Computads and Multitopic Sets

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Abstract

We compare *computads* (as defined in [15], [16], [3]) with *multitopic sets* (cf. [5]- [7]). Both these kinds of structures have n-dimensional objects (called *n-cells* for computads and *n-pasting diagrams* for multitopic sets), for each natural number n. In both cases, the set of n-dimensional objects is freely generated by one of its subsets. The computads form a subclass of the more familiar collection of *ω-categories* while multitopic sets are of a more novel nature, being based on an iteration of free *multicategories*. Multitopic sets have been devised as a vehicle for a definition of the concept of *weak ω-category*. Our main result states that the category of multitopic sets is equivalent to that of *many-to-one* computads, which is a certain full subcategory of the category of all computads.

Introduction and preliminaries

The notion of *free structure* has penetrated all parts of modern algebra. It has the following abstract generalization. Given categories $\mathbf{C}$ and $\mathbf{S}$ and a functor $U : \mathbf{C} \to \mathbf{S}$, we say that an object A of $\mathbf{C}$ is *free* with respect to U iff for some object I and arrow $\iota : I \to UA$ in $\mathbf{S}$, the following *universal property* holds: for every object B of $\mathbf{C}$ and arrow $\phi : I \to UB$ of $\mathbf{S}$, there is a *unique* $\mathbf{C}$-arrow $f : A \to B$ such that the following diagram commutes:

$$\begin{tikzcd}
I \arrow[rr, "\iota"] \arrow[dr, "\phi"'] & & UA \arrow[dl, dotted, "Uf"] \\
& UB &
\end{tikzcd}$$

We say that I *generates* the free object A (*via* the arrow ι). In the familiar cases, the objects of $\mathbf{C}$ and $\mathbf{S}$ are mathematical structures, I is a substructure of UA with ι being

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the *inclusion* map, and the elements of (the universe of) I are called *generators* of the free structure A.

For example, if $\mathbf{C}$ is the *category of commutative rings*, $\mathbf{S}$ the *category of sets* and U the *forgetful functor*, the free ring generated by a set X is nothing but the ring $\mathbb{Z}[X]$ of polynomials with integral coefficients and *indeterminates* from the set X. Borrowing terminology from this example, we will usually refer to the generators of a free structure as *indeterminates* or, in short, *indets*.

Another familiar example is that of a *free category* generated by a *directed graph* (see, e.g. [10], §7 of Chapter II). In this case, $\mathbf{C}$ is the category Cat of (small) categories, $\mathbf{S}$ is the category $Grph$ of directed graphs and U is, again, the forgetful functor.

The notion of free category has been generalized to higher dimensional categories by Street, leading to the concept of *computad* which is central for the present work (cf. [15] and [16] for the 2-dimensional case and [3] for the general definition).

To fix our notations, we now recall the structure of higher dimensional categories. An n-dimensional category or, in short, an *n-category* $\mathbf{C}$ has a set of *k-cells* C_k, for each $k \leqslant n$. For $k > 0$, it has domain and codomain functions $d, c : C_k \to C_{k-1}$; thus, a k-cell u is envisaged as an arrow $du \xrightarrow{u} cu$, linking its domain $du \in C_{k-1}$ to its codomain $cu \in C_{k-1}$. For $k \geqslant 2$, we also require du and cu to be *parallel*, meaning that $ddu = dcu$ and $cdu = ccu$, i.e. du and cu have the same domain and the same codomain. For the sake of uniformity, we say that any two 0-cells are parallel, so that we can say that $du \parallel cu$ whenever $u \in C_k$, with $k > 0$. If u is an l-cell and $k < l$ we let $d^{(k)}u$ be the k-cell obtained from u by $l - k$ successive applications of the domain function d; the k-cell $c^{(k)}u$ is defined similarly. The n-category $\mathbf{C}$ is also equipped with *partial composition operations* $\bullet_k$ for $k < n$. If $u, v \in C_l$ and $k < l$, then $u \bullet_k v$ is an l-cell that is defined iff $d^{(k)}u = c^{(k)}v$. Finally, with each l-cell u, $l < n$, $\mathbf{C}$ has an *identity* $(l+1)$-*cell* $u \xrightarrow{1_u} u$. The concepts that we mentioned, satisfy certain axioms. For a precise definition, see [9], as well as section 1 below.

An ω-category is one that has n-cells for each natural number $n < \omega$ (as customary in set theory, ω is the first infinite ordinal number). An n-category can be seen as an ω-category in which all cells of dimension $> n$ are identities. An ω-functor $F : \mathbf{C} \to \mathbf{C}'$ between ω-categories is a map from the cells of $\mathbf{C}$ to those of $\mathbf{C}'$ that preserves the ω-categorical structure. The category ωCat of ω-categories is the one that has the small ω-categories as objects and the ω-functors as arrows.

For an N-category $\mathbf{C}$, with $N \leqslant \omega$, and $n < N$, let $\mathbf{C}_n$ be the n-category whose k-cells are the same as those of $\mathbf{C}$, for all $k \leqslant n$. $\mathbf{C}_n$ is called the nth truncation of $\mathbf{C}$. If $\mathbf{A}$ is an n-category, we say that $\mathbf{C}$ *extends* $\mathbf{A}$ iff $\mathbf{C}_n = \mathbf{A}$.

Fix an $(n-1)$-category $\mathbf{A}$. An *extension* of $\mathbf{A}$ will be any n-category that extends $\mathbf{A}$. A *pre-extension* (I, d, c) of $\mathbf{A}$ will be a set I together with functions $d, c : I \to A_{n-1}$, such that $dx \parallel cx$ for $x \in I$ (remember that A_{n-1} is the set of $(n-1)$-cells of $\mathbf{A}$). For the sake of this preliminary discussion, let us introduce the categories $Ext(\mathbf{A})$ and $Preext(\mathbf{A})$; the former has the extensions of $\mathbf{A}$ as objects while its arrows are the ω-functors that extend the identity functor on $\mathbf{A}$. The latter category has the pre-extensions of $\mathbf{A}$ as objects and the structure preserving maps as arrows. There is an obvious *forgetful* functor $U : Ext(\mathbf{A}) \to Preext(\mathbf{A})$. An extension $\mathbf{B}$ of $\mathbf{A}$ is called *free* if it is a free object of $Ext(\mathbf{A})$ with respect to U, in the sense of the definition that opened this introduction. This concept is at the heart of Street's definition. An ω-category $\mathbf{A}$ is called a *computad* iff $\mathbf{A}_{n+1}$ is a free extension of $\mathbf{A}_n$, for each $n < \omega$.

In the first part of this paper (sections 1-6), we concentrate on the study of free extensions of finite dimensional categories. We start by presenting a construction of a free extension, using a method familiar from universal algebra (cf. e.g. [4]): given a pre-extension (I, d, c) of an $(n-1)$-category $\mathbf{A}$, we set up a formal *equational language* $\mathcal{C}$ which has *terms* denoting all the cells that can be constructed from the elements of I by applications of the partial operations defined among n-cells in an extension of $\mathbf{A}$. The language $\mathcal{C}$ has also *a deductive system* that allows us to prove equalities among terms. Two terms are called *equivalent* if their equality is provable in $\mathcal{C}$. The elements of the free extension constructed by this method, will be the equivalence classes of $\mathcal{C}$-terms.

The same method could be used to construct the free ring generated by a set of indeterminates X. However, a simplifying circumstance occurs in this case. The terms of the corresponding formal language are algebraic expressions that use indeterminates, constants for 0 and 1, binary operation symbols $+$, $\cdot$, $-$ as well as parentheses. Each such term t can be proven to be equal to a polynomial, which is *unique* (assuming that the monomials that are the terms of the polynomial occur in a canonical ordering induced by a given ordering of the set of indeterminates). We shall call this polynomial the *reduced* form of the term t. This situation allows us to replace the equivalence class of t by the unique polynomial which is the common reduced form of the members of this class. The free structure generated by X becomes, in this way, a *term model*, i.e. a structure whose elements are individual terms, rather than equivalence classes. This is how the polynomial ring $\mathbb{Z}[X]$ is obtained.

Can the free extension $\mathbf{X}$ of an n-category $\mathbf{B}$, generated by a given pre-extension, be also construed as a term model? In other words, can we substitute each equivalence class of terms by a "canonical" representative, a common "reduced form" of its elements? *Under certain conditions*, the answer to this question is positive. This result is just one corollary, a side benefit, of the study that we conduct in sections 3-6. We are now going to describe the content of these sections, in rough terms.

Assume that $\mathbf{B}$ itself is a *free* extension of an $(n-1)$-category $\mathbf{A}$, generated by a set I of n-dimensional indets (i.e., generated by a pre-extension of the form (I, d, c)) and let $\mathbf{X}$ be any extension of $\mathbf{B}$. Call an $(n+1)$-cell of $\mathbf{X}$, $u \in X_{n+1}$, *many-to-one* iff its codomain cu is an indet, i.e. $cu \in I$. We define, in section 3, a partial binary operation, called *placed composition*, between the many-to-one cells of $\mathbf{X}$. As it turns out, the many-to-one cells of $\mathbf{X}$, together with the operation of partial composition yield a structure $\mathbb{C}_{\mathbf{X}}$ which is a *multicategory.* The abstract notion of multicategory, described in section 4, is a generalization, introduced in [6], of a notion due to Lambek (cf. [8]). Free multicategories do have *term models*, as shown in [6] and briefly sketched in section 5. The main technical result of this paper, theorem 6.1, states that if $\mathbf{X}$ is a free extension of $\mathbf{B}$, generated by a set J of many-to one indets, then the multicategory $\mathbb{C}_{\mathbf{X}}$ is also free (and, actually, generated by the same set of indets J).

As we stated already, a computad is obtained by starting with a barren set and iterating the free extension construction indefinitely. If, at each stage, the generating indets are many-to-one cells, then we get a *many-to-one* computad. These are the objects of a category $m/1Comp$ described in section 7. The many-to-one cells of a many-to-one computad $\mathbf{A}$ together with the (partial) operations of placed composition and the domain/codomain functions, form a structure $S_{\mathbf{A}}$. This structure is a *multitopic set*, an abstract notion introduced in [7]. Roughly speaking, a multitopic set is a structure obtained by iterating indefinitely the construction of free multicategory. The precise setup, as well as the description of the category *mltSet* of multitopic sets, are presented also in section 7. In section 8 we show, using the results of section 6, that actually, *all* multitopic sets are of the form $S_{\mathbf{A}}$ for some many-to-one computad $\mathbf{A}$. We then infer that the categories $m/1Comp$ and *mltSet* are equivalent. More colorfully said, *multitopic sets are the same as many-to-one computads*. This is the main result of our paper.

Multitopic sets have been introduced in the sequence of papers [5], [6], [7] as a vehicle for producing the "right" definition for the notion of *weak higher dimensional category*. This approach was inspired by an earlier attempt of Baez and Dolan (cf. [1], [2]). See [9] for a survey of the competing definitions of weak higher dimensional categories, including the one of [11], based on multitopic sets. Our main result shows that the definition of [11] could be rephrased using the more familiar notion of many-to-one computad.

An alternative approach for defining weak higher dimensional categories, based on a concept called *dendrotopic sets*, has been devised by Palm in [14]. In addition, Palm shows that the category of dendrotopic sets is equivalent to that of many-to-one computads, thus concluding that the categories of multitopic sets and of dendrotopic sets are also equivalent.

We conclude the preliminaries by recalling one more notation. If u is a k-cell of an ω-category $\mathbf{A}$ and $k < n$, then we let $1_u^{(n)}$ be the n-cell obtained from u by $n-k$ successive applications of the $x \mapsto 1_x$ operation.

1 Free extensions

Let (I, d, c) be a pre-extension of an $(n-1)$-category $\mathbf{A}$, meaning, as we recall, that I is a set and $d, c : I \to A_{n-1}$ are functions such that $dx \parallel cx$ for each $x \in I$. As in the introduction, the elements of I will be called *n-indets*. We should think of an n-indet x as denoting an arbitrary n-cell belonging to an ω-category *extending* $\mathbf{A}$ (i.e. an ω-category whose $(n-1)$th truncation is $\mathbf{A}$), having domain and codomain dx and cx. We now define an equational language $\mathcal{C} = \mathcal{C}(\mathbf{A}, I, d, c)$, dealing with the n-cells obtained from the (cells denoted by) n-indets, by repeated compositions. The symbols of $\mathcal{C}$ will be the n-indets, the *composition symbols* $\bullet_k$, for $k < n$, as well as the *identity symbols* 1_a, for each $(n-1)$-cell $a \in A_{n-1}$. Besides these, $\mathcal{C}$ will employ left and right parentheses as *auxiliary* symbols.

Definition 1.1. The set $\mathcal{T}(\mathcal{C})$ of $\mathcal{C}$-terms and the domain and codomain functions d,$c : \mathcal{T}(\mathcal{C}) \to A_{n-1}$ are defined as follows:

1. Every n-indet x is a $\mathcal{C}$-term with dx, cx as specified by the given functions d,$c : I \to A_{n-1}$.
2. For each $a \in A_{n-1}$, 1_a is a $\mathcal{C}$-term with $d1_a = c1_a = a$.
3. If t, s are $\mathcal{C}$-terms and $d^{(k)}t = c^{(k)}s$, then $(t) \bullet_k (s)$ is a $\mathcal{C}$-term (the parentheses around t and s insure unique readability; usually, we just write $t \bullet_k s$) and we have

$$d(t \bullet_k s) = \begin{cases} ds & \text{if } k = n-1 \\ dt \bullet_k ds & \text{if } k < n-1 \end{cases}$$

$$c(t \bullet_k s) = \begin{cases} ct & \text{if } k = n-1 \\ ct \bullet_k cs & \text{if } k < n-1 \end{cases}$$

4. There are no $\mathcal{C}$-terms besides those mentioned in 1-3.

The meaning of the $\mathcal{C}$-terms should be clear. If $\mathbf{X}$ is an ω-category extending $\mathbf{A}$ and if $\varphi : I \to X_n$ is an *assignment* which is *correct*, meaning that $d\varphi x = dx$ and $c\varphi x = cx$ for all $x \in I$, then we can *evaluate* any $\mathcal{C}$-term t *under* the said assignment and get the value $val_\varphi(t) \in X_n$. Remember that when saying that $\mathbf{X}$ extends $\mathbf{A}$, we mean that $\mathbf{A} = \mathbf{X}_{n-1}$ (the $(n-1)$th truncation of $\mathbf{X}$). More generally, if $\mathbf{X}$ is *any* ω-category, $F : \mathbf{A} \to \mathbf{X}$ an ω-functor and $\varphi : I \to X_n$ an assignment that is *consistent* with F, in the sense that $d\varphi x = Fdx$, $c\varphi x = Fcx$ for $x \in I$, we can evaluate t *under* F, φ and get the value $val_{F,\varphi}(t) \in X_n$. The formal definition runs as follows.

Definition 1.2. Under the assumptions that we just mentioned, we define the function $val = val_{F,\varphi} : \mathcal{T}(\mathcal{C}) \to X_n$, by induction on $\mathcal{C}$-terms:

1. $val(x) = \varphi x$, for $x \in I$.
2. $val(1_a) = 1_{Fa}$, for $a \in A_{n-1}$.
3. $val(t \bullet_k s) = val(t) \bullet_k val(s)$.

If $\mathbf{A} = \mathbf{X}_{n-1}$ and φ is a correct assignment, we let $val_\varphi = val_{i_\mathbf{A},\varphi}$, where $i_\mathbf{A}$ is the inclusion ω-functor of $\mathbf{A}$ into $\mathbf{X}$.

It may so happen, that for terms t and s we have $val_{F,\varphi}(t) = val_{F,\varphi}(s)$ for *all* F and φ. This occurs whenever t and s *must* be equal in virtue of the axioms of ω-category. We can describe this situation precisely, by setting up a deductive system for proving equality of terms. This is done in the definition below, which completes the presentation of the equational logical system $\mathcal{C} = \mathcal{C}(\mathbf{A}, I, d, c)$. Let us mention that the axioms of the notion of ω-category are the associativity, exchange and identity axioms of this definition.

Definition 1.3. We define the deductive system $\mathcal{C}$ as follows, where, in the axioms and rules below, t, s, w, t_1, s_1 are arbitrary $\mathcal{C}$-terms and all compositions are supposed to be well defined (according to definition 1.1).

Axioms.

1. $t = t$ (equality axioms).
2. $(t \bullet_k s) \bullet_k w = t \bullet_k (s \bullet_k w)$ (associativity axioms).
3. $(t \bullet_k t_1) \bullet_l (s \bullet_k s_1) = (t \bullet_l s) \bullet_k (t_1 \bullet_l s_1)$, where $l < k < n$ (exchange axioms).
4. $1_b^{(n)} \bullet_k t = t = t \bullet_k 1_a^{(n)}$, where $k < n$, $c^{(k)}t = b$ and $d^{(k)}t = a$.
   Also, $1_a \bullet_k 1_b = 1_{a \bullet_k b}$, where $a, b \in A_{n-1}$, $d^{(k)}a = c^{(k)}b$ (identity axioms).

Rules.

1. $\dfrac{t = s}{s = t} \qquad \dfrac{t = s \quad s = w}{t = w}$ (equality rules)
2. $\dfrac{t = s}{t \bullet_k w = s \bullet_k w} \qquad \dfrac{t = s}{w \bullet_k t = w \bullet_k s}$ (congruence rules)

We will write '$\vdash t = s$' or, sometimes, $\vdash_{\mathcal{C}} t = s$ to indicate that $t = s$ is provable in this system.

Is this system complete? In other words are we sure that, whenever $\nvdash t = s$, there are $\mathbf{X}$,F and φ for which $val(t) \neq val(s)$? The *positive* answer to this question, follows from the existence of *free* extensions.

Theorem 1.4. *Given $\mathbf{A}$, I, d, c as above, there exists an n-category $\mathbf{A}[I]$ satisfying:*

1. *$\mathbf{A}[I]$ is an extension of $\mathbf{A}$, i.e its $(n-1)$-th truncation is $\mathbf{A}$, $\mathbf{A}[I]_{n-1} = \mathbf{A}$.*
2. *Each $x \in I$ is an n-cell of $\mathbf{A}[I]$ with domain dx and codomain cx.*
3. *$\mathbf{A}[I]$ has the following* universal property*: if $\mathbf{X}$ is any ω-category extending $\mathbf{A}$ and $\varphi : I \to X_n$ a function satisfying that $d\varphi x = dx$ and $c\varphi x = cx$ for $x \in I$, then there is a* unique *ω-functor $G : \mathbf{A}[I] \to \mathbf{X}$ such that $Ga = a$ when a is a cell of $\mathbf{A}$ and $Gx = \varphi x$ for $x \in I$.*

Moreover, $\mathbf{A}[I]$ *has the following* strong universal property*: whenever* $\mathbf{X}$ *is an* ω*-category,* $F : \mathbf{A} \to \mathbf{X}$ *an* ω*-functor and* $\varphi : I \to X_n$ *a function such that* $d\varphi x = Fdx$*,* $c\varphi x = Fcx$ *for* $x \in I$*, there is a* unique ω*-functor* $G : \mathbf{A}[I] \to \mathbf{X}$ *such that* $Ga = Fa$ *whenever* a *is a cell of* $\mathbf{A}$ *and* $Gx = \varphi x$ *for* $x \in I$*.*

Remark. The universal property means that $\mathbf{A}[I]$ is a free extension of $\mathbf{A}$ in the sense explained in the introduction. The *strong* universal property means that $\mathbf{A}[I]$ is free with respect to a forgetful functor $U_1 : \mathbf{C}_1 \to \mathbf{S}_1$, where $\mathbf{C}_1$ is ωCat while $\mathbf{S}_1$ is a category whose objects are pairs $\langle \mathbf{B}, (Z, d, c)\rangle$ with $\mathbf{B}$ an $(n-1)$-category and (Z, d, c) a pre-extension of $\mathbf{B}$ (the interested reader should have no problems in identifying the arrows of $\mathbf{S}_1$ and the definition of U_1).

Proof. As outlined in the introduction, the n-cells of $\mathbf{A}$ will be *equivalence classes* of $\mathcal{C}$-terms, under a suitable equivalence relation.

Claim 1.5. (a) *If we define, for* $\mathcal{C}$*-terms* t *and* s*,* $t \approx s$ *iff* $\vdash t = s$*, then* $\approx$ *is an equivalence relation that is a congruence with respect to* $\bullet_k$*,* $k < n$*.*

(b) *If* $\vdash t = s$ *then* $dt = ds$*,* $ct = cs$*.*

(c) *If* $\vdash t = s$ *then* $val_{F,\varphi}(t) = val_{F,\varphi}(s)$*, for all* F *and* φ*.*

Proof. (a) is immediate (congruence with respect to $\bullet_k$ means that $t \approx s$ implies $t \bullet_k w \approx s \bullet_k w$ and $w \bullet_k t \approx w \bullet_k s$).

(b) and (c) are easily checked by induction on proofs. □

We can now describe the n-category $\mathbf{A}[I]$. The cells of $\mathbf{A}[I]$ of dimension $\leqslant n-1$ are those of $\mathbf{A}$, while the n-cells are the equivalence classes $t/{\approx}$ for $t \in \mathcal{T}(\mathcal{C})$, where $d(t/{\approx}) = dt$, $c(t/{\approx}) = ct$ and $(t/{\approx}) \bullet_k (s/{\approx}) = t \bullet_k s/{\approx}$.

Claim 1.5(a)(b), insures that the definitions of c, d and $\bullet_k$ are correct, and the axioms of our deductive system insure that we defined, indeed, an n-category. However, we wanted the elements of I to be n-cells of $\mathbf{A}[I]$ and what we have, instead, is that $x/{\approx}$ is a such, for every $x \in I$. To correct this, we only have to *identify* x with $x/{\approx}$. To be sure that we do not make unwanted identifications in this way, we have to check that $x \not\approx y$, whenever $x \neq y$ for $x, y \in I$. This is easily seen, however. It should be clear when do we say that an indet x *occurs* in a term $t \in \mathcal{T}(\mathcal{C})$. A straightforward verification shows that the following is true.

Claim 1.6. *If* $\vdash t = s$ *then any indet* $x \in I$ *occurs in* t *iff it occurs in* s*. Hence, if* x *and* y *are* distinct *indets, then* $\nvdash x = y$*, which means that* $x \not\approx y$*.*

This shows that we can, indeed, identify x with $x/\approx$ and assume that the elements of I are n-cells of $\mathbf{A}[I]$.

To conclude the proof, it is enough to show that $\mathbf{A}[I]$ has the *strong universal property* stated in part *3* of 1.4. Given an ω-functor $\mathbf{A} \xrightarrow{F} \mathbf{X}$ and a function $\varphi : I \to X_n$ such that $d\varphi x = Fdx,\ c\varphi x = Fcx$, we define $G : \mathbf{A}[I] \to \mathbf{X}$ by

$$Ga = Fa \text{ for } a \text{ a cell of } \mathbf{A} \text{ and } G(t/\approx) = val_{F,\varphi}(t) \text{ for } t \in \mathcal{T}(\mathcal{C}).$$

Claim 1.5(c) implies that this definition is correct and definition 1.2 of $val_{F,\varphi}(-)$ insures that G is an ω-functor. It follows immediately that G extends both, F and φ and that any such G *has* to be defined as above. Thus, G is unique and we have proved 1.4. $\square$

If we now let $i_{\mathbf{A}} : \mathbf{A} \to \mathbf{A}[I]$ be the inclusion functor and φ be the inclusion function from I into the n-cells of $\mathbf{A}[I]$, then an easy induction on terms shows that $val_{i_{\mathbf{A}},\varphi}(t) = t/\approx$. This fact yields immediately the following.

