e^{-\lambda} d\mu_t(\lambda) = \lim_{t\to\infty} t^\alpha \int e^{-t\lambda} d\mu(\lambda) = a. \qquad (*)$$

Note that by definition of the Gamma function $\frac{1}{\Gamma(\alpha+1)} \int e^{-\lambda} \alpha\lambda^{\alpha-1} d\lambda = 1$. We therefore define the measure $d\nu(\lambda) := \alpha\lambda^{\alpha-1} d\lambda$ and get

$$\lim_{t\to\infty} \int e^{-\lambda} d\mu_t(\lambda) = \frac{a}{\Gamma(\alpha+1)} \int e^{-\lambda} d\nu(\lambda).$$

Consider the space $\mathcal{B} := \operatorname{span}\{g_s : \mathbb{R}^+ \to \mathbb{R}^+ : \ g_s(\lambda) = e^{-\lambda s}, \quad s \in (0, +\infty)\}$. By performing a change of variables one checks

$$\lim_{t\to\infty} \int h(\lambda) d\mu_t(\lambda) = \frac{a}{\Gamma(\alpha+1)} \int h(\lambda) d\nu(\lambda)$$

for all $h \in \mathcal{B}$. By the locally compact spaces version of the Stone-Weirstrass Theorem one has that $\mathcal{B}$ is dense in $C_0^\infty(\mathbb{R}^+) = \{f \in C^\infty(\mathbb{R}^+) : \ f(\lambda) \text{ vanishes as } \lambda \to \infty\}$. Let $f \in C_c^\infty(\mathbb{R}^+)$. Since $f(\lambda)e^\lambda \in C_0^\infty(\mathbb{R}^+)$, then there exists a sequence $h_j \in C_0^\infty(\mathbb{R}^+)$ with $\lim_{j\to\infty} h_j = e^\lambda f$. Note that for each j we get

$$\lim_{t\to\infty} \int h_j(\lambda) e^{-\lambda} d\mu_t(\lambda) = \frac{a}{\Gamma(\alpha+1)} \int h_j(\lambda) e^{-\lambda} d\nu(\lambda).$$

In order to interchange $\lim_{j\to\infty}$ and $\lim_{t\to 0}$ we use that according to $(*)$ the measures $e^{-\lambda} d\mu_t$ are uniformly bounded. We proved

$$\lim_{t\to\infty} \int f(\lambda) d\mu_t(\lambda) = \frac{a}{\Gamma(\alpha+1)} \int f(\lambda) d\nu(\lambda).$$

In particular this equality holds for $f = \chi_{[0,1]}$ and it is easy to check that acceding to our definitions of the measures μ_t and ν

$$\lim_{t\to\infty} \int \chi_{[0,1]}(\lambda) d\mu_t(\lambda) = \frac{a}{\Gamma(\alpha+1)} \int \chi_{[0,1]}(\lambda) d\nu(\lambda).$$

is equivalent to the conclusion we desired to prove. $\square$

Let ω_n be the volume of the unite ball in $\mathbb{R}^n$,

$$\omega_n := \frac{2\pi^{n/2}}{n\Gamma(n/2)}.$$

Our aim is to prove the following Theorem:

Theorem 72 (Weyl's asymptotic formula)**.** *Let M be a compact Riemannian manifold with eigenvalues $0 = \lambda_0 < \lambda_1 \leq \dots$, each distinct eigenvalue repeated according to its multiplicity.*
Then for $N(\lambda) := \#\{j : \lambda_j \leq \lambda\}$, we have

$$N(\lambda) \sim \frac{\omega_n}{(2\pi)^n} vol_g(M) \lambda^{n/2}, \qquad \lambda \to \infty.$$

In particular,

$$\lambda_j \sim \frac{\sqrt{2\pi}}{(\omega_n vol_g(M))^{2/n}} j^{2/n}, \qquad j \to \infty.$$

Proof.
For the measure $\mu = \sum \delta_{\lambda_j}$, Proposition 69 asserts that

$$\int_0^\infty e^{-t\lambda}\, d\mu(\lambda) \sim \frac{1}{(4\pi)^{\frac{n}{2}}} \mathrm{vol}_g(M) t^{-n/2}.$$

Using Karamata's theorem on μ and $\alpha = n/2$ we obtain

$$N(\lambda) = \int_0^\lambda d\mu(\lambda) \sim \frac{\mathrm{vol}_g(M)}{(4\pi)^{n/2}\Gamma(n/2+1)} \lambda^{n/2} = \frac{\omega_n \mathrm{vol}_g(M)}{(2\pi)^n} \lambda^{n/2}.$$

□

7.9 Isospectral manifolds

In this section we prove that if (M, g_M) and (N, g_N) are compact Riemannian manifolds which share the same eigenvalues then they must have the same dimension, same volume and same total curvature.

Theorem 73. *If (M, g_M) and (N, g_N) are isospectral compact Riemannian manifolds, then*

$$dim\, M = dim\, N, \qquad vol_{g_M}(M) = vol_{g_N}(N) \qquad and \qquad \int_M R_{g_M}\omega_{g_M} = \int_N R_{g_N}\omega_{g_N}.$$

Proof. Let $\lambda_0 \leq \lambda_1 \leq \dots$ be the eigenvalues of both Δ_{g_M} and Δ_{g_N}. Then

$$\sum_{j=0}^\infty e^{-\lambda_j t} = \frac{1}{(4\pi t)^{\dim M/2}} \left(\sum_{j=0}^k t^j \int_M u_j^{g_M}(x,x)\, \omega_g(x) + O(t^{k+1}) \right)$$

and

$$\sum_{j=0}^\infty e^{-\lambda_j t} = \frac{1}{(4\pi t)^{\dim N/2}} \left(\sum_{j=0}^k t^j \int_M u_j^{g_N}(x,x)\, \omega_g(x) + O(t^{k+1}) \right).$$

