The word problem for computads

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Contents:

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Introduction

(A) The origins of the paper

Computads were introduced by Ross Street, for dimension 2 as early as 1976 in [S1] and in general, in [S3]. Albert Burroni's paper [Bu] calls computads "polygraphs". Jacques Penon's paper [Pe] is important for us, since it contains the full syntactical definition of computads (reproduced with small changes in section 7 below), which will be used to formulate the problem in the title of the present paper.

My interest in computads stems from their role in the definition and theory of weak higher-dimensional categories. This role came to be realized as an afterthought.

In [He/M/Po], the definition of "opetopic set" introduced by John Baez and James Dolan [Bae/D] is reworked into what we called "multitopic set". In [He/M/Po], it was shown, among other, that the category of multitopic sets is, up to equivalence, the same as the category of presheaves on a category called the category of *multitopes*. In [M3], inspired by the second part of the Baez/Dolan definition of "opetopic category", but also following my earlier work [M1], [M2] on logic with dependent sorts, I proposed a definition of "the large multitopic category of all small multitopic categories"; the small multitopic categories constitute the zero-cells in said large multitopic category.

Already at the time of our joint work with Claudio Hermida and John Power, we had the feeling that multitopic sets were related to computads, in fact, that they were essentially identical with the "many-to-one" computads, ones whose indeterminates (free generating cells) have codomains that are themselves indeterminates (although, I must confess, at the time I did not really understand the notion of computad). The paper [Ha/M/Z] established this result, in the form of a pair of adjoint functors between the category of multitopic sets on the one hand, and the category of small ω-categories on the other, under which the left adjoint functor, from the first of the above categories to the second, being faithful and full on isomorphisms, has, as its essential and full-on-isomorphisms image, the category of many-to-one computads.

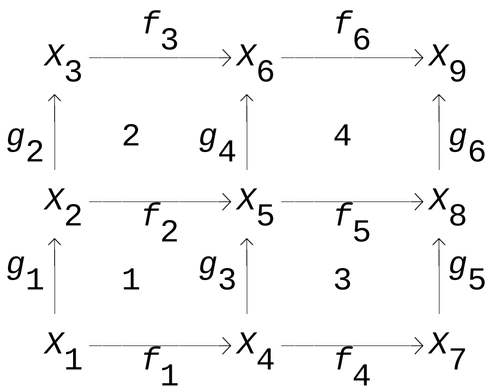
This result represented an advance inasmuch the fairly complicated, albeit combinatorially explicit, original definition in [He/M/Po] of multitopic sets became a conceptually simple one. On the other hand, it is to be noted that the fact that the category of many-to-one computads is a presheaf category, and the implied equivalent concept to the notion of multitope, do not become obvious by merely looking at many-to-one computads. At the present stage of our knowledge, said fact needs, for its proof, the detour via the original theory of multitopic sets in [He/M/Po].

The basic perspective of the present paper is a reversal of the above chronology. The notions "multitopic set" and "multitope" are seen here as the result of a combinatorial/algebraic analysis of the notion of many-to-one computad. The paper attempts to extend said analysis to *all* computads.

(B) Computads as the algebraic notion of higher-dimensional diagram.

The notion of *computad* is, as far as I am concerned, nothing but *the* precise notion of *higher-dimensional categorical diagram*. To explain this, I start earlier, with an informal introduction to the notion of (strict!) ω-*category*. (In this paper, no "weak" category theory appears at all.)

Consider the following ordinary categorical diagram:



consisting of objects X_i and arrows f_j , g_k in some category. The reader will agree when we say this:

(*) **if** the four small squares 1, 2, 3, 4 commute,
then, as a consequence, the big outside square will commute as well.

Having agreed on this, one may ask what are the *general laws* behind this, and countless other similar and/or more complicated facts. The answer is: the laws codified in the definition of notion of " *ω-category*".

To motivate that definition, the starting point is to adopt the position that "there is no bare equality": every equality is mediated by some data that we -- conveniently or not -- forget when we simply assert the fact of an equality. (At the Minneapolis (IMA) meeting on higher-dimensional categories in June 2004, John Baez gave, as the introductory talk to the conference, a brilliant lecture with this theme.) That position dictates that we, abstractly and theoretically, introduce data that are responsible for the commutativity of the four numbered squares above, in the way of fillers, 2-dimensional cells, or 2-arrows, as follows:

$$\begin{array}{ccccc} X_3 & \xrightarrow{f_3} & X_6 & \xrightarrow{f_6} & X_9 \\ g_2\uparrow & a_2\searrow & g_4\uparrow & a_4\searrow & \uparrow g_6 \\ X_2 & \xrightarrow[f_2]{} & X_5 & \xrightarrow[f_5]{} & X_8 \\ g_1\uparrow & a_1\searrow & g_3\uparrow & a_3\searrow & \uparrow g_5 \\ X_1 & \xrightarrow[f_1]{} & X_4 & \xrightarrow[f_4]{} & X_7 \end{array} \qquad (1)$$