Corollary 1.7. *The deductive system $\mathcal{C}$ is complete, namely, if $\nvdash t = s$, then for some $\mathbf{X}$, F and φ we have $val_{F,\varphi}(t) \neq val_{F,\varphi}(s)$.*

Remark. As easily seen, the *universal property* of 1.4, part *3*, determines $\mathbf{A}[I]$ uniquely up to an isomorphism (actually, up to a *unique* isomorphism that is the identity for the cells of $\mathbf{A}$ and for the elements (n-indets) of I). It follows that the universal property actually implies the *strong* universal property.

An n-category $\mathbf{B}$ will be called a *free extension* of $\mathbf{A}$ iff it extends $\mathbf{A}$ and for some $I \subset B_n$, $\mathbf{B}$ has the universal property of $\mathbf{A}[I]$ (and hence, it is isomorphic to $\mathbf{A}[I]$, as just remarked). We also say, in this situation, that $\mathbf{B}$ is freely generated by the set I (an abbreviated terminology that suppresses $\mathbf{A}$).

An important convention. A 0-*category* $\mathbf{B}$ consists of the set B_0 of its 0-cells, and nothing more. Thus, a 0-category is just a barren set (this is a customary point of view). An ω-functor from such a $\mathbf{B}$ to any ω-category $\mathbf{X}$ is just a function from B_0 to the set X_0 of 0-cells of $\mathbf{X}$. We will say that *any* 0-category $\mathbf{B}$ is *freely generated* by the set B_0 of its 0-cells. This is justified because the obvious universal property holds trivially. Also, we will sometimes refer to the 0-cells of *any* ω-category as 0-*indets.*

This terminology will turn out to be convenient in the sequel, as it will allow the inclusion of the case $n = 0$ in several statements.

We conclude this section with a remarkable property of free extensions. As the statement and, even more so, the proof, involve some technical details, the reader may wish to skip this on first reading and return to it when it is invoked in later sections.

In analogy with the notion of free group, one might expect that the same free extension of $\mathbf{A}$ might be generated by several distinct sets of n-indets. In many important instances, this is not so, however. As it turns out, under certain conditions, the set of n-indets of a free extension is uniquely determined.

Definition 1.8. An n-cell u of an ω-category $\mathbf{X}$ is *indecomposable* if whenever $u = v \bullet_k w$, with $k < n$, then either $u = 1_a^{(n)}$ or $v = 1_a^{(n)}$, where $a = d^{(k)}v = c^{(k)}w$.

Identity cells are, in general, decomposable in many obvious ways. For example, if u, v are *non-identity* m-cells such that $u \bullet_{m-1} v = a$ is defined, then $1_a = 1_u \bullet_{m-1} 1_v$, showing that 1_a is *decomposable*. More generally, if $l < k < m$ and $a = u \bullet_l v$, where u, v are k-cells, then it is easy to see that $1_a^{(m)} = 1_u^{(m)} \bullet_l 1_v^{(m)}$. We will consider this kind of decompositions of k-identities to be trivial. A formal definition, which is wider in a certain respect, will now be given. A cell of the form $1_a^{(m)}$, with a a k-cell and $m > k$, will be called a *k-identity of dimension m*.

Definition 1.9. A k-identity e of dimension $m > k$ is called *essentially indecomposable* if whenever $e = u \bullet_l v$ with $l \leqslant k$, then both u and v are k-identity cells of dimension m.

Remark. In the case of $l = k < m$, the condition of essential indecomposability just means that if $e = 1_a^{(m)} = s \bullet_k w$ with a of dimension k, then $s = w = 1_a^{(m)}$.

Definition 1.10. An n-category $\mathbf{X}$ is *well-behaved* if, for all $k < m \leqslant n$, all k-identities of dimension m are essentially indecomposable.

Notice that any 0-category is trivially well-behaved. Also, as *free categories*, i.e. free extensions of 0-categories, have a very simple structure (cf. e.g. section 7 of chapter I in [10]) and are easily seen to be well-behaved. The remarkable result that we want to prove is the following.

Theorem 1.11. *If $\mathbf{A}$ is a well-behaved $(n-1)$-category and I is a set of n-indets over it, then for any n-cell x of $\mathbf{A}[I]$, $x \in I$ iff x is indecomposable and is* not *an identity cell. Furthermore, $\mathbf{A}[I]$ is also well behaved.*

Thus, an n-dimensional extension $\mathbf{B}$ of a well-behaved $(n-1)$-category $\mathbf{A}$ is free iff it is freely generated by the set of its non-identity indecomposable cells.

For $n = 1$, this theorem is easily checked, due to the above mentioned simple structure of free categories. For $n > 1$, the proof involves a deeper analysis of the deductive system $\mathcal{C}$. We begin with a definition.

Definition 1.12.
1. A term $t \in \mathcal{T}(\mathcal{C})$ is called *constant* iff no variable occurs in t.
2. t is called an *identity* iff for some $(n-1)$-cell a of $\mathbf{A}$, $\vdash t = 1_a$. An identity t is called a *k-identity* (where $k < n$) iff $\vdash t = 1_a^{(n)}$ for some k-cell a of $\mathbf{A}$.

3. A term t is called *indecomposable* iff whenever $\vdash t = s \bullet_k w$ (with $k < n$) then one of s, w is a k-identity.

Thus, a term t is indecomposable iff $t/\approx$ is an indecomposable n-cell of $\mathbf{A}[I]$.
The following simple statement implies immediately the "if" direction of 1.11.

Proposition 1.13. *If $t \in \mathcal{T}(\mathcal{C})$ is an indecomposable term, then t is an identity or $\vdash t = x$ for some variable $x \in I$.*

Proof. By induction on t. If t is an identity or a variable then we have nothing to prove. If $t = s \bullet_k w$, then either s is a k-identity and then $\vdash t = w$ or w is an identity and $\vdash t = s$; in either case, the claim follows by the induction hypothesis. ☐

Next, we point out a very simple fact.

Claim 1.14. *A term t is constant iff it is an identity.*

Proof. By induction on t. If t is an identity or an n-indet, this is immediate. If $t = s \bullet_k w$ and t is constant then so are s, w, hence, by the induction hypothesis, we can find $(n-1)$-cells a, b such that $\vdash s = 1_a$, $\vdash w = 1_b$.

If $k = n-1$, then we have $a = d1_a = ds = cw = c1_b = b$, hence $\vdash t = s \bullet_n w = 1_a \bullet_n 1_a = 1_a$, so t is an identity. If $k < n-1$, then $d^{(k)}s = d^{(k)}1_a = d^{(k)}a$ and likewise, $c^{(k)}w = c^{(k)}b$. As $s \bullet_k w$ is defined, we have that $d^{(k)}a = c^{(k)}b$, hence $a \bullet_k b$ is defined and we have, by one of the identity axioms, $\vdash t = 1_a \bullet_k 1_b = 1_{a \bullet_k b}$. this completes the proof of the "only if" direction of the claim. The "if" direction follows immediately by claim 1.6. ☐

This allows us to infer the "Furthermore" part of 1.11.

Claim 1.15. *If $\mathbf{A}$ is well-behaved then so is $\mathbf{A}[I]$.*

Proof. We have to show that if $t \in \mathcal{T}(\mathcal{C})$ is a k-identity and $\vdash t = s \bullet_k w$, for $k < n$, then both s and w are k-identities. Indeed, in this case, t, s, w must all be constant, hence identities, by 1.14. So, assume that $\vdash t = 1_a^{(n)}$ and $\vdash s = 1_u, w = 1_v$ with a being a k-cell and u, v being $(n-1)$-cells of $\mathbf{A}$. If $k = n-1$, we immediately infer that $a = u = v$. If $k < n-1$, then $dt = ds \bullet_k dw$ which means that $1_a^{(n-1)} = u \bullet_k v$ and as $1_a^{(n-1)}$ is a k-identity in $\mathbf{A}$, it is essentially indecomposable, which means that u, v are also k-identities hence, so are s and w. ☐

Each occurrence of a composition symbol $\bullet_k$ in a term $t \in \mathcal{T}(\mathcal{C})$ has a definite *scope* which is a subterm of t of the form $s \bullet_k w$.

Definition 1.16. An occurrence of $\bullet_k$ with scope $s \bullet_k w$ in a term t is called *inessential* iff *either* one of s, w is a k-identity *or* both, s and w are identities.

To put it more colorfully, a composition occurrence in t is inessential iff it can be "wiped out" by the use of one of the *identity* axioms of our deductive system. The next lemma, which is crucial for the proof of 1.11, says, in effect, that this is the only way in which a composition symbol can be made to disappear from a term of $\mathcal{T}$.

Lemma 1.17. *Under the assumption of 1.11, if* $\vdash t = s$ *and one of* t, s *has only inessential occurrences of composition, then so does the other .*

Proof. By induction on proofs. We have to show, first, that the statement of the lemma is true for all the axioms and, second, that if the statement is true for the premise, or the premises, of a rule then it is true for its conclusion as well.

We start with the associativity axioms. Let $(t \bullet_k s) \bullet_k w = t \bullet_k (s \bullet_k w)$ be a such and assume, e.g., that the left hand side only inessential compositions. This means that the same is true for t, s, w, so all we have to show is that the two $\bullet_k$-occurrences indicated on the right are inessential. As the rightmost indicated occurrence of $\bullet_k$ on the left hand side is inessential, we have three cases, and we examine each of them separately.

Assume first that $t \bullet_k s$ is a k-identity. If so, then so are t and s and this implies that the two indicated occurrences of $\bullet_k$ on the right are inessential. Second, assume that w is a k-identity. But then, the left hand side of the axiom is provably equal to $t \bullet_k s$ and by assumption, this occurrence of $\bullet_k$ is also inessential and we easily conclude that the compositions on the right hand side are also inessential. Finally, if both $t \bullet_k s$ and w are identities, then so are t and s and all compositions on the right are inessential. This completes the examination of the associativity axiom.

The case of exchange axioms is more complex. The argumentation is not hard, but is somewhat tedious. Consider the instance

$$(t \bullet_k t_1) \bullet_l (s \bullet_k s_1) = (t \bullet_l s) \bullet_k (t_1 \bullet_l s_1)$$

where $l < k \leqslant n$.

Assume first that the *left* side has no essential composition occurrences. Then, certainly, t, t_1, s, s_1 have no such occurrences and so, all we have to show this that, on the *right* side, the three indicated composition occurrences are inessential. As $\bullet_l$ on the left is inessential, *either* both terms that it binds are identities *or* one of these terms that is an l-identity. In the first case, t, t_1, s, s_1 are all identities and hence, all compositions on the right are inessential. In the second case assume, e.g., that $t \bullet_k t_1$ is an l-identity. As $l < k$, any l-identity is also a k-identity hence, by 1.15, is essentially indecomposable and both t and t_1 are k-identities. But then, $\vdash t \bullet_k t_1 = t = t_1$ and so, t and t_1 are actually l-identities. Taking into consideration that the second $\bullet_k$ on the left is also inessential, we now easily conclude that all compositions on the right are inessential.

Now assume that the right side of the exchange axiom has no essential composition occurrences and let's show that the three compositions indicated on the left are also inessential.

Either both terms bound by $\bullet_k$ on the right are identities *or* one of these is a k-identity. The first case is, again, trivial, so let us consider the second. Assume, e.g., that $t \bullet_l s$ is a k-identity and hence, is essentially indecomposable. Then t and s are both k-identities and, as the second $\bullet_k$ on the right is inessential, we conclude immediately that all compositions on the left are inessential.

Checking the statement of the lemma for the other axioms is trivial and so is for the rules. $\square$

Proof of 1.11. It remains to show that any indeterminate $x \in I$ is indecomposable. assume that $\vdash x = t \bullet_k s$. As the basic term x has no essential compositions, it follows by 1.17 that $\bullet_k$ on the right is inessential. By 1.6, x must occur in $t \bullet_k s$, which means that t and s cannot be both constant, i.e., by 1.14 cannot be both identities. We conclude that one of t, s must be a k-identity. $\square$

2 Indet occurrences

The notion of occurrence of an indet in (a $\mathcal{C}$-term denoting) an n-cell u of $\mathbf{A}[I]$ is surprisingly complex and will be discussed in the present section.

We start by pointing out that the same indet may occur several times in a term denoting u. A simple example: assuming that $a = dx = cy$ and $b = dy = cx$, the $\mathcal{C}$-term $t = (x \bullet_{n-1} y) \bullet_{n-1} x$ denotes an n-cell $a \xrightarrow{u} b$, the composite of the diagram

$$a \xrightarrow{x} b \xrightarrow{y} a \xrightarrow{x} b,$$

and the indet x has two distinct occurrences in t. As we shall see, in certain situations we will be interested in replacing *one* of these occurrences of x by a cell v of dimension n or *higher* (!), such that $d^{(n-1)}v = a$ and $c^{(n-1)}v = b$. Therefore, we must have a mean of indicating a particular occurrence of an indet x in an n-cell u. One solution could be to arrange the occurrences of the indets in a sequence, in the order in which they occur in t. In the example that we just considered, we are speaking of the sequence $\langle x, y, x \rangle$. *Unfortunately*, the same cell u is denoted by several terms and the order of indet occurrences may vary from one such term to the other. For instance, the terms $t = (x \bullet_k x_1) \bullet_l (y \bullet_k y_1)$ and $s = (x \bullet_l y) \bullet_k (x_1 \bullet_l y_1)$ denote the same n-cell, where $l < k < n$. *Fortunately*, whenever t and s denote the same cell, i.e. whenever $\vdash t = s$, the same indets occur in both, each occurring the same number of times in t and in s and, *moreover*, each proof of $t = s$ yields, in an obvious way, a one-to-one correspondence between the indet occurrences in t and those in s.

To deal with this situation, we start by attaching to each n-cell u an *indexed set* $\langle u \rangle$ of indet occurrences; this is a function $\langle u \rangle : |\langle u \rangle| \to I$ whose domain is a finite set $|\langle u \rangle|$. An indet $x \in I$ has an occurrence in u iff it is in the range of $\langle u\rangle$ and if this is the case, then the number of occurrences of x in u is the cardinality of the set $\{r \in |\langle u\rangle| : \langle u\rangle(r) = x\}$.

It will be useful to assume that all the domains $|\langle u\rangle|$ are subsets of a given infinite set $\mathcal{N}$. Following [6], we let $I^{\#}$ be the category whose objects are the finite indexed subsets of I (i.e. the functions from finite subsets of $\mathcal{N}$ into I) and arrows are defined in the obvious way. Let us mention, for further use, that in $I^{\#}$, $\langle u_1 \bullet_k u_2\rangle$ is a *coproduct* of $\langle u_1\rangle$ and $\langle u_2\rangle$, with several possible pairs of coprojections $\kappa_i : |\langle u_i\rangle| \to |\langle u_1 \bullet_k u_2\rangle|$, $i = 1, 2$.

Remark. The choice of the finite set $|\langle u\rangle|$ is totally arbitrary, apart from the fact that the number of its elements should equal that of *distinct* occurrences of indets in u and we may, if we wish so, *reparametrize* $\langle u\rangle$, meaning that we replace its domain $|\langle u\rangle|$ by any subset of $\mathcal{N}$ of the same cardinality.

$\langle u\rangle$ is just an abstract object that carries the basic information about the indets occurring in u and the number of occurrences of each. We still have to attach every $r \in |\langle u\rangle|$ to a particular occurrence of $x = \langle u\rangle(r)$ in u. This is done with the help of an *indet-occurrence specification* or, in short, a *specification* for u. Such a specification is given by a $\mathcal{C}$-term t denoting u (i.e. such that $u = t/\approx$) together with a one-to-one function θ whose domain is $|\langle u\rangle|$ and such that for each $r \in |\langle u\rangle|$, $\theta(r)$ will be a place in the string of symbols t, in which $x = \langle u\rangle(r)$ occurs. This occurrence will be referred to as the *r-occurrence* of x in u, *as specified by* θ. We will denote $\theta : |\langle u\rangle| \to t$, to indicate that θ is a specification as described.

As mentioned above, every $\mathcal{C}$-proof π of $t = s$ generates a bijection between the indet occurrences in t and those in s. This is a one-to-one function $\chi = \chi_\pi$ which maps every location in the string of symbols t at which a certain indet occurs to a location in s occupied by the same indet. We denote this situation by $\chi : t \to s$. We let the reader figure out the obvious definition of χ_π. With the help of this notion, we can now define when two specifications for u are the same.

Definition 2.1. Two specifications $\theta_i : |\langle u\rangle| \to t_i$, $i = 1, 2$, are called *equivalent* if there exists a $\mathcal{C}$-proof π of $t_1 = t_2$ such that $\theta_2 = \chi_\pi \theta_1$.

Remarks. 1. If every indet that occurs in u, occurs there precisely *once*, then we have a unique specification $\theta : |\langle u\rangle| \to t$, for every t denoting u. If this is the case, then any two specifications for u are equivalent. If, however, there are indets with multiple occurrences in u, then there are several possible specifications for u into the same t. In this case, u may have *inequivalent* specifications. To show how delicate the issue of indet occurrences may be, let us also mention that we might have two *distinct* specifications $\theta_i : |\langle u\rangle| \to t$ into the same t that are *equivalent*! Indeed, as remarked by Eckmann and Hilton, if a is a 0-cell and $u, v : 1_a \to 1_a$ are 2-cells, then one can prove that

$$u \bullet_0 v = v \bullet_0 u = u \bullet_1 v = v \bullet_1 u.$$

Thus, if we let x be a 2-indet with $dx = cx = 1_a$, then substituting x for u and v in, e.g. the proof of the first equality, we get a non-trivial $\mathcal{C}$-proof π of $x \bullet_0 x = x \bullet_0 x$ which yields a χ_π that interchanges the two occurrences of x.

2. An alternative, more picturesque and less formal, point of view is this. A specification $\theta : |\langle u \rangle| \to t$ actually *relabels* the distinct occurrences of any indet x by different symbols $x', x'', \ldots$. In this way, we transform t into a term $t^\star$, all of whose indets have unique occurrences. Any $\mathcal{C}$-proof of $t = s$ yields a suitable relabelling $s^\star$ and a $\mathcal{C}$-proof of $t^\star = s^\star$. In this way, by looking at the proof, we can follow the rearrangement in s of the indets that occur in t.

We now *choose*, for every n-cell u of $\mathbf{A}$, a *preferred* specification $\theta_u : |\langle u \rangle| \to t_u$. From this point on, when speaking of *the* r-occurrence of $x = \langle u \rangle(r)$ in u, we will mean the occurrence specified by θ_u. Occasionally, we might have to use another term t denoting u, and in such a case, it should always be considered together with a proof π of $t_u = t$. Then, the above mentioned r-occurrence in u is also the one specified by the equivalent specification $\theta = \chi_\pi \theta_u : |\langle u \rangle| \to t$. One typical context in which such a situation occurs naturally, will now be described.

If u_1, u_2 are k-composable n-cells of $\mathbf{A}[I]$, then $u_1 \bullet_k u_2$ is denoted by both $t_{u_1 \bullet_k u_2}$ and $t_{u_1} \bullet_k t_{u_2}$. Let's write, for simplicity, $u_1 \bullet_k u_2 = u$, $t_u = t$, $t_{u_i} = t_i$, $i = 1, 2$. Select a proof π of the equality $t = t_1 \bullet_k t_2$. It will yield a map $\chi_\pi : t \to t_1 \bullet_k t_2$. Let ι_i be the "embedding" of the term t_i into $t_1 \bullet_k t_2$, $i = 1, 2$. By this we mean that ι_i maps every location in the string of symbols t_i into the corresponding location in the the larger string $t_1 \bullet_k t_2$. Remember that $\langle u \rangle = \langle u_1 \bullet_k u_2 \rangle$ is a coproduct of u_1 and u_2 in the category $I^\#$. A pair of coprojections κ_i, $i = 1, 2$, will be called *appropriate* (with respect to the selected proof π), if the following diagrams commute:

$$\begin{tikzcd}
{|\langle u_1 \rangle|} \arrow[rr, "\theta_1"] \arrow[d, "\kappa_1"'] & & t_1 \arrow[d, "\iota_1"] \\
{|\langle u \rangle|} \arrow[r, "\theta"] & t \arrow[r, "\chi_\pi"] & t_1 \bullet_k t_2 \\
{|\langle u_2 \rangle|} \arrow[rr, "\theta_2"'] \arrow[u, "\kappa_2"] & & t_2 \arrow[u, "\iota_2"']
\end{tikzcd}$$

where θ, θ_1, θ_2, are the *preferred* specifications for u, u_1, u_2. As all maps in this diagram are one-to-one, we immediately conclude that, given u_1, u_2 and π, there is a unique pair of appropriate coprojections κ_1, κ_2. These coprojections will relate each indet occurrence in u_i to the corresponding one in $u = u_1 \bullet_k u_2$.

An important convention. As we remarked already, we can reparametrize any given u by changing its domain at will. We will use this flexibility and *always assume* that, whenever we are considering the cell $u \bullet_k v$, the index sets $|\langle u\rangle|$, $|\langle v\rangle|$ were so chosen as to be *disjoint* and to have $|\langle u \bullet_k v\rangle| = |\langle u\rangle|\dot{\cup}|\langle v\rangle|$ (where the customary notation "$\dot{\cup}$" comes to emphasize that the two terms of the union are disjoint sets), with the *inclusion* maps of $|\langle u\rangle|$, $|\langle v\rangle|$ in $|\langle u \bullet_k v\rangle|$ being *appropriate* coprojections. This convention will simplify notations in the sequel.

3 Placed composition

In this section, we will assume that $\mathbf{B}$ is an n-category freely generated by a set I of indets. This means that $n > 0$ and $\mathbf{B} = \mathbf{A}[I]$, where $\mathbf{A} = \mathbf{B}_{n-1}$, the $(n-1)$th truncation of $\mathbf{B}$ or else, $n = 0$ and $I = B_0$. Let $\mathbf{X}$ be an ω-category extending $\mathbf{B}$.

We are going to describe several operations involving cells of dimension $\geqslant n$ of the ω-category $\mathbf{X}$. The most important, for the present article, is the operation of *placed composition*, that will be presented later in this section.

The first operation to be described is the *n-cell replacement* operation. If u is an n-cell of $\mathbf{X}$, $r \in |\langle u\rangle|$, $\langle u\rangle(r) = x \in I$ and if v is any n-cell of $\mathbf{X}$ parallel to x, then we can *replace* the r-occurrence of the n-indet x in u by the n-cell v, producing an n-cell $u\Box_r v$, as result. Notice that $u\Box_r v$ is parallel to u. Let us recall that an n-cell v is parallel to x iff $n > 0$ and $dv = dx$, $cv = cx$ or else, $n = 0$ (as any two 0-cells are considered to be parallel).

We can generalize this operation by allowing v to be any cell of dimension$\geqslant n$, provided that, if $n > 0$ then $d^{(n-1)}v = dx$ and $c^{(n-1)}v = cx$. Indeed, if u is an n-cell and v an m-cell, where $k < n < m$, such that $d^{(k)}u = c^{(k)}v$ it is customary to define $u \bullet_k v = 1_u^{(m)} \bullet_k v$. Similarly, $u \bullet_k v = u \bullet_k 1_v^{(m)}$, if u is of dimension m and v of dimension n. These operations that yield an m-cell when applied to cells of dimensions n and m, are called *whiskerings*. As any n-cell u is obtained from indets by means of compositions, we conclude that it makes sense to replace the r-occurrence of x in u, by any cell v of $\mathbf{X}$ of dimension $m \geqslant n$, provided that $d^{(n-1)}v = dx$ and $c^{(n-1)}v = cx$. The result is an m-cell $u\Box_r v$ and this kind of replacement will be called a *generalized whiskering* operation. The n-cell replacement operation is just the generalized whiskering, restricted to n-cells.

For $n = 0$, the generalized whiskering operations is trivial: if u is a 0-cell, then it is an indet, and $u\Box_r v = v$, hence $u\Box_r -$ is the identity function on the set of all cells of $\mathbf{X}$.