It follows immediately that $dim\, M = dim\, N := n$. Next, note that

$$\frac{1}{(4\pi t)^{n/2}}\left(\int_M u_0^{g_M}(x,x)\,\omega_g(x) - \int_N u_0^{g_N}(x,x)\,\omega_g(x)\right) =$$
$$= \frac{1}{(4\pi t)^{n/2}}\left[\sum_{j=1}^{k} t^j \left(\int_M u_j^{g_M}(x,x)\,\omega_g(x) - \int_N u_j^{g_N}(x,x)\,\omega_g(x)\right) + O(t^{k+1})\right]$$

yields

$$\int_M u_0^{g_M}(x,x)\,\omega_g(x) = \int_N u_0^{g_N}(x,x)\,\omega_g(x)$$

and since $u_0^{g_M}(x,x) = u_0^{g_N}(x,x) = 1$, we have $vol_{g_M}(M) = vol_{g_N}(N)$. Repeating the same argument it follows that if (M, g_M) and (N, g_N) are isospectral compact Riemannian manifolds, then for all j

$$\int_M u_j^{g_M}(x,x)\,\omega_g(x) = \int_N u_j^{g_N}(x,x)\,\omega_g(x).$$

In particular, since $u_1(x,x) = \frac{1}{6}R_g(x)$ we have that (M, g_M) and (N, g_N) have the same total curvature. □

We next prove that in the case of compact surfaces isospectrality implies a strong result:

Corollary 74. *If (M, g_M) and (N, g_N) are isospectral compact Riemannian surfaces, then M and N are diffeomorphic.*

Proof. By Gauss-Bonnet Theorem

$$\int_M R_{g_M}\,\omega_{g_M} = 8\pi(1 - \gamma_M)$$

where γ_M is the genus of M. The same result holds for N. We then use that

$$\int_M R_{g_M}\,\omega_g = \int_N R_{g_N}\,\omega_g$$

which yields $\gamma_M = \gamma_N$. The result follows from the fact that two orientable surfaces with the same genus are diffeomorphic. □

Theorem 75. *Suppose (M, g_M) and (N, g_N) are isospectral compact Riemannian manifolds of dimension $n = 2, 3, 4, 5$. If (M, g_M) has constant sectional curvature, then so does (N, g_N).*

Remark 76. In dimension $n = 6$ the result also holds provided we ask the sectional curvatures of (M, g_M) to be strictly positive.

Theorem 77. *If (M, g) is isospectral to (S^2, g_{S^2}), then (M, g) is isometric to (S^2, g_{S^2}).*

CHAPTER 8

Eigenfunctions

Write (x, ξ) for the coordinates of a particle in phase space. That is, x denotes position and ξ is the momentum. Let $H : T^*M \to \mathbb{R}$ be the Hamiltonian

$$H(x, \xi) = \frac{1}{2}|\xi|_g^2 + V(x).$$

As discussed in the Introduction, the classical Hamiltonian equations

$$\begin{cases} \frac{\partial x_j}{\partial t} &= \frac{\partial H}{\partial \xi_j} \\ \frac{\partial \xi_j}{\partial t} &= -\frac{\partial H}{\partial x_j}, \end{cases}$$

describe the motion of a particle with kinetic energy $\frac{1}{2}|\xi|^2$ and potential energy $V(x)$. The idea of Schrödinger was to model the behavior of the electron by a wave-function φ_j that solves the problem

$$\big(-\frac{h^2}{2}\Delta_g + V\big)\varphi_j = E_j(h)\varphi_j.$$

Here h is Planck's constant, a very small number $h \sim 6.6 \times 10^{-34}\, m^2 kg/s$. If we choose a system free of potential energy, $V = 0$, then the limit $h \to 0$ is equivalent to the high frequency limit $\lambda \to \infty$. When working with normalized eigenfunctions, $\|\varphi_j\|_2 = 1$, one has that for any $A \subset M$

$$\int_A |\varphi_j(x)|^2\, \omega_g(x) = \mathbb{P}\big(\text{particle of energy } E_j(h)/h \text{ belongs to } A\ \big).$$

The time evolution of a particle in the initial state u_0 is given by

$$u(x, t) = e^{-i\frac{t}{h}(-\frac{h^2}{2}\Delta_g + V)}u(x).$$

Note that for all t

$$\left| e^{-i\frac{t}{h}(-\frac{h^2}{2}\Delta_g+V)}\varphi_j(x)\right|^2 dx = |e^{-i\frac{tE_j(h)}{h}}\varphi_j(x)|^2 dx = |\varphi_j(x)|^2 dx$$

and so eigenstates are stationary states.
We start studying the behavior of solutions φ_λ to the equation $\Delta_g\varphi_\lambda = \lambda\varphi_\lambda$ on balls $B(x_0, r)$ for r small. They are not necesarilly solutions on the whole manifold. Sometimes local results on smalls balls can be extended to solutions on all of M by covering arguments.

8.1 Local properties of Eigenfunctions

It can be shown that on small length scales comparable to wavelegth scale $1/\sqrt{\lambda}$ eigenfunctions behave like harmonic functions. Fix an atlas over M. Then, in local coordinates at $x_0 \in M$ the Laplace equation is given by

$$-\frac{1}{\sqrt{\det g(x)}}\sum_{i,j=1}^{n} \partial_{x_i}\left(g^{ij}\sqrt{\det g}\,\partial_{x_j}\varphi_\lambda\right)(x) = \lambda\varphi_\lambda(x) \qquad x \in B_{\epsilon/\sqrt{\lambda}}(x_0).$$

We rescale this problem to the unit ball. That is, we set $r = \epsilon/\sqrt{\lambda}$ and given any function u on M we write u_r for the rescaled function $u_r(x) = u(rx)$. Then, the Laplace equation becomes

$$-\sum_{i,j=1}^{n} \partial_{x_i}\left(g_r^{ij}\sqrt{\det g_r}\,\partial_{x_j}\varphi_{\lambda,r}\right)(x) = \epsilon^2\sqrt{g_r(x)}\varphi_{\lambda,r}(x) \qquad x \in B_1(x_0).$$

Consider the operator

$$L = -\sum_{i,j=1}^{n} \partial_{x_i}\left(g_r^{ij}\sqrt{\det g_r}\,\partial_{x_j}\right) - \epsilon^2\sqrt{g_r}.$$

Then the Laplace equation becomes

$$L\varphi_{\lambda,r} = 0 \qquad \text{on } B_1(0).$$

For $\varepsilon > 0$ small enough, the operator L is close to be the Euclidean Laplacian and $\varphi_{\lambda,r}$ is close to being a harmonic function. This property is extensively used in the works of H. Donelly, C. Fefferman and N. Nadirashvilli.

