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Bipartite Graphs Associated with Pell, Mersenne and Perrin Numbers

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Abstract

In this paper, we consider the relationships between the numbers of perfect matchings (1-factors) of bipartite graphs and Pell, Mersenne and Perrin Numbers. Then we give some Maple procedures in order to calculate the numbers of perfect matchings of these bipartite graphs.

1 Introduction

The well-known integer sequences (e.g., Fibonacci, Pell) provide invaluable opportunities for exploration, and contribute handsomely to the beauty of mathematics, especially number theory [1, 2].

The Pell sequence $\{P(n)\}$ is defined by the recurrence relation, for $n \geq 2$

$$P(n) = 2P(n-1) + P(n-2) \tag{1}$$

with $P(0) = 0$ and $P(1) = 1$ [3]. The number $P(n)$ is called nth Pell number. The Pell sequence is named as $A000129$ in [4].

The Mersenne sequence $\{M(n)\}$ is defined by the recurrence relation, for $n \geq 2$

$$M(n) = 2M(n-1) + 1 \tag{2}$$

with $M(0) = 0$ and $M(1) = 1$ [5]. The number $M(n)$ is called nth Mersenne number. The Mersenne sequence is named as $A000225$ in [4].

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The Perrin sequence $\{R(n)\}$ is defined by the recurrence relation, for $n > 2$

$$R(n) = R(n-2) + R(n-3)$$

with $R(0) = 3$, $R(1) = 0$ $R(2) = 2$. The number $R(n)$ is called nth Perrin number [6]. The Perrin sequence is named as $A001608$ in [4].

The first few values of these sequences can be seen at the following table:

n	0	1	2	3	4	5	6	7	8	9	10	...
$P(n)$	0	1	2	5	12	29	70	169	408	985	2378	...
$M(n)$	0	1	3	7	15	31	63	127	255	511	1023	...
$R(n)$	3	0	2	3	2	5	5	7	10	12	17	...

The investigation of the properties of bipartite graphs was begun by König. His work was motivated by an attempt to give a new approach to the investigation of matrices on determinants of matrices. As a practical matter, bipartite graphs form a model of the interaction between two different types of objects. For example; social network analysis, railway optimization problem, marriage problem, etc [7]. The enumeration or actual construction of perfect matching of a bipartite graph has many applications, for example, in maximal flow problems and in assignment and scheduling problems arising in operational research [8]. The number of perfect matchings of bipartite graphs also plays a significant role in organic chemistry [9].

A bipartite graph G is a graph whose vertex set V can be partitioned into two subsets V_1 and V_2 such that every edge of G joins a vertex in V_1 and a vertex in V_2. A perfect matching (or 1 -factor) of a graph is a matching in which each vertex has exactly one edge incident on it. Namely, every vertex in the graph has degree 1. Let $A(G)$ be adjacency matrix of the bipartite graph G and $\mu(G)$ denote the number of perfect matchings of G. Then, one can find the following fact in [8]: $\mu(G) = \sqrt{\operatorname{per}(A(G))}$.

Let G be a bipartite graph whose vertex set V is partitioned into two subsets V_1 and V_2 such that $|V_1| = |V_2| = n$. We construct the bipartite adjacent matrix $B(G) = (b_{ij})$ of G as following: $b_{ij} = 1$ if and only if G contains an edge from $v_i \in V_1$ to $v_j \in V_2$, and otherwise $b_{ij} = 0$. Then, the number of perfect matchings of bipartite graph G is equal to the permanent of its bipartite adjacency matrix [8].

The permanent of an $n \times n$ matrix $A = (a_{ij})$ is defined by

$$\operatorname{per}(A) = \sum_{\sigma \epsilon S_n} \prod_{i=1}^{n} a_{i\sigma(i)}$$

where the summation extends over all permutations σ of the symmetric group S_n. The permanent of a matrix is analogous to the determinant, where all of the signs used in the Laplace expansion of minors are positive. One can find the basic properties and more applications of permanents [8, 9, 10, 11, 12, 13].

Permanents have many applications in physics, chemistry and electrical engineering. Some of the most important applications of permanents are via graph theory. A more difficult problem with many applications is the enumeration of perfect matchings of a graph [8]. Therefore, counting the number of perfect matchings in bipartite graphs has been very popular problem.