For $n > 0$, given *parallel* $(n-1)$-cells $a, b \in X_{n-1}$ of $\mathbf{X}$, we let $\mathbf{X}(a, b)$ be the ω-category whose k-cells are those $(n+k)$-cells $v \in X_{n+k}$ that satisfy $d^{(n-1)}v = a$, $c^{(n-1)}v = b$. With this notation, we see that, for $u \in X_n$ and $r \in |\langle u\rangle|$, $x = \langle u\rangle(r)$, $u\Box_r -$ is a function from the set of cells of $\mathbf{X}(dx, cx)$ to the cells of $\mathbf{X}(du, cu)$. As a clue to a precise definition of this function, we note that the following three conditions should be met.

1. If u is an n-cell of $\mathbf{X}$, $r \in |\langle u\rangle|$ and $\langle u\rangle(r) = x$ then $u\Box_r v$ is defined iff $d^{(n-1)}v = dx$ and $c^{(n-1)}v = cx$. If this is the case, then $d^{(n-1)}(u\Box_r v) = du$ and $c^{(n-1)}(u\Box_r v) = cu$.

2. If $u = x \in I$ (remember that we identified x with the n-cell $x/\approx$) then $u \Box_r v = v$ (where, of course, $|\langle u \rangle| = \{r\}$).

3. $(u' \bullet_k u'') \Box_r v = \begin{cases} (u' \Box_r v) \bullet_k u'' & \text{if } r \in |\langle u' \rangle| \\ u' \bullet_k (u'' \Box_r v) & \text{if } r \in |\langle u'' \rangle| \end{cases}$

   (remember that, by the convention established at the end of section 2, we have $|\langle u' \bullet_k u'' \rangle| = |\langle u' \rangle| \dot{\cup} |\langle u'' \rangle|$).

We want to associate with every n-cell u an indexed set of partial functions $\langle u \Box_r - \rangle_{r \in |\langle u \rangle|}$ so as to have conditions 1-3 met. One might think that these conditions can be used to define the partial function $u \Box_r -$ by recursion on the n-cell u. However, the same composite u might be represented in more than one way as a composition of two other cells. Conditions 1-3 allow us to define, by recursion on the $\mathcal{C}$*-term* t, a partial function $t \Box_r -$ and we still have to show that all terms denoting a given u yield the same function. This can be done by induction on proofs. However, we prefer another route.

We will use the universal property of $\mathbf{A}[I]$ (cf. theorem 1.4) and construe the mapping $u \mapsto \langle u \Box_r - \rangle_{r \in |\langle u \rangle|}$ as a functor into an n-category $\mathbf{W}$.

Definition 3.1. $\mathbf{W}$ is the n-category satisfying the following requirements:

1. $\mathbf{W}_{n-1} = \mathbf{A}$, i.e. the k-cells of $\mathbf{W}$ are those of $\mathbf{A}$ for $k < n$.

2. The n-cells of $\mathbf{W}$ are pairs $U = (u, \langle H_r \rangle_{r \in |\langle u \rangle|})$, with u an n-cell of $\mathbf{X}$ and H_r a function from the set of cells of $\mathbf{X}(dx_r, cx_r)$ to the set of cells of $\mathbf{X}(du, cu)$, where $x_r = \langle u \rangle(r)$. The domain and codomain are $dU = du$, $cU = cu$.

3. For $a \in A_{n-1} = W_{n-1}$, the identity over a in $\mathbf{W}$ will be $(1_a, \langle \, \rangle)$ (where $\langle \, \rangle$ is, of course, the empty indexed set of functions).

4. if U is as above and $V = (v, \langle K_r \rangle_{r \in |\langle v \rangle|})$ is such that $d^{(k)}U = d^{(k)}u = c^{(k)}v = c^{(k)}V$, then we have
$$U \bullet_k V = (u \bullet_k v, \langle L_r \rangle_{r \in |\langle u \bullet_k v \rangle|}),$$
   where
$$L_r(-) = \begin{cases} H_r(-) \bullet_k v & \text{if } r \in |\langle u \rangle| \\ u \bullet_k K_r(-) & \text{if } r \in |\langle v \rangle| \end{cases}$$

A straightforward verification shows that $\mathbf{W}$ is, indeed, an n-category.

For $x \in I$, seen as an n-cell of $\mathbf{A}[I]$ with $|\langle x \rangle| = \{r\}$, we have that $\varphi x =_{def} (x, \langle H_r \rangle)$ is an n-cell of $\mathbf{W}$, where $H_r = id_{\mathbf{X}(dx,\, cx)}$ is the identity function from $\mathbf{X}(dx, cx)$ to itself. Thus we defined a function $\varphi : I \to W_n$ and we have $d\varphi x = dx$, $c\varphi x = cx$. By theorem 1.4, there is a unique ω-functor $G : \mathbf{A}[I] \to \mathbf{W}$ such that $Ga = a$, for a a cell of $\mathbf{A}$ and $Gx = \varphi x$, for $x \in I$.

Claim 3.2. *For every n-cell u of $\mathbf{A}[I]$, the first component of $Gu \in W_n$ is u itself.*

Proof. Let $\Pi : \mathbf{W} \to \mathbf{A}[I]$ be defined as $\Pi a = a$ for a a cell of $\mathbf{A} = \mathbf{W}_{n-1}$ and $\Pi U = u$ for $U = (u, \langle H_r \rangle_{r \in |\langle u \rangle|}) = u$. Then Π is an ω-functor, hence so is the composite $\Pi G : \mathbf{A}[I] \to \mathbf{A}[I]$ and we must have that $\Pi G = \mathbf{1}_{\mathbf{A}[I]}$, the identity ω-functor on $\mathbf{A}[I]$, because, by 1.4 there is a unique functor $\mathbf{A}[I] \to \mathbf{A}[I]$ which is the identity for the cells of $\mathbf{A}$ and for the indets $x \in I$. $\square$

Definition 3.3. For u an n-cell of $\mathbf{X}$, if $Gu = (u, \langle H_r \rangle_{r \in |\langle u \rangle|})$, then we define $u \square_r - = H_r(-)$, for $r \in |\langle u \rangle|$.

It follows immediately that the partial functions $u \square_r -$ satisfy conditions 1-3 stipulated just before definition 3.1. Actually, conditions 1-3 determine these functions uniquely, as summed up in the following statement.

Theorem 3.4. *Given $\mathbf{X}_n = \mathbf{A}[I]$, there exists a unique system of partial functions $\{u \square_r - : u \in X_n,\ r \in |\langle u \rangle|\}$ satisfying conditions 1-3.*

Proof. The *existence* of a system as stipulated has been just proven, so we have only to prove *uniqueness*. This done by *induction on n-cells*. Let us emphasize that, while *definitions by recursion* on n-cells cells require special caution, as we just saw, *proofs by induction* are unproblematic, as the set of n-cells of $\mathbf{X}$, being the same as the set of n-cells of $\mathbf{A}[I]$, is the least that contains the indets and the identity n-cells and is closed under composition. We are now going to see a first instance of such a proof.

Assuming that $\{u *_r - : u \in X_n,\ r \in |\langle u \rangle|\}$ is another system of functions satisfying 1-3, an induction on u shows that $u *_r - = u \square_r -$. We leave the straightforward argument to the reader. Many more instances of proofs by induction on cells will be met soon. $\square$

Remark. All these involved statements are relevant for the case $n > 0$ only. If $n = 0$ then every n-cell is an indet and $u \square_r -$ is always the identity function.

It is well known and easily seen that the whiskering operations are *functorial* in the following sense: if x, v, u are n-cells such that $u = x \bullet_k v$ for some $k < n$, then the function $- \bullet_k v : \mathbf{X}(dx, cx) \to \mathbf{X}(du, cu)$ is an ω-functor (and, of course, a similar statement holds for $v \bullet_k -$). The same is true for *generalized* whiskering.

Theorem 3.5. *If $\mathbf{X}_n = \mathbf{A}[I]$, $u \in X_n$, $r \in |\langle u \rangle|$ and $\langle u \rangle(r) = x$ then the function $u \square_r - : \mathbf{X}(dx, cx) \to \mathbf{X}(du, cu)$ is an ω-functor.*

Proof. By induction on u.

If u is an indet $x \in I$, then $u \square_r -$ is an identity map and there is nothing to prove.

u cannot be an identity, as $|\langle u \rangle| \neq \emptyset$.

If $u = u' \bullet_k u''$ and, say, $r \in |\langle u' \rangle|$, then $u' \square_r -$ is an ω-functor by the induction hypothesis, hence so is the composition $u \square_r - = (u' \square_r -) \bullet_k u''$ of the ω-functors $- \bullet_k u''$ and $u' \square_r -$. $\square$

If u, v are n-cells of $\mathbf{X}$, then so is $u \square_r v$, if defined. Again, an easy proof by induction on u, will show that $\langle u \square_r v\rangle$ is a coproduct of $\langle u\rangle \setminus r$ (i.e. $\langle u\rangle$ restricted to $|\langle u\rangle| - \{r\}$) and $\langle v\rangle$. The coprojections of this coproduct are induced by those of the $\bullet_k$ operations involved, and if we stick to our convention of choosing disjoint index sets for the arguments of these composition operations, we will always have that $|\langle u \square_r v\rangle| = (|\langle u\rangle| - \{r\}) \dot\cup |\langle v\rangle|$, with the *inclusion* maps being the *induced* coprojections. Again, this will greatly simplify notations in the sequel.

Theorem 3.6. *If u is an n-cell then:*

1. *("Commutativity") If r, $q \in |\langle u\rangle|$, $r \neq q$ such that $u \square_r v$, $u \square_q w$ are defined where v, w are also n-cells, then $(u \square_r v) \square_q w = (u \square_q w) \square_r v$.*
2. *("Associativity") If $r \in |\langle u\rangle|$, and $u \square_r v$ is defined, v an n-cell, $q \in |\langle v\rangle|$ and $v \square_q w$ is defined with w a cell of dimension $\geqslant n$, then*
$$(u \square_r v) \square_q w = u \square_r (v \square_q w).$$
3. *(Identity rule) If $r \in |\langle u\rangle|$ and $\langle u\rangle(r) = x$ then $u \square_r x = u$.*

Proof. By induction on u. We sketch the proofs of parts *1,2* and leave the proof of *3* to the reader.

Proof of part *1*: As $|\langle u\rangle|$ is assumed to have at least two distinct elements, u is neither an indet nor an identity. Assume that $u = u' \bullet_k u''$. Then $|\langle u\rangle| = |\langle u'\rangle| \dot\cup |\langle u''\rangle|$. If r, q belong to different summands, e.g. if $r \in |\langle u'\rangle|$, $q \in |\langle u''\rangle|$, then both sides of the stipulated equality are seen to be equal to $(u' \square_r v) \bullet_k (u'' \square_q w)$ (this case doesn't require any induction hypothesis). If both r and q belong to the same summand, e.g. $r, q \in |\langle u'\rangle|$ then the statement follows from the induction hypothesis for u'.

Proof of *2*: If u is an indet, then both sides equal $v \square_q w$. If $u = u' \bullet_k u''$ and, say, $r \in |\langle u'\rangle|$ then the left side equals $((u' \square_r v) \square_q w) \bullet_k u''$, while the right one equals $(u' \square_r (v \square_q w)) \bullet_k u''$ and the statement follows from the induction hypothesis for u'. $\square$

Assume that, not only is $\mathbf{X}_n = \mathbf{B}$ a free extension of $\mathbf{B}_{n-1} = \mathbf{A}$, but also $\mathbf{X}_{n+1}$ is a free extension of $\mathbf{X}_n$. Let's say that $\mathbf{X}_{n+1} = \mathbf{X}_n[J] = \mathbf{B}[J]$, for a set J of $(n+1)$-indets. This situation will be encountered from section 6 on. If so, then we can define generalized whiskering functors for n-cells, as well as for $(n+1)$-cells. The following simple technical lemma, linking these two kinds of operations, will be useful later.

Lemma 3.7. *Assume that $\mathbf{X}$ is an ω-category as just described. If we have $w \in X_n$, $q \in |\langle w\rangle|$, $u \in X_{n+1}$, $r \in |\langle u\rangle|$ and v is any cell of $\mathbf{X}$ of dimension $m \geqslant n+1$ then the following equality holds, provided that the expressions involved are defined:*

$$(w \square_q u) \square_r v = w \square_q (u \square_r v).$$

Proof. By induction on w. If w is an n-indet, then $w\square_q-$ is an identity functor, and there is nothing to prove. w cannot be an identity, as $|\langle w\rangle| \neq \emptyset$. Assume that $w = w' \bullet_k w''$ and , e.g., $q \in |\langle w'\rangle|$. Then

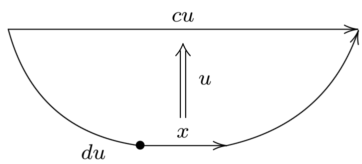
$$(w\square_q u)\square_r v = ((w'\square_q u) \bullet_k w'')\square_r v.$$

By the induction hypothesis, $(w'\square_q u)\square_r v = w'\square_q(u\square_r v)$, and we conclude

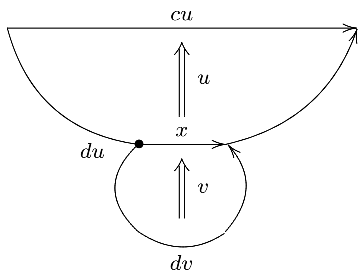
$$= (w'\square_q(u\square_r v)) \bullet_k w'' = (w' \bullet_k w'')\square_q(u\square_r v) = w\square_q(u\square_r v).$$

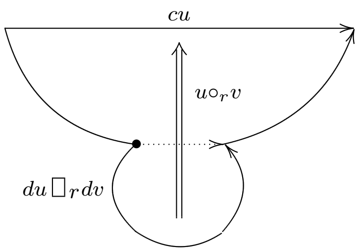
$\square$

We now go *one dimension higher* and define the operations of *placed composition* that involve $(n+1)$-cells of $\mathbf{X}$. Let u be such a cell. Its domain du is an n-cell of $\mathbf{X}$, hence of $\mathbf{A}[I]$. Assume that $r \in |\langle du\rangle|$ and $\langle du\rangle(r) = x \in I$. Schematically, the situation may be represented as in the figure below, where we indicated the r-occurrence of x in du.



Let, in addition, v be another $(n+1)$-cell of $\mathbf{X}$ with codomain $cv = x$. The two cells can be represented as in the figure at left below and it is a natural thought to combine the two cells into a single one, $u \circ_r v$, whose domain will be $du\square_r dv$, the result of replacing the r-occurrence of x in du by dv. The new cell is represented schematically in the figure at right and is called the *placed composition of u and v at r*.





What is the precise definition of placed composition? The cells u and v cannot be composed as they are, because the domain of u doesn't match the codomain of v. This, however, can be corrected with the help of the generalized whiskering functor $du\square_r-$. Indeed, as we have $dv \xrightarrow{v} cv = x = \langle du\rangle(r)$, we get, after applying $du\square_r-$,

$$du\square_r dv \xrightarrow{du\square_r v} du\square_r cv = du\square_r x = du$$

and thus, $du\square_r v$ is an $(n+1)$-cell with codomain du, matching the domain of u. This motivates the following definition:

Definition 3.8. For u, $v \in X_{n+1}$ with $r \in |\langle du\rangle|$ and $\langle du\rangle(r) = x = cv$, we define the *placed composition of* u *and* v *at* r to be the $(n+1)$-cell

$$u \circ_r v = u \bullet_n (du\square_r v)$$

with domain $du\square_r dv$ and codomain cu.

Again, we can generalize this operation further, by allowing v to be any **X**-cell of dimension $\geqslant n+1$ such that $c^{(n)}v = x$. Definition 3.8 makes sense for such a v, with $\bullet_n$ indicating a whiskering, and produces a cell $u \circ_r v$, of dimension equal to that of v, which will be called the placed *whiskering* of u and v at r.

Remark concerning the case $n = 0$. In this situation, du is a 0-cell, i.e. an indet, so that $du\square_r -$ is the identity function, $|\langle u\rangle|$ is a singleton, say $\{r\}$, and the placed composition is defined only when $c^{(0)}v = du$ and we have, therefore, $u\square_r v = u \bullet_0 v$.

The placed whiskering operations in general, and placed compositions in particular, have properties similar to those of the operations of replacement and generalized whiskering.

Theorem 3.9. *If* u *is an* $(n+1)$*-cell then:*

1. *("Commutativity") If* $r, q \in |\langle du\rangle|$, $r \neq q$ *and* v, w *are* $(n+1)$*-cells for which* $u \circ_r v$, $u \circ_q w$ *are defined, then* $(u \circ_r v) \circ_q w = (u \circ_q w) \circ_r v$.
2. *("Associativity") If* $r \in |\langle du\rangle|$, v *is an* $(n+1)$*-cell such that* $u \circ_r v$ *is defined,* $q \in |\langle dv\rangle|$ *and* w *is any* **X***-cell of dimension* $\geqslant n+1$ *with* $v \circ_q w$ *defined, then* $(u \circ_r v) \circ_q w = u \circ_r (v \circ_q w)$.
3. *(Identity rules) If* $\langle du\rangle(r) = x$ *then* $u \circ_r 1_x = u$. *If* $cv = x$ *and* $|\langle x\rangle| = \{r\}$, *then* $1_x \circ_r v = v$.

Proof. Proof of part *1*: We have

$$\begin{aligned}(u \circ_r v) \circ_q w = (u \bullet_n (du\square_r v)) \circ_q w = u \bullet_n (du\square_r v) \bullet_n (d(u \bullet_n (du\square_r v))\square_q w) = \\ = u \bullet_n (du\square_r v) \bullet_n (d(du\square_r v)\square_q w) = u \bullet_n (du\square_r v) \bullet_n ((du\square_r dv)\square_q w)\end{aligned}$$

(remember that $du\square_r -$ is functorial, therefore $d(du\square_r -) = du\square_r d-$)

In the same way, $(u \circ_q w)\square_r v = u \bullet_n (du\square_q w) \bullet_n ((du\square_q dw)\square_r v)$, hence the desired conclusion will follow from the following:

Lemma 3.10. *If u' is an n-cell, $r, q \in |\langle u' \rangle|$, $r \neq q$ and v, w are $(n+1)$-cells satisfying $cv = \langle u' \rangle(r)$, $cw = \langle u' \rangle(q)$ then*

$$(u' \Box_r v) \bullet_n ((u' \Box_r dv) \Box_q w) = (u' \Box_q w) \bullet_n ((u' \Box_q dw) \Box_r v)$$

Proof. By induction on u'. u' can be neither an indet nor an identity, so assume that $u' = u'_1 \bullet_k u'_2$, $k < n$.

Case 1: r, q belong to the same one of $|\langle u'_1 \rangle|$, $|\langle u'_2 \rangle|$, e.g. $r, q \in |\langle u'_1 \rangle|$. Then

$$\begin{aligned}(u' \Box_r v) \bullet_n ((u' \Box_r dv) \Box_q w) &= ((u'_1 \Box_r v) \bullet_k u'_2) \bullet_n (((u'_1 \Box_r dv) \Box_q w) \bullet_k u'_2) = \\ &= ((u'_1 \Box_r v) \bullet_n ((u'_1 \Box_r dv) \Box_q w)) \bullet_k u'_2,\end{aligned}$$

where the second equality is just an instance of the exchange axiom (axiom 3 of definition 1.3). To see this, one should notice that the $\bullet_k$ compositions stand for whiskerings and, therefore, u'_2 is just short for $1_{u'_2}$.

Similarly, $(u' \Box_q w) \bullet_n ((u' \Box_q dw) \Box_r v) = ((u'_1 \Box_q w) \bullet_n ((u'_1 \Box_q dw) \Box_r v)) \bullet_k u'_2$ and the equality follows from the induction assumption for u'_1.

Case 2: $r \in |\langle u'_1 \rangle|$, $q \in |\langle u'_2 \rangle|$. Then,

$$\begin{aligned}(u' \Box_r v) \bullet_n ((u' \Box_r dv) \Box_q w) &= ((u'_1 \bullet_k u'_2) \Box_r v) \bullet_n (((u'_1 \bullet_k u'_2) \Box_r dv) \Box_q w) = \\ = ((u'_1 \Box_r v) \bullet_k u'_2) \bullet_n ((u'_1 \Box_r dv) \bullet_k (u'_2 \Box_q w)) &= ((u'_1 \Box_r v) \bullet_n (u'_1 \Box_r dv)) \bullet_k (u'_2 \bullet_n (u'_2 \Box_q w)) = \\ &= (u'_1 \Box_r v) \bullet_k (u'_2 \Box_q w),\end{aligned}$$

where, again, the equality before the last is an instance of the exchange axiom, while the last equality follows by identity axioms (the first line of axiom 4, definition 1.3), taking into consideration that $u'_1 \Box_r dv$, u'_2 are just short for $1_{u'_1 \Box_r dv}$, $1_{u'_2}$, respectively .

A similar computation shows that $(u' \Box_q w) \bullet_n ((u' \Box_q dw) \Box_r v)$ equals $(u'_1 \Box_r v) \bullet_k (u'_2 \Box_q w)$ as well. No need for any induction hypothesis for this case. $\Box$

The proof of part *1* is now complete.

Proof of part *2*: A computation shows that

$$u' \circ_r (v \circ_q w) = u' \bullet_n (du \Box_r v) \bullet_n (du \Box_r (dv \Box_q w)), \text{ while}$$

$$(u' \Box_r v) \Box_q w = u' \bullet_n (du \Box_r v) \bullet_n ((du \Box_r dv) \Box_q w)$$

and the desired equality follows by part *2* of theorem 3.6.

The proof of *3* is easy (for the first statement, one should only notice that $du \Box_r 1_x = 1_{du}$). $\Box$

Theorem 3.4 and definition 3.8 show that the operations of placed composition are uniquely determined by the ω-categorical composition operations $\bullet_k$. The next statement describes the behavior of a $\bullet_k$ composition operation when one of its arguments is a placed composition. It will allow us to show, in section 6, that under certain conditions a *converse* also holds, namely, the placed compositions determine uniquely the ω-categorical ones.

Proposition 3.11. *If $\mathbf{X}_n = \mathbf{A}[I]$, then the following identities hold, where u, u', u'', v, v', v'' are $(n+1)$-cells of $\mathbf{X}$ such that the left hand side expressions are defined, then:*

1. $u \bullet_k (v' \circ_r v'') = (u \bullet_k v') \circ_r v''$, *for $k \leqslant n$.*

2. $(u' \circ_r u'') \bullet_k v = (u' \bullet_k v) \circ_r u''$, *for $k < n$.*

Proof. Part *1*: As $c(v' \circ_r v'') = cv'$, we see that $u \bullet_k v'$ is defined and, as $|\langle u \bullet_k v\rangle| = |\langle u\rangle| \dot{\cup} |\langle v'\rangle|$, the right hand side expression is defined, whenever the left is.

If $k = n$, then we have $u \bullet_n (v' \circ_r v'') = u \bullet_n (v' \bullet_n (dv' \Box_r v'')) = (u \bullet_n v') \bullet_n (dv' \Box_r v'')$ and the desired identity follows once we notice that $dv' = d(u \bullet_n v')$.

If $k < n$, then $u \bullet_k (v' \circ_r v'') = u \bullet_k (v' \bullet_n (dv' \Box_r v) = (u \bullet_n 1_{du}) \bullet_k (v' \bullet_n (dv' \Box_r v)$. We can now use an instance of the exchange axiom and conclude that $u \bullet_k (v' \circ_r v'') = (u \bullet_k v') \bullet_n (1_{du} \bullet_k (dv' \Box_r v'')) = (u \bullet_k v') \bullet_n (du \bullet_k (dv' \Box_r v'')) = (u \bullet_k v') \bullet_n ((du \bullet_k dv') \Box_r v'')$ (notice that the second $\bullet_k$ in the third expression represents a whiskering) and the desired identity follows if we notice that $du \bullet_k dv' = d(u \bullet_k v')$.

