Mathematics and painting

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Mathematics and painting are interrelated in many ways. At a technical level mathematics can be used to enhance our appreciation of paintings. This was recently demonstrated beautifully by Taylor, Micolich, and Jones in their analysis of Jackson Pollock's drip paintings, which led them to the conclusion that Pollock's paintings are in a way figurative pictures. Relations also exist at a more basic level. Painting and mathematics can share subject matter, as is illustrated here by a discussion of the mathematical concepts 'open' and 'closed', which are related to paintings by Pollock, Kandinsky, Turner, and van Gogh. It is suggested in conclusion that mathematics and painting are so closely related and have so many similarities that it is reasonable to consider them simply as two different but complementary ways of visualising aspects of the concrete or abstract reality in which we are embedded.

Mathematics and fine art painting are two examples of the human consciousness striving to comprehend reality – not just the immediate physical reality around us, but reality in its broadest sense. Relations and parallels between these two disciplines are therefore to be expected. The artist as well as the mathematician is involved in attempting to make sense of the world. Both reflect on the structure of reality and try to define and extract elements of that structure, sometimes abstract, sometimes concrete. The artist might investigate ways to express, and thereby also define, a psychological mood, for example the horror so effectively captured in Edvard Munch's 1893 painting 'The Scream'.[1] There is clearly a bridge in that painting, but Munch was not interested in how to paint a bridge as a bridge. Instead his aim was to reveal to us the quintessential features of horror. In other cases artists are very much interested in the more figurative aspects of how to represent a specific object, in, say, the technicalities of how to make a painted surface look like a seascape. In a similar way the mathematician sometimes tries to extract the conceptual essence of a certain property. An algebraist might investigate the very nature of the operation 'addition' by extracting that specific arithmetical property of the natural numbers and studying the operation of addition in its pure form in the context of abstract group theory. Just as Munch made the abstract concept of horror his subject matter, in this case the algebraist makes the abstract concept of combining elements the subject matter of the investigation. In contrast, other branches of mathematics will, like the figurative painter, be interested in specifics. For example one might, as Einstein did, investigate what is the most precise way of describing a physical phenomenon, or at least an idealised one, such as light rays travelling past massive objects in space.

In this perspective painters and mathematicians are working very much along the same lines. They have to relate their productions to reality. Physical, concrete reality can put constraints on creativity in a very specific way when the task is, through either painting or mathematics, to represent some specific object. But often the constraints on creativity experienced by the creative mind, whether the artist's or the mathematician's, are not so much physically explicit as they are constraints arising from the need to produce a description which is coherent and consistent. In principle the painter can literally throw the paint onto the canvas in any way physically possible. And as we will see below, Jackson Pollock did precisely this. The mathematician can also make any definitions and any abstract constructions conceivable. But neither the painting nor the mathematical construction will make sense if it is not coherent and consistent. These requirements are tough to define in a functional way, but it is probably easier to do so in the case of mathematics than it is for painting. We will now attempt to make the relationship between mathematics and painting more explicit by discussing some specific examples. We shall touch on relations at several levels, technical as well as conceptual.

There are several obvious relations between painting and mathematics. One can obviously use mathematical geometry to analyse paintings, figurative or abstract, in terms of shapes such as points and lines, circles and triangles. Kandinsky sought to develop an analytical theory of painting in terms of fundamental geometrical forms and their emotional value. Kandinsky's aim was to establish a theory to aid the conceptual understanding of a painting in much the same way as has been done for music.[2]

Mathematics can also be used in a more concrete and direct way to analyse paintings. One may think of the theory of perspective applied to figurative painting, or the use of concepts like fractals for the comprehension of abstract paintings. As an example of the latter we will briefly discuss work that analyses Jackson Pollock's drip paintings in terms of fractal geometry.[3] The investigation is technical, nevertheless the analysis does add qualitatively to an appreciation of the artistic content of Pollock's paintings.