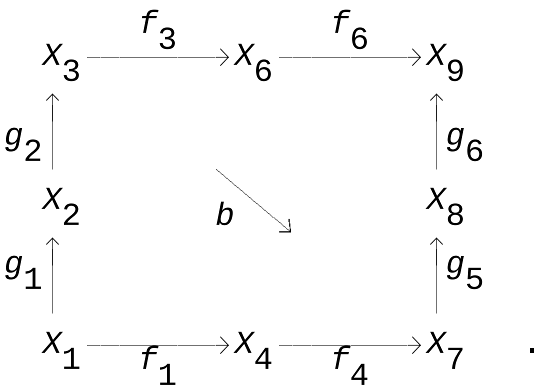
We think e.g. of a_2 as a (2-)*arrow* with *domain* g_2f_3, and *codomain* f_2g_4. (We use "geometric order"; g_2f_3 is what usually is denoted by $f_3 \circ g_2$, or also $g_2\#f_3$.) We even have given up the symmetry in the idea of the equality $g_2f_3 = f_2g_4$, and think of g_2f_3 being transformed into f_2g_4 in some general way, that way being denoted by a_2.

A 2-arrow must have a domain and a codomain that are ordinary (1-) arrows, which are parallel: they share their domain and their codomain, which are 0-*cells*. The idea here is that a 2-arrow as a transformation does not have any effect on 0-cells: it must leave them alone; transforming 0-cells is the responsibility of 1-cells.

The "if-then" statement (*) above becomes an *operation* that, applied to the four arguments a_1, a_2, a_3, a_4 results in a transformation, say b, of $g_1g_2f_3f_6$ into $f_1f_4g_5g_6$:

$$b\colon g_1g_2f_3f_6 \longrightarrow f_1f_4g_5g_6 \qquad (2)$$

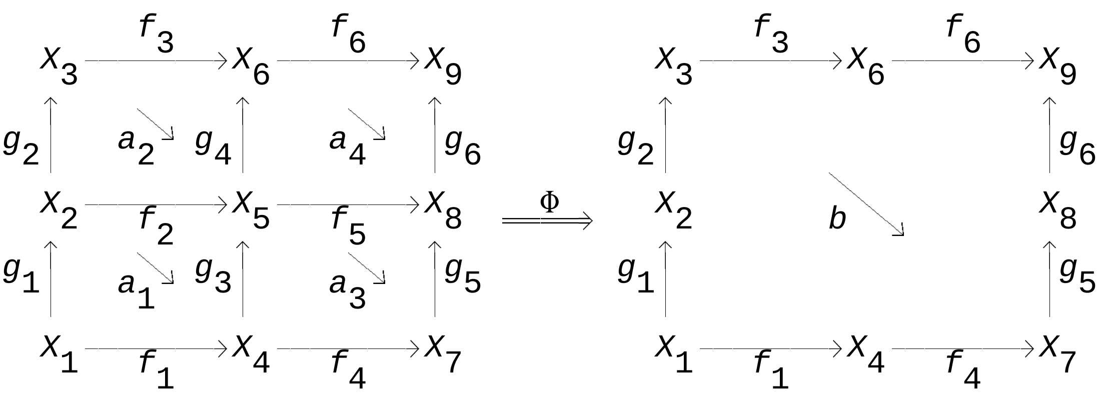
depicted as



The above procedure of introducing 2-dimensional arrows into diagrams to represent evidence, or proof, of a commutativity is closely related to the similar procedures in proof theory, especially categorical proof theory; see for example [L/Sc], section I.1, "Propositional calculus as a deductive system".

The concept of ω-category (" ω- " here anticipates the need for passing to ever higher dimensions after 0, 1 and 2 that have appeared so far) will have, on the one hand, some algebraically codified *primitive operations* that let us obtain b out of a_1, a_2, a_3, a_4 by repeatedly applying those operations, and on the other, certain *laws* that ensure that no matter in what order we apply the primitive operations to the four arguments, the result is always the same: b is *well-defined* as *the* composite of a_1, a_2, a_3, a_4 without any further qualification.

Just as the commutativity of 1-dimensional diagrams, that is, the equality of composite 1-arrows, has "given rise" to 2-dimensional diagrams, the contemplation of the equality of composite 2-cells (possible 2-commutativities) gives rise to 3-arrows, and 3-dimensional diagrams. For instance, the fact that the composite of a_1, a_2, a_3, a_4 *equals* b is mediated by a 3-cell



(3)

Of course, the process does not stop at dimension 3, and we see the need for a concept of *ω-category* in which there are arrows (cells) of arbitrary non-negative integer dimensions (but none of dimension ω or ∞).

Since the examples like the ones we considered clearly encompass a large variety, especially when one contemplates arbitrarily high dimensions, it is a highly non-obvious fact that a satisfactory concept of ω-category is possible at all. It is not a priori clear that there is a neatly defined set of primitive operations whose combinations account for all the desired compositions of cells; and it is not a priori clear that there is a neat set of laws that ensure facts like the one above of b being well-defined as *the* composite of a_1, a_2, a_3, a_4. It is therefore a kind of miracle that in fact we do have a good notion of ω-category. It is the basic general aim of the present paper and its projected sequels to investigate the ways and means of this "miracle".