Part *2*: To see that the right hand side is defined if the left is, notice that $d(u' \circ_r u'') = du' \Box_r du'' \parallel du'$, hence $d^{(k)}(u' \circ_r u'') = d^{(k)} u'$. The proof of the identity is similar to that of the case $k < n$ of part *1*. □

Theorems 3.9 and 3.6 point out common properties of the placed composition operations on one hand, and the n-cell replacement ones, on the other. Actually, these two families of operations are particular instances of a general concept that forms the subject of the next section.

4 Multicategories

The notion of multicategory that we are about to present, has been introduced in [6] and extends a notion defined previously, under the same name, by Lambek (cf. [8]). It is an abstract concept that, as we just hinted, displays the common features of the placed composition operations, on one hand, and the n-cell replacement ones, on the other.

A multicategory has a set of *objects* and a set of *arrows*. Each arrow u has a *source* Su and a *target* Tu. Su is an indexed set of objects, a function from a finite set of indices $|Su|$ into the set of objects. The multicategory has also partial *multicomposition* operations, which we denote $\odot_r$, r being any index. If u, v are arrows then $u \odot_r v$ is defined whenever

$r \in |Su|$ and the target of v is "appropriate" (in a sense to be made precise shortly) for the object $Su(r)$ that occurs in the r-position in the source of u. If such is the case, we will say that v is *multicomposable* (or, r-*multicomposable*) *into* u.

One kind of examples of multicategories is based on the operations of placed compositions playing the role of multicompositions. In this context, the objects are the n-indets while the arrows are certain $(n+1)$-cells. The source of an arrow u will be $\langle du \rangle$ and its target will be cu. Thus, the target of v is "appropriate" for the object $Su(r) = \langle du \rangle(r)$ iff it *equals* it.

The situation is a bit different in a multicategory based on the n-cell replacements. The objects are, again, the n-indets and the arrows are the n-cells, the source of u being $\langle u \rangle$. This time, the r-multicomposition of v into u will be defined iff we have the equality of ordered pairs $(dv, cv) = (dx, cx)$ where $x = Su(r)$. We will call (dx, cx) the *type* of the object x and let the target of v be $Tv = (dv, cv)$. Hence, in this case, the target of v is "appropriate" for the object $Su(r) = \langle u \rangle(r)$ iff it equals its type.

In preparation for a formal definition, let us specify a few conventions and notations. As we mentioned already, given a set O, we let $O^{\#}$ be the category whose objects are finite indexed sets of elements of O, i.e. functions from finite subsets of a given infinite set of indices $\mathcal{N}$, and arrows defined in the obvious way (see also [6]). Recall that, given an object f of $O^{\#}$, $f : |f| \to O$, we allow ourselves to *reparametrize* f replacing, at will, the domain $|f|$ by any subset of $\mathcal{N}$ of equal cardinality. To be more precise, if $s \subset \mathcal{N}$ and $\sigma : s \to |f|$ is a bijection, then we regard $f' = f\sigma : s \to O$ as being *the same* as f. Of course, when we do this, we also identify the $O^{\#}$-arrows from and to f with the corresponding maps (e.g. $\gamma : g \to f$ should be identified with $\gamma' = \sigma^{-1}\gamma : g \to f'$). Finally, if $x \in O$, we let $\langle x \rangle$ be the object of $O^{\#}$ whose domain is a singleton and whose range is $\{x\}$.

Definition 4.1. A multicategory $\mathbb{C}$ consists of;

1. An *object system*, which is a triple $\Omega = \Omega(\mathbb{C}) = (O, \dot{O}, (-)^{\cdot})$ where O is a set of *objects*, $\dot{O}$ a set of *object types* and $(-)^{\cdot} : O \to \dot{O}$ a map that associates with every $x \in O$ its *type* $\dot{x} \in \dot{O}$. We say that $\mathbb{C}$ is *based* on Ω. If $O = \dot{O}$ and $(-)^{\cdot}$ is the identity, then Ω is called a *simple* object system and is denoted $\Omega = (O)$.

2. A set $A = A(\mathbb{C})$ of *arrows* together with *source* and *target* functions $S : A \to Ob(O^{\#})$ and $T : A \to \dot{O}$.

3. Partial *multicomposition operations* that associate with each pair of arrows u, $v \in A$ and each $r \in |Su|$ such that $Tv = (Su(r))^{\cdot}$, an arrow $u \odot_r v$ such that $S(u \odot_r v)$ is a coproduct of $Su \setminus r$ and Sv with *specified coprojections* and $T(u \odot_r v) = Tu$ (following our practice, we will always assume that Su, Sv have been so reparametrized as to have $|S(u \odot_r v)| = (|Su| \setminus \{r\}) \dot{\cup} |Sv|$ with the inclusion maps being the specified coprojections).

$u \odot_r v$ will be referred to as the *multicomposition of* v *into* u *at place* r.

4. An *identity* arrow 1_x, for each $x \in O$, such that $S(1_x) = \langle x \rangle$, $T(1_x) = \dot{x}$.

These components are subject to the following conditions:

(a) (Identity rules) If $Tu = \dot{x}$ then $1_x \odot_r u = u$, where, of course, $|S1_x| = \{r\}$. If $Su(r) = x$, then $u \odot_r 1_x = u$.

(b) ("Commutativity") If $r, q \in |Su|$, $r \neq q$, $Tv = (Su(r))^{\cdot}$ and $Tw = (Su(q))^{\cdot}$ then $(u \odot_r v) \odot_q w = (u \odot_q w) \odot_r v$.

(c) ("Associativity") If $r \in |Su|$, $Tv = (Su(r))^{\cdot}$, $q \in |Sv|$ and $Tw = (Sv(q))^{\cdot}$ then $(u \odot_r v) \odot_q w = u \odot_r (v \odot_q w)$.

We now reexamine the examples that motivated this definition.
As it turns out, there are *two* important examples based on placed composition.

The *first (and* main) *example*: If $\mathbf{B}$ is an n-category generated by a set I of indets, as we considered in section 3, and $\mathbf{X}$ is an $(n+1)$-category extending $\mathbf{B}$, i.e. $\mathbf{X}_n = \mathbf{B} = \mathbf{A}[I]$, then we define the multicategory $\mathbb{C} = \mathbb{C}_{\mathbf{X}}$ of *placed-composition*, whose object system is simple, with set of objects I. The set of arrows will be $A = \{u : u \in X_{n+1},\ cu \in I\}$, i.e. the set of those $(n+1)$-cells of $\mathbf{X}$ that were called *many-to-one* in the introduction. For $u \in A$, $Su = \langle du \rangle$ and $Tu = cu$. The multicomposition operation at place r will be, of course, $\circ_r$. Finally, for $x \in I$, the identity arrow will be the identity *cell* 1_x.

Remark concerning the terminology. An arbitrary $(n+1)$-cell $u \in X_{n+1}$ can be seen as *linking* between the finite indexed sets of n-indets $\langle du \rangle$ and $\langle cu \rangle$. In general, both these indexed sets have (finitely) *many* components. If it so happens that $cu \in I$, i.e. $\langle cu \rangle$ contains just *one* component, then it is only natural to say that u is a *many-to-one* cell.

A moment of thought will show that we do not *have* to take the arrows to be just the many-to-one $(n+1)$-cells. By deciding that *all* $(n+1)$-cells of $\mathbf{X}$ are arrows we get another example of multicategory based on placed composition.

The *second example* of multicategory: We enlarge the placed-composition multicategory $\mathbb{C}_{\mathbf{X}}$ into an *extended* placed-composition multicategory $\mathbb{C}^+ = \mathbb{C}^+_{\mathbf{X}}$ whose set of objects O is still the set of n-indets I, but the set of arrows A equals X_{n+1}, the set of *all* $(n+1)$-cells of $\mathbf{X}$. To accommodate this situation, the object system of $\mathbb{C}^+_{\mathbf{X}}$ is *not* simple anymore. The set of object types is $\dot{O} = B_n$, the set of all n-cells of $\mathbf{B} = \mathbf{A}[I]$ and $(-)^{\cdot}$ is the inclusion map. The source and the target of u are $Su = \langle du \rangle$ and $Tu = cu$. The multicomposition operations and the identity arrows are defined as in the case of $\mathbb{C}_{\mathbf{X}}$.

The definition of $\mathbb{C}^+_{\mathbf{X}}$ is made possible by the fact that, in the abstract concept of multicategory, the map $(-)^{\cdot} : O \to \dot{O}$ is not necessarily onto $\dot{O}$. Hence, we might have arrows whose target is not the type of any object; such arrows cannot be multicomposed into any other arrow (but, of course, other arrows can be multicomposed *into* it). This possibility was not ruled out in [6], but it seems that it had no relevance in that paper. It is, however, useful in the present work as the notion of extended placed-composition multicategory will turn out to be valuable in section 6 below.

Remark concerning the case $n = 0$. In this case, $\mathbf{X}$ is a 1-category (i.e., just an ordinary category) and all its 1-cells are many-to-one. Furthermore, as we remarked after definition 3.8, placed composition is the same as categorical composition and so, we have in this case that $\mathbb{C}_{\mathbf{X}} = \mathbb{C}^+_{\mathbf{X}} = \mathbf{X}$. Hence, an ordinary category, can be seen at the same time as a multicategory of a very particular kind. Actually, the ordinary categories are precisely those multicategories whose object system is simple and the source of any arrow is a singleton.

We now turn to the replacement context.

Third example: Given $\mathbf{B} = \mathbf{A}[I]$ of dimension $n > 0$, we construct a multicategory $\mathbb{R} = \mathbb{R}_{\mathbf{B}}$ of *cell replacement* as follows. The set of objects of $\mathbb{R}$ will be $O = I$, the set of n-indets. The set of types $\dot{O} = \{(du, cu) : u \text{ an } n\text{-cell of } \mathbf{A}\}$ and for $x \in O = I$, $\dot{x} = (dx, cx)$. The set of arrows A will be B_n, the set of n-cells of $\mathbf{B} = \mathbf{A}[I]$ and for $u \in A$, $Su = \langle u \rangle$, $Tu = (du, cu)$. The placed multicomposition operation at r will be $\boxdot_r$ and for $x \in O = I$, the identity arrow 1_x will be x itself.

We now define the obvious notions of morphisms of object systems and of multicategories.

Definition 4.2.

1. A *morphism* $\gamma : \Omega \to \Lambda$ between object systems $\Omega = (O, \dot{O}, (-)^{\cdot})$ and $\Lambda = (L, \dot{L}, (-)^{\cdot})$ is a pair of functions $\gamma = (\gamma_o, \gamma_t)$, where $\gamma_o : O \to L$, $\gamma_t : \dot{O} \to \dot{L}$ and we have, for $x \in O$, $(\gamma_o x)^{\cdot} = \gamma_t \dot{x}$. Thus, if Ω is simple, then $\gamma_o = \gamma_t$ and, if such is the case, we denote $\gamma = \gamma_o$.
2. A *morphism* $\chi : \mathbb{C} \to \mathbb{D}$, where $\mathbb{C}$, $\mathbb{D}$ are multicategories, is a pair $\chi = (\chi_\Omega, \chi_a)$ such that:
   - **i.** $\chi_\Omega : \Omega(\mathbb{C}) \to \Omega(\mathbb{D})$ is a morphism of object systems.
   - **ii.** $\chi_a : A(\mathbb{C}) \to A(\mathbb{D})$ and for each $u \in A(\mathbb{C})$, $\chi_a Tu = T\chi_a u$ and there is a bijection $\theta_u : |Su| \to |S\chi_a u|$ such that $Su = (S\chi_a u)\theta_u$ (and we will usually assume that an appropriate reparametrization has been made, so that θ_u is an identity map).
   - **iii.** If $u, v \in A(\mathbb{C})$ and $u \odot_r v$ is defined, then $\chi_a(u \odot_r v) = (\chi_a u) \odot_r (\chi_a v)$
   - **iv.** $\chi_a 1_x = 1_{\chi_o x}$, for $x \in O$.

Remark. Stipulation **iii** has been made under the assumption that the θ bijections of **ii** are identity maps. Otherwise, we have to say that $\chi(u \odot_r v) = (\chi u) \odot_{r'} (\chi v)$, where $r' = \theta_u r$ and must add obvious requirements concerning the links between θ_u, θ_v, $\theta_{u \odot_r v}$

and the coprojections related to the sources $S(u \odot_r v)$, $S((\chi u) \odot_r (\chi v))$. For example, if the coprojections are inclusion maps, as we usually assume, then we must just require that $\theta_{u \odot_r v} = \theta_u \dot{\cup} \theta_v$.

5 Free multicategories

We follow a path analogous to the one taken in section 1. We will design a language that allows to specify arrows built from given *indeterminates* by means of multicompositions in a multicategory. Given an object system $\Omega = (O, \dot{O}, (-)^{\cdot})$, let J be a set of *arrow*-indeterminates, together with source and target functions $S : J \to Ob(O^{\#})$, $T : J \to \dot{O}$. The elements of J will be also called a-*indets* or, simply, *indets*, and will denote arbitrary arrows in a multicategory based on Ω. We will define an equational language $\mathcal{M} = \mathcal{M}(\Omega, J, S, T)$. The symbols of $\mathcal{M}$ will be the a-indets, the *multicomposition* symbols $\odot_r$, for $r \in \mathcal{N}$, the *identity* symbols 1_x for $x \in O$, as well as left and right parentheses, as auxiliary symbols.

Definition 5.1. The set $\mathcal{T}(\mathcal{M})$ of $\mathcal{M}$-terms and the source and target functions $S : \mathcal{T}(\mathcal{M}) \to Ob(O^{\#})$, $T : \mathcal{T}(\mathcal{M}) \to \dot{O}$ are defined as follows:

1. Each indet $f \in J$ is an $\mathcal{M}$-term with Sf, Tf as specified by the given source and target functions.
2. For each $x \in I$, 1_x is an $\mathcal{M}$-term with $S1_x = \langle x \rangle$ and $T1_x = \dot{x}$.
3. If t, s are $\mathcal{M}$-terms and $r \in |St|$, $Ts = (St(r))^{\cdot}$, then $(t) \odot_r (s)$ is an $\mathcal{M}$-term (usually written just as $t \odot_r s$), with $T(t \odot_r s) = Tt$ and $S(t \odot_r s)$ being a coproduct, with *specified coprojections*, of $St \setminus r$ and Ss. We will follow our simplifying practice and assume that St, Ss have been so reparametrized as to have $|S(t \odot_r s)| = (|St| - \{r\}) \dot{\cup} |Ss|$, with the inclusion maps being the specified coprojections.
4. There are no $\mathcal{M}$-terms besides those mentioned in 1-3.

The semantics of the $\mathcal{M}$-terms is analogous to that of the $\mathcal{C}$-terms of section 1. For $\mathbb{C}$ a multicategory based on Ω and an assignment $\varphi : J \to A(\mathbb{C})$ which is *correct*, in the sense that $S\varphi f = Sf$, $T\varphi f = Tf$, one defines the value $val(t) = val_{\varphi}(t) \in A(\mathbb{C})$ of any term $t \in \mathcal{T}(\mathcal{M})$, under the assignment φ. More generally, if $\gamma : \Omega \to \Omega(\mathbb{C})$ is a morphism of object structures for any multicategory $\mathbb{C}$ and $\varphi : J \to A(\mathbb{C})$ an assignment that is *consistent* with γ (in the sense that $S\varphi f = \gamma Sf$, $T\varphi f = \gamma Tf$) , we can evaluate t *under* γ, φ and get $val_{\gamma,\varphi}(t) \in A(\mathbb{C})$. The definition of the evaluation function $val_{\gamma,\varphi}$ is most natural and similar to definition 1.2, so that we do not present it formally.

Next, we define the *axioms* and *rules* of the equational logic $\mathcal{M}$ as we did in definition 1.3:

Definition 5.2. The deductive system $\mathcal{M}$ has the following axioms and rules, where, t, s, w are arbitrary $\mathcal{M}$-terms and all multicompositions are supposed to be well defined (according to definition 5.1).

Axioms.

1. $t = t$ (equality axioms).

2. $1_x \odot_r t = t$ and $t \odot_r 1_x = t$ (identity axioms).

3. $(t \odot_r s) \odot_q w = (t \odot_q w) \odot_r s$, if $r \neq q$ (commutativity axioms).

4. $(t \odot_r s) \odot_q w = t \odot_r (s \odot_q w)$ (associativity axioms).

Rules.

1. $\dfrac{t = s}{s = t} \qquad \dfrac{t = s \quad s = w}{t = w}$ (equality rules).

2. $\dfrac{t = s}{t \odot_r w = s \odot_r w} \qquad \dfrac{t = s}{w \odot_r t = w \odot_r s}$ (congruence rules).

Again, we will write '$\vdash t = s$' or, sometimes, '$\vdash_{\mathcal{M}} t = s$', to indicate that $t = s$ is provable in system $\mathcal{M}$.

As in section 1, we are now able to prove the existence of *free* multicategories.

Theorem 5.3. *Given Ω, J, S, T, there exists a multicategory $\Omega[J]$ based on Ω, with $J \subset A(\Omega[J])$, such that for $f \in J$, Sf, Tf are the source and target of f in $\Omega[J]$ and the following* universal property *holds:*

Whenever $\mathbb{C}$ is a multicategory based on Ω and $\varphi : J \to A(\mathbb{C})$ a function such that $S\varphi f = Sf$, $T\varphi f = Tf$ for all $f \in J$, there is a unique *morphism $\chi : \Omega[J] \to \mathbb{C}$ which is the identity on objects and object-types and satisfies $\chi f = \varphi f$ for $f \in J$.*

Moreover, $\Omega[J]$ has also the following strong universal property*: whenever $\mathbb{C}$ is any multicategory, $\gamma : \Omega \to \Omega(\mathbb{C})$ a morphism of object systems and $\varphi : J \to A(\mathbb{C})$ a function such that $S\varphi f = \gamma Sf$, $T\varphi f = \gamma Tf$, there is a* unique *morphism $\chi : \Omega[J] \to \mathbb{C}$ extending both, γ and φ in the sense that $\chi_\Omega = \gamma$ and $\chi_a f = \varphi f$ for $f \in J$.*

Remark. Here we used abbreviated notations, that will be adopted in the sequel. We wrote just χ for χ_a and , likewise, γ for γ_o or γ_t, as the subscripts are understood for the context. Also, when applying a function to a finite sequence (like in γSf), we understand that the function is applied to each component of the sequence.

First proof (Sketch). As in the proof of 1.4, we define, for $\mathcal{M}$-terms t, s, $t \approx s$ iff $\vdash t = s$, and take the arrows of $\Omega[J]$ to be equivalence classes $t/\approx$, of $\mathcal{M}$-terms, identifying $f \in J$ with $f/\approx$. The details are similar to those of the proof of 1.4. In particular, $\chi(t/\approx) = val_{\gamma,\varphi}(t)$. $\square$

The multicategory $\Omega[J]$ will be called *free* or, more specifically, *freely generated by J over Ω*. This terminology is justified, as both universal properties show that $\Omega[J]$ is a free object with respect to suitable functors U, in the sense described in the introduction.

An important example. Let $\Omega_0 = (\{0, 1\})$ be the simple object system having $\{0, 1\}$ as set of objects and object types. Given *any* set J, make J it into a set of a-indets over Ω_0 by letting $Sx = \langle 0\rangle$ and $Tx = 1$, for each $x \in J$, and consider the multicategory $\Omega_0[J]$. A moment of thought will show that there are no non trivial arrow compositions in this multicategory and hence its set of arrows will contain, besides the two identity arrows $1_0, 1_1$, only the elements of J. We can, therefore, identify the set J with the free multicategory $\Omega_0[J]$. Hence, *any barren set can be viewed as a free multicategory.*

The notion of a-*indet occurrence in an arrow* $u \in A(\Omega[J]$, can be developed precisely as we did in section 2 for the similar notion of indet occurrence in an n-cell of an n-category which is a free extension of its $(n-1)$th truncation. Thus, each u as above has a finite indexed set $\langle u\rangle : |\langle u\rangle| \to J$ of a-indet occurrences and $\langle u \odot_r v\rangle$ is a coproduct of $\langle u\rangle$ and $\langle v\rangle$ with specified *appropriate* coprojections and we will always assume that the index sets $|\langle u\rangle|$, $|\langle v\rangle|$ were so chosen as to have $|\langle u \odot_r v\rangle| = |\langle u\rangle| \dot\cup |\langle v\rangle|$, with the inclusion maps being the appropriate coprojections.

As we mentioned in the introduction, there is, *however*, a basic difference between free extensions, on one hand, and free multicategories on the other. The latter is *simpler*, in the sense that the free multicategory $\Omega[J]$ can also be described as a true *term* model, whose arrows are certain terms (and *not* equivalence classes of terms) in ‘Polish’ notation. This is the way free multicategories are constructed in [6] and we reproduce the description here.

Second proof of 5.3 (Sketch). The arrows of $\Omega[J]$ will be certain strings of elements of $O\dot\cup J$. *For the following construction only*, it will be useful to depart from the convention adopted elsewhere in this paper and to assume, first, that the index set $\mathcal{N}$ is the set of natural numbers and, second, that for an arrow u, the finite set $|Su|$ will always be of the form $[k] = \{0, .., k-1\}$, for some natural number k (thus, the objects of $O^{\#}$ will be strings of symbols (i.e. elements) from O). We also assume that each $f \in J$ has a uniquely specified source, with no reparametrizations allowed. By the way, theses are the conventions adopted throughout [6]. As a result, the specified coprojections associated with multicompositions will no longer be assumed to be inclusion maps.

Definition of $\mathcal{A} = A(\Omega[J])$ and of the target function T:

1. If $x \in O$ then $x \in \mathcal{A}$ and $Tx = \dot{x}$.

2. If $f \in J$, $|Sf| = [k]$, $u_r \in \mathcal{A}$ and $Tu_r = (Sf(r))^{\cdot}$ for $r < k$, then $u = fu_0u_1..u_{k-1} \in \mathcal{A}$ and $Tu = Tf$ (here, u is the concatenation of the one symbol string f and the strings $u_0, u_1, .., u_{k-1}$).

3. There are no arrows in $\mathcal{A}$ besides those mentioned in 1-2.

The elements of $\mathcal{A}$ will sometimes be called *reduced $\mathcal{M}$-terms* or, simply, *reduced terms.*

Definition of the source function, multicomposition and identity arrows:
For $u \in \mathcal{A}$, Su will be the *substring* of u consisting of the O-symbols only.
If $u, v \in \mathcal{A}$, $Su(r) = x \in O$ and $Tv = \dot{x}$, then the rth O-symbol occurrence in the string u is an occurrence of x and $u \odot_r v$ will be the string obtained from u by substituting the said occurrence of x by an occurrence of v. Thus, if $u = u'xu''$ with x indicating the said O-symbol occurrence, then $u \odot_r v = u'vu''$ (this explicit way of writing, should be useful when checking that the multicategory laws are fulfilled for this definition). The specified coprojections associated with this multicomposition are obvious.
Finally, for $x \in O$, 1_x will be x itself.

We leave the reader the tedious but routine task of checking that we did, indeed, construct a multicategory.
In order to have $J \subset \mathcal{A}$, we have to identify $f \in J$ with $fx_0x_1..x_{k-1}$, where $x_r = Sf(r)$ for $r < k$.
Finally, the universal property of $\Omega[J]$ is also routinely checked, using the fact that $fu_0u_1..u_{k-1} = (..((f \odot_{k-1} u_{k-1}) \odot_{k-2} u_{k-2})..) \odot_0 u_0$. $\Box$

As an immediate corollary of this second proof of 5.3, we conclude a simple but important statement. We say that a multicategory $\mathbb{C}$ is a *submulticategory* of $\mathbb{C}'$, $\mathbb{C} \subset \mathbb{C}'$, iff $O(\mathbb{C}) \subset O(\mathbb{C}')$, $\dot{O}(\mathbb{C}) \subset \dot{O}(\mathbb{C}')$, $A(\mathbb{C}) \subset A(\mathbb{C}')$ and the inclusion maps of the components of $\mathbb{C}$ into those of $\mathbb{C}'$ form a multicategory morphism $\chi : \mathbb{C} \to \mathbb{C}'$. We also say, in such a situation, that $\Omega(\mathbb{C})$ is an object *subsystem* of $\Omega(\mathbb{C}')$, $\Omega(\mathbb{C}) \subset \Omega(\mathbb{C}')$.