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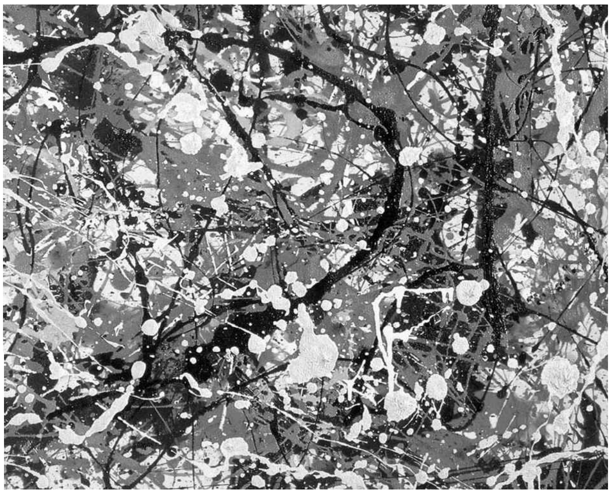
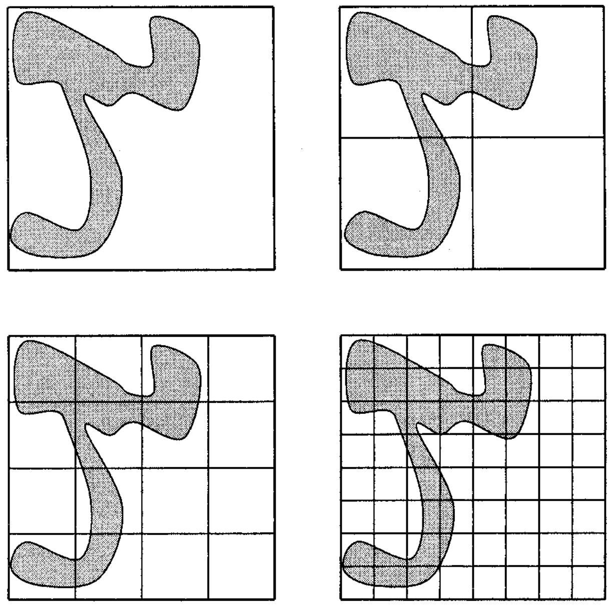


1 Detail of Jackson Pollock's 1949 painting 'Number 8' (Neuberger Museum, State University of New York)



2 The 'box counting' method, used to establish the fractal dimension of a plane shape

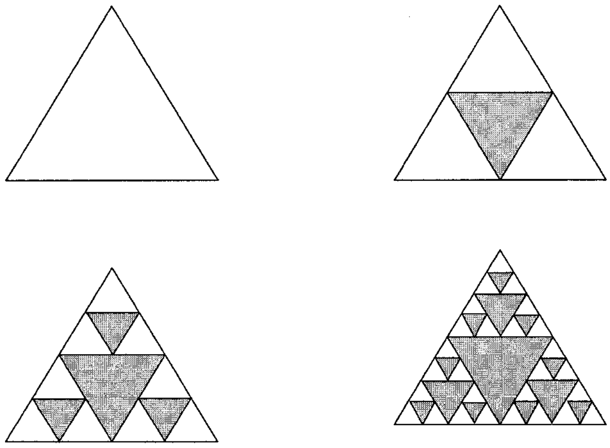
Kandinsky is concerned with the conceptual or spiritual content of shapes, in much the same way as Itten's colour theory investigates the conceptual and psychological mood of different colours.[4] Kandinsky shares this concentration on geometrical shape with the analysis done by Taylor, Micolich, and Jones – though the latter analysis is strictly objective and factual. We will now discuss another relation between mathematics and painting that goes beyond geometry and focuses on shared categories. To illustrate this relation we will first discuss the concept of 'open' and 'closed' sets and will then go on to look at paintings by Pollock, Kandinsky, Turner, and van Gogh from this perspective.

Turning first to the fractal analysis of Pollock's paintings reported by Taylor, Micolich, and Jones,[3] the most interesting result of their investigation is a quantitative argument for why Pollock's drippings are so attractive and pleasing to the eye. The reason they propose is that Pollock in reality was a naturalistic painter, in the sense that he (probably subconsciously) created structures with the same geometrical character as we find in nature. To appreciate this conclusion, and to illustrate an example of a recent mathematical analysis of an artistic painting, we first need to discuss some properties of geometrical fractals.

When one looks at one of Pollock's drip paintings, for example 'Number 8' of 1949 (Fig. 1), one sees paint scattered in an apparently arbitrary way across the canvas. But gaze at the picture for a little while and one starts to see more than just sprinkled paint. The painting begins to look like something we have seen somewhere before. Perhaps it reminds us of staring into the bushes and shrubs of the undergrowth in a dense wood. How can this be? The picture produces a sense of something organic. Can this be quantified? It turns out that the paint doesn't cover the surface of the canvas in a homogeneous way, but rather resembles the spatial organisation of fractals. Taylor, Micolich, and Jonas measured the mass dimension of the region covered by paint using the 'box counting' method.