There is another, perhaps even more convincing, way of approaching the concept of ω-category. This argues that the totality of (small) n-(dimensional) categories, properly construed, is an $(n+1)$-category; therefore, if we want to freely form "arbitrary totalities", we need n-categories for all n. (Let me note that the process stops at ω: the totality of (small) ω-categories is, in a natural way, an ω-category again, not an $(\omega+1)$-category.) However, in this second argumentation, when carried out with proper care, we find a similar step of replacing an *equality* by a *transformation*. In fact, this latter thinking, when followed to its logical conclusion, gives rise to the notion of *weak ω-category*, a concept that we do not discuss in this paper. The present paper sticks to the formal or *syntactical* role of higher dimensional diagrams, and it does not need the consideration of "totalities".

In an ω-category in which the diagram (3) lives, there are many cells (infinitely many if we consider the identities of all dimensions required by the concept of ω-category). In particular, we have the composite 1-cell g_2f_3 "on the same level" as the generating arrows g_2, f_3, etc. The concept that makes the distinction between "generating cell" and "composite cell" is the concept of *computad*. This is a conceptually very simple notion; it can be stated as *levelwise free ω-category*.

Imagine a structure, a typical computad, that can be taken to be essentially identical with the diagram (3). We want the *elements* of this structure to be exactly the named items in the diagram: the 0-cells X_i ($i=1, ..., 9$), the 1-cells f_j ($j=1, ..., 6$), g_k ($k=1, ..., 6$), the 2-cells a_l (l=l, ..., 4), b , and the 3-cell Φ . However, to account for the structure itself, we need to consider various composites of the elements. We decide to form the composites *freely*.

To begin with, we take the free category $\mathbf{X}^1$ on the ordinary graph consisting of said 0-cells and 1-cells. To incorporate the 2-cells and their composites, we need the operation of *freely adjoining* the mentioned 2-cells as *indeterminates* to $\mathbf{X}^1$, *with the appropriate preassigned domains and codomains* given as certain (composite) 1-cells in the category $\mathbf{X}^1$.

This process of free adjunction is very familiar from algebra. The ring of polynomials $R[X, Y, ...]$ is obtained from the ring R , by freely adjoining the indeterminates $X, Y, ...$. The definition, via a universal property, is too familiar to be quoted here. The 2-category $\mathbf{X}^2$ obtained by the free adjunction of the appropriate 2-cells to $\mathbf{X}^1$, with the specified domains and codomains in $\mathbf{X}^1$, is defined by a similar universal property. The only additional complication is that the adjoined 2-cell a_2 , to have an example, is constrained to have the specified domain g_2f_3 and codomain f_2g_4 given in $\mathbf{X}^1$ already. In the section I.5, "Polynomial categories", of [L/Sc], we find a similar situation in which an arrow with preassigned domain and codomain is freely adjoined to Cartesian closed category. (The definition of computad is given in section 5 , based on section 4.)

When we adjoin an indeterminate u to an ω-category $\mathbf{X}$ in which we have specified $\mathrm{d}u$ and $\mathrm{c}u$ in $\mathbf{X}$, to get $\mathbf{X}[u]$, we usually assume that $\mathrm{d}u$ and $\mathrm{c}u$ are parallel: they have the same domain and codomain. However, this is only a "reasonability assumption". The definition through the appropriate universal property works without this assumption. The canonical map $F:\mathbf{X}\to\mathbf{X}[u]$ will naturally produce the equality $F(\mathrm{dd}u)=F(\mathrm{dc}u)=\mathrm{d}_{\mathbf{X}[u]}(u)$. Thus, F can be injective only if said parallelism condition is satisfied. As we will see, in that case, F is indeed injective.

The composite of a_1, a_2, a_3, a_4 will be a definite 2-arrow $\vec{a}$ in $\mathbf{X}^2$. Of course, this is a major point of the construction, and it has to be ascertained specifically. That is, we have to define, using the primitive operations of " ω-category", a specific 2-cell that we will take, by definition, to be the composite of a_1, a_2, a_3, a_4 . This we will not do here, since we do not have the formalism of ω-category yet. However, once we have done this, the resulting 2-cell

$\vec{a}$ will have domain and codomain as b does in (2); in other words, $\vec{a}$ will be parallel to b.

Finally, the whole structure -- a computad -- is obtained by freely adjoining the 3-dimensional indeterminate Φ to X^2, with the stipulation that $\mathrm{d}(\Phi)$ is to be the composite $\vec{a}$, and $\mathrm{c}(\Phi)$ is b.

We have outlined the definition of a particular ω-category, in fact, a computad, that we take to be the structure representing the pasting diagram (3). It gives a good idea of the general notion of *computad*.

It turns out (see sections 4 and 5) that, when we define a computad to be an ω-category without additional data as we did in the example, we are able to recover the indeterminates in the computad from its ω-category structure as the elements that are *indecomposable* in a natural sense. Thus, it is not necessary to carry the indeterminates as data for the structure.

I consider the notion of computad as being *identical* to the notion of higher dimensional diagram, or pasting diagrams.