Eigenfunctions φ_λ in small length scales are an analogue to polynomials of degree $\sqrt{\lambda}$. Actually, polynomials and eigenfunctions have many properties in common:

- Order of vanishing.
- Local growth.
- Local structure of nodal sets.

The frequency function of a given function $u \in C^\infty(M)$ measures the local growth rate of u. Let u be any *harmonic* function on $B_1(0) \subset \mathbb{R}^n$. For $a \in B_1(0)$ and $0 < r \leq 1-|a|$, define the frequency of u at a in the ball $B_r(a)$ by

$$N_u(a, r) = r\frac{D_u(a, r)}{H_u(a, r)},$$

where

$$D_u(a,r)=\int_{B_r(a)}|\nabla u|^2dx, \qquad H_u(a,r)=\int_{\partial B_r(a)}u^2d\sigma.$$

It can be shown that $N_u(a,r)$ is a monotone non-dedreasing function of $r\in(0,1-|a|)$ for any $a\in B_1(0)$. Supose u is a harmonic homogeneous polynomial of degree k. Then we may write u on polar coordinates $u(r,\omega)=c_kr^k\phi_k(\omega)$ where ϕ_k is a spherical harmonic of degree k on S^{n-1}. Then,

$$N_u(0,r)=\frac{k\,c_k^2\,r^{2k}}{c_k^2\,r^{2k}}=k=\text{degree}(u).$$

Since by integration by parts one can show that $\frac{d}{dr}H_u(a,r)=\frac{n-1}{r}H_u(a,r)+2D_u(a,r)$, we get

$$\frac{d}{dr}\log\left(\frac{H_u(a,r)}{r^{n-1}}\right)=\frac{2N_u(a,r)}{r}.$$

We therefore obtain that for all $0<R<\frac{1}{2}(1-|a|)$,

$$\frac{H_u(a,2R)}{(2R)^{n-1}}=\frac{H_u(a,R)}{R^{n-1}}\ \exp\Big(\int_R^{2R}2N_u(0,r)/r\,dr\Big)\le\frac{H_u(a,R)}{R^{n-1}}\ 4^{N(a,1-|a|)}$$

In particular,

$$\frac{H_u(a,\lambda R)}{(\lambda R)^{n-1}}=\lambda^{2N_u(0,1)}\frac{H_u(a,R)}{R^{n-1}} \qquad 1\le\lambda\le 2 \qquad (*)$$

Integrating with respect to R this gives

$$\frac{1}{\text{vol}(B_{\lambda R}(0))}\int_{B_{\lambda R}(0)}u^2dx\le\lambda^{2N_u(0,1)}\frac{1}{\text{vol}(B_R(0))}\int_{B_R(0)}u^2dx \qquad 0\le R\le 1/2,\ \ 1\le\lambda\le 2.$$

Using this one can prove that the order of vanishing $\nu_u(a)$ of u at $a\in B_{1/4}(0)$ is bounded

$$\nu_u(a)\le CN_u(0,1)+c(n).$$

Using that $\varphi_{\lambda,r}(x)=\varphi_\lambda(rx)$ is almost harmonic the previous order of vanishing estimate can be extended to manifolds. But in order to do that one has to extend the definition of the frequency function. It turns out that on a compact manifold (M,g) the natural extension of the frequency function of a harmonic map $u\in C^\infty(M)$ is

$$N_u(a,r)=r\frac{D_u(a,r)}{H_u(a,r)},$$

where

$$D_u(a,r)=\int_{B_r(a)}\mu|\nabla u|^2dx, \qquad H_u(a,r)=\int_{\partial B_r(a)}\mu u^2d\sigma$$

with

$$\mu(x)=\frac{g_{ij}(x)x_ix_j}{|x|^2}.$$

One then gets the followin result on the order of vanishing of φ_λ:

Theorem 78 (Order of vanishing)**.** *Let (M, g) be a compact manifold of dimension n. Then there exists $C > 0$ such that*

$$\nu_{\varphi_\lambda}(a) \leq C\sqrt{\lambda} \qquad \textit{for all } \ a \in M.$$

Using that $\varphi_{\lambda,r}(x) = \varphi_\lambda(rx)$ is almost harmonic the previous doubling estimate can be extended to the following result.

Theorem 79 (Doubling estimate)**.** *Let φ_λ be a global eigenfunction for Δ_g on (M, g) compact. Then there exists $C > 0$ and $r_0 > 0$ such that for all $0 < r < r_0$*

$$\frac{1}{vol(B_{2r}(a))} \int_{B_{2r}(a)} |\varphi_\lambda|^2 \omega_g \leq e^{C\sqrt{\lambda}} \frac{1}{vol(B_r(a))} \int_{B_r(a)} |\varphi_\lambda|^2 \omega_g.$$

In addition, for $0 < r' < r$,

$$\max_{x \in B_r(p)} |\varphi_\lambda(x)| \leq \left(\frac{r}{r'}\right)^{C\sqrt{\lambda}} \max_{x \in B_{r'}(p)} |\varphi_\lambda(x)|.$$

8.1.1 Gradient estimates

Harmonic functions on $\mathbb{R}^n$ satisfy many other nice properties. For instance, they satisfy Bernstein type estimates. Let u be harmonic on $\mathbb{R}^n$. Then

$$\sup_{x \in B_r(a)} |\nabla u(x)| \leq \frac{C}{r} \sup_{x \in B_{2r}(a)} |u(x)|$$

and

$$\sup_{x \in B_r(a)} |\nabla u(x)|^2 \leq \frac{C}{r^{n+2}} \int_{B_r(a)} |u(x)|^2 dx.$$

The translation of these results to the setting of eigenfunctions on compact manifolds is the following.