One can find so many studies on the relationship between the number of perfect matchings of bipartite graphs and the well-known integer sequences [14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26].

In this paper, we define three $n \times n$ $(0,1)$-matrices which correspond to the adjacency matrices of some bipartite graphs. Then we show that the numbers of perfect matchings of these bipartite graphs are equal to Pell, Mersenne and Perrin numbers, respectively. Finally, we give some Maple procedures regarding our calculations.

2 Main Results

Let $A = [a_{ij}]$ be an $m \times n$ real matrix with row vectors $\alpha_1, \alpha_2, ..., \alpha_m$. We say A is contractible on column (resp. row) k if column (resp. row) k contains exactly two nonzero entries. Suppose A is contractible on column k with $a_{ik} \neq 0 \neq a_{jk}$ and $i \neq j$. Then the $(m-1) \times (n-1)$ matrix $A_{ij:k}$ obtained from A by replacing row i with $a_{jk}\alpha_i + a_{ik}\alpha_j$ and deleting row j and column k is called the contraction of A on column k relative to rows i and j. If A is contractible on row k with $a_{ki} \neq 0 \neq a_{kj}$ and $i \neq j$, then the matrix $A_{k:ij} = \left[A^T_{ij:k}\right]^T$ is called the contraction of A on row k relative to columns i and j. We say that A can be contracted to a matrix B if either $B = A$ or there exist matrices $A_0, A_1, ..., A_t$ $(t \geq 1)$ such that $A_0 = A$, $A_t = B$, and A_r is a contraction of A_{r-1} for $r = 1, ..., t$ [10].

Brualdi and Gibson [10] proved the following result about the permanent of a matrix.

Lemma 2.1. *Let A be a nonnegative integral matrix of order n for $n > 1$ and let B be a contraction of A. Then*

$$\text{per}A = \text{per}B. \tag{3}$$

Let H_n be an $n \times n$ $(0,1)$-matrix having form

$$H_n = \begin{pmatrix} 1 & 1 & 0 & 0 & \cdots & \cdots & \cdots & 0 \\ 1 & 1 & 0 & 1 & 0 & & & \vdots \\ 0 & 1 & 1 & 1 & \ddots & \ddots & & \vdots \\ \vdots & 0 & 1 & \ddots & \ddots & \frac{1+(-1)^j}{2} & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \frac{1+(-1)^j}{2} & \ddots & 0 \\ \vdots & & & \ddots & \ddots & \ddots & \ddots & \frac{1+(-1)^n}{2} \\ \vdots & & & & 0 & 1 & 1 & \frac{1+(-1)^n}{2} \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 & 1 \end{pmatrix} \tag{4}$$

where

$$h_{ij} = \begin{cases} 1, & \text{if } j-i=-1 \text{ or } j-i=0, \\ \frac{1+(-1)^j}{2}, & \text{if } j-i=1 \text{ or } j-i=2, \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 2.2. *Let $G(H_n)$ be the bipartite graph with bipartite adjacency matrix H_n given by (4). Then, the number of perfect matchings of $G(H_n)$ is $\left\lfloor \frac{n+2}{2} \right\rfloor$th Pell number $P\left(\left\lfloor \frac{n+2}{2} \right\rfloor\right)$, where $\lfloor x \rfloor$ is the largest integer less than or equal to x.*

Proof. Let H_n^r be the rth contraction of the matrix H_n, $1 \le r \le n-2$. By definition of H_n, the matrix H_n can be contracted on column 1 so that

$$H_n^1 = \begin{pmatrix} 2 & 0 & 1 & 0 & \cdots & \cdots & \cdots & 0 \\ 1 & 1 & 1 & 0 & 0 & & & \vdots \\ 0 & 1 & 1 & 0 & \ddots & \ddots & & \vdots \\ \vdots & 0 & 1 & \ddots & \ddots & \frac{1+(-1)^j}{2} & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \frac{1+(-1)^j}{2} & \ddots & 0 \\ \vdots & & & \ddots & \ddots & \ddots & \ddots & \frac{1+(-1)^n}{2} \\ \vdots & & & & 0 & 1 & 1 & \frac{1+(-1)^n}{2} \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 & 1 \end{pmatrix}.$$