An analysis, using combinatorial, algebraic or geometric means, may provide descriptions amounting to equivalent definitions of smaller or larger classes of computads. In fact, such descriptions are one of the main areas of the theory of computads. As I mentioned above, the paper [Ha/M/Z] is part of this area.

The notion of a diagram being pastable (composable), the focus of the attention in the theory of pasting diagrams, is implicit in the concept of computad, since a computad always carries within itself all possible (free) compositions of the indeterminates (elements of the diagram). Of course, this does not mean that the problem of pastability of given candidates of pasting diagrams, given in some combinatorial or other manner, is solved automatically by using computads. The value of computads is mainly in their ability to provide mathematically satisfactory definitions of intuitive concepts -- such as pastability --, which then can be analyzed in any manner that comes to mind.

The important papers [J], [Po1], [Po2], [Ste]) give with various combinatorial, algebraic and geometric definitions of classes of pasting diagrams. They make connections to computads to varying degrees. In ongoing and future work (e.g. [M4]), I revisit the results of the existing theory of pasting diagrams in the spirit of computads.

(C) Concrete presheaf categories.

In part (A) above, we mentioned two results, both asserting the equivalence of certain categories. The first said that $\mathtt{MltSet}$, the category of multitopic sets, is equivalent of $\mathtt{Mlt}^{\wedge} = \mathtt{Set}^{\mathtt{Mlt}^{\mathrm{op}}}$, with $\mathtt{Mlt}$ the category of multitopes. The second said that $\mathtt{MltSet}$ is equivalent to $\mathtt{Comp}_{\mathtt{m/1}}$, the category of many-to-one computads.

It turns out that in both cases, what is proved is stronger than what is stated. In both cases, we have equivalences of *concrete* categories.

A *concrete category* is a category $\boldsymbol{A}$ together with an "underlying-set" functor $|\text{-}|:\boldsymbol{A}\to\mathtt{Set}$. *Equivalence* of concrete categories $(\boldsymbol{A},\ |\text{-}|_{\boldsymbol{A}})$, $(\boldsymbol{B},\ |\text{-}|_{\boldsymbol{B}})$ is equivalence of categories compatibly with the underlying-set functors: we require the existence of a functor $\Phi:\boldsymbol{A}\to\boldsymbol{B}$ that is an equivalence of categories, such that the following diagram of functors:

$$\begin{array}{ccc} \boldsymbol{A} & \xrightarrow{\quad\Phi\quad} & \boldsymbol{B} \\ & \searrow \;\cong\; \swarrow & \\ |\text{-}|_{\boldsymbol{A}} & \mathtt{Set} & |\text{-}|_{\boldsymbol{B}} \end{array}$$

commutes up to an isomorphism: there is an isomorphism $\varphi:\ |\text{-}|_{\boldsymbol{A}} \xrightarrow{\cong} |\text{-}|_{\boldsymbol{B}}\circ\Phi$.

All three of the above-mentioned categories $\mathtt{MltSet}$, $\mathtt{Mlt}^{\wedge}$, $\mathtt{Comp}_{\mathtt{m/1}}$ are equipped with canonical underlying-set functors.

A multitopic set, an object of $\mathtt{MltSet}$, consists of n*-cells*, for each $n\in\mathbb{N}$; we have an a priori underlying-set functor $|\text{-}|:\mathtt{MltSet}\longrightarrow\mathtt{Set}$. (In the notation of [He/M/Po] , Part 3, p. 83, for the multitopic set S , $|S| = \coprod_{k\in\mathbb{N}} C_k$; the elements of C_k are called the k*-cells* of S . In [Ha/M/Z], §1 gives an alternative, possibly more conceptual introduction to multitopic sets. On p.51 loc.cit., it is pointed out that in dimension 0 , a detail in the definition in [He/M/Po] is to be corrected.)

For any small category $\boldsymbol{C}$, we take the presheaf category $\hat{\boldsymbol{C}}=\mathtt{Set}^{\boldsymbol{C}^{\mathrm{op}}}$ to be equipped with the underlying-set functor $|\text{-}|:\hat{\boldsymbol{C}}\to\mathtt{Set}$ defined as $|A| = \coprod_{U\in\mathrm{Ob}(\boldsymbol{C})} A(U)$.

We have the underlying-set functor $|\text{-}|:\mathtt{Comp}\to\mathtt{Set}$ which assigns to each computad $\boldsymbol{X}$ the set $|\boldsymbol{X}|$ of all indeterminates of $\boldsymbol{X}$. $\mathtt{Comp}_{\mathtt{m/1}}$ is a concrete category with the underlying-set functor the restriction of that for $\mathtt{Comp}$.

It turns out that both equivalences

$$\mathtt{MltSet}\simeq\hat{\mathtt{Mlt}}$$

$$\mathtt{Comp}_{\mathtt{m/1}}\simeq\mathtt{MltSet}$$

are in fact concrete, that is, compatible with said underlying-set functors. As a corollary, we have the concrete equivalence

$$\mathtt{Comp}_{\mathtt{m/1}}\simeq\hat{\mathtt{Mlt}} .$$

This statement is meaningful even if we do not know the precise definition of $\mathtt{Mlt}$; it says that $\mathtt{Comp}_{\mathtt{m/1}}$ is (concrete-equivalent to) a *concrete presheaf category*.