Theorem 80 (Bernstein inequalities)**.** *Let (M, g) be a compact manifold. Then,*

$$\sup_{x \in B_r(a)} |\nabla \varphi_\lambda(x)| \leq \frac{C\lambda^{\frac{1}{4}}}{r} \sup_{x \in B_{2r}(a)} |\varphi_\lambda(x)|, \qquad r \leq C_2 \lambda^{-1/8}$$

and

$$\int_{B_r(a)} |\nabla \varphi_\lambda(x)|^2 \omega_g(x) \leq \frac{C\lambda}{r^2} \int_{B_r(a)} |\varphi_\lambda(x)|^2 \omega_g(x).$$

8.1.2 Positive mass on subsets of M

Suppose that (M,g) is compact and that φ_λ is a global eigenfunction, $\Delta_g\varphi_\lambda = \lambda\varphi_\lambda$. Then, if $A\subset M$,

$$\int_A |\varphi_\lambda(x)|^2\omega_g(x) \geq c\, e^{-C\sqrt{\lambda}}.$$

As an illustration, the highest weight spherical harmonics Y_ℓ^ℓ decay at a rate $e^{-c\sqrt{\lambda}d_g(x,\gamma)}$ away from a stable elliptic orbit γ, where $\lambda = \ell(\ell+1)$.

A **semi-classical lacuna** is an open subset $A\subset M$ for which there exist a sequence $\{\varphi_{\lambda_{j_k}}\}$ of L^2-normalized eigenfunctions and constants $c, C>0$ so that

$$\int_A |\varphi_{\lambda_{j_k}}(x)|^2\omega_g(x) \leq c\, e^{-C\sqrt{\lambda_{j_k}}}.$$

Another descriptive term is exponential trough. Lacunae are also known as classically forbidden regions. For instance, on the sphere the sequences $\{Y_m^\ell\}$ with $m/\ell \to E$ concentrates on an invariant annulus $K_E\subset S^2$, which is known as the "classically allowed region". One can show that for this sequence $|Y_m^\ell(x)| \leq e^{-\ell d_A(x,K_E)}$, where d_a denotes the Agmon distance.
It is unknown whether semi-classical lacunas can occur on (M,g) with classically chaotic (i.e. highly ergodic) geodesic flows. In contrast, we have the following result:

Theorem 81 (Quantum ergodicity). *If (M,g) is a compact manifold with ergodic geodesic flow then there exists a density one subsequence of eigenfunctions $\{\varphi_{j_k}\}_k$ such that for any $A\subset M$*

$$\lim_{k\to\infty}\int_A |\varphi_{j_k}(x)|^2\,\omega_g(x) = \frac{vol_g(A)}{vol_g(M)}.$$

By density one subsequence it is meant that $\inf_m \frac{\#\{k\colon j_k\leq m\}}{m} = 1$. This result is due to Schnirelman (1973) finished by Colin de Verdiere (1975).

Remark. On arithmetic surfaces the above result holds for the entire sequence of eigenfunctions. This is known as Quantum Unique Ergodicity.

8.2 Global properties of eigenfunctions

8.2.1 L^∞-norms

Weyl's law says that

$$N(\lambda) = \#\{\text{eigenvalues } \leq \lambda\} \sim \frac{\omega_n \mathrm{vol}_g(M)}{(2\pi)^n}\lambda^{n/2}$$

and one can further prove what's known as **local Weyl's law**

$$\sum_{\lambda_j\leq\lambda} |\varphi_{\lambda_j}(x)|^2 = \frac{\omega_n \mathrm{vol}_g(M)}{(2\pi)^n}\lambda^{n/2} + R(\lambda,x)$$

with $R(\lambda, x) = O(\lambda^{\frac{n-1}{2}})$ uniformly in $x \in M$. In particular,

$$\sum_{\lambda_j = \lambda} |\varphi_j(x)|^2 = O(\lambda^{\frac{n-1}{2}})$$

uniformly in $x \in M$. In particular, the following result holds:

Theorem 82. *Let (M, g) be a compact Riemannian manifold of dimension n. Then, if $\Delta_g \varphi_\lambda = \lambda \varphi_\lambda$,*

$$\|\varphi_\lambda\|_{L^\infty} = O(\lambda^{\frac{n-1}{4}}).$$

Let $V_\lambda = \ker(\Delta_g - \lambda)$ and set

$$\Pi_\lambda(x, y) := \sum_{j:\ \lambda_j \leq \lambda} \varphi_{\lambda_j}(x) \varphi_{\lambda_j}(y).$$

We define the **coherent state** at $x \in M$ by

$$\Phi_\lambda^x(y) := \frac{\Pi_\lambda(x, y)}{\sqrt{\Pi_\lambda(x, x)}}, \qquad y \in M.$$

The coherent states are extremes for the L^∞-norms. Indeed,

$$|\varphi_\lambda(x)| = \left| \int_M \Pi_\lambda(x, y) \varphi_\lambda(y) \omega_g(y) \right| \leq \sqrt{\int_M |\Pi_\lambda(x, y)|^2 \omega_g(y)} \leq \sqrt{\Pi_\lambda(x, x)} = |\Phi_\lambda^x(x)|.$$

Remark. On the sphere, the zonal spherical harmonic Y_0^ℓ of degree ℓ is the coherent state Φ_λ^x where x is the north pole and $\lambda = \ell(\ell + 1)$.

The L^∞-norm estimate is very rarely sharp. Here we show that if the upper bound is attained, then there must be a recurrent point on M. For $x \in M$ consider the set $\mathcal{L}_x$ of $\xi \in S_x^* M$ that are the initial velocities of geodesic loops that start at x. That is, $\mathcal{L}_x := \{\xi \in S_x^* M :\ \exp_x(T\xi) = x \text{ for some } T\}$. We write $|\mathcal{L}_x|$ for its measure. For example, on (S^2, g_{S^2}) we have $|\mathcal{L}_x| = 2\pi$ for x being the south or north pole.

Note that we know $\sup\{\|\varphi\|_{L^\infty} :\ \varphi \in V_\lambda,\ \|\varphi\| = 1\} = O(\lambda^{\frac{n-1}{4}})$.

Theorem 83. *Suppose*

$$\sup\{\|\varphi\|_{L^\infty} :\ \varphi \in V_\lambda,\ \|\varphi\| = 1\} \quad \textit{is not} \quad o(\lambda^{\frac{n-1}{4}}).$$

Then, there exists $x \in M$ for which $|\mathcal{L}_x| > 0$. In particular, if g is analytic, then all the geodesics loops at x must return to x at the same time.