Proposition 5.4. *If $\Omega \subset \Omega'$ and J, J' are sets of a-indets over Ω, Ω' such that $J \subset J'$ and the source and target functions on J are the restrictions of those on J', then $\Omega[J] \subset \Omega[J']$.*

Strictly speaking, the $\mathcal{A}$-terms are *not* $\mathcal{M}$-terms, but can be easily *translated* into terms of the latter kind. Indeed, the last remark of the second proof of 5.3 implies that each $u \in \mathcal{A}$ is the value $val_\varphi(u^\star)$ of a recursively defined $\mathcal{M}$-term $u^\star$, where $\varphi : J \to \mathcal{A}$ is the inclusion map of J into $\mathcal{A}$ (actually, the map $u \mapsto u^\star$ is primitive recursive).
The $\mathcal{M}$-terms $u^\star$ are of a special form. Call an $\mathcal{M}$-term t *normal*, if t is an identity term or else, is of the form

$$t = (..((f \odot_{k-1} t_{k-1}) \odot_{k-2} t_{k-2})..) \odot_0 t_0$$

with $f \in J$, $|Sf| = [k]$ and t_0, .., t_{k-2}, t_{k-1} normal terms (we still cling to the convention of the second proof of 5.3, according to which the index sets are initial segments of the natural numbers, and each a-indet has a uniquely specified source). Obviously, $u^\star$ is a normal $\mathcal{M}$-term for all $u \in \mathcal{A}$. Conversely, every normal $\mathcal{M}$-term can be seen to be $u^\star$ for a *unique*

$u \in \mathcal{A}$. Thus, the free multicategory $\Omega[J]$ can be described as a *term* model whose arrows are the normal $\mathcal{M}$-terms.

It follows that every $\mathcal{M}$-term t is $\mathcal{M}$-provably equivalent to a unique normal term $\hat{t}$ (namely, the only normal term satisfying $\hat{t} \in t/\approx$). It is not hard to establish this fact directly and to show that the function $t \mapsto \hat{t}$ is primitive recursive. Incidentally, this implies that we have a primitive recursive algorithm for deciding whether $t = s$ is $\mathcal{M}$-provable or not, for given t, s. This fact is usually described as saying that *the word problem for $\mathcal{M}$* is decidable.

These circumstances allow a simpler treatment of the notion of a-indet occurrence, as we can define $\langle u \rangle$ *canonically*, as the sequence of a-indets arranged in the order in which they occur in the unique *normal* $\mathcal{M}$-term that denotes u. Still, we prefer to think of $|\langle u \rangle|$ as a finite indexed set with domain $|\langle u \rangle| \subset \mathcal{N}$, which can be reparametrized to our convenience.

We now return to the analogy that exists, nevertheless, between free extensions of $(n-1)$-categories on one hand, and free multicategories on the other. Given an arrow $u \in A(\Omega[J])$ and $r \in |\langle u \rangle|$, with $\langle u \rangle(r) = f \in J$, if v is another arrow such that $Sv = Sf$, $Tv = Tf$, we can *replace* the r-occurrence of f in u by an occurrence of v and get an arrow $u \boxdot_r v$. The precise definition is worked our similarly to that of cell replacement, as done in section 3.

Theorem 5.5. *There is a unique system $\{u \boxdot_r - \, : \, u \in A(\Omega[J]), r \in |\langle u \rangle|\}$ of partial functions , satisfying the following conditions:*

1. *If $\langle u \rangle(r) = f \in J$ then $u \boxdot_r v$ is defined iff $v \parallel f$, meaning that $Sv = Sf$, $Tv = Tf$. If this is the case, then $u \boxdot_r v \in A(\Omega[J])$ and $S(u \boxdot_r v) = Su$, $T(u \boxdot_r v) = Tu$.*

2. *If $u = f \in J$ and $|\langle u \rangle| = \{r\}$, then $u \boxdot_r v = v$.*

3. *If $u = u' \odot_j u''$ then*

$$u \boxdot_r v = \begin{cases} (u' \boxdot_r v) \odot_j u'' & \text{if } r \in |\langle u' \rangle| \\ u' \odot_j (u'' \boxdot_r v) & \text{if } r \in |\langle u'' \rangle| \end{cases}$$

Proof. (Sketch) The uniqueness is easily seen by induction on u.

Let us use the following notations: $A = A(\Omega[J])$ and $A(Su, Tu) = \{v \in A : v \parallel u\}$, for $u \in A$. We construe the function $u \mapsto \langle u \boxdot_r - \rangle_{r \,\in\, |\langle u \rangle|}$ as a morphism into a multicategory $\mathbb{W}$, whose definition is based on the idea that was used also in definition 3.1:

1. The object system is $\Omega(\mathbb{W}) = \Omega$.

2. The arrows are pairs $U = (u, \langle H_r \rangle_{r \,\in\, |\langle u \rangle|})$, where $H_r : A(Sf_r, Tf_r) \to A(Su, Tu)$, $f_r = \langle u \rangle(r)$. Also, $SU = Su$, $TU = Tu$.

3. If $x \in O$ then the identity arrow over x in $\mathbb{W}$ is $(1_x, \langle\rangle)$.

4. If U is as above and $V = (v, \langle K_r \rangle_{r \,\in\, |\langle v \rangle|})$, $j \in |SU| = |Su|$ and $TV = Tv = (Su(j))^{\cdot} = (SU(j))^{\cdot}$ then $U \odot_j V = (u \odot_j v, \langle L_r \rangle_{r \,\in\, |\langle u \,\odot_j\, v \rangle|})$ where

$$L_r(-) = \begin{cases} H_r(-) \odot_j v & \text{if } r \in |\langle u \rangle| \\ u \odot_j K_r(-) & \text{if } r \in |\langle v \rangle| \end{cases}$$

It is easy to verify that $\mathbb{W}$ is, indeed, a multicategory. We can define $\varphi : J \to A(\mathbb{W})$ by letting $\varphi f = (f, \langle H_r \rangle)$, where $|\langle f \rangle| = \{r\}$ and $H_r = id_{A(Sf,\, Tf)}$, the identity map of $A(Sf, Tf)$ onto itself. By the universal property of $\Omega[J]$, there is a unique morphism $\chi : \Omega[J] \to \mathbb{W}$ which is the identity on the object system and extends φ. As in 3.2, we see that for $u \in A$, we have $\chi_a u = (u, \langle H_r \rangle_{r \,\in\, |\langle u \rangle|})$ and we define $u \boxdot_r - = H_r(-)$. $\square$

Given Ω and J as above, one can define a multicategory $\mathbb{D} = \mathbb{D}_{\Omega,\, J}$ of *arrow replacement* as follows:

$\Omega(\mathbb{D}) = (O_{\mathbb{D}}, \dot{O}_{\mathbb{D}}, (-)^{\cdot}_{\mathbb{D}})$, where $O_{\mathbb{D}} = J$, $\dot{O}_{\mathbb{D}} = \{(Su, Tu) : u \in A\}$ and $\dot{f} = (Sf, Tf)$.

The arrows of $\mathbb{D}$ are those of $\Omega[J]$, while the source and target functions are defined by $S_{\mathbb{D}} u = \langle u \rangle$, $T_{\mathbb{D}} u = (Su, Tu)$. The multicomposition operation at $r \in |\langle u \rangle|$ is $u \boxdot_r -$ and the identity arrow over $f \in J$ is f itself.

The proof that $\mathbb{D}$ is a multicategory is similar to that of theorem 3.6.

A morphism $\chi : \Omega[J] \to \Omega'[J']$ between *free* multicategories is said to be *indet preserving* if $\chi f \in J'$ whenever $f \in J$. If χ is such a morphism then it easy to see that, for every $u \in A(\Omega[J])$ there is a bijection $\theta : |\langle u \rangle| \to |\langle \chi u \rangle|$ such that $\chi(\langle u \rangle(r)) = \langle \chi u \rangle(\theta r)$. We will assume that an appropriate reparametrization was made such that $|\langle u \rangle| = |\langle \chi u \rangle|$ and θ is the identity. If so, then we have the following useful statement:

Proposition 5.6. *If a morphism $\chi : \Omega[J] \to \Omega'[J']$ preserves indets, then it preserves also arrow replacement. This means that for $u, v \in A(\Omega[J]$, if $u \boxdot_r v$ is defined the so is $(\chi u) \boxdot_r (\chi v)$ and $\chi(u \boxdot_r v) = (\chi u) \boxdot_r (\chi v)$.*

Proof. A straightforward induction on u. $\square$

We now return to the comparison between the languages of composition and multicomposition. As we saw, $\mathcal{M}$-terms have normal forms and two terms are $\mathcal{M}$-provably equal iff they have the same normal form. Is a similar result true for $\mathcal{C}$-terms? It does not seem to be so, especially in view of [12]. However, in the restricted *many-to-one* situation, the $\mathcal{C}$ and $\mathcal{M}$ equational logics can be linked to each other in a beneficial way that displays useful similarities. This is the subject of the next section.

6 Comparing $\mathcal{M}$ and $\mathcal{C}$ in the many-to-one case

Consider, again, an n-category $\mathbf{B}$ generated by a set I of n-indets. In this section we make the following

Assumption. J is a set of many-to-one indets over $\mathbf{B} = \mathbf{A}[I]$. In other words, J is a set together with domain and codomain functions $d, c : J \to B_n$ such that $cf \in I$ for all $f \in J$ (and, of course, $df \parallel cf$).

Thus, the indets in J denote arbitrary many-to-one cells in ω-categories extending $\mathbf{B}$. Once we have such a J, we can construct three distinct structures:

First, there is the free $(n+1)$-category $\mathbf{X} = \mathbf{B}[J]$, which is the n-category $\mathbf{B}$ augmented by the set X_{n+1} of the $(n+1)$-cells generated from J.

Second, we have the multicategory $\mathbb{C}_{\mathbf{X}}$ based on the the simple object system Ω with set of objects $O = \dot{O} = I$. The arrows of $\mathbb{C}_{\mathbf{X}}$ are, as we recall, the *many-to-one* $(n+1)$-cells of of $\mathbf{X}$ and the source and target functions are $Su = \langle du \rangle$, $Tu = cu$. In particular, *all* indets $f \in J$ are arrows of $\mathbb{C}_{\mathbf{X}}$.

Finally, we construct the free multicategory $\Omega[J]$ generated by J over the *same* object system Ω on which $\mathbb{C}_{\mathbf{X}}$ is based. The arrows of $\Omega[J]$ can be construed either as equivalence classes $t/\approx$ of $\mathcal{M}$-terms or, else, as reduced $\mathcal{M}$-terms $u \in \mathcal{A}$.

By the universal property of $\Omega[J]$, there is a unique morphism $\chi : \Omega[J] \to \mathbb{C}_{\mathbf{X}}$ which is the identity on both, the set of objects (and object-types) O and the set of indets J. This map deserves a closer look. As remarked at the end of the proof of 5.3, for any $\mathcal{M}$-term t, $\chi(t/\approx) = val_{i_J}(t)$, where i_J is the inclusion map of J into the set of arrows of $\mathbb{C}_{\mathbf{X}}$, which is nothing but the set of many-to-one $(n+1)$-cells of $\mathbf{X}$. Thus, χ maps every arrow of $\Omega[J]$, which is described by an $\mathcal{M}$-term, to a many-to-one $(n+1)$-cell of $\mathbf{X}$, which is described by a $\mathcal{C}$-term. Actually, by carefully following the proofs of 1.4 and 5.3, one can exhibit a primitive recursive function that sends every term $t \in \mathcal{T}(\mathcal{M})$ to a term $\tilde{t} \in \mathcal{T}(\mathcal{C})$ such that $\chi(t/\approx) = \tilde{t}/\approx$. The function $t \mapsto \tilde{t}$ is, therefore, a *translation* of $\mathcal{M}$-terms into $\mathcal{C}$-terms.

The considerations above point to the fact that the map χ is a very important one. It deserves a special notation and name.

Notation. If $\chi : \Omega[J] \to \mathbb{C}_{\mathbf{X}}$ is the unique morphism of multicategories that is the identity on $O = I$ and on J, then we denote $\chi = [\![-]\!]$. This morphism will be referred to as *the canonical morphism of* $\Omega[J]$ *into* $\mathbb{C}_{\mathbf{X}}$.

Thus, we have $\chi u = \chi_a u = [\![u]\!]$ for $u \in A(\Omega[J])$ and $[\![x]\!] = x$, $[\![f]\!] = f$ for $x \in I$, $f \in J$.

As we remarked in section 4, if $n = 0$ then the category $\mathbf{X}$ is the same as the multicategory $\mathbb{C}_{\mathbf{X}}$ and, as in the present case $\mathbf{X}$ is a *free* category, it is also identical with the free multicategory $\Omega[J]$. Moreover, the canonical morphism $[\![-]\!]$ is the identity map.

In the case $n > 0$, however, the situation is much more complex and interesting. Not every $(n+1)$-cell of $\mathbf{X}$ is of the form $[\![u]\!]$ for some arrow u of $\Omega[J]$, simply because the latter is always a many-to-one cell. But are all many-to-one $(n+1)$-cells of $\mathbf{X}$ of the form $[\![u]\!]$? Furthermore, is the $[\![-]\!]$ map one-to-one? In other words, is $[\![u]\!] \neq [\![u']\!]$ whenever $u \neq u'$? The answer to both these questions is positive, as it follows from the following statement which is the main technical result of this paper:

Theorem 6.1. $[\![-]\!] : \Omega[J] \to \mathbb{C}_{\mathbf{X}}$ *is an isomorphism of multicategories.*

Thus, if $\mathbf{X}$ is an $(n+1)$-category *freely generated* by a set J, then $\mathbb{C}_{\mathbf{X}}$ is a multicategory *freely generated* by the same set J. As a result, we have the following corollary that will be extremely useful in the sequel.

Corollary 6.2. *Assume that* $\mathbf{X}$, $\mathbf{B}$ *and* J *are as above. If* $\mathbf{Z}$ *is any other* $(n+1)$*-category extending* $\mathbf{B}$ *and* $\chi : \mathbb{C}_{\mathbf{X}} \to \mathbb{C}_{\mathbf{Z}}$ *is a morphism of multicategories which is the identity on objects and satisfies, for all* $x \in J$, $d\chi_a x = dx$, $c\chi_a x = cx$, *then there is a* unique ω*-functor* $F : \mathbf{X} \to \mathbf{Z}$ *which is the identity on the cells of* $\mathbf{B}$ *and extends* χ, *in the sense that* $Fu = \chi_a u$ *whenever* u *is a many-to-one* $(n+1)$*-cell of* $\mathbf{X}$ *(which means that* u *is also an arrow of* $\mathbb{C}_{\mathbf{X}}$*). If* $\mathbf{Z}$ *is also a free extension of* $\mathbf{B}$ *and* χ *is an isomorphism, then* F *is an isomorphism as well.*

The significance of the last statement of this corollary is that in a free extension $\mathbf{X}$ of $\mathbf{B}$ generated by many-to-one indets, the *many-to-one* $(n+1)$-cells of $\mathbf{X}$ (i.e. the arrows of $\mathbb{C}_{\mathbf{X}}$) determine the entire $(n+1)$-cell structure of $\mathbf{X}$.

Proof. Due to the freeness of the $(n+1)$-category $\mathbf{X}$, there is a unique ω-functor $F : \mathbf{X} \to \mathbf{Z}$ which is the identity on the $\mathbf{B}$-cells and such that $Fx = \chi_a x$ for $x \in J$. All we have to show is that F extends χ_a on *all* many-to-one $(n+1)$ cells of $\mathbf{X}$. As these cells are also the arrows of $\mathbb{C}_{\mathbf{X}}$ and, by 6.1, $\mathbb{C}_{\mathbf{X}}$ is a free multicategory, we may prove that $Fu = \chi_a u$ by induction on the arrows of $\mathbb{C}_{\mathbf{X}}$. If u is an indet or an identity, there is nothing to prove. To handle the induction step $u = u' \circ_r u''$, notice first that for any n-cell w of $\mathbf{B}$ and $r \in |\langle w \rangle|$, F preserves the generalized whiskering operation $w \square_r -$. This is seen by induction on w, using conditions 1-3 of 3.4 which, as stated by that theorem, characterize the generalized whiskering operations. Once this is done, we infer

$$Fu = F(u' \circ_r u'') = F(u' \bullet_n (du' \square_r u'')) = Fu' \bullet_n F(du' \square_r u'') = Fu' \bullet_n (du' \square_r Fu'')$$

By the induction assumption, $Fu' = \chi_a u'$, $Fu'' = \chi_a u''$. Also, as F is the identity on $\mathbf{B}$-cells, we have $du' = Fdu' = dFu' = d\chi_a u'$, hence we can go on with our sequence of equalities and conclude

$$= \chi_a u' \bullet_n (d\chi_a u' \square_r \chi_a u'') = \chi_a u' \circ_r \chi_a u'' = \chi_a (u' \circ_r u'') = \chi_a u.$$

The last statement of the corollary now follows immediately. If $\mathbf{Z}$ is free as well, then we have also a unique ω-functor $G : \mathbf{Z} \to \mathbf{X}$ which is the identity on $\mathbf{B}$-cells and extends χ_a^{-1}. Hence, both GF, FG are identity functors, as they are identities on the cells of $\mathbf{B}$ as well as on the *many-to-one* $(n+1)$-cells (which include the $(n+1)$-indets). $\square$

Before turning to the proof of 6.1, let's point out the significance of this theorem at the level of $\mathcal{M}$-terms. If $t \in \mathcal{T}(\mathcal{M})$ then $t/\!\approx$ is an arrow of $\Omega(J)$. Let's denote $[\![t/\!\approx]\!] = [\![t]\!]$. The significance of $[\![t]\!]$ is clear: t describes a way of constructing an arrow from a-indets and identity arrows by means of repeated multicomposition operations; $[\![t]\!] \in A(\mathbb{C}_{\mathbf{X}}) \subset X_{n+1}$ is the $(n+1)$-cell described by t when we interpret the a-indets as the corresponding $(n+1)$-indets in $\mathbf{X}$, while the multicomposition operations $\odot_r$ are interpreted as the $(n+1)$-cell placed compositions $\circ_r$. Theorem 6.1 states, first, that t and s denote *distinct* cells $[\![t]\!] \neq [\![s]\!]$, whenever $\nvdash_{\mathcal{M}} t = s$. Furthermore, 6.1 tells us that an $(n+1)$-cell $u \in X_{n+1}$ is of the form $[\![t]\!]$ for some $t \in \mathcal{T}(\mathcal{M})$ *iff* u is a many-to-one cell.

Theorem 6.1 will follow from a stronger and somewhat surprising one that will be stated after the preliminary discussion below.

The multicategory $\mathbb{C}_{\mathbf{X}}$ has the extension $\mathbb{C}^+_{\mathbf{X}}$ based on the object system $\Omega^+ = (I, B_n, i_I)$, where i_I is the inclusion map of I into the set B_n of all n-cells of $\mathbf{B} = \mathbf{A}[I]$. If $\Omega[J]$ is, indeed, isomorphic to $\mathbb{C}_{\mathbf{X}}$, then it must have an extension based on Ω^+ which is isomorphic to $\mathbb{C}^+_{\mathbf{X}}$ and we now set out to identify such an extension. The set $A(\mathbb{C}^+_{\mathbf{X}})$ of arrows of $\mathbb{C}^+_{\mathbf{X}}$ is also the set of *all* $(n+1)$-cells of $\mathbf{X}$ and has the following characterization that will assist us in our endeavor:

$A(\mathbb{C}^+_{\mathbf{X}})$ is the least set of arrows containing the indets and the $(n+1)$-*identity cells* (of $\mathbf{X}$) and closed under the placed composition operations $\circ_r$.

(As we use this fact only as a guiding principle, we will not give a full proof, but only indicate how a categorical composition $\bullet_k$ can be expressed by means of multicategorical composition in a simple case: assuming that u and v are many-to-one $(n+1)$-cells such that $u \bullet_k v$ is defined for some $k < n$, then $u \bullet_k v = (1_{x \bullet_k y} \circ_2 v) \circ_1 u$, where $x = cu$, $y = cv$ and 1, 2 are the indices indicating the occurrences of x, y in $x \bullet_k y$.)

We conclude that the set of arrows of $\mathbb{C}_{\mathbf{X}}$, i.e. the set of many-to-one $(n+1)$-cells of $\mathbf{X}$, fails to encompass all $(n+1)$-cells, just because it lacks the identity cells 1_w for the n-cells $w \in B_n \setminus I$ that are not n-indets. Likewise, the multicategory $\Omega[J]$ lacks arrows that would naturally correspond to the same identity cells. This observations leads us to the idea of augmenting J by adding *new* a-indets that will denote these missing items. To be more precise:

We extend the set of a-indets J over Ω to a set J^+ of a-indets over Ω^+ by letting $J^+ = J \dot\cup \{e_w : w \in B_n \setminus I\}$ with the source and target functions extended by setting $Se_w = \langle w \rangle$ and $Te_w = w$. The *new* indets e_w will be called, also, *predeterminates* or, in short, *predets*. From a *syntactical* point of view, the predets are indets like all the others, but *semantically* they are predetermined to denote identity cells or arrows.

Consider the multicategory $\Omega^+[J^+]$ freely generated by J^+ over Ω^+. It extends the free multicategory $\Omega[J]$, cf. 5.4. Let $\varphi : J^+ \to X_{n+1} = A(\mathbb{C}^+_{\mathbf{X}})$ be defined by $\varphi f = f$ for $f \in J$ and $\varphi e_w = 1_w$ for $w \in B_n \setminus I$. By the universal property of free multicategories, there is a unique morphism $\chi : \Omega^+[J^+] \to \mathbb{C}^+_{\mathbf{X}}$ which is the identity on the object system Ω^+ and such that $\chi g = \varphi g$ for $g \in J^+$. We denote, for any $u \in A(\Omega^+[J^+])$, $\chi u = [\![u]\!]^+$. The main property of the map $[\![-]\!]^+$ is that $[\![u \odot_r v]\!]^+ = [\![u]\!]^+ \circ_r [\![v]\!]^+$. Using this, it is easy to infer that $[\![-]\!]^+$ extends the canonical morphism $[\![-]\!]$. This means that $[\![u]\!]^+ = [\![u]\!]$ whenever $u \in A(\Omega[J])$.

We can now state the stronger result to which we alluded above.

Theorem 6.3. $[\![-]\!]^+ : \Omega^+[J^+] \to \mathbb{C}^+_{\mathbf{X}}$ *is an isomorphism of multicategories.*

An unexpected feature of this statement is that $\mathbb{C}^+_{\mathbf{X}}$ turns out to be a free multicategory some of whose generating arrows are, at the same time, identity cells in a related category.

To get a better grasp of the significance of this result, it will be useful to have a closer look at the structure of the arrows of $\Omega^+[J^+]$. To shorten terminology, these arrows will be called J^+-arrows, while those of $\Omega[J]$ will be referred to as J-arrows.