A concrete presheaf category is, of course, in particular, a presheaf category; thus it is a very "good" category. Let us see what the "concreteness" in the equivalence says, in addition.

The concrete equivalence of $(\boldsymbol{A},|\text{-}|)$ and $\hat{\boldsymbol{C}}$ means that we have an equivalence functor $F:\boldsymbol{A}\xrightarrow{\simeq}\hat{\boldsymbol{C}}$, and a natural bijection

$$|A| \cong \coprod_{U\in\mathrm{Ob}(\boldsymbol{C})} (FA)(U) . \qquad (A\in\boldsymbol{A}) .$$

Thus, up to a natural bijection, we have a *classification* of the elements of an object A (the elements of the set $|A|$) of $\boldsymbol{A}$ into mutually disjoint classes $(FA)(U)$, the classes being labelled with a fixed set of *types*, the objects U of $\boldsymbol{C}$. This classification is functorial: it is compatible with the arrows of $\boldsymbol{A}$. Moreover, we have arrows between the types, the *type-arrows*, that account for the complete structure of the category $\mathbf{A}$: an arrow $A \xrightarrow{f} B$, essentially a natural transformation, is given by a system of maps

$(FA)(U) \xrightarrow{f_U} (FB)(U)$, one for each type U , that are jointly compatible with the type-arrows.

It turns out that the equivalence type a concrete presheaf category $\hat{\mathbf{C}}$ determines $\mathbf{C}$ up to *isomorphism*; we say that $\mathbf{C}$ is the *shape category* of any concrete category that is concretely equivalent to $\hat{\mathbf{C}}$.

Given a concrete category $\mathbf{A}=(\mathbf{A}, |-|)$, we identify a category, denoted by $\mathbf{C}^*[\mathbf{A}]$, which *is* the shape category of $\mathbf{A}$ in case $\mathbf{A}$ turns out to be a concrete presheaf category. Here is the definition.

$\mathrm{El}(\mathbf{A})$ denotes the category of elements of the functor $|-|:\mathbf{A}\to\mathrm{Set}$; its objects are pairs $(A, a) = (A\in\mathbf{A}, a\in |A|)$, and they are called *elements* of $\mathbf{A}$.

An element (A, a) is said to be *principal* if it is A is *generated* by a , in the sense that whenever $f:(B, b)\to(A, a)$ is an arrow in $\mathrm{El}(\mathbf{A})$ such that $f:A\to B$ is a monomorphism in $\mathbf{A}$, then f is an isomorphism. The element (A, a) is *primitive* if it is principal, and for any principal (B, b) , any arrow $f:(B, b)\to(A, a)$ must be an isomorphism.

The shape category $\mathbf{C}^*[\mathbf{A}]$ has objects that are in a bijective correspondence with the isomorphism types of primitive elements (A, a) . Moreover, if the primitive elements (A, a) , (B, b) are (represent) objects of $\mathbf{C}^*[\mathbf{A}]$, then an arrow $(A, a)\to(B, b)$ in $\mathbf{C}^*[\mathbf{A}]$ is the same as an arrow $A\to B$ in the category $\mathbf{A}$. Thus, there is a full and faithful forgetful functor $\mathbf{C}^*[\mathbf{A}]\to\mathbf{A}$.

Furthermore, we can spell out a set of conditions, some of them involving the primitive elements of $\mathbf{A}$, that are jointly necessary and sufficient for $\mathbf{A}$ to be a concrete presheaf category.
The first group, (i), of the conditions says that $\mathbf{A}$ is small cocomplete, $|-|:\mathbf{A}\to\mathrm{Set}$

preserves small colimits, and reflects isomorphisms.

The second group contains four conditions.

The first, (ii)(a), says that the set of isomorphism types of primitive elements is (indexed by a) small (set).

The second, (ii)(b), says that every element is the *specialization of a primitive element*:

for every element (A, a) of $\mathbf{A}$, there is a primitive element (U, u) together with a map $f:(U, u)\longrightarrow(A, a)$ in $\mathtt{El}(\mathbf{A})$.

Here, (U, u) is said to be a *type for* (A, a), f a *specializing map* for (A, a).

The third condition, (ii)(c), says that, for any element (A, a), with any given primitive (U, u), there is at most one specializing map $(U, u)\rightarrow(A, a)$.

Finally, the last one, (ii)(d), says that if the primitive elements (U, u), (V, v) are both types for (A, a), then they are isomorphic: $(U, u)\cong(V, v)$.

All the above facts concerning concrete presheaf categories are established as parts of standard category theory; they are easy, but form a basic setting for the first of the two main lines of inquiry in the paper, the investigation of the category $\mathtt{Comp}$ and certain of its full subcategories as to which of the above conditions are satisfied in them. $\mathtt{Comp}_{\mathtt{m/1}}$ is one of those full subcategories, and, by what we know from previous work, it satisfies every one of said conditions.