The proof of this result is based on the study of $R(\lambda, x)$. Indeed, if $|\mathcal{L}_x| = 0$, then $R(\lambda, x) = o_x(\lambda^{\frac{n-1}{2}})$.

It follows that there are topological restrictions for M to have a real analytic metric such that some sequence of eigenfunctions has the maximal sup-norms. Among all possible surfaces, only the sphere possesses such a metric. In addition, a metric on S^2

with ergodic geodesic flow can never exhibit the maximal growth rate achieved by zonal harmonics on (S^2, g_{S^2}).

When the geodesic flow is chaotic, the random wave conjecture predicts that eiegenfunctions should behave like Gaussian random functions. In particular, one should have $\|\varphi_\lambda\|_{L^\infty} = O(\sqrt{\log\lambda})$. This is very likely to be true in most chaotic systems but not for all of them. Indeed, It has been shown that on some special arithmetic hyperbolic quotients the $O(\sqrt{\log\lambda})$ doesn't hold. The best result known to date if that on manifolds with no conjugate points $\|\varphi_\lambda\|_{L^\infty} = O(\lambda^{\frac{n-1}{4}}/\log\lambda)$.

8.2.2 L^p-norms.

In general L^p-norms are hard to compute. For general L^p-norms one has the following general result due to C. Sogge.

Theorem 84. *Let (M, g) be compact Riemannian manifold and let φ_λ be a normalized eigenfunction of eigenvalue λ. Then, for all $2 \leq p \leq \infty$*

$$\|\varphi_\lambda\|_{L^p} = O(\lambda^{\frac{\delta(p)}{2}})$$

where

$$\delta(p) = \begin{cases} n(\frac{1}{2} - \frac{1}{p}) - \frac{1}{2} & \text{if } \frac{2(n+1)}{n-1} \leq p \leq \infty, \\ \frac{n-1}{2}(\frac{1}{2} - \frac{1}{p}) & \text{if } 2 \leq p \leq \frac{2(n+1)}{n-1}. \end{cases}.$$

The upper bounds are saturated on the round sphere. For $p > \frac{2(n+1)}{n-1}$, zonal (rotationally invariant) spherical harmonics saturate the L^p-bounds. For $2 \leq p \leq \frac{2(n+1)}{n-1}$ the bounds are saturated by highest weight spherical harmonics, i.e. Gaussian beam along a stable elliptic geodesic.

The zonal has high L^p norm due to its high peaks on balls of radius $1/\sqrt{\lambda}$. The balls are so small that they do not have high L^p norms for small p. The Gaussian beams are not as high but they are relatively high over an entire geodesic.

Sogge's result holds for $p \geq 2$. For the L^1 we have the obvious bound $\|\varphi_\lambda\|_{L^1} \leq \|\varphi_\lambda\|_{L^2} \leq 1$. It is interesting however to observe that we can use the L^p bounds for $p \geq 2$ to get a lower bound on $\|\varphi_\lambda\|_{L^1}$.

Theorem 85. *Let (M, g) be compact Riemannian manifold and let φ_λ be a normalized eigenfunction of eigenvalue λ. Then, there exists c such that*

$$\|\varphi_\lambda\|_{L^1} \geq c\lambda^{-\frac{n-1}{8}}.$$

Proof. Fix $2 < p \leq \frac{2(n+1)}{n-1}$. Sogge's L^p bounds give that there exists $C_p > 0$ for which

$$\|\varphi_\lambda\|_{L^p} \leq C_p\lambda^{\frac{n-1}{4}(\frac{1}{2}-\frac{1}{p})}.$$

By Hölder's inequality,

$$\|\varphi_\lambda\|_{L^2}^{\frac{1}{\theta}} \leq \|\varphi_\lambda\|_{L^1}\|\varphi_\lambda\|_{L^2}^{\frac{1}{\theta}-1}$$

for $\theta = \frac{p}{p-1}\left(\frac{1}{2} - \frac{1}{p}\right) = \frac{p-2}{2(p-1)}$. It follows that

$$1 = \|\varphi_\lambda\|_{L^2}^{\frac{1}{\theta}} \leq \|\varphi_\lambda\|_{L^1}\|\varphi_\lambda\|_{L^2}^{\frac{1}{\theta}-1} \leq \|\varphi_\lambda\|_{L^1}\left(C_p\lambda^{\frac{n-1}{4}(\frac{1}{2}-\frac{1}{p})}\right)^{\frac{1}{\theta}-1}.$$

The result follows from observing that $\frac{n-1}{4}\left(\frac{1}{2} - \frac{1}{p}\right)\left(\frac{1}{\theta} - 1\right) = \frac{n-1}{8}$. □

Symmetry of L^p-norms. Jakobson and Nadirashvilli studied the relation between the L^p norm of the negative and positive parts of eigenfunctions. Indeed, for χ the indicator function, set

$$\varphi_\lambda^+ := \varphi \cdot \chi_{\{\varphi_\lambda \geq 0\}}, \qquad \text{and} \qquad \varphi_\lambda^- := \varphi_\lambda \cdot \chi_{\{\varphi_\lambda \leq 0\}}.$$

Theorem 86. *Let (M, g) be a compact smooth manifold. Then, for any $p \in \mathbb{Z}^+$ there exists $C_p > 0$ such that for any non constant eigenfunction φ_λ of the Laplacian*

$$\frac{1}{C_p} \leq \frac{\|\varphi_\lambda^+\|_{L^p}}{\|\varphi_\lambda^-\|_{L^p}} \leq C_p$$

for all λ.

8.3 Zeros of eigenfunctions

Let (M, g) be a compact Riemannian manifold. Given any function $\phi \in C^\infty(M)$ we define its **nodal set**

$$\mathcal{N}_\phi := \{x \in M : \ \phi(x) = 0\}.$$

Each connected component of the complement of $\mathcal{N}_\phi$ is called a **nodal domain**.

We continue to write $\lambda_1 \leq \lambda_2 \leq \ldots$ for the eigenvalues of the Laplacian repeated according to multiplicity (for any initial problem). Write $\varphi_1, \varphi_2, \ldots$ for the corresponding L^2-normalized eigenfunctions.