The box counting method works as follows. A geometrical shape is covered by a grid consisting of boxes of linear dimension L. As L is made smaller, more and more boxes are needed to cover the entire shape. The number of boxes of area L^2 needed to cover all of the geometrical form is denoted by $N(L)$. In Fig. 2 we have $L=1$ corresponding to $N(L)=1$, $L=1/2$ corresponding to $N(L)=3$, $L=1/4$ corresponding to $N(L)=9$, and $L=1/8$ corresponding to $N(L)=32$. The fractal dimension of the object is said to be D if the number of boxes needed increases as $N(L)\propto L^{-D}$ when L is varied to successively smaller values. The fractal dimension D is equal to the dimension of the plane, that is $D=2$ for ordinary geometrical shapes that homogeneously fill part of the plane, for example a region of the plane consisting of all the points inside some closed contour. The fractal dimension is also equal to 2 for other types of homogeneous structures. Consider for instance the set of points constructed by randomly sprinkling a large number of dots on a canvas of area A. Assume that each point is equally likely to go anywhere on the canvas or in other words sprinkle with uniform probability. The uniform probability will ensure that there are dots in all regions of the canvas. When we do box counting we will need to cover the entire canvas with boxes for each choice of box size $L\times L$, hence we will need $N(L)=A/L^2$ boxes, i.e. the fractal dimension is $D=2$.

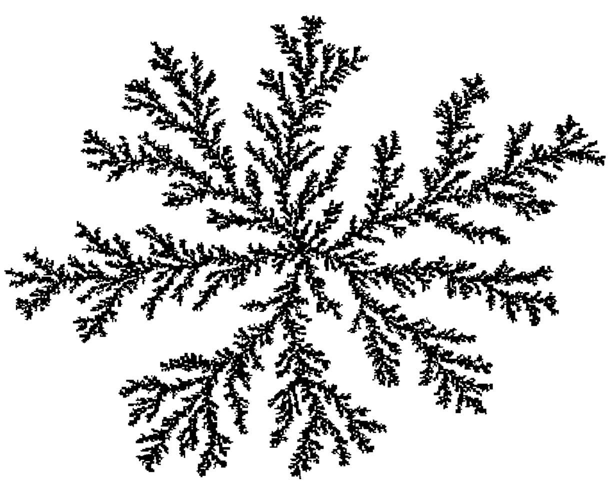
The fractal dimension of a fractal structure on the plane is always smaller than or equal to 2. To have $D<2$, the structure must contain holes of all sizes. This is illustrated in Fig. 3, which shows the construction by iteration of the regular fractal known as the Sierpinski gasket. This object is constructed by iteratively removing central triangles, from the



3 **Iterative construction of a regular fractal, the Sierpinski gasket**

starting point of an equilateral triangle as shown, the remaining point set (after infinitely many iterations) having $D = \ln 3/\ln 2 \approx 1.585$. Fractals certainly don't need to be regular. Figure 4 shows an example of a diffusion limited aggregation fractal constructed in a computer simulation. One notices that each small sub-branch looks very much like one of the bigger branches, so, as in the Sierpinksi gasket, the same theme is repeated at different length scales.

When Taylor, Micolich, and Jonas did box counting on the areas covered by paint in Jackson Pollock's paintings they found two different fractal dimensions. The fractal dimension corresponding to boxes of size L less than about 3–5 cm was found to be around $D = 1.65$, and for larger L they found $D = 1.96$. These numbers didn't change much from one Pollock painting to another. The fractal dimension found for the short scales is clearly smaller than 2 and hence the paintings are at this scale genuinely fractal. At longer length scales the fractal dimension $D = 1.96$ is so close to 2 that the paint can be said to cover the canvas in an almost homogeneous manner. Nevertheless, the fractal aspects of these drip paintings do suggest why the paintings somehow look familiar and remind us of organic structures.