It is relatively easy to show that $\mathtt{Comp}$ itself satisfies (i) and (ii)(a); see the work leading up to section 6. One of the main results of the paper that $\mathtt{Comp}$ satisfies (ii)(b) ; *every element of* $\mathtt{Comp}$ *has at least one type*. The proof of this result requires the more substantial tools of the paper developed in sections 8 , 9 and 11. An easy example shows that (ii)(c) fails in $\mathtt{Comp}$ (see section 6). I do not know if (ii)(d) is satisfied or not by $\mathtt{Comp}$.

Let $\mathcal{C}$ be a *sieve* in $\mathtt{Comp}$, that is a full subcategory of $\mathtt{Comp}$ for which if B is in $\mathcal{C}$, and $A\rightarrow B$ is any arrow, then A is in $\mathcal{C}$. ($\mathtt{Comp}_{\mathtt{m/1}}$ is an example for a sieve in $\mathtt{Comp}$). $\mathbf{C}$ is regarded as a concrete category with the underlying-set functor inherited from $\mathtt{Comp}$. It is then immediate that the notions of principal element, primitive element, and type for an element for $\mathcal{C}$ become the direct restrictions of those for $\mathtt{Comp}$. More precisely, for (A, a) in $\mathtt{El}(\mathcal{C})$, (A, a) is principal resp. primitive for the concrete category $\mathcal{C}$ just in case it is principal resp. primitive for $\mathtt{Comp}$. Moreover, obviously, for an element (A, a) of $\mathcal{C}$, any (U, u) is a type of (A, a) in the sense of the concrete category $\mathtt{Comp}$ if and only if (U, u) is a type of (A, a) in the context of the concrete category $\mathcal{C}$.

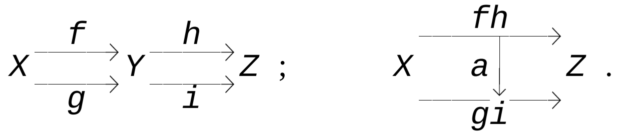
Thus, for a sieve $\mathcal{C}$ in $\mathtt{Comp}$, to say that it is a concrete presheaf category, is to say that it satisfies (i) -- which is ensured by assuming that $\mathcal{C}$ is closed under colimits in $\mathtt{Comp}$ --, and that the conditions (ii)(c) and (ii)(d) are satisfied by primitive elements of $\mathtt{Comp}$ that belong to $\mathcal{C}$.

An additional simplification is provided by the fact that a principal element (A, a) of $\mathtt{Comp}$ is determined by the underlying computad A ; a is the unique indeterminate of maximal dimension in A ; it is denoted by m_A . We call A a *computope* if (A, m_A) is primitive. If $\mathcal{C}$ is a sieve in $\mathtt{Comp}$, and as a concrete category is a concrete presheaf category, then its shape category $\mathbf{C}^*[\mathcal{C}]$ is the skeletal category of the computopes that are in $\mathcal{C}$. Furthermore, it is a *one-way category* (all non-identity arrows $A \to B$ have $\mathtt{dim}(A)<\mathtt{dim}(B)$), which makes it amenable to the manipulations of logic with dependent sorts ([M1], [M2]).

In particular, multitopes can be identified with many-to-one computopes: an elegant, albeit fairly abstract, definition of "multitope".

Here is an example illustrating the role of computopes.

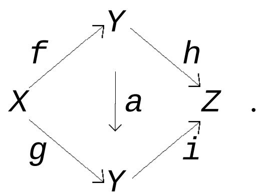
Consider the diagram



(4)

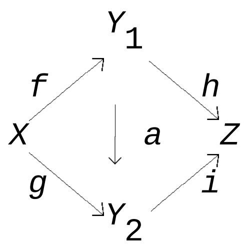
(recall the use of geometric order in compositions). It is clear how to interpret (4) as a computad: once again, the elements (indeterminates) of the computad are exactly the distinct elements named by single letters in the figure. (4) is a principal computad; its main cell ($m_{(4)}$) is a .

In drawing the diagram, we had the inconvenience of Y being in the way of placing the 2-cell a ; this made us repeat parts and denote some composites (the last should not be done ...). We would do better drawing the same as follows:



(5)

This repeats the 0-cell Y , but this is "all right". It seems right to say that (5) shows the real shape of the diagram (4). Of course, as a computad, (5) is identical to (4). However, we have the diagram -- computad --



(6)

"without repetition" of indeterminates; in fact, it is easy to see that (6) is a computope. We also have the obvious computad map $f:(6)\to(5)$ that, in particular, collapses Y_1 and Y_2 to Y . f is a specializing map for (5) (using the terminology introduced above), and (6) is *the* type for (5). In this case, it is easy to see that the type is unique up to isomorphism (condition (ii)(d) above), and it is obvious that the specializing map is unique (condition (ii)(c) above). We are inclined to say that (6) is in fact *the shape* of (5) (and (4)). (5) is obtained from the shape by *labelling*, in particular, labelling the spots Y_1 , Y_2 both by the same item Y .

"Computope" is the mathematical concept of shape of (principal, in particular finite) higher-dimensional diagrams. The specializing maps are the labellings of shapes to get the general diagrams.