Theorem 87 (Courant's nodal domain Theorem)**.** *The number of nodal domains of φ_k is strictly smaller than $k + 1$.*

Proof. Suppose that φ_k has at least $k+1$ nodal domains $D_1, \ldots, D_{k+1}, \ldots$ and define

$$\psi_j = \begin{cases} \varphi_k \mid_{D_j} & \text{on } D_j, \\ 0 & \text{else} \end{cases}.$$

There exists $\phi = \sum_{j=1}^k a_j\psi_j \in H_1(M)$ orthogonal to $\varphi_1, \ldots, \varphi_{k-1}$ by the same argument in Theorem 58. Then,

$$\lambda_k \leq \frac{D_g(\phi, \phi)}{\|\phi\|_g^2}.$$

Also,

$$D_g(\phi, \phi) = \sum_{ij=1}^k a_i a_j \langle \tilde{\Delta}_g \psi_i, \psi_j \rangle_g,$$

and

$$\langle \Delta_g \psi_i, \psi_j \rangle_g = \int_M \tilde{\Delta}_g \psi_i \cdot \psi_j \, \omega_g = \int_{D_j} \tilde{\Delta}_g \psi_i \cdot \psi_j \, \omega_g$$
$$= \int_{D_j} \psi_i \cdot \Delta_g \varphi_k \, \omega_g + 0$$
$$= \delta_{ij} \int_{D_j} \varphi_k \cdot \Delta_g \varphi_k \, \omega_g$$
$$= \lambda_k \delta_{ij} \int_{D_j} \varphi_k^2 \, \omega_g.$$

Therefore,

$$D_g(\phi, \phi) = \sum_{j=1}^{k} \lambda_k \int_{D_j} a_j^2 \varphi_k^2 \, \omega_g \leq \lambda_k \|\phi\|_g^2.$$

We proved

$$\frac{D_g(\phi, \phi)}{\|\phi\|_g^2} = \lambda_k$$

and so it follows that ϕ is an eigenfunction of eigenvalue λ_k. Since ϕ vanishes on an open set D_{k+1} we conclude from the unique continuation principle that $\phi = 0$, which is a contradiction. □

Important remarks.

- φ_1 always has constant sign.
- The multiplicity of λ_1 is 1: Otherwise, if φ_2 is an eigenfunction for λ_1 we would know that φ_2 has constant sign. Since φ_1 has constant sign as well we have a contradiction from the fact that $\langle \varphi_1, \varphi_2 \rangle_g = 0$.
- φ_2 has precisely two nodal domains and φ_k has at least two nodal domains for all $k \geq 2$: Otherwise φ_k has constant sign but $\langle \varphi_1, \varphi_k \rangle_g = 0$.

Theorem 88. *For any (M, g) there exists a constant $C > 0$ so that every ball of radius bigger that $C/\sqrt{\lambda}$ contains a zero of any eigenfunction φ_λ.*

Proof. Fix $x \in M$ and $r > 0$. Suppose that φ_λ has no zeros in $B_r(x)$. Then, there must exist a nodal domain D_λ of φ_λ such that $B_r(x) \subset D_\lambda$. Consider now the eigenvalue problem on D_λ

$$(*) \begin{cases} \Delta_g \phi = \mu \phi & \text{in } D_\lambda \\ \phi(x) = 0 & x \in \partial D_\lambda. \end{cases}.$$

It follows that φ_λ is an eigenfunction for $(*)$. In addition, since φ_λ doesn't change sign in D_λ we must have that it is the first eigenfunction for $(*)$. Let us write $\lambda_1(D_\lambda)$ for the corresponding eigenvalue. By Domain monotonicity, since $B_r(x) \subset D_\lambda$, we get

$$\lambda = \lambda_1(D_\lambda) \leq \lambda_1(B_r(x)).$$

To finish the proof one needs to show that $\lambda_1(B_r(x)) \leq C^2/r^2$ for some constant $C > 0$.

Consider the Euclidean metric $g_e(y) := g(x)$ for all $y \in B_{ar}(x; g)$ where $a \in (0, 1)$ is chosen so that $B_{ar}(x; g_e) \subset B_r(x; g)$. By Domain monotonicity we get

$$\lambda_1(B_r(x; g)) \leq \lambda_1(B_{ar}(x; g)).$$

In addition, by comparing Rayleigh's quotients we get that there exists $C_1 > 0$ making

$$\lambda_1(B_{ar}(x; g)) \leq C_1 \lambda_1(B_{ar}(x; g_e)).$$

On the other hand, by explicit computations using the Bessel functions on euclidean balls it is possible to get

$$\lambda_1(B_{ar}(x; g_e)) \leq \frac{C_2}{r^2}.$$

It follows that $\lambda_1(B_r(x; g)) \leq \frac{C_1 C_2}{r^2}$. □

One way of measure the local asymmetry of the nodal sets is the following result due to Mangoubi.

Proposition 89. *Let (M, g) be a compact manifold of dimension n. Then there exists $C > 0$ with*

$$\frac{vol_g\big(\{\varphi_\lambda > 0\} \cap B_{2r}(x)\big)}{vol_g(B_{2r}(x))} \geq C\lambda^{-\frac{n-1}{2}}$$

for all $x \in M$ and $r > 0$ such that $\{\varphi_\lambda = 0\} \cap B_r(x) \neq 0$.

Using this result Mangoubi showed the following theorem on the inner radius of the nodal domains. By inner radius of D, inrad(D) we mean the largest r such that there exists a ball of radius r that can be inscribed in D.

Theorem 90. *Let (M, g) be a compact manifold of dimension n. Then there exists $C_1, C_2 > 0$ such that for D_λ nodal domain of φ_λ one has*

$$\frac{C_1}{\lambda^{\alpha(n)}} \leq inrad(D_\lambda) \leq \frac{C_2}{\sqrt{\lambda}}$$

where $\alpha(n) = \frac{1}{4}(n-1) + \frac{1}{2n}$.

Note that on surfaces this means that the inner radius of D_λ is comparable to $1/\sqrt{\lambda}$.