Claim 6.4. *A J^+-arrow u is a J-arrow iff $Tu \in I$. Consequently, if $u = u' \odot_r u''$ then u'' is always a J-arrow.*

Proof. The "only if" direction is immediate. For the "if" direction, assume that $Tu \in I$ and prove by induction on arrows that u is a J-arrow. If u is an indet, then it cannot be a predet, hence is a J-arrow. If u is an identity, it must be 1_x, where $x = Tu$. If $u = u' \odot_r u''$ then $Tu = Tu' \in I$ and $Tu'' \in I$ as well since otherwise, u'' could not possibly be composed into another arrow. Therefore, both u' and u'' are J-arrows, by the induction hypothesis, hence so is u. □

We can now show that 6.3 implies immediately our important theorem 6.1.
Proof of 6.1. All we have to show is that $[\![-]\!]$ is a one-to-one mapping from the arrows of $\Omega[J]$, i.e. the J-arrows, *onto* those of $\mathbb{C}_{\mathbf{X}}$. But this follows immediately from the fact that, by 6.3, $[\![-]\!]^+$ is bijective. As $[\![-]\!]^+$ is the identity on the object system Ω^+, it will map bijectively the arrows of $\Omega^+[J^+]$ whose targets belong to I onto those of $\mathbb{C}^+_{\mathbf{X}}$ with the same property. □

Proof of 6.3. The advantage of working with the multicategory $\mathbb{C}^+_{\mathbf{X}}$, rather than $\mathbb{C}_{\mathbf{X}}$, is that its arrows have an additional structure embodied by the partial categorical composition operations. If $[\![-]\!]^+$ is, indeed, an isomorphism then its inverse map will induce a similar additional structure on the arrows of $\Omega^+[J^+]$ and we ought to be able to identify it.

We will define a new $(n+1)$-category $\mathbf{Y}$ such that $\mathbf{Y}_n = \mathbf{B}$ and $Y_{n+1} = A(\Omega^+[J^+])$. Thus, in particular, $J \subset Y_{n+1}$ and we will show that, on one hand, $\mathbf{Y}$ *is freely generated over* $\mathbf{B}$ *by* J and hence, $\mathbf{Y}$ is isomorphic to $\mathbf{X} = \mathbf{B}[J]$, while, on the other hand, $\Omega^+[J^+]$ is *identical* with $\mathbb{C}^+_{\mathbf{Y}}$. From this follows that $\Omega^+[J^+]$ is isomorphic to $\mathbb{C}^+_{\mathbf{X}}$ and it will be very easy to show that the canonical morphism $[\![-]\!]^+$ is the isomorphism that we exhibited.

By setting $\mathbf{Y}_n = \mathbf{B}$, we already defined the $\leqslant n$-dimensional structure of $\mathbf{Y}$. Also, as we decided that the $(n+1)$-dimensional cells of $\mathbf{Y}$ are the arrows of $\Omega^+[J^+]$, all that remains to be done is to define the domain/codomain functions for $(n+1)$-cells, the $(n+1)$-dimensional identity cells and the compositions of $(n+1)$-cells at all dimensions$\leqslant n$.

The domain/codomain functions of $\mathbf{Y}$ will be denoted $\hat{d}$, $\hat{c}$ and are defined simply by $\hat{d}u = d[\![u]\!]^+$, $\hat{c}u = c[\![u]\!]^+$. Thus, we get $\hat{d}u, \hat{c}u \in B_n = Y_n$ and $\hat{d}u \parallel \hat{c}u$, as required. Also, for $k < n$, we have $\hat{d}^{(k)}u = d^{(k)}[\![u]\!]^+ = d^{(k)}\hat{d}u, \quad \hat{c}^{(k)}u = c^{(k)}[\![u]\!]^+ = c^{(k)}\hat{c}u$, where d, c are the domain/codomain functions in $\mathbf{B}$. Remember that $[\![-]\!]^+$ is the identity on object systems, hence it preserves sources and targets. As the source and target of $[\![u]\!]^+$, as an arrow of $\mathbb{C}^+_{\mathbf{X}}$, are $\langle d[\![u]\!]^+\rangle$ and $c[\![u]\!]^+$, we infer the following useful equalities: $Su = \langle \hat{d}u\rangle$ and $Tu = \hat{c}u$, for all $u \in Y_{n+1}$. Also, $\hat{d}(u \odot_r v) = \hat{d}u \Box_r \hat{d}v$ and $\hat{c}(u \odot_r v) = \hat{c}u$, as is easily seen.

The identity cells are easy to define: if $w = x \in I$, then the identity over w will be the identity arrow 1_x and if $w \in B_n \setminus I$ then the identity cell over w will be the predet e_w. We introduce a helpful notation: for $w \in B_n$, we let $\varepsilon_w = 1_x$ if $w = x \in I$ and $\varepsilon_w = e_w$ when $w \notin I$. Thus, the identity cell over $w \in B_n = Y_n$ will be, in any case, ε_w.

Before going on, let us remark that, as a consequence of 6.4, the set of all J^+-arrows is the least set $P \subset A(\Omega^+[J^+])$ such that: (a) P contains all predets and identity arrows (in other words, $\varepsilon_w \in P$ for all $w \in B_n$) and (b) $u \odot_r v \in P$ whenever $u \in P$ and v is a J-arrow such that $u \odot_r v$ is defined. This observation will allow us to prove statements by induction on J^+-arrows.

We now turn to the definition of the composition operations of $\mathbf{Y}$, which will be denoted $\hat{\bullet}_k$, for $k \leqslant n$. We have to define these only for cells of dimension $n+1$. This is done through the following two claims that are strongly suggested by proposition 3.11.

Claim 6.5. *There is a unique partial binary operation $\hat{\bullet}_n$ over Y_{n+1}, satisfying the following requirements:*

1. *$u\hat{\bullet}_n v$ is defined iff $\hat{d}u = \hat{c}v$.*

2. *$\hat{d}(u\hat{\bullet}_n v) = \hat{d}v$ and $\hat{c}(u\hat{\bullet}_n v) = \hat{c}u$.*

3. *$u\hat{\bullet}_n \varepsilon_{\hat{d}u} = u$.*

4. *$u\hat{\bullet}_n(v' \odot_r v'') = (u\hat{\bullet}_n v') \odot_r v''$.*

Proof. The uniqueness of $u\hat{\bullet}_n v$ follows easily by induction on v. We have to show, for every $u \in Y_{n+1}$, the existence of the partial function $u\hat{\bullet}_n(-)$.

Case 1: $\hat{d}u = x \in I$. In this case, $Su = \langle \hat{d}u\rangle = \langle x\rangle$ and $\hat{d}u = \hat{c}v$ iff $Tv = \hat{c}v = x$ and we can define $u\hat{\bullet}_n v = u \odot_r v$, where, of course, $|Su| = \{r\}$. Conditions *2-4* are easily verified.

Case 2: $\hat{d}u = w_0 \notin I$. We use the *strong* universal property of $\Omega^+[J^+]$. Let $\gamma : \Omega^+ \to \Omega^+$ be such that γ_o is the identity and $\gamma_t w = w$ for $w \neq w_0$, while $\gamma_t w_0 = \hat{c}u = Tu$. It is easily seen that this γ is a morphism of object systems. Next, let $\varphi : J^+ \to A(\Omega^+[J^+])$

be defined as $\varphi g = g$ for $g \in J^+ \setminus \{e_{w_0}\}$ and $\varphi e_{w_0} = u$. Then φ is *consistent* with γ, in the sense that $S\varphi g = \gamma S g$ and $T\varphi g = \gamma T g$, hence there is a *unique* morphism $\chi : \Omega^+[J^+] \to \Omega^+[J^+]$ extending γ and φ. Obviously, the restriction of χ to $\Omega[J]$ is the identity. We now define, for v such that $\hat{c}v = Tv = w_0$, $u\hat{\bullet}_n v = \chi v$ and have to show that conditions *2*-*4* are met. *3* and *4* are easily verified and condition *2* is proven by induction on v. We indicate only the induction step for $\hat{d}$: if $v = v' \odot_r v''$, then $u\hat{\bullet}_n v = (u\hat{\bullet}_r v') \odot_r v''$, by condition *4*. Hence, $\hat{d}(u\hat{\bullet}_n v) = \hat{d}(u\hat{\bullet}_n v') \square_r \hat{d}v''$ and, by the induction hypothesis this equals $\hat{d}v' \square_r \hat{d}v'' = \hat{d}(v' \odot_r v'') = \hat{d}v$. $\square$

Claim 6.6. *For every $k < n$, there is a unique partial binary operation $\hat{\bullet}_k$ on Y_{n+1}, satisfying the following:*

1. *$u\hat{\bullet}_k v$ is defined iff $\hat{d}^{(k)}u = \hat{c}^{(k)}v$.*
2. *$\hat{d}(u\hat{\bullet}_k v) = \hat{d}u\hat{\bullet}_k\hat{d}v$ and $\hat{c}(u\hat{\bullet}_k v) = \hat{c}u\hat{\bullet}_k\hat{c}v$ (where, of course, the composition $\hat{\bullet}_k$ of n-cells in $\mathbf{Y}$ is the same as $\bullet_k$ in $\mathbf{B}$).*
3. *$\varepsilon_w\hat{\bullet}_k\varepsilon_{w'} = \varepsilon_{w\hat{\bullet}_k w'}$.*
4. *$u\hat{\bullet}_k(v' \odot_r v'') = (u\hat{\bullet}_k v') \odot_r v''$.*
5. *$(u' \odot_r u'')\hat{\bullet}_k v = (u'\hat{\bullet}_k v) \odot_r u''$.*

Proof. Again, the uniqueness of $\hat{\bullet}_k$ satisfying *1*-*5* is easily established by an induction on u and v, so we have to show only the existence.

It would be nice to produce an argument that uses solely the universal (or strong universal) property of $\Omega^+[J^+]$, as we did in the proof of 6.5. Unfortunately, we did not find a such, yet. The proof that we are presenting uses the concrete description of the J^+-arrows as equivalence classes of $\mathcal{M}^+$-terms, where, of course, $\mathcal{M}^+$ stands for the multicomposition language $\mathcal{M}(\Omega^+, J^+, S, T)$ which is appropriate for $\Omega^+[J^+]$. Thus, we will define, first, $t\hat{\bullet}_k s$ for $\mathcal{M}^+$-*terms* t, s satisfying $\hat{d}^{(k)}t = \hat{c}^{(k)}s$, such that conditions *2*-*5* will be met (here and in the sequel, we abuse notation slightly, by letting $\hat{d}t = \hat{d}(t/\approx)$ and so on). Then we will show that $\approx$ is a congruence relation with respect to $\hat{\bullet}_k$ and conclude by setting $u\hat{\bullet}_k v = t\hat{\bullet}_k s$ for $u = t/\approx$, $v = s/\approx$.

We will define, by recursion on the $\mathcal{M}^+$-term t, the partial function $t\hat{\bullet}_k(-)$. Assume that $\hat{d}^{(k)}t = \hat{c}^{(k)}s$.

If t is an identity or a predet, i.e. $t = \varepsilon_w$ for $w \in B_n$, we define $t\hat{\bullet}_k s$ by recursion on s:

$$t\hat{\bullet}_k s = \begin{cases} \varepsilon_{w\hat{\bullet}_k w'} & \text{if } s = \varepsilon_{w'} \\ \varepsilon_{w\hat{\bullet}_k x} \odot_r f & \text{if } s = f \in J,\ \hat{c}f = Tf = x,\ |\langle x\rangle| = \{r\} \\ (t\hat{\bullet}_k s') \odot_r s'' & \text{if } s = s' \odot_r s'' \end{cases}$$

(where, in the middle case $s = f \in J$, $|\langle x\rangle|$ represents the second summand in $|\langle w\rangle|\dot{\cup}|\langle x\rangle| = |\langle w\hat{\bullet}_k x\rangle| = |S\varepsilon_{w\hat{\bullet}_k x}|$).

As we proceed with this recursion, we prove by induction on s that condition *2* is fulfilled, i.e. $\hat{d}(t\hat{\bullet}_k s) = \hat{d}t\hat{\bullet}_k\hat{d}s$ and $\hat{c}(t\hat{\bullet}_k s) = \hat{c}t\hat{\bullet}_k\hat{c}s$. The basis of this induction, i.e. the cases in which t is an identity or a predet or an indet, are easily handled using the fact that $[\![\varepsilon_w]\!]^+ = 1_w$, hence $\hat{d}\varepsilon_w = w$. Let us turn to the case of s being a multicomposition, which is the induction step. We have:

$$\hat{d}(t\hat{\bullet}_k s) = \hat{d}((t\hat{\bullet}_k s') \odot_r s'') = d[\![(t\hat{\bullet}_k s') \odot_r s'']\!]^+ = d([\![(t\hat{\bullet}_k s')]\!]^+ \circ_r [\![s'']\!]^+) = d[\![t\hat{\bullet}_k s']\!]^+ \Box_r d[\![s'']\!]^+$$

and the induction hypothesis tells us that $d[\![(t\hat{\bullet}_k s')]\!]^+ = \hat{d}(t\hat{\bullet}_k s') = \hat{d}t\hat{\bullet}_k\hat{d}s' = d[\![t]\!]^+\hat{\bullet}_k d[\![s']\!]^+$, so that we can continue the evaluation of $\hat{d}(t\hat{\bullet}_k s)$, keeping in mind that, $\hat{\bullet}_k$ is the same as the ordinary $\bullet_k$ for $\mathbf{Y}$-cells of dimension $\leqslant n$:

$$\hat{d}(t\hat{\bullet}_k s) = (d[\![t]\!]^+\hat{\bullet}_k d[\![s']\!]^+) \Box_r d[\![s'']\!]^+ = d[\![t]\!]^+\hat{\bullet}_k (d[\![s']\!]^+ \Box_r d[\![s'']\!]^+) = d[\![t]\!]^+\hat{\bullet}_k d[\![s' \odot_r s'']\!]^+ = \hat{d}t\hat{\bullet}_k\hat{d}s.$$

The proof that the same is true for the codomain function$\hat{c}$ is similar and somewhat simpler. It uses the fact that $\hat{c}s = Ts = T(s' \odot_r s'') = Ts' = \hat{c}s'$.

This completes the definition of the $t\hat{\bullet}_k(-)$ function when t is an identity or a predet.

If t is an indet $f \in J$, $Tf = \langle x\rangle$ then we know that $\vdash_{\mathcal{M}^+} t = 1_x \odot_r f = \varepsilon_x \odot_r t$ and, as we have already defined the partial function $\varepsilon_x\hat{\bullet}_k(-)$, we may let $t\hat{\bullet}_k s = (\varepsilon_x\hat{\bullet}_k s) \odot_r t$.

Finally, if t is a multicomposition, $t = t' \odot_r t''$ then we let $t\hat{\bullet}_k s = (t'\hat{\bullet}_k s) \odot_r t''$.

We leave the reader the verification of condition *2* in these other two cases.

Conditions *3-5* are obviously met for the $\hat{\bullet}_k$ operation thus defined for $\mathcal{M}^+$-terms. It remains to show that $\approx$ is a congruence relation with respect to this operation.

To show that $\vdash t = t_1$ implies $\vdash t\hat{\bullet}_k s = t_1\hat{\bullet}_k s$, we proceed by induction on the proof of $t = t_1$.

If $t = t_1$ is an $\mathcal{M}^+$-axiom, we have to examine five cases (as there are two kinds of identity axioms). These cases range from trivial to very easy, *except* (somewhat surprisingly) for the left identity axioms of the form $t = 1_x \odot_r t$. We have to show that $\vdash t\hat{\bullet}_k s = (1_x\hat{\bullet}_k s) \odot_r t$ and we do this by induction on t. Notice that, in this case, t has to be a J-arrow, as $Tt = x \in I$. If t is an indet $f \in J$, then we have by definition that $t\hat{\bullet}_k s = (\varepsilon_x\hat{\bullet}_k s) \odot t$, so there is nothing to prove (remember that $\varepsilon_x = 1_x$). If t is an identity, it has to be 1_x and $t\hat{\bullet}_k s = (1_x\hat{\bullet}_k s) \odot_r t$ becomes an instance of a right identity axiom. Finally, if $t = t' \odot_q t''$, where $q \neq r$, then we have:

$$\vdash (1_x\hat{\bullet}_k s) \odot_r t = (1_x\hat{\bullet}_k s) \odot_r (t' \odot_q t'') = ((1_x\hat{\bullet}_k s) \odot_r t') \odot_q t'',$$

by an instance of the associativity axiom. However, by the induction hypothesis we also have

$$\vdash (1_x\hat{\bullet}_k s) \odot_r t' = t'\hat{\bullet}_k s,$$

from which we infer, using the congruence rule,

$$\vdash ((1_x \hat{\bullet}_k s) \odot_r t') \odot_q t'' = (t' \hat{\bullet}_k s) \odot_q t'' = t \hat{\bullet}_k s,$$

the last equality holding by the definition of $t\hat{\bullet}_k s$ for the case of t being a multicomposition.

If $t = t_1$ is the conclusion of an inference rule of $\mathcal{M}^+$ then the desired equality follows immediately from the induction hypothesis.

The proof that $\vdash s = s_1$ implies $\vdash t\hat{\bullet}_k s = t\hat{\bullet}_k s_1$ is similar, once we established that, for $s = f \in J$, with $Tf = x$, $|\langle x\rangle| = \{r\}$, we have $\vdash t\hat{\bullet}_k s = (t\hat{\bullet}_k \varepsilon_x) \odot_r s$. This is done by induction on t and presents no difficulties.

The proof of the claim is now complete. $\square$

Claim 6.7. *The structure* $\mathbf{Y}$ *that we just described, is an* $(n+1)$*-category.*

Proof. We have to verify the axioms for $(n+1)$-cells only.

Verifying the exchange law $(u_1' \hat{\bullet}_k u_1'') \hat{\bullet}_l (u_2' \hat{\bullet}_k u_2'') = (u_1' \hat{\bullet}_l u_2') \hat{\bullet}_k (u_1'' \hat{\bullet}_l u_2'')$, when $l < k \leqslant n$ and the expression on the left is defined (which implies that so is the one on the right). We have to distinguish two cases:

Case 1: $k = n$. We reason by induction on u_1'', u_2''. If they are both ε's, i.e. $u_i'' = \varepsilon_{\hat{d}u_i'}$ for $i = 1, 2$, then, by 6.5, part *3*, the left side of the desired equality is nothing but $u_1' \hat{\bullet}_l u_2'$; as to the right side, it is $(u_1' \hat{\bullet}_l u_2') \hat{\bullet}_n (\varepsilon_{hdu_1'} \hat{\bullet}_l \varepsilon_{\hat{d}u_2'})$ and is seen to be equal to the same, because by 6.6, parts *3* and *2*,

$$\varepsilon_{\hat{d}u_1'} \hat{\bullet}_l \varepsilon_{\hat{d}u_2'} = \varepsilon_{\hat{d}u_1' \hat{\bullet}_l \hat{d}u_2'} = \varepsilon_{\hat{d}(u_1' \hat{\bullet}_l u_2')}.$$

If any of the u''s is a multicomposite, then the exchange axiom follows from the induction hypothesis, using the connection between the $\hat{\bullet}$ and $\odot$ operations, as displayed in 6.5, part *4* and 6.6, parts *4,5*.

Case 2: $k < n$. If all four cells are ε's, i.e. $u_i' = \varepsilon_{w_i'}$, $u_i'' = \varepsilon_{w_i''}$ then, by part *3* of 6.6, all we have to show is

$$\varepsilon_{(w_1' \hat{\bullet}_k w_1'') \hat{\bullet}_l (w_2' \hat{\bullet}_k w_2'')} = \varepsilon_{(w_1' \hat{\bullet}_l w_2') \hat{\bullet}_k (w_1'' \hat{\bullet}_l w_2'')}$$

and this follows by the exchange law in $\mathbf{B} = \mathbf{Y}_n$. Otherwise, if any of the cells is a $\odot$-composite, then the equality follows easily from the induction hypothesis, using again the connections between $\hat{\bullet}$ and $\odot$.

The verification of the associative law is similar, and somewhat simpler.

The identity laws:

To verify the left identity law for $\hat{\bullet}_n$, we have to show that $\varepsilon_{\hat{c}v} \hat{\bullet}_n v = v$. We do this by induction on v. If v is an ε, we must have $\hat{d}v = \hat{c}v$ and $v = \varepsilon_{\hat{c}v}$ and the desired conclusion follows by 6.5, part *3*. If $v = v' \odot_r v''$, then $\hat{c}v = Tv = T(v' \odot_r v'') = Tv' = \hat{c}v'$ and using the induction hypothesis as well as part *4* of 6.5, we conclude that

$$\varepsilon_{\hat{c}v} \hat{\bullet}_n v = (\varepsilon_{\hat{c}v'} \hat{\bullet}_n v') \odot_r v'' = v' \odot_r v'' = v.$$

The right identity law for $\hat{\bullet}_n$ is part *3* of 6.5.

The left identity law for $\hat{\bullet}_k$, with $k < n$, is $\varepsilon_w \hat{\bullet}_k v = v$, provided that $w = 1_a^{(n)}$ where $a = c^{(k)}\hat{c}v$. The proof is by induction on v. If $v = \varepsilon_{w'}$, then $\varepsilon_w \hat{\bullet}_k v = \varepsilon_{w \hat{\bullet}_k w'}$ and as $w' = \hat{c}v$, we have that $a = d^{(k)}w'$, hence $w \hat{\bullet}_k w' = w'$, by the left identity law in $\mathbf{B} = \mathbf{Y}_n$, and the desired law follows. If $v = v' \odot_r v''$, then the conclusion follows easily from the induction hypothesis, once we notice that $\hat{c}v = Tv = T(v' \odot v'') = Tv' = \hat{c}v'$.

The right identity law is $u \hat{\bullet}_k \varepsilon_{w'} = u$, where $w' = 1_a^{(n)}$ with $a = d^{(k)}\hat{d}u$. The proof, by induction on u is similar, except that for the induction step $u = u' \odot_r u''$, we have to notice that $\hat{d}u = d[\![u]\!]^+ = d([\![u']\!]^+ \circ_r [\![u'']\!]^+) = \hat{d}[\![u']\!]^+ \square_r \hat{d}[\![u'']\!]^+$ and hence, $d^{(k)}\hat{d}u = d^{(k)}\hat{d}u'$. $\square$

As $\mathbf{Y}$ is an $(n+1)$-category whose nth truncation is free over its $(n-1)$th truncation, we may define in it generalized whiskering operations $w \widehat{\square}_r -$ for $w \in Y_n$, as described in section 3. Once we did that, we can also define partial placed composition operations $\hat{\circ}_r$ by the formula

$$u \hat{\circ}_r v = u \hat{\bullet}_n (\hat{d}u \widehat{\square}_r v) \text{ for } u, v \in Y_{n+1},\ r \in |\langle u \rangle|,\ \hat{c}v = \langle \hat{d}u \rangle(r),$$

as in definition 3.8. Not surprisingly, $\hat{\circ}_r$ turns out to be the same with the multicomposition operation $\odot_r$ of $\Omega^+[J^+]$.

Claim 6.8. *If* $\mathbf{Y}$ *is the* $(n+1)$*-category described above, then:*

1. *If* $w \in Y_n$, $r \in |\langle w \rangle|$ *and* $v \in Y_{n+1}$ *are such that* $w \widehat{\square}_r v$ *is defined, then* $w \widehat{\square}_r v = \varepsilon_w \odot_r v$.
2. *For* $u, v \in Y_{n+1}$ *and* $r \in |\langle u \rangle|$ *such that* $u \odot_r v$ *is defined, we have* $u \odot_r v = u \hat{\bullet}_n (\hat{d}u \widehat{\square}_r v)$. *Hence,* $\odot_r = \hat{\circ}_r$.

Proof. Part *1*: by induction on the *n-cell* w.

If $w = x \in I$, then $w \widehat{\square}_r v = v = 1_x \odot_r v = \varepsilon_w \odot_r v$.

w cannot be an identity cell, as $|\langle w \rangle| \neq \emptyset$.