4 **A diffusion limited aggregation fractal constructed by computer (Thomas Rage)**

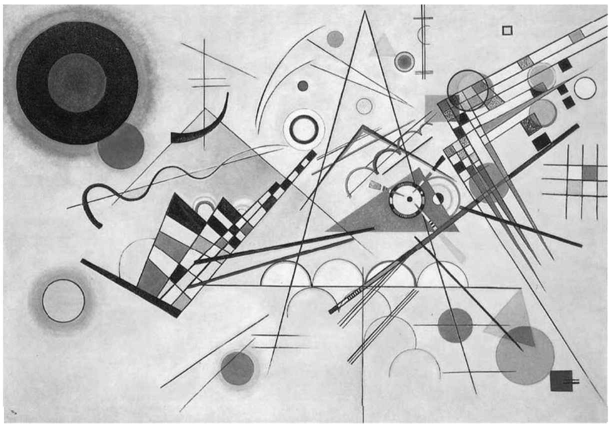
To elaborate on this point, Taylor, Micolich, and Jonas compared Pollock's paintings to snow covered ground vegetation and forest canopy and pointed out the structural similarity arising from the lack of a clear characteristic scale in all three cases. The fractal analysis thus reveals one aspect of the beauty of Pollock's paintings, namely that they are in fact naturalistic in the sense that he managed to develop techniques that allowed him to reconstruct characteristic features of naturally occurring spatial structures. He probably didn't think explicitly along such lines, but might very well intuitively have followed Picasso's maxim that good painting consists of smearing paint onto the canvas until it looks and feels exactly right.

After this example of a successful application of mathematics to analysing paintings we turn now to relations between painting and mathematics at the more fundamental abstract level. One can immediately think of a number of concepts which are studied and employed by painters as well as by mathematicians, for instance 'open' versus 'closed', 'continuity', and 'dimensionality', to mention a few. To illustrate the point we will concentrate on the terms 'open' and 'closed'. First we will briefly recapitulate the mathematical definition of open and closed sets, later turning to comparisons of paintings by Pollock and Kandinsky and by Turner and van Gogh and arguing that the contrasts between these pairs of pictures are well described in terms of open versus closed.

Let us first consider an open set. To be specific we will use the set of all real numbers larger than 0 and smaller than 1, call it M. We write $M =]0,1[$. The brackets point away from the 0 and the 1 in order to indicate that these two numbers are *not* included in the set M. One important property of an open set is that one can have sequences of numbers all within the set and still these numbers can approach a number outside the set. Let us elaborate on this point. Consider numbers r_n, n being equal to 1, 2, 3, etc., where all r_n are in M: we write this as $r_n \in M$. Assume that as n is made larger and larger, i.e. as n approaches infinity (written as $n' \to \infty$), the set of numbers approaches a certain number r, said to be the limit of the sequence r_n, and denoted by $r = \lim_{n \to \infty} r_n$. What we said above is that for an open set, one can have $r_n \in M$ for all n, but that nevertheless the limit r can be outside M (written as $r \notin M$).

An example will illustrate this property of an open set. Let $r_n = 1/(1 + n)$ for $n = 1, 2, 3$, etc. Clearly for any choice of n we have $r_n \in M$ since $0 < r_n < 1$. On the other hand it is also clear that $r_n = 1/(1 + n) \to 0$ as n becomes larger and larger, so the limit r of the sequence r_n is $r = 0$. But since zero is not an element in M we have an example of a sequence of numbers all within M which converges towards something not included in M. One can in a way sense from within M that something which in a natural way belongs to M is not included in M.

We can slightly change our set and thereby turn it into a closed set. All we have to do is to include the two endpoints of the interval under consideration.



5 Wassily Kandinsky's 'Composition 8' (1923) (Solomon R. Guggenheim Museum, New York)

6 Van Gogh's 'Room in Arles' (1889) (Musee d'Orsay, Paris)

We denote this slightly altered set as $\bar{M} = [0,1]$, that is we turn the brackets round to indicate that 0 as well as 1 now belong to this new set, i.e. $\bar{M}$ consists of all real numbers from 0 to 1, *including* 0 and 1. Although $\bar{M}$ is in a way only slightly different from M, the two sets do differ in one very significant property. It is now *impossible* to construct a sequence of numbers r_n which all belong to $\bar{M}$ with a limit r which is outside $\bar{M}$. We see immediately that our previous sequence $r_n = 1(1+n)$ fulfils this criterion, since $r_n \to 0$ and 0 is included in $\bar{M}$. So in this sense the closed set $\bar{M}$ is self-contained, whereas the open set M is not. When we follow number sequences inside M, we may need to refer to numbers not included in M in order to define limits. Not so when exploring the set $\bar{M}$: one will never sense from wandering around inside $\bar{M}$ that there is an exterior world.