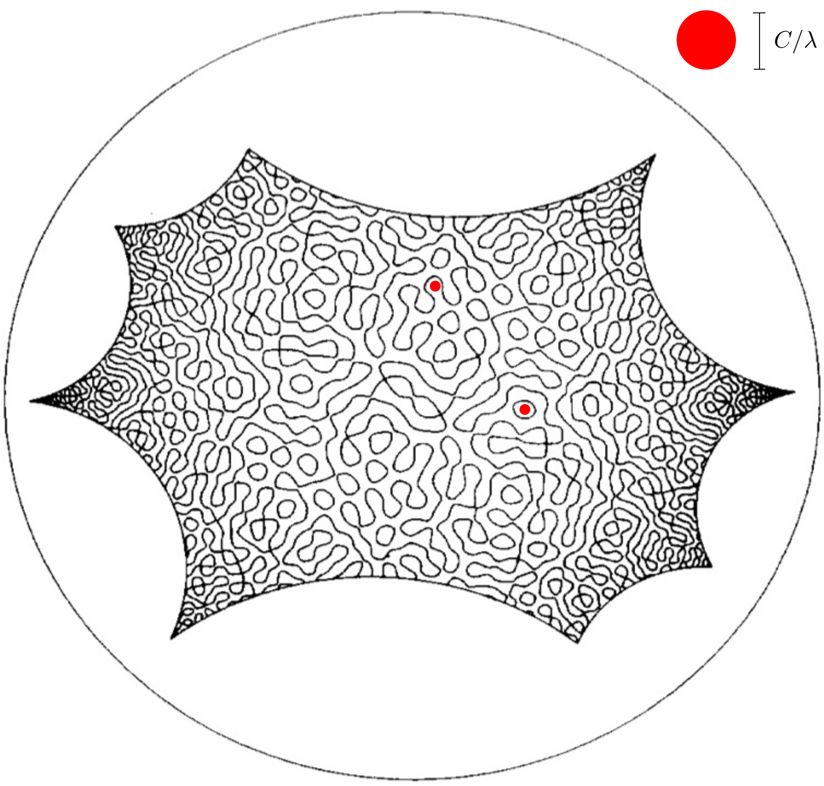


Figure: Nodal set of a bitorus

Hausdorff measure. We continue to write $n = \mathrm{dim} M$. Let $A \subset M$ be any subset and set

$$\mathcal{H}^d_\delta := \inf \left\{ \sum_{j=1}^{\infty} (\mathrm{diam}(U_j))^d \; : \; A \subset \cup_j U_j, \; \mathrm{diam} U_j < \delta \right\},$$

and

$$\mathcal{H}^d_\delta(A) := \lim_{\delta \to 0} \mathcal{H}^d_\delta(A).$$

Since $\mathcal{H}^d_\delta(A)$ is monotone decreasing with δ we have that $\mathcal{H}^d(A)$ is well defended. However, it can be infinite. It can be shown that for $A \subset M$ borel set one has $\mathcal{H}^n(A)$ is proportional to $\mathrm{vol}_g(A)$. Also, if $\gamma \subset M$ is a curve, one has that $\mathcal{H}^1(\gamma)$ is propositional to the length of γ. Similarly, $\mathcal{H}^0(A)$ is the number of points in A.

Uhlembeck proved that 0 is a regular value of the eigenfunctions for a generic set of metrics. In particular, generically, the nodal set $\{\varphi_\lambda = 0\}$ is a smooth hypersurface. In addition, it was proved by Baer that for $x_0 \in M$ there exists local coordinates $(x_1, \dots, x_n)$ at $x_0 = (0, \dots, 0)$ such that an eigenfunction φ can be rewritten as

$$\varphi(x) = v(x) \Big(x_1^k + \sum_{j=0}^{k-1} x_1^j \, u_j(x_2, \dots, x_n) \Big)$$

where φ vanishes to order k at x_0, u_j vanishes to order $k-j$ at $(x_2, \dots, x_n) = (0, \dots, 0)$, and $v(x) \neq 0$ close to p. It follows that nodal sets are rectifiable and therefore $\mathcal{H}^{n-1}(\mathcal{N}_{\varphi_\lambda}) < \infty$.

In 1978 J.Brüning proved that on surfaces there exists $C > 0$ for which $\mathcal{H}^{n-1}(\mathcal{N}_{\varphi_\lambda}) \geq \lambda^{\frac{1}{2}}$. Later, in 1982, S. T. Yau conjectured that on any compact n-dimensional manifold there exist constants $C, c > 0$ for which

$$c\lambda^{\frac{1}{2}} \leq \mathcal{H}^{n-1}(\mathcal{N}_{\varphi_\lambda}) \leq C\lambda^{\frac{1}{2}}.$$

Yau's conjecture was proved for n-dimensional manifolds with real analytic metrics by H. Donelly and C. Fefferman in 1988. For manifolds with smooth metrics the conjecture remains open.

The best known upper bound is due to R. Hardt and L. Simon (1989):

Theorem 91. *Let (M, g) be a compact n-dimensional Riemannian manifold. Then, there exists $C > 0$ for which*

$$\mathcal{H}^{n-1}(\mathcal{N}_{\varphi_\lambda}) \leq \lambda^{C\sqrt{\lambda}}.$$

The best known lower bound to date is due to T. Colding and P. Minicozzi (2010):

Theorem 92. *Let (M, g) be a compact n-dimensional Riemannian manifold. Then, there exists $c > 0$ for which*

$$c\lambda^{\frac{3-n}{4}} \leq \mathcal{H}^{n-1}(\mathcal{N}_{\varphi_\lambda}).$$

The same result was obtained in a very neat proof by H. Hezari and C.Sogge. This proof is heavily based on a result by C.Sogge and S. Zelditch which we prove next.