Since the matrix H_n^1 can be contracted on column 1 and $P(2)=2$, $P(1)=1$

$$H_n^2 = \begin{pmatrix} 2 & 3 & 0 & 0 & \cdots & \cdots & \cdots & 0 \\ 1 & 1 & 0 & 1 & 0 & & & \vdots \\ 0 & 1 & 1 & 1 & \ddots & \ddots & & \vdots \\ \vdots & 0 & 1 & \ddots & \ddots & \frac{1+(-1)^j}{2} & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \frac{1+(-1)^j}{2} & \ddots & 0 \\ \vdots & & & \ddots & \ddots & \ddots & \ddots & \frac{1+(-1)^n}{2} \\ \vdots & & & & 0 & 1 & 1 & \frac{1+(-1)^n}{2} \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} P(2) & P(2)+P(1) & 0 & 0 & \cdots & \cdots & \cdots & 0 \\ 1 & 1 & 0 & 1 & 0 & & & \vdots \\ 0 & 1 & 1 & 1 & \ddots & \ddots & & \vdots \\ \vdots & 0 & 1 & \ddots & \ddots & \frac{1+(-1)^j}{2} & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \frac{1+(-1)^j}{2} & \ddots & 0 \\ \vdots & & & \ddots & \ddots & \ddots & \ddots & \frac{1+(-1)^n}{2} \\ \vdots & & & & 0 & 1 & 1 & \frac{1+(-1)^n}{2} \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 & 1 \end{pmatrix}.$$

Furthermore, the matrix H_n^2 can be contracted on column 1 and taking into account (1), so that

$$H_n^3 = \begin{pmatrix} P(3) & 0 & P(2) & 0 & \cdots & \cdots & \cdots & 0 \\ 1 & 1 & 1 & 0 & 0 & & & \vdots \\ 0 & 1 & 1 & 0 & \ddots & \ddots & & \vdots \\ \vdots & 0 & 1 & \ddots & \ddots & \frac{1+(-1)^j}{2} & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \frac{1+(-1)^j}{2} & \ddots & 0 \\ \vdots & & & \ddots & \ddots & \ddots & \ddots & \frac{1+(-1)^n}{2} \\ \vdots & & & & 0 & 1 & 1 & \frac{1+(-1)^n}{2} \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 & 1 \end{pmatrix}.$$

Continuing this process, we derive the rth contraction of H_n as: If r is odd,

$$H_n^r = \begin{pmatrix} P\left(\frac{r+1}{2}+1\right) & 0 & P\left(\frac{r+1}{2}\right) & 0 & \cdots & \cdots & \cdots & 0 \\ 1 & 1 & 1 & 0 & 0 & & & \vdots \\ 0 & 1 & 1 & 0 & \ddots & \ddots & & \vdots \\ \vdots & 0 & 1 & \ddots & \ddots & \frac{1+(-1)^j}{2} & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \frac{1+(-1)^j}{2} & \ddots & 0 \\ \vdots & & & \ddots & \ddots & \ddots & \ddots & \frac{1+(-1)^n}{2} \\ \vdots & & & & 0 & 1 & 1 & \frac{1+(-1)^n}{2} \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 & 1 \end{pmatrix}$$

and if r is even,

$$H_n^r = \begin{pmatrix} P\left(\frac{r}{2}+1\right) & P\left(\frac{r}{2}+1\right)+P\left(\frac{r}{2}\right) & 0 & 0 & \cdots & \cdots & \cdots & 0 \\ 1 & 1 & 0 & 1 & 0 & & & \vdots \\ 0 & 1 & 1 & 1 & \ddots & \ddots & & \vdots \\ \vdots & 0 & 1 & \ddots & \ddots & \frac{1+(-1)^j}{2} & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \frac{1+(-1)^j}{2} & \ddots & 0 \\ \vdots & & & \ddots & \ddots & \ddots & \ddots & \frac{1+(-1)^n}{2} \\ \vdots & & & & 0 & 1 & 1 & \frac{1+(-1)^n}{2} \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 & 1 \end{pmatrix}$$

for $3 \le r \le n-3$. Notice that if n is odd (even) then $r = n-3$ is even (odd). Consequently,