Finally, if $w = w' \hat{\bullet}_k w''$, assume, e.g., that $r \in |\langle w' \rangle|$. Then $w \widehat{\square}_r v = (w' \widehat{\square}_r v) \hat{\bullet}_k w'' = (\varepsilon_{w'} \odot_r v) \hat{\bullet}_k w''$, where the last equality holds by the induction hypothesis. In these equalities, $\hat{\bullet}_k$ represents a whiskering, which means that w'' is just short for $\varepsilon_{w''}$ (which is the identity cell over w'' in $\mathbf{Y}$). Taking this into consideration, we can go on and conclude that $w \widehat{\square}_r v = (\varepsilon_{w'} \odot_r v) \hat{\bullet}_k \varepsilon_{w''} = (\varepsilon_{w'} \hat{\bullet}_k \varepsilon_{w''}) \odot_r v = \varepsilon_{w' \hat{\bullet}_k w''} \odot_r v = \varepsilon_w \odot_r v$.

Part *2*: $u \hat{\bullet}_n (\hat{d}u \widehat{\square}_r v) = u \hat{\bullet}_n (\varepsilon_{\hat{d}u} \odot_r v) = (u \hat{\bullet}_n \varepsilon_{\hat{d}u}) \odot_r v = u \odot_r v$. $\square$

Following our plan for the proof of 6.1, we now show the following.

Claim 6.9. *The* $(n+1)$*-category* $\mathbf{Y}$ *is freely generated by* $J \subset Y_{n+1} = A(\Omega^+[J^+])$ *over* $\mathbf{B} = \mathbf{Y}_n$.

Proof. Let $\mathbf{Z}$ be an ω-category extending $\mathbf{B}$ and $\varphi : J \to Z_{n+1}$ a map such that $d\varphi f = \hat{d}f$, $c\varphi f = \hat{c}f$ for $f \in J$ (here and in the sequel, d and c represent the domain/codomain functions $d_{\mathbf{Z}}$, $c_{\mathbf{Z}}$ of the ω-category $\mathbf{Z}$). We have to show the existence of a unique ω-functor G extending both, the identity functor on $\mathbf{B}$ and φ. This amounts to specifying the function that sends each element $u \in Y_{n+1} = A(\Omega^+[J^+])$ to Gu, which is an $(n+1)$-cell of $\mathbf{Z}$ and proving that there is just one such function that makes G into an ω-functor.

At this point, it is useful to remember that the $(n+1)$-cells of $\mathbf{Z}$ are the arrows of the extended multicategory $\mathbb{C}^+_{\mathbf{Z}}$ which is based on the object system Ω^+ as well. Our proof will proceed as follows.

First, we extend the function φ to $\varphi^+ : J^+ \to Z_{n+1}$, by sending the predets to the corresponding identity cells. By the universal property of $\Omega^+[J^+]$, there is a unique morphism of multicategories $\chi : \Omega^+[J^+] \to \mathbb{C}^+_{\mathbf{Z}}$, which is the identity on Ω^+ and extends φ^+.

Next, we show that the function χ_a, operating on arrows, preserves domains, codomains, identity cells as well as ω-categorical compositions (i.e. $\chi_a(u \hat{\bullet}_k v) = \chi_a u \bullet_k \chi_a v$ for $k \leqslant n$, where $\bullet_k$ is the composition in $\mathbf{Z}$). This last fact follows readily from claims 6.5, 6.6, proposition 3.11 and the fact that χ_a preserves multicomposition. Hence, by setting $Gu = \chi_a u$ for $u \in Y_{n+1}$, we get an ω-functor as desired.

Finally, claim 6.8 implies that any G as above preserves multicomposition, hence it originates from the unique morphism χ that we just described. This proves the uniqueness of G.

In the rest of this claim's proof we are elaborating on these three steps.

If we define $\varphi^+ f = \varphi f$ for $f \in J$ and $\varphi e_w = 1_w (\in Z_{n+1})$ for $w \in B_n \setminus J$, we get a function that preserves sources and targets. Indeed, $S\varphi^+ f = S\varphi f = \langle d\varphi f \rangle = \langle \hat{d} f \rangle = Sf$ for $f \in J$ and $S\varphi^+ e_w = S1_w = \langle d1_w \rangle = \langle w \rangle = Se_w$ for $w \in B_n \setminus J$ (notice the ambiguous use of S as denoting source in $\Omega^+[J^+]$ as well as in $\mathbb{C}^+_{\mathbf{Z}}$). A similar computation shows that φ^+ preserves targets. The conclusion is that we can apply the universal property of $\Omega^+[J^+]$ and infer the existence of the morphism χ mentioned above.

We have to show that for $u \in A(\Omega^+[J^+]) = Y_{n+1}$, $d\chi_a u = \hat{d}u$ and $c\chi_a u = \hat{c}u$. We do this by induction on the *arrow* u. If $u = f \in J$, then $\chi_a u = \varphi f$ and there is nothing to prove. If $u = \varepsilon_w$ for $w \in B_n$, then $\chi_a u = 1_w$ and $d\chi_a u = w = \hat{d}\varepsilon_w = \hat{d}u$ and similarly for codomains. As to the induction step: if $u = u' \odot_r u''$, then $\chi_a u = \chi_a u' \circ_r \chi_a u''$ (where $\circ_r$ is $(n+1)$-cell placed composition in $\mathbf{Z}$). The induction hypothesis is that $d\chi_a u' = \hat{d}u'$, $d\chi_a u'' = \hat{d}u''$, hence $d\chi_a u = d\chi_a u' \square_r d\chi_a u'' = \hat{d}u' \square_r \hat{d}u'' = \hat{d}(u' \odot_r u'') = \hat{d}u$, where $\square_r$ is cell replacement in *both* $\mathbf{Z}$ and $\mathbf{X}$, as we have $\mathbf{Z}_n = \mathbf{X}_n$. The preservation of codomains is proven by a similar, but simpler, computation.

It is very easy to see that χ_a preserves identities. We still have the task of proving that $\chi_a(u\hat{\bullet}_k v) = \chi_a u \bullet_k \chi_a v$, for $k \leqslant n$.

For $k = n$, we prove this by induction on v. If v is an identity or a predet, then we must have $v = \varepsilon_{\hat{d}u}$ and the equality is trivial. If $v = v' \odot_r v''$, then $\chi_a(u\hat{\bullet}_n v) = \chi_a((u\hat{\bullet}_n v') \odot_r v'') = \chi_a(u\hat{\bullet}_n v') \circ_r \chi_a v''$. By using the induction hypothesis $\chi_a(u\hat{\bullet}_n v') = \chi_a u \bullet_n \chi_a v'$ and then proposition 3.11, we can go on and conclude that $\chi_a(u\hat{\bullet}_n v) = (\chi_a u \bullet_n \chi_a v') \circ_r \chi_a v'' = \chi_a u \bullet_n (\chi_a v' \circ_r \chi_a v'') = \chi_a u \bullet_n \chi_a(v' \odot_r v'') = \chi_a u \bullet_n \chi_a v$.

If $k < n$, then we show by induction on u that $(u\hat{\bullet}_k v) = \chi_a u \bullet_k \chi_a v$, for all v for which the left hand side is defined (and hence, so is the right). For $u = \varepsilon_w$, this is done by induction on v, much in the style of the calculation that we just completed (the main difference being that this time we use 6.6, rather than 6.5). For $u = u' \odot_r u''$, we use 6.6 again, as well as the induction hypothesis for u' and 3.11.

By letting $Ga = a$ for a a cell of $\mathbf{B} = \mathbf{Y}_n = \mathbf{Z}_n$ and $Gu = \chi_a u$ for $u \in Y_{n+1}$, we complete the proof of the *existence* of G.

To show *uniqueness*, assume that $G : \mathbf{Y} \to \mathbf{Z}$ is an ω-functor as desired. We have to prove that G must be induced by the morphism χ as described above. For this, suffices to show that G preserves multicomposition, meaning that $G(u \odot_r v) = Gu \circ_r Gv$. This is quite trivial, though: on one hand, we know from 6.8 that the multicompositions $\odot_r$ are the same as the cell replacements $\hat{\circ}_r$ in the ω-category $\mathbf{Y}$; on the other hand, any ω-functor like G, between two extensions of the n-category $\mathbf{B}$ which extends the identity on $\mathbf{B}$, clearly preserves placed compositions between $(n+1)$-cells.

The proof of 6.9 is now complete. $\square$

It follows that $\mathbf{Y}$ is isomorphic to $\mathbf{X} = \mathbf{B}[J]$ by a unique isomorphism that extends the identity functions on $\mathbf{B}$ and J. We are now able to infer immediately the following fact that we stated when outlining the proof of 6.3.

Claim 6.10. *The multicategories $\Omega^+[J^+]$ and $\mathbb{C}^+_{\mathbf{Y}}$ are identical.*

Proof. Obviously, the two multicategories have the same object system Ω^+, the same set of arrows $A(\Omega^+[J^+]) = Y_{n+1}$ and the same source and target functions $Su = \langle \hat{du} \rangle$, $Tu = \hat{c}u$. Further, they have the same identity arrows 1_x, for $x \in I$. By 6.8, they also have the same multicomposition operations $\odot_r = \hat{\circ}_r$. $\square$

Concluding the proof of 6.3: The unique ω-functor $K : \mathbf{Y} \longrightarrow \mathbf{X}$ extending the identity maps on both $\mathbf{B}$ and J is an isomorphism that induces an isomorphism of multicategories $\kappa : \mathbb{C}^+_{\mathbf{Y}} = \Omega^+[J+] \longrightarrow \mathbb{C}^+_{\mathbf{X}}$. In addition, κ maps the indets e_w, $w \in B_n \setminus I$, which are also identity cells in $\mathbf{Y}$, to the corresponding identity cells 1_w in $\mathbf{X}$. Hence, κ must be the canonical morphism $[\![-]\!]^+$. $\square$

We now mention one more remarkable fact. The elements of the set $A(\Omega^+[J^+] = Y_{n+1}$ are, at the same time, the arrows of the *free* multicategory $\Omega^+[J^+]$ and the $(n+1)$-cells of the *free* extension $\mathbf{Y}$ of the n-category $\mathbf{B}$. Therefore, we can define on this set *two* replacement operations, the multicategorical $\boxdot_r$ (cf. 5.5) and the $(n+1)$-categorical $\widehat{\Box}_r$ (cf. 3.4). Are these operations the same? Certainly not, because we might encounter u, v and $r \in |\langle u \rangle|$ such that, for $f = \langle u \rangle(r)$, we have $Tv = Tf$ (which also means that $\hat{c}v = \hat{c}f$) and $Sv = Sf$ (which is the same as $|\langle \hat{d}v \rangle| = |\langle \hat{d}f \rangle|$), but $\hat{d}v \neq \hat{d}f$. In such a case, $u \boxdot_r v$ is defined, while $u \widehat{\Box}_r v$ is not. However, when both expressions are defined, they are the same.

Claim 6.11. *If $u, v \in Y_{n+1}$ and $r \in |\langle u \rangle|$ are such that $u \widehat{\Box}_r v$ is defined, then $u \widehat{\Box}_r v = u \boxdot_r v$.*

Proof. By induction on the *arrow* u of $\Omega^+[J^+]$. If u is an indet, then both expressions equal v. If $u = u' \odot_q u''$ then, by part *2* of 6.8, $u = u' \hat{\bullet}_n (\hat{d}u' \widehat{\Box}_q u'')$. If $r \in |\langle u'' \rangle|$ then

$$u \widehat{\Box}_r v = u' \hat{\bullet}_n ((\hat{d}u' \widehat{\Box}_q u'') \widehat{\Box}_r v) = u' \hat{\bullet}_n (\hat{d}u' \widehat{\Box}_q (u'' \widehat{\Box}_r v)) = u' \hat{\bullet}_n (\epsilon_{\hat{d}u'} \odot_q (u'' \boxdot_r v))$$

where the second equality follows by 3.7, while the third uses the induction hypothesis for u'' as well as part *1* of 6.8. Employing parts *4,3* of 6.5, we go on and conclude

$$= (u' \hat{\bullet}_n \epsilon_{\hat{d}u'}) \odot_q (u'' \boxdot_r v) = u' \odot_q (u'' \boxdot_r v) = (u' \odot_q u'') \boxdot_r v = u \boxdot_r v.$$

The case $r \in |\langle u' \rangle|$ is similar and simpler. It uses the identity $\hat{d}(u' \widehat{\Box}_r v) = \hat{d}u'$ (cf. condition *1* of 3.4). $\Box$

Of course, the same claim is true for the isomorphic $(n+1)$-category $\mathbf{X}$ as well. By this we mean that the operation of $(n+1)$-cell replacement in the free $(n+1)$-category $\mathbf{X}$ is the *same* with arrow replacement in the free multicategory $\mathbb{C}_{\mathbf{X}}$, whenever the former is defined.

We stated in the introduction that our results imply that, *under certain conditions*, the free extension $\mathbf{X} = \mathbf{B}[J]$ can be construed as a *term model.* We conclude this section by outlining a proof of this fact.

Proposition 6.12. *Under the assumptions of this section, there is a primitive recursive function $(-)^\nu : \mathcal{T}(\mathcal{C}) \to \mathcal{T}(\mathcal{C})$ which associates with every $\mathcal{C}$-term t another $\mathcal{T}$-term t^ν such that for all $t, s \in \mathcal{T}(\mathcal{C})$, we have that $\vdash t = s$ iff $t^\nu = s^\nu$.*

This means that, in the construction of the free extension $\mathbf{X} = \mathbf{B}[J]$, we can substitute the term t^ν for the equivalence class $t/{\approx}$.

Proof. (Sketch) By following our proofs of 6.5 and 6.6, it is not hard to see that there exists a primitive recursive function $t \mapsto t'$ that associates with any $\mathcal{C}$-term t a $\mathcal{M}^+$-term t', such that $[\![t'/{\approx}]\!]^+ = t/{\approx}$ (hint: one clause in the recursive definition of $(-)'$ is $(t_1 \bullet_k t_2)' = t_1' \hat{\bullet}_k t_2'$, with $t_1' \hat{\bullet}_k -$ defined as in the proof of 6.6).

Next, another primitive recursive function takes any $\mathcal{M}^+$-term s to a $\mathcal{C}$-term $s^\sharp$ such that $[\![s]\!]^+ = s^\sharp/{\approx}$.

Finally, take $t^\nu = ((t')^\star)^\sharp$, where $(t')^\star$ is the unique *normal* $\mathcal{M}^+$-term equivalent to t' (cf. the discussion that follows proposition 5.4). $\Box$

7 Computads and multitopic sets

The notion of *computad* that we are going to present, was first defined by Street. A computad is a special kind of ω-category which is obtained by starting with a 0-category, i.e. a barren set, taking a free extension of it which is a 1-category, i.e. an ordinary category, then taking a free extension of it which is a 2-category and so on, *ad infinitum*. The precise definition is very simply stated.

Definition 7.1. An ω-category $\mathbf{A}$ is called a *computad* if for every $n < \omega$, $\mathbf{A}_{n+1}$ is a free extension of $\mathbf{A}_n$.

Thus, if $\mathbf{A}$ is a computad then there exists, for every $n < \omega$, a set $I_{n+1} \subset A_{n+1}$ of $(n+1)$-indets, such that $\mathbf{A}_{n+1} = \mathbf{A}_n[I_{n+1}]$. For the sake of uniformity, we also set $I_0 = A_0$ and refer, sometimes, to 0-cells as 0-*indets*. A simple proof by induction, using theorem 1.11, shows that for each $n > 0$, $\mathbf{A}_n$ is well behaved (cf. definition 1.10) and that an n-cell u is an n-indet iff it is a non-identity cell indecomposable in the sense of 1.8. Thus, the sets of indets of a computad are *uniquely* determined.

Definition 7.2. An ω-functor $F : \mathbf{A} \to \mathbf{A}'$ between computads $\mathbf{A}$ and $\mathbf{A}'$ is called a *computad functor* iff it preserves indets namely, Fu is an indet whenever u is. The category *Comp*, whose objects are the computads and arrows the computad functors, will be called the *category of computads*.

Obviously, *Comp* is a non-full subcategory of the category ωCat of ω-categories.

It is not hard to see that a computad functor preserves not only ω-categorical, but also computad structure:

Proposition 7.3. *Assume that $F : \mathbf{A} \to \mathbf{A}'$ is a computad functor and u an n-cell of $\mathbf{A}$, $n > 0$.*

1. *There is a bijection $\theta : |\langle u\rangle| \to |\langle Fu\rangle|$ such that, for $r \in |\langle u\rangle|$, $F(\langle u\rangle(r)) = \langle Fu\rangle(\theta r)$. We will always assume, as we may, that due to an appropriate reparametrization, θ is the identity.*

2. *F preserves the generalized whiskering operations. This means that $F(u\Box_r v) = (Fu)\Box_r(Fv)$ whenever $r \in |\langle u\rangle|$ and $u\Box_r v$ is defined.*

3. *F preserves the placed composition operations, meaning that $F(u \circ_r v) = (Fu) \circ_r (Fv)$, whenever $r \in |\langle u\rangle|$ and $u \circ_r v$ is defined.*

Proof. As $\mathbf{A}_n$ is a free extension of $\mathbf{A}_{n-1}$, we can prove statements by induction on n-cells. Parts *1,2* are easily seen by induction on u and then, part *3* follows immediately, because $\circ_r$ is defined, in 3.8, in terms of operations that are preserved by F, namely categorical composition, generalized whiskering and the domain function. $\Box$

In view of the results of section 6, we take a special interest in the case in which all indets are many-to-one.

Definition 7.4. A *many-to-one* computad is one in which the codomain of any $(n+1)$-indet is an n-indet, for all $n \in \omega$. The full subcategory $m/1Comp$ of $Comp$, whose objects are the many-to-one computads, will be called the *category of many-to-one computads.*

As we learned from corollary 6.2, if $\mathbf{A}$ is a many-to-one computad then for each n, the many-to-one $(n+1)$-cells of $\mathbf{A}$ determine the structure of all $(n+1)$-cells. Let us pursue this line of thought and take a closer look at the set of all many-to-one cells of $\mathbf{A}$. Following our practice, we consider all 0-cells to be *indets* and, for convenience, we declare them to be many-to-one cells. All 1-cells are many-to-one, but for $n \geqslant 2$, only *some* n-cells are many to one.

The set of many-to-one cells of a many-to-one computad $\mathbf{A}$ is *not* closed under the ω-categorical composition operations $\bullet_k$ and yet, this set enjoys remarkable closure properties. First of all, if u is a many-to-one cell, then so are its domain and codomain (assuming, of course, that u has positive dimension). Indeed, cu is an indet, hence is many-to-one, and du is parallel to cu, hence is many-to-one as well. Next, the many-to-one cells are closed under the *placed* composition operations. Thus, the many-to-one cells form a complex structure that deserves a special name. We arrive thus, in a natural way, to the notion of *multitopic set* that was introduced in [7].

Given a many-to-one computad $\mathbf{A}$ define, for $n > 0$, $\mathbb{C}_n = \mathbb{C}_{\mathbf{A}_n}$. In other words, $\mathbb{C}_n$ is the multicategory whose arrows are the many-to-one n-cells of $\mathbf{A}$, and whose objects (and object types) are the $(n-1)$-indets. By 6.1, $\mathbb{C}_n$ is a free multicategory generated by the n-indets. For the sake of completeness, we also let $\mathbb{C}_0$ be the barren set A_0 of 0-cells, viewed as a free multicategory (as indicated in the "important example" following 5.5). Thus, we have a sequence $(\mathbb{C}_n)_{n \in \omega}$, of free multicategories, such that the generating a-indets of $\mathbb{C}_n$ are at the same time the objects (and object types) of $\mathbb{C}_{n+1}$. There is an additional structural item that links these multicategories, as we have the domain and codomain functions $d, c : A(\mathbb{C}_{n+1}) \to A(\mathbb{C}_n)$. The structure $S = S_{\mathbf{A}}$ consisting of the sequence $(\mathbb{C}_n)_{n \in \omega}$ and the functions d, c, will be called the *multitopic set associated with the many-to-one computad* $\mathbf{A}$.

We now reproduce the definition of the abstract notion of multitopic set from [7].

We start with a preliminary definition that will describe the connection between the multicategories $\mathbb{C}_n$ and $\mathbb{C}_{n+1}$ mentioned above.

Definition 7.5. Given a free multicategory $\mathbb{C} = \Omega[J]$, we say that $\widehat{\mathbb{C}}$ is a *free extension* of $\mathbb{C}$ via the functions d and c iff the following conditions are met:

1. $\widehat{\Omega} = \Omega(\widehat{\mathbb{C}}) = (J)$. In other words, $\widehat{\mathbb{C}}$ is based on the simple object system whose objects are the a-indets that generate $\mathbb{C}$.

2. $\widehat{\mathbb{C}} = \widehat{\Omega}[\widehat{J}]$, meaning that $\widehat{\mathbb{C}}$ is freely generated by a set of a-indets $\widehat{J} \subset A(\widehat{\mathbb{C}})$.

3. d and c are functions $d : A(\widehat{\mathbb{C}}) \to A(\mathbb{C})$, $c : A(\widehat{\mathbb{C}}) \to J$, such that for $u \in A(\widehat{\mathbb{C}})$, $Su = \langle du \rangle$ and $Tu = cu$. Furthermore, $du \parallel cu$, meaning that $Sdu = Scu$, $Tdu = Tcu$. Also, for $x \in J$, $d1_x = c1_x = x$.

4. For $u, v \in A(\widehat{\mathbb{C}})$ and $r \in |Su|$ such that the multicomposition $u \widehat{\odot}_r v$ is defined in $\widehat{\mathbb{C}}$, we have $d(u \widehat{\odot}_r v) = du \boxdot_r dv$ and $c(u \widehat{\odot}_r v) = cu$ (where $\boxdot_r$ is the replacement operation in $\mathbb{C}$ as defined by theorem 5.5).

We are now ready to define:

Definition 7.6. A *multitopic set* S consists of sequences $\mathbb{C}_n = \mathbb{C}_n(S)$ of multicategories and $d_n = d_n(S)$, $c_n = c_n(S)$ of functions, $n \in \omega$, such that the following conditions are met:

1. $\mathbb{C}_0$ is a barren set viewed as a free multicategory.

2. $\mathbb{C}_{n+1}$ is a free extension of $\mathbb{C}_n$ via the functions d_n, c_n, for all $n \in \omega$.

3. For $n \in \omega$, we have $d_n d_{n+1} = d_n c_{n+1}$, $c_n d_{n+1} = c_n c_{n+1}$ (*globularity conditions*).

Remark. If S is a multitopic set, then each $\mathbb{C}_n = \mathbb{C}_n(S)$ is a multicategory based on a *simple* object system, as it follows from definition 7.5.

If $\mathbf{A}$ is a many-to-one computad, then the structure $S_{\mathbf{A}}$ is a multitopic set in the sense of this definition, when d_n, c_n are the domain/codomain functions of the ω-category $\mathbf{A}$ restricted to the set $A(\mathbb{C}_{n+1})$ of the many-to-one $(n+1)$-cells of $\mathbf{A}$. This is easily seen, thanks to the remark following claim 6.11 (applied to the free $(n+1)$-category $\mathbf{X} = \mathbf{A}_{n+1}$). As we shall see in the next section, *every* multitopic set is (isomorphic to) some $S_{\mathbf{A}}$.

Following the notation of [7], we shall write $d = d_n(S)$, $c = c_n(S)$, as the subscripts are understood from the context. Thus, the globularity conditions become $dd = dc$ and $cd = cc$.

Other notations and terminology from [7] that we will use are as follows. The set of generating a-indets of $\mathbb{C}_n(S)$ will be $C_n = C_n(S)$ (its elements are called "*n-cells*" in [7], but we shall not adopt this terminology here, as it would be confusing in our context, that mentions so often n-cells in ω-categories). The set of arrows of $\mathbb{C}_n$ is $P_n = P_n(S)$ and its members are called *n-pasting diagrams*, because they can be naturally given a diagrammatic representation (cf. [5]). Notice that $P_0 = C_0$.