Proposition 93. *Let (M,g) be a compact n-dimensional Riemannian manifold. For any $f \in C^2(M)$*

$$\int_M (\Delta_g f - \lambda f)\,|\varphi_\lambda|\,\omega_g = -2\int_{\mathcal{N}_{\varphi_\lambda}} f|\nabla_g\varphi_\lambda|\,\sigma_g.$$

Choosing $f = (1+\lambda\varphi_\lambda^2 + |\nabla_g\varphi_\lambda|_g^2)^{\frac{1}{2}}$ and using the Sobolev bounds $\|\varphi_\lambda\|_{H_s} = O(\lambda^{s/2})$ H. Hezari and C.Sogge proved as a corollary of Proposition 93 that there exists $c>0$ making

$$c\sqrt{\lambda}\,\|\varphi_\lambda\|_{L^1}^2 \le \mathcal{H}^{n-1}(\mathcal{N}_{\varphi_\lambda}). \tag{8.1}$$

Inequality (8.1) cannot be improved on general manifolds since it is saturated by zonal harmonics for which one can check that $\mathcal{H}^{n-1}(\mathcal{N}_{Y_0^\ell}) \sim \sqrt{\lambda}$.

Combining (8.1) with the L^1- lower bound $c\lambda^{\frac{1-n}{8}} \le \|\varphi_\lambda\|_{L^1}$ that we proved in Theorem 85 we get the proof of Theorem 92.We proceed to prove Proposition 93.

Proof of Proposition 93. Given λ write $D_+^1(\lambda),\dots,D_+^{N_+(\lambda)}(\lambda)$ for all the positive nodal domains of φ_λ, and analogously write $D_-^1(\lambda),\dots,D_-^{N_-(\lambda)}(\lambda)$ for the negative ones. We then have

$$M = \bigcup_{j=1}^{N_+(\lambda)} D_+^j(\lambda) \cup \bigcup_{j=1}^{N_-(\lambda)} D_-^j(\lambda) \cup \mathcal{N}_{\varphi_\lambda}.$$

Suppose 0 is a regular value of φ_λ. Then, all the nodal domains have smooth boundary. In particular,

$$\begin{aligned}\int_{D_+^j(\lambda)} (\Delta_g f - \lambda f)\,|\varphi_\lambda|\,\omega_g &= \int_{D_+^j(\lambda)} (\Delta_g f - \lambda f)\,\varphi_\lambda\,\omega_g \\ &= \int_{D_+^j(\lambda)} f(\Delta_g - \lambda)\varphi_\lambda\,\omega_g + \int_{\partial D_+^j(\lambda)} f\,\partial_\nu\varphi_\lambda\,\sigma_g - \int_{\partial D_+^j(\lambda)} \varphi_\lambda\,\partial_\nu f\,\sigma_g \\ &= -\int_{\partial D_+^j(\lambda)} f|\nabla\varphi_\lambda|\,\sigma_g.\end{aligned}$$

In the last equality we used that $\partial_\nu\varphi_\lambda = -|\nabla_g\varphi_\lambda|$ and that $D_+^j(\lambda)$ is a positive nodal domain and so ν and $\nabla_g\varphi_\lambda$ point in opposite directions. Similarly,

$$\int_{D_-^k(\lambda)} (\Delta_g f - \lambda f)\,|\varphi_\lambda|\,\omega_g = -\int_{\partial D_-^k(\lambda)} f|\nabla\varphi_\lambda|\,\sigma_g.$$

Adding these identities over j and k and using that $\mathcal{N}_{\varphi_\lambda} = \cup_j\partial D_+^j(\lambda) = \cup_k\partial D_+^k(\lambda)$ we get the desired result. If 0 is not a regular value, a version of Green's identities for domains with rough boundaries yields the same formula.

8.4 Random wave conjecture

In 1977, M. Berry proposed that random linear combinations of planar waves of a fixed high frequency λ in dimension n

$$W_\lambda(x) := \frac{1}{\sqrt{N}}\sum_{i=1}^{N} a_i\cos(k_i\cdot x + \varepsilon_i)\,, \quad x\in\mathbb{R}^n \tag{8.2}$$

serve as a model for high frequency wavefunctions in any quatum system in which the underlying classical dynamics is chaotic. In equation (8.2) the coefficients a_i are independent standard Gaussian random variables, k_i are uniformly distributed on the sphere of radius λ in $\mathbb{R}^n$, and ε_i are independent and uniformly distributed in $(0, 2\pi]$. R. Aurich, A. Becker, R. Schubert, and M. Taglieber show that

$$\limsup_{\lambda \mapsto \infty} \frac{\|W_{\lambda\infty}\|}{\sqrt{n \log \lambda}} \leq 3$$

almost surely.
On compact manifolds one cannot reproduce this construction. Actually, we know that eigenvalues are generically simple so it is pointless to even think about considering random linear combinations of eigenfunctions with a fixed eigenvalue. S. Zelditch then proposed to consider random linear combinations of eigenfunctions with eigenvalues in a window $(\lambda, \lambda + 1]$. These combinations are known as Gaussian random waves:

A *Gaussian random wave of frequency* λ on (M, g) is a random function $\phi_\lambda \in \mathcal{H}_\lambda$ defined by

$$\phi_\lambda := \sum_{\lambda_j \in (\lambda, \lambda+1]} a_j \varphi_j,$$

where $a_j \sim N(0, k_\lambda^{-2})$ are independent and identically distributed.

Remark 94. The normalizing constant k_λ is chosen so that $\mathbb{E}(\|\phi_\lambda\|_2) = 1$. Also, the law of ϕ_λ is independent of the choice of a particular orthonormal basis.

Aside from L^∞-norms, much work has been done on the distribution of the nodal sets for random waves. For instance, S. Zelditch shows that the $n-1$ dimensional Hausdorff measureof the nodal set satisfies Yau's conjecture in average

$$c\sqrt{\lambda} \leq \mathbb{E}(\mathcal{H}^{n-1}(\mathcal{N}_{\phi_\lambda})) \leq C\sqrt{\lambda}$$

for some constants $c, C > 0$ as long as (M, g) is either aperiodic or Zoll.
The random wave model predicts that the behavior of deterministic sequences of L^2-normalized eigenfunctions φ_{λ_j} as $\lambda_j \mapsto \infty$ should coincide with the behavior of the random plane waves W_λ. It has been shown by N. Burq and G. Lebeau that

$$\|\phi_{\lambda_j}\|_\infty = O\left(\sqrt{\log \lambda_j}\right)$$

and

$$\|\phi_{\lambda_j}\|_p = O(1).$$