$$H_n^{n-3} = \begin{cases} \begin{pmatrix} P\left(\frac{n-1}{2}\right) & P\left(\frac{n-1}{2}\right)+P\left(\frac{n-1}{2}-1\right) & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} & \text{if } n \text{ is odd,} \\ \begin{pmatrix} P\left(\frac{n}{2}\right) & 0 & P\left(\frac{n}{2}-1\right) \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} & \text{if } n \text{ is even.} \end{cases}$$

which, by contraction of H_n^{n-3} on column 1 and taking into account (1), gives

$$H_n^{n-2} = \begin{cases} \begin{pmatrix} P\left(\frac{n+1}{2}\right) & 0 \\ 1 & 1 \end{pmatrix}, & \text{if } n \text{ is odd,} \\ \begin{pmatrix} P\left(\frac{n}{2}\right) & P\left(\frac{n}{2}\right)+P\left(\frac{n}{2}-1\right) \\ 1 & 1 \end{pmatrix}, & \text{if } n \text{ is even.} \end{cases} \tag{5}$$

By applying the equation (3) to the expression (5) and taking into account (1), we obtain

$$\mathrm{per}H_n = \mathrm{per}H_n^{n-2} = \begin{cases} P\left(\frac{n+1}{2}\right), & \text{if } n \text{ is odd,} \\ P\left(\frac{n+2}{2}\right), & \text{if } n \text{ is even,} \end{cases}$$

which is deduced that $\mathrm{per}H_n = P\left(\left\lfloor \frac{n+2}{2} \right\rfloor\right)$. So, the proof is completed. □

Let K_n be an $n \times n$ $(0,1)$-matrix having form

$$K_n = \begin{pmatrix} 1 & 0 & 1 & 0 & \cdots & \frac{1-(-1)^j}{2} & \cdots & \frac{1-(-1)^n}{2} \\ 1 & 1 & 1 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & 1 & 1 & 0 & 0 & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & 1 & 1 & \frac{1-(-1)^j}{2} & \ddots & \vdots \\ \vdots & & & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & & & \ddots & \ddots & 1 & \frac{1-(-1)^n}{2} \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 & 1 \end{pmatrix} \tag{6}$$

where

$$k_{ij} = \begin{cases} 1, & \text{if } j-i=-1 \text{ or } j-i=0, \\ \frac{1-(-1)^j}{2}, & \text{if } i=1 \text{ or } j-i=1, \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 2.3. *Let $G(K_n)$ be the bipartite graph with bipartite adjacency matrix K_n given by (6). Then, the number of perfect matchings of $G(K_n)$ is $\left\lfloor \frac{n+1}{2} \right\rfloor$th Mersenne number $M\left(\left\lfloor \frac{n+1}{2} \right\rfloor\right)$, where $\lfloor x \rfloor$ is the largest integer less than or equal to x.*

Proof. Let K_n^r be the rth contraction of B_n for $1 \le r \le n-3$. By applying successive contractions to the matrices K_n^r for $1 \le r \le n-3$ according to their first columns, we get

$$K_n^{n-2} = \begin{cases} \begin{pmatrix} M\left(\frac{n-1}{2}\right) & M\left(\frac{n-1}{2}\right)+1 \\ 1 & 1 \end{pmatrix}, & \text{if } n \text{ is odd,} \\ \begin{pmatrix} M\left(\frac{n}{2}\right) & 0 \\ 1 & 1 \end{pmatrix}, & \text{if } n \text{ is even.} \end{cases} \tag{7}$$

By applying the equation (3) to the expression (7) and taking into account (2), we obtain

$$\mathrm{per}K_n = \mathrm{per}K_n^{n-2} = \begin{cases} M\left(\frac{n+1}{2}\right), & \text{if } n \text{ is odd,} \\ M\left(\frac{n}{2}\right), & \text{if } n \text{ is even,} \end{cases}$$

which is deduced that $\mathrm{per}K_n = M\left(\left\lfloor \frac{n+1}{2} \right\rfloor\right)$. So, it is desired. $\square$

In [23, Theorem 2], we can reach the following result regarding the relationship between Perrin numbers and the permanent of a certain upper Hessenberg matrix.