A multitopic set S is called *n-dimensional* iff $C_k(S) = \emptyset$ for all $k > n$; this condition implies that all pasting diagrams of dimension $> n$ are *identities.* An n-dimensional multitopic set is determined by the *finite* sequence $\langle \mathbb{C}_k \rangle_{k \leqslant n}$ of its first $n+1$ components. If S is any multitopic set, its nth truncation will be n-dimensional multitopic set S_n with $\mathbb{C}_k(S_n) = \mathbb{C}_k(S)$, for $k \leqslant n$. Obviously, for a many-to-one computad $\mathbf{A}$, the nth truncation of $S_{\mathbf{A}}$ is $S_{\mathbf{A}_n}$.

Next, we define the obvious notion of *morphism* of multitopic sets.

Definition 7.7. A morphism $\Phi : S \to S'$ between multitopic sets S and S' is a sequence $\langle \phi_n \rangle_{n<\omega}$ of maps $\phi_n : P_n \to P'_n$ (where here and in the sequel, unprimed notations, like P_n refer to components of S, while their primed counterparts, like P'_n, refer to S'), that preserve the multitopic structure, meaning that for each $n < \omega$:

1. ϕ_n maps indets to indets, i.e., $\phi_n x \in C'_n$ whenever $x \in C_n$.
2. If $\tilde{\phi}_n$ is the restriction of ϕ_n to C_n, then the pair $\chi = (\tilde{\phi}_n, \phi_{n+1})$ is a morphism of multicategories from $\mathbb{C}_{n+1}$ to $\mathbb{C}'_{n+1}$.
3. For $u \in P_{n+1}$, we have $d\phi_{n+1}u = \phi_n du$ and $c\phi_{n+1}u = \phi_n cu$ (notice the context sensitivity of the notation for the domain/codomain functions: d, c refer to S' on the left sides of the equations, and to S on the right).

Notation. For a morphism Φ as above and for $u \in P_n$, we denote $\Phi u = \phi_n u$. Thus, Φ can be viewed as one single, dimension preserving, function from the pasting diagrams of S to those of S'.

Obviously, if S is an n-dimensional multitopic set and $\Phi : S \to S'$ is a morphism then the components ϕ_k of Φ for $k > n$ are trivial, and Φ is determined by its first $n+1$ components and we write $\Phi = \langle \phi_k \rangle_{k \leqslant n}$. One useful instance of this is the following: if $\Phi = \langle \phi_n \rangle_{n<\omega} : S \to S'$ is a morphism of multitopic sets, the so is $\Phi_n : S_n \to S'_n$, where $\Phi_n = \langle \phi_k \rangle_{k \leqslant n}$. Φ_n will be called the *nth truncation* of Φ.

Remark. Morphisms of multitopic sets are determined by their values on indets. These values can be chosen arbitrarily, subject to certain restrictions that insure the preservation of domains/codomains. More explicitly, a stepwise process of building a multitopic morphism goes as follows. We start by choosing $\phi_0 : C_0 \to C'_0$ *arbitrarily.* Assuming that we have already constructed ϕ_k, for $k \leqslant n$ such that $\langle \phi_k \rangle_{k \leqslant n}$ is a morphism from S to S', we start the construction of ϕ_{n+1} by choosing a function $\tilde{\phi}_{n+1} : C_{n+1} \to C'_{n+1}$ *arbitrarily*, subject to the restriction that $d'\tilde{\phi}_{n+1} f = \phi_n df$ and similarly for the codomain function. There is a *unique* morphism $\chi : \mathbb{C}_{n+1} \to \mathbb{C}'_{n+1}$ such that $\chi x = \phi_n x$ and $\chi f = \tilde{\phi}_{n+1} f$ for $x \in C_n$, $f \in C_{n+1}$. We define $\phi_{k+1} u = \chi u$ for $u \in P_{k+1}$. Then ϕ_{k+1} extends $\tilde{\phi}_{n+1}$, and we know that it satisfies condition 3 of 7.7 for $u \in C_{n+1}$. Using 5.6, we can show that the same condition is fulfilled for *all* $u \in P_{n+1}$.

The composition of morphisms of multitopic sets is again such a morphism. Also, for a multitopic set S, the sequence of identity maps $id_n : P_n(S) \to P_n(S)$ is a morphism from S to itself. Hence we may define a new category:

Definition 7.8. The category *mltSet*, whose objects are the multitopic sets and arrows their morphisms, is called the *category of multitopic sets.*

Can we extend the function $\mathbf{A} \mapsto S_{\mathbf{A}}$ to a functor? We can, and actually, much more is true.

Theorem 7.9. *The function that associates the multitopic set $S_{\mathbf{A}}$ to any many-to-one computad $\mathbf{A}$ can be extended to a functor $S_{-} : m/1Comp \to mltSet$ which is full and faithful.*

Proof. Given a computad functor $F : \mathbf{A} \to \mathbf{A}'$ between many-to-one computads $\mathbf{A}$ and $\mathbf{A}'$, we have to define a morphism $S_F : S_{\mathbf{A}} \to S_{\mathbf{A}'}$ of multitopic sets. We set $S_F = \langle \phi_n \rangle_{n<\omega}$ where ϕ_n is the restriction of F to the set of many-to-one n-cells of $\mathbf{A}$, which is the same with the set $P_n = P_n(S_{\mathbf{A}})$ of the n-pasting diagrams of $\mathbb{C}_n = \mathbb{C}_{\mathbf{A}_n}$. As F is a computad map, it maps indets to indets and, therefore, condition *1* of 7.7 is fulfilled. Conditions *2-3* are also satisfied, as it follows by 7.3. Thus, $S_F = \langle \phi_n \rangle_{n<\omega}$ is, indeed, a morphism of multitopic sets, according to 7.7. The functoriality of $F \mapsto S_F$ is readily verified.

The functor S_{-} is *faithful*. Indeed, if $S_F = S_G$ then we show by induction on n that $F_n = G_n$, where $F_n, G_n : \mathbf{A}_n \to \mathbf{A}'_n$ are the restrictions of F, G to the nth truncation of $\mathbf{A}$. The case $n = 0$ is trivial, because F_0, G_0 are both the 0th component of $S_F = S_G$. If $n > 0$, then F_n, G_n extend F_{n-1}, G_{n-1} respectively and by the induction hypothesis, $F_{n-1} = G_{n-1}$. Thus, as $\mathbf{A}_n$ is a free extension of $\mathbf{A}_{n-1}$, to infer that F_n and G_n are equal, we have only to show that they are equal on the set of *n-indets*, which equals $C_n \subset P_n$. This is clear, however, as the restrictions of F_n, G_n to P_n are, both, equal to the nth component of $S_F = S_G$.

Finally, we can show that S_{-} is *full*. Given a morphism $\Phi : S_{\mathbf{A}} \to S_{\mathbf{A}'}$ we define by induction the sequence $\langle F_n \rangle_{n<\omega}$ of truncations of an ω-functor $F : \mathbf{A} \to \mathbf{A}'$ such that $S_F = \Phi$. We start by letting $F_0 = \phi_0$. Once we have F_n, we let F_{n+1} be the unique $(n+1)$-functor $H : \mathbf{A}_{n+1} \to \mathbf{A}'_{n+1}$ that extends F_n and satisfies $Hf = \phi_{n+1} f$ for f an $(n+1)$-indet (by 7.3, it follows that $Hu = \phi_{n+1} u$ for u any many-to-one $(n+1)$-cell of $\mathbf{A}$). $\square$

8 Multitopic sets are equivalent to many-to-one computads

Definition 8.1. We say that Σ is an *assignment* of a multitopic set S into a many-to-one computad $\mathbf{A}$, and denote this as $\Sigma : S \to \mathbf{A}$, iff $\Sigma : S \to S_{\mathbf{A}}$ is a morphism of multitopic sets.

Remark. If $\Sigma : S \to \mathbf{A}$ is an assignment and $F : \mathbf{A} \to \mathbf{A}'$ is a computad functor in $m/1Comp$ then the composite function $\Theta = F\Sigma$ is an assignment $\Theta : S \to \mathbf{A}'$.

Roughly speaking, an assignment Σ is determined by its values on the a-indets that generate S. By this we mean that once we know the nth component σ_n, σ_{n+1} is uniquely determined by the values $\sigma_{n+1} f \in A_{n+1}$ for $f \in C_{n+1}(S)$. These values can be chosen arbitrarily, apart from the conditions that domains and codomain should be preserved (i.e. $d\sigma_{n+1}f = \sigma_n df$ and similarly for codomains).

As we shall see, theorem 6.1 implies that every multitopic set is (isomorphic to) $S_{\mathbf{A}}$, for some many-to-one computad $\mathbf{A}$. Actually, we prove somewhat more:

Proposition 8.2. *For every multitopic set S there is a many-to-one computad $\langle S\rangle$ and an assignment $\langle S\rangle^* : S \to \langle S\rangle$ such that:*

1. *$\langle S\rangle^*$ is an isomorphism of multitopic sets.*

2. *For any assignment $\Sigma : S \to \mathbf{B}$ into a many-to-one computad $\mathbf{B}$, there is a* unique *computad functor $F : \langle S\rangle \to \mathbf{B}$ such that $\Sigma = F\langle S\rangle^*$.*

Before proving 8.2, let us state two important corollaries. The first one is the main result of this article.

Theorem 8.3. *The categories $m/1Comp$ and $mltSet$ are equivalent. Actually, the functor $S_- : m/1Comp \to mltSet$ is an equivalence of categories.*

Proof. By 7.9, S_- is full and faithful. By 8.2 part *1*, S_- is essentially surjective on objects, i.e., for every object S of $mltSet$, there is an object $\mathbf{A}$ of $m/1Comp$ such that S is isomorphic to $S_{\mathbf{A}}$ (we mean, of course, that $\mathbf{A} = \langle S\rangle$). These conditions mean that S_- is an equivalence of categories. □

The second corollary states that $\langle -\rangle$ and $\langle -\rangle^*$ are functorial. To explain the functoriality of the second of these functions, we have to define one more category.

Definition 8.4. The category *Ass* of *assignments* is defined as follows. The *objects* are the assignments $\Sigma : S \to \mathbf{A}$ from multitopic sets to many-to-one computads. An *arrow* with domain $\Sigma : S \to \mathbf{A}$ and codomain $\Sigma' : S' \to \mathbf{A}'$ will be a pair (Φ, F) consisting of a morphism $\Phi : S \to S'$ and a computad functor $F : \mathbf{A} \to \mathbf{A}'$, such that the following diagram commutes:

$$\begin{tikzcd} S \arrow[r, "\Sigma"] \arrow[d, "\Phi"'] & \mathbf{A} \arrow[d, "F"] \\ S' \arrow[r, "\Sigma'"'] & \mathbf{A}' \end{tikzcd}$$

Thus, if S is a multitopic set, then $\langle S\rangle^* : S \to \langle S\rangle$ is an object of the category *Ass*.

Theorem 8.5. *$\langle -\rangle$ and $\langle -\rangle^*$ can be expanded to functors $\langle -\rangle : mltSet \to m/1Comp$ and $\langle -\rangle^* : mltSet \to Ass$ such that, for any morphism $\Phi : S \to S'$ in mltSet, we have $\langle \Phi\rangle^* = (\Phi, \langle \Phi\rangle)$.*

Remark. The last condition means that the following diagram commutes:

$$\begin{tikzcd} S \arrow[r, "\langle S\rangle^*"] \arrow[d, "\Phi"'] & \langle S\rangle \arrow[d, "\langle\Phi\rangle"] \\ S' \arrow[r, "\langle S'\rangle^*"'] & \langle S'\rangle \end{tikzcd}$$

Proof. We have to define arrows $\langle\Phi\rangle$, $\langle\Phi\rangle^*$ in $m/1Comp$ and Ass, respectively. The composite function $\Sigma = \langle S'\rangle^*\Phi$ is an assignment from S to the many-to-one computad $\langle S'\rangle$. By 8.2 part *2*, there is a *unique* computad functor $F : \langle S\rangle \to \langle S'\rangle$ such that $\Sigma = F\langle S\rangle^*$. We now define the arrows $\langle\Phi\rangle = F : \langle S\rangle \to \langle S'\rangle$ of $m/1Comp$ and $\langle\Phi\rangle^* = (\Phi, F) : \langle S\rangle^* \to \langle S'\rangle^*$ of Ass. It is easy to verify that we have thus defined the desired functors. □

Proof of 8.2. We define, by induction, the truncations of $\langle S\rangle = \mathbf{A}$ and of $\langle S\rangle^* = \Phi$. To be more precise, we will define sequences $\langle\mathbf{A}_n\rangle_{n<\omega}$ and $\langle\phi_n\rangle_{n<\omega}$ such that the following conditions are fulfilled:

a. $\mathbf{A}_n$ is an n-dimensional many-to-one computad.

b. $\mathbf{A}_{n+1} = \mathbf{A}_n[C_{n+1}]$, which means that $\mathbf{A}_{n+1}$ is a free extension of $\mathbf{A}_n$ generated by a set of many-to-one $(n+1)$-indets which is identical with the set $C_{n+1} = C_{n+1}(S)$ of a-indets that generate $\mathbb{C}_{n+1} = \mathbb{C}_{n+1}(S)$ over $\mathbb{C}_n$, as indicated in definitions 7.5 and 7.6.

c. $\phi_n : P_n \to A_n$ and $\Phi_n = \langle\phi_k\rangle_{k\leqslant n}$ is an isomorphism $\Phi_n : S_n \to S_{\mathbf{A}_n}$ of n-dimensional multitopic sets such that $\phi_n x = x$ for $x \in C_n$.

d. Condition *2* of 8.2 is fulfilled with S_n, $\mathbf{A}_n$ and Φ_n replacing S, $\langle S\rangle$ and $\langle S\rangle^*$, respectively.

Once this is done, we will take $\langle S\rangle$ and $\langle S\rangle^*$ having $\langle\mathbf{A}_n\rangle_{n<\omega}$ and $\langle\Phi_n\rangle_{n<\omega}$ as sequences of truncations.

As the basis of the induction, we set $\mathbf{A}_0 = P_0 = C_0$ and take ϕ_0 to be the identity function.

Assume that we defined already $\mathbf{A}_n$ and $\Phi_n = \langle\phi_k\rangle_{k\leqslant n}$.

Defining $\mathbf{A}_{n+1}$. Let us define functions $d', c' : C_{n+1} \to A_n$ by letting $d'f = \phi_n df$ and $c'f = \phi_n cf$, for $f \in C_{n+1}$. The functions $d, c : C_{n+1} \to P_n$ are closely related to their primed counterparts. Indeed, as ϕ_n is the identity on indets, we have $\langle d'f\rangle = \langle df\rangle$; moreover, $c'f = cf$, as $cf \in C_n$ and hence, $\phi_n cf = cf$. Because of these considerations, *we shall denote these newly defined functions by* d, c, *rather than* d', c'. Using the fact that, by induction hypothesis, Φ_n is an isomorphism between the multitopic sets S_n and $S_{\mathbf{A}_n}$, we infer that df, cf are *parallel* as n-cells of $\mathbf{A}_n$ and therefore, C_{n+1} together with the functions $d, c : C_{n+1} \to A_n$ becomes a set of $(n+1)$-*indets* over $\mathbf{A}_n$. We now define $\mathbf{A}_{n+1} = \mathbf{A}[C_{n+1}]$, and thus fulfill condition **b.** above.

Defining ϕ_{n+1}. By 7.5 and 7.6, we have $\mathbb{C}_{n+1} = \Omega[C_{n+1}]$, where Ω is the *simple* object system with set of objects C_n. The same Ω is also the object system of the multicategory $\mathbb{C}_{\mathbf{A}_{n+1}}$ whose arrows are the many-to-one $(n+1)$-cells of $\mathbf{A}_{n+1}$. The indets $f \in C_{n+1}$ are arrows of $\mathbb{C}_{n+1}$ as well as of $\mathbb{C}_{\mathbf{A}_{n+1}}$, and have the *same* source and target, $Sf = \langle df \rangle$ and $Tf = cf$, in both multicategories. At this point of the proof, we use our main technical result 6.1 and conclude that the *canonical* morphism $[\![-]\!] : \Omega[C_{n+1}] \to \mathbb{C}_{\mathbf{A}n+1}$ (i.e. the unique morphism that is the identity on both, C_n and C_{n+1}) is an *isomorphism.* We *define* $\phi_{n+1} : P_{n+1} \to A_{n+1}$ by $\phi_{n+1}u = [\![u]\!]$.

Verifying condition **c.** The pair $(\widetilde{\phi_n}, \phi_{n+1})$ (cf. the notation used in 7.7) is the same with $[\![-]\!]$, hence it is an isomorphism of multicategories. We have to prove, in addition, that $d\phi_{n+1}u = \phi_n du$, for *all* $u \in P_{n+1}$. We show this by induction on $(n+1)$ pasting diagrams. To begin with, this is given for $u \in C_{n+1}$ and immediate for identities. For the induction step, we use the fact that ϕ_{n+1} preserves multicomposition and infer:

$$d\phi_{n+1}(u \odot_r v) = d(\phi_{n+1}u \circ_r \phi_{n+1}v)) = d\phi_{n+1}u \,\square_r\, d\phi_{n+1}v =$$

Using the induction hypothesis as well as the fact that, by proposition 5.6, ϕ_n preserves arrow replacement, we go on and conclude

$$= \phi_n du \,\square_r\, \phi_n dv = \phi_n(du \,\boxdot_r\, dv) = \phi_n d(u \odot_r v).$$

Verifying condition **d.** Given an assignment $\Sigma : S_{n+1} \to \mathbf{B}$, let $\Sigma' : S_n \to \mathbf{B}$ be its restriction to S_n. By the induction hypothesis, we have a computad functor $F' : \mathbf{A}_n \to \mathbf{B}$ such that $\Sigma' = F'\Phi_n$. F' has a unique extension $F : \mathbf{A}_{n+1} = \mathbf{A}_n[C_{n+1}] \to \mathbf{B}$ such that $Ff = \Sigma f$ for $f \in C_{n+1}$. To show that the assignments Σ and $F\Phi_{n+1}$ from S_{n+1} to $\mathbf{B}$ are equal, we have only to show that they induce the same multicategory morphism from $\mathbb{C}_{n+1} = \Omega[C_{n+1}]$ to $\mathbb{C}_{\mathbf{B}_{n+1}}$. To this end, it suffices to show that they are equal on the indets in C_n and C_{n+1} and this is readily seen. Indeed, for $x \in C_n$, this follows from $\Sigma' = F'\Phi_n$, while for $f \in C_{n+1}$, $\Sigma f = Ff = F\phi_{n+1}f$. Thus, condition *2* of 8.2 is established. $\square$

9 Concluding remarks

A noteworthy result of [7] says that the category *mltSet* of multitopic sets is a *presheaf* category, i.e. it is equivalent to the category $Set^{Mlt^{op}}$ of the *contravariant* functors from a certain category *Mlt*, called the category of *multitopes*, into the category of sets *Set*. Thus, from our main result 8.3, we infer that *the category* $m/1$*Comp of many-to-one multitopic sets is a presheaf category* as well. This is a remarkable fact, since it is known that the category *Comp* of all computads is *not* a presheaf category, as shown in [13].

The objects of Mlt, as described in [7], are the same as the pasting diagrams of the *terminal* multitopic set. An alternative description of Mlt was given recently by the third named author of this paper, cf. [17].

As a corollary of our proposition 6.12, we infer that *the word problem for many-to-one computads is solvable.* The meaning of this statement is, roughly, as follows. A computad $\mathbf{A}$ is determined by the sequence $(I_n)_{n \in \omega}$ of sets of indets of the various dimensions. One can set up a large language which has terms denoting the cells of $\mathbf{A}$. This language has a hierarchical structure, being built in consecutive stages. In the initial stage we have a language $\mathcal{C}_0$ whose terms are the indets $x \in I_0$. Once the nth stage language $\mathcal{C}_n$ is defined, we take the next one to be $\mathcal{C}_{n+1} = \mathcal{C}(\mathbf{A}_n, I_{n+1}, d, c)$ whose terms are defined as in definition 1.1, with one difference: the values of the domain/codomain functions dt, ct of a $\mathcal{C}_{n+1}$-term t are $\mathcal{C}_n$*-terms*, rather than n-cells of $\mathbf{A}$. The meaning of $\mathcal{C}_{n+1}$-terms is clear, once the semantics of $\mathcal{C}_n$ is understood. Each $\mathcal{C}_n$ comes with its deduction system, similar to the one defined in 1.3. The word problem for $\mathcal{A}$ is to find an algorithm for deciding whether $t = s$ is $\mathcal{C}_n$-provable or not, for given $\mathcal{C}_n$ terms t, s. As we mentioned already, 6.12 implies that we have such an algorithm, actually a *primitive recursive* one, for $\mathbf{A}$ a many-to-one computad.

After a first draft of the present work has been completed, the second named author proved that *the word problem for arbitrary computads is solvable* as well., cf. [12]. His algorithm is very different from the present one. It is *not* based on the existence of term models and actually, we do not know if a result similar to 6.12 is true for arbitrary, not necessarily many-to-one, free extensions.

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References

[1] John C. Baez and James Dolan, *Higher-dimensional algebra and topological quantum field theory*, J. Math. Phys. **36** (1995), no. 11, 6073–6105.

[2] ———, *Higher-dimensional algebra. III. n-categories and the algebra of opetopes*, Adv. Math. **135** (1998), no. 2, 145–206.

[3] M. A. Batanin, *Computads for finitary monads on globular sets*, Higher category theory (Evanston, IL, 1997), Contemp. Math., vol. 230, Amer. Math. Soc., Providence, RI, 1998, pp. 37–57.

[4] George Grätzer, *Universal algebra*, second ed., Springer-Verlag, New York, 1979.

[5] Claudio Hermida, Michael Makkai, and John Power, *On weak higher dimensional categories. I. 1*, J. Pure Appl. Algebra **154** (2000), no. 1-3, 221–246, Category theory and its applications (Montreal, QC, 1997).

[6] ———, *On weak higher-dimensional categories. I.2*, J. Pure Appl. Algebra **157** (2001), no. 2-3, 247–277.

[7] ———, *On weak higher-dimensional categories. I. 3*, J. Pure Appl. Algebra **166** (2002), no. 1-2, 83–104.

[8] Joachim Lambek, *Deductive systems and categories. II. Standard constructions and closed categories*, Category Theory, Homology Theory and their Applications, I (Battelle Institute Conference, Seattle, Wash., 1968, Vol. One), Springer, Berlin, 1969, pp. 76–122.

[9] Tom Leinster, *A survey of definitions of n-category*, Theory Appl. Categ. **10** (2002), 1–70 (electronic).

[10] Saunders Mac Lane, *Categories for the working mathematician*, second ed., Graduate Texts in Mathematics, vol. 5, Springer-Verlag, New York, 1998.

[11] Michael Makkai, *The multitopic omega-category of all multitopic omega-categories*, Report, McGill University, http://www.math.mcgill.ca/makkai/, 1999.

[12] ———, *The word problem for computads*, Report, McGill University, http://www.math.mcgill.ca/makkai/, 2005.

[13] Michael Makkai and Marek Zawadowski, *The category of 3-computads is not cartesian closed*, J. Pure Appl. Algebra **to appear** (2008).

[14] Thorsten Palm, *Dendrotopic sets*, Galois theory, Hopf algebras, and semiabelian categories, Fields Inst. Commun., vol. 43, Amer. Math. Soc., Providence, RI, 2004, pp. 411–461.

[15] Ross Street, *Limits indexed by category-valued 2-functors*, J. Pure Appl. Algebra **8** (1976), no. 2, 149–181.

[16] ———, *Categorical structures*, Handbook of algebra, Vol. 1, North-Holland, Amsterdam, 1996, pp. 529–577.

[17] Marek Zawadowski, *Multitopes are the same as principal ordered face structures*, Report, Warsaw University, http://duch.mimuw.edu.pl/˜zawado/papers.htm, 2008.