Let us with this notion of open versus closed in mind compare Pollock's 'Number 8' and Kandinsky's 'Composition 8' (Fig. 5). Pollock's painting is 'open' in the sense that at every point on the canvas one is visually sucked in, and we experience a sense of falling through the painting from this side of the canvas into another world on the other side. Every point on the canvas hints at the outside world. The painting is like a two-dimensional piece of a three-dimensional something, say a thicket. The painting is clearly not self-contained, the subject matter continues beyond the canvas not only at its edges but at every point, by forcing the viewer's attention in the direction transverse to the canvas. 'Composition 8' by Kandinsky is entirely different. If we focus our attention on an arbitrary section of this painting we don't need to worry about a world outside the painting. There are a few lines reaching the edge of the canvas but most of the painting is neatly arranged within the frame. There is no feeling that the constructed geometrical forms extend or reach out in a direction transverse to the canvas. This is a closed painting.

An examination of 'open' and 'closed' paintings is not restricted to non-figurative art, as a comparison of van Gogh's 'Room in Arles' (Fig. 6) with Turner's 'Snowstorm' (Fig. 7) will make clear. Everything in van Gogh's painting is closed – even the door! Every shape and surface is bounded by a thick dark border whose purpose it is to close each subsection off from the others and to make the entire picture solidly cut off from the exterior world. In contrast Turner's painting is simply too energetic, too expansive to be contained within the space of the canvas. As with Pollock's 'Number 8' we are drawn through Turner's canvas and we know that the storm continues with wild force far beyond the little extract captured on the canvas. We are in no doubt that a world exists beyond the edges of the canvas, nor are we in any doubt about what this world is like, namely cold and windy. Van Gogh on the contrary gives us no clue as to what is happening outside his 'Room in Arles', if that is there *is* a world outside this room – we can't know for sure, we aren't even able to catch a glimpse of what might be outside the window.

There are of course many more concepts investigated and employed both by mathematicians and painters, however we will not pursue this relation in any more detail here. Instead we will finish with a few remarks concerning the importance of such a relation between art and mathematics. Perhaps the most interesting implication is concerned with how we perceive the very nature of mathematics. Often mathematics is considered as belonging to an entirely different realm

7 J. M. W. Turner's 'Snowstorm' (1842) (Tate Britain, London)

of human enterprise to the arts. The quantitative and exact nature of mathematics tends to make people think of it as a useful technical device rather than a soul enriching exercise. But to mathematicians, mathematics is much more than mindless juggling with numbers. The brief discussion above of the terms 'closed' and 'open' perhaps gives a flavour of something more far reaching, something conceptually satisfying, in fact something not at all alien to an artist's struggle to reflect reality.

The mathematician must, like the artist, decide from among the infinities of entities constituting reality which to extract and focus investigation on. Mathematical concepts have to be invented, created, or (perhaps) discovered, much in the same way that Braque and Picasso invented, created, or discovered cubism. Was the imaginary unit i ($\sqrt{-1}$) invented, created, or discovered? It was invented and created in the sense that before mathematicians decide to consider the square root of -1, no number had ever been considered which would lead to a negative number when squared. Yet the introduction of $\sqrt{-1}$ appears to be more like a discovery when we realise that laws of nature, for example quantum mechanics, function according to mathematics that relies on the use of $\sqrt{-1}$. Surely in some sense the rules of quantum mechanics have always existed, even if they were only discovered by humans about a century ago. Did cubism exist before Picasso, Braque, and Cézanne? It did in the sense that spheres, cones, and cylinders have existed always, so that in theory it has always been possible to resolve the shapes of physical objects in terms of these three forms, but before the cubists nobody noticed that. So the cubists 'discovered' this feature of physical reality.

At this rather abstract level isn't mathematics related to all other forms of human intellectual activity? Perhaps it is, but nevertheless the relation between mathematics and painting is a particularly close one: the two activities sometimes share subject matter, mathematics can be used to analyse paintings, and obviously painting – or drawing – can be used to investigate mathematics, not least geometry. Mathematics and painting may therefore be considered quite simply as two different, complementary ways of visualising the concrete or abstract reality in which we are embedded.

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