I should note that the fact that $\mathrm{Comp}_{m/1}$ is a concrete presheaf category does not become obvious by what has been said above: although we know, by previous work, that conditions (ii)(c) and (d) hold true for many-to-one computads, I do not have a direct proofs of these facts. Despite this circumstance, I think it is be possible, by further developing the methods of this paper, to show that further significant categories of computads are concrete presheaf categories.

Perhaps it is not superfluous to state that my interest in higher-dimensional diagrams, hence, in computads in general, stems from the view that they should constitute the language for talking in a flexible way about matters *within* weak higher dimensional categories. Although the many-to-one computads are sufficient for defining a suitable concept of higher-dimensional weak category, a flexible language to develop mathematics in the context of a suitable weak higher-dimensional category, in analogy to mathematics developed in a topos, one needs higher-dimensional diagrams in general.

(D) The word problem for computads

For a fully explicit, computationally adequate, implementation of higher-dimensional diagrams -- that is, computads -- we need a notational system to represent, not only the indeterminates, but also the *pasting diagrams*, or *pd*'s, i.e., all composite cells, in the computad. After all, we must input the information about the domain and the codomain (arbitrary pd's in general) of each indeterminate.

The formalism of ω-categories provides such a notational system; as usual with free constructions, we can denote all cells of the ω-category freely generated by indeterminates by using a system of *words* derived directly from said formalism. This method is familiar from algebra, for instance, in the study of free groups, or more generally, groups given by presentations. In the case of computads, there is a new element, namely, the necessity to consider the condition of a word being *well-formed*. This becomes clear on the conditional nature of composition: one needs the precondition that a domain be equal to a codomain for the composite to be well-formed. However, having realized that we have to talk about well-formedness, the system of words is naturally defined. In this paper, this is done in section 7, following Jacques Penon's system in [Pe].

Similarly to what happens in the algebra of groups, the pd's in a computad will be identified with *equivalence classes* of words, rather than with words simply; the laws of ω-category will make certain pairs of words *equivalent*, that is, denote the same pd in the computad. The question how to see if two words are equivalent naturally arises, and one wants to know if the *word problem is solvable*: whether or not there is an decision method, efficient if possible, to decide for any two words if they are equivalent. Only in possession of such a decision method can we hope to have a reasonably general way of handling higher-dimensional diagrams computationally.

One of the main results of this paper is that *the word problem for computads* in general *is solvable*. After preparations, the main part of the work of the proof is done in section 10.

The motivation for this result also came from the situation of the many-to-one computads. In [Ha/M/Z] and independently, in [Pa], there is a description of the ω-category, in fact, a typical many-to-one computad, generated by a multitopic set, in which the general cells of the ω-category are given as multitopic pd's of the multitopic set "with niches". (In [Ha/M/Z], this is given as the left-adjoint of a pair of adjoint functors between `MltSet` and ω`Cat`, the right adjoint of which is a *multitopic nerve* functor). The construction provides a *normal form* for words denoting the pd's of the many-to-one computad. Starting from any many-to-one word, its normal form is computable, and two words are equivalent iff their normal forms are identical; the word problem of many-to-one computads is solvable as a consequence.

The solution of the word problem for general computads starts in a similar manner, with reducing an arbitrary well-formed word to a "pre"-normal form. The question of equivalence of pre-normal words is still non-trivial, but it is simpler than that for raw words, and it is eventually manageable, although the decision procedure as it stands at present uses searches through fairly large finite sets, and therefore it is quite unfeasible.

(E) The contents and the methods of the paper

The paper separates into two parts, one that uses, and the other that does not use, words. Sections 7 and 10 use, and are about, words. The other sections do not mention words, or use results based on words, at all.

The elementary theory of equivalence to a concrete presheaf category is explained in section 1.

Here, and elsewhere, the proofs that were found boring or less than easily readable were put into appendices. On the other hand, the paper, taken as a whole, is more than usually self-contained.

Sections 2 and 3 contain the generally accepted definitions of ω-graph and ω-category. Compare [Str2].

Sections 4 and 5 contain the concepts underlying the definition of "computad", and the basic results concerning these concepts. The approach is leisurely and the proofs are mostly routine. Section 4 explores the operation of adjoining indeterminates to a general ω-category, and the iteration of this operation. Section 5 defines computad as an ω-category obtained by iterated adjunctions of indeterminates to the empty ω-category. The emphasis in section 5 is on the properties of the category `Comp` of all computads, and the way this category resembles "good" categories such as presheaf categories.

The one element of section 5 that seems to be novel is the concept of the *content* of a pd in a computad: this is a multiset of the indeterminates occurring in the pd, counting the multiplicity (number of occurrences) of each indeterminate.

The definition of the content function was a non-trivial matter, and in fact, it is not entirely successful. One of the main intuitive requirements would be that in case of a computope A, the multiplicity of each occurring indeterminate in m_A is equal to 1. Our definition of the content function definitely does not satisfy this; and I do not know if it is possible to give such a definition, also having the other desired properties.

Despite its drawbacks, the content function is an efficient tool for the main purposes of the paper. One needed property is its invariance under equivalence. Its verbal description sounds as if it is defined for words, by a direct count of occurrences. However, such a definition would not give something that is invariant under equivalence of word, that is well-defined for pd's. As a matter of fact, the definition of the content is not done using words at all.

Another crucial property of "content" is its "linear" behaviour under maps of computads; see 5.(12)(ix).

Section 6 was fairly completely described under (C).

Section 7 displays the system of words for computads in complete (and straight-forward) detail.
Sections 8 and 9 contain the main mathematical novelty in the paper. I propose a kind of normal form, the *expanded form*, for compound expressions (words) in the language of

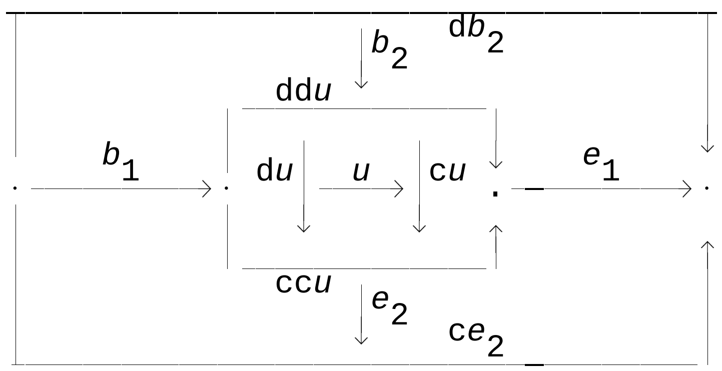
ω-categories. The expanded form is constrained in two ways. The first is that it admits only restricted instances of the ω-category operations. Specifically, the operation $a\#_k b$ is allowed for cells a of dimension m and b of dimension n only if $k=\min(m,n)-1$. Since k is determined by a and b, its notation is not necessary; we write $a \cdot b$ for $a\#_k b$.

The second constraint is that the expanded form allows the operations only in a certain order. For instance, denoting a 4-cell using a single indeterminate 4-cell u, is allowed only in the form of an *atom*

$$b_3 \cdot (b_2 . (b_1 \cdot u \cdot e_1) \cdot e_2) \cdot e_3$$

where b_i and e_i are cells of dimensions i, $i=1, 2, 3$. Of course, it is required that the composites be well-defined. "Bigger" 4-cells are obtained in the form of *molecules*, which are $\cdot$-composites of atoms.

The picture for a 3-atom $b_2 . (b_1 \cdot u \cdot e_1) \cdot e_2$ is

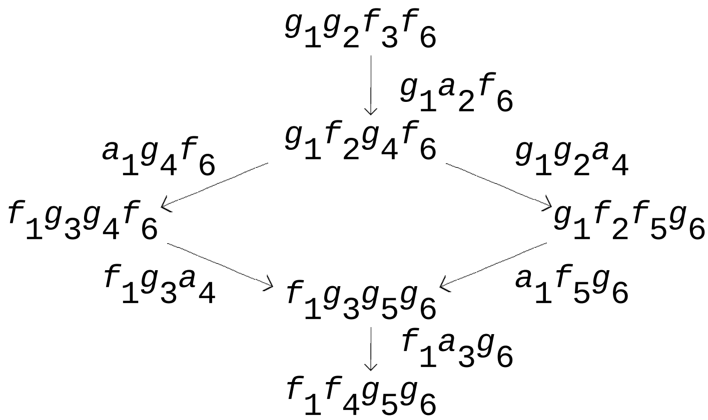


The success of the expanded form to account for all expressions rests with certain features of the constrained dot-operations. Section 8 shows that the operations obey laws that are of a nature that is more familiar from algebra than the laws in the generally accepted definition of ω-category. In particular, we have an associative law involving three variables (similarly to the usual definition), a distributive law, also involving three variables, and a "commutative law" involving two variables. It is shown that the usual operations, with their laws, are recoverable from the dot-operations with their postulated laws, effectively providing a new definition of ω-category, equivalent to the original one. In particular, the distributive law let's one distribute a lower dimensional cell over the composite of higher-dimensional cells as in

$$a \cdot (b \cdot e) = (a \cdot b) \cdot (a \cdot e) ,$$

where $\dim(a)<\dim(b)$, $\dim(a)<\dim(e)$, and the expressions are well-defined. It is mainly this that allows to reduce an arbitrary word to the form of a molecule.

The definition of ω-category through the dot-operations and the expanded form are very natural, and they readily come to mind when one discusses examples. For instance, the composite of the diagram in (1) has two molecular forms, both shown in



as the two equal composites from top to bottom.

The expanded form is used in section 9, the heart of the paper. This provides a reduction of the structure of an $(n+1)$-dimensional computad to that of a "collapsed" n-dimensional one, whereby the only thing, beyond the n-computad, left to discuss for the description of the $(n+1)$-computad is the effect of the commutative law on interchanging $(n+1)$-dimensional atoms.

I note that the results of section 8 and 9 are stated without referring to words. They have immediate variants involving words, which are stated and used as the main tools for the solution of the word problem in section 9. The same results, without reference to words, are used to establish certain finiteness lemmas, which are needed, in a natural fashion, to limit certain searches to finite sets, and to establish the decision procedure for the equivalence of words in section 10. In both sections 9 and 10, the content function of section 5 is crucial.

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