Theorem 2.4. *Let $B_n = (b_{ij})$ be the $n \times n$ matrix such that $b_{ij} = 2$ if and only if $i = 1$ and $j = 1$, $b_{ij} = 3$ if and only if $i = 1$ and $j = 2$, $b_{ij} = 1$ if and only if $j - i = -1$ or $i > 1$ and $j - i = 1$, or $i > 1$ and $j - i = 2$ and otherwise 0. Clearly,*

$$B_n = \begin{pmatrix} 2 & 3 & 0 & 0 & \cdots & \cdots & 0 \\ 1 & 0 & 1 & 1 & 0 & & \vdots \\ 0 & 1 & 0 & 1 & 1 & \ddots & \vdots \\ \vdots & 0 & 1 & 0 & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & 1 & 1 \\ \vdots & & & \ddots & 1 & 0 & 1 \\ 0 & \cdots & \cdots & \cdots & 0 & 1 & 0 \end{pmatrix}. \tag{8}$$

Then the permanent of B_n is the $(n+1)$st Perrin number $R(n+1)$.

Let $S_n = (s_{ij})$ be the $n \times n$ $(0,1)$ -matrix defined by $s_{ij} = 1$ if and only if $|j - i| = 1$ or $j - i = 2$. Let $T_n = (t_{ij})$ be the $n \times n$ tridiagonal $(0,1)$-matrix with $t_{11} = t_{22} = 1$. Let $U_n = (u_{ij})$ be the $n \times n$ $(0,1)$-matrix with $u_{35} = 1$. Then we can give the following theorem.

Theorem 2.5. *Let $G(L_n)$ be the bipartite graph with bipartite adjacency matrix $L_n = S_n + T_n + U_n$ for $n \geq 3$. Then, the number of perfect matchings of $G(L_n)$ is $(n-1)$st Perrin number $R(n-1)$.*

Proof. Let L_n^r be the rth contraction of the matrix L_n, $1 \leq r \leq n-2$. By definition of L_n, the matrix L_n can be contracted on column 1 so that

$$L_n^1 = \begin{pmatrix} 2 & 2 & 1 & 0 & \cdots & \cdots & \cdots & 0 \\ 1 & 0 & 1 & 0 & 0 & & & \vdots \\ 0 & 1 & 0 & 1 & 1 & 0 & & \vdots \\ \vdots & 0 & 1 & 0 & 1 & 1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & & \ddots & \ddots & 0 & 1 & 1 \\ \vdots & & & & 0 & 1 & 0 & 1 \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 & 0 \end{pmatrix}.$$

If the matrix L_n^1 can be contracted on column 1, then

$$L_n^2 = \begin{pmatrix} 2 & 3 & 0 & 0 & \cdots & \cdots & \cdots & 0 \\ 1 & 0 & 1 & 1 & 0 & & & \vdots \\ 0 & 1 & 0 & 1 & 1 & 0 & & \vdots \\ \vdots & 0 & 1 & 0 & 1 & 1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & & \ddots & \ddots & 0 & 1 & 1 \\ \vdots & & & & 0 & 1 & 0 & 1 \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 & 0 \end{pmatrix} \tag{9}$$

which is equal to B_{n-2}, where B_n is the matrix defined by (8). By applying the equation (3) to the expression (9) and taking into account Theorem 2.4, we obtain

$$\text{per}L_n = \text{per}L_n^2 = \text{per}B_{n-2} = R\,(n-1)\,,$$

which is desired. □

Appendix A. The following Maple procedure calculates the numbers of perfect matchings of bipartite graph $G(H_n)$ given in Theorem 2.2.

```
restart:
with(LinearAlgebra):
permanent:=proc(n)
local i,j,r,h,H;
h:=(i,j)->piecewise(j-i=-1,1,j-i=0,1,j-i=1,(1+(-1)^j)/2, j-i=2, (1+
(-1)^j)/2, 0);
H:=Matrix(n,n,h):
for r from 0 to n-2 do
print(r,H):
for j from 2 to n-r do
H[1,j]:=H[2,1]*H[1,j]+H[1,1]*H[2,j]:
od:
H:=DeleteRow(DeleteColumn(Matrix(n-r,n-r,H),1),2):
od:
print(r,eval(H)):
end proc:with(LinearAlgebra):
permanent(n);
```

Appendix B. The following Maple procedure calculates the numbers of perfect matchings of bipartite graph $G(K_n)$ given in Theorem 2.3.

```
restart:
with(LinearAlgebra):
permanent:=proc(n)
local i,j,r,k,K;
k:=(i,j)->piecewise(i=1,(1-(-1)^j)/2, j-i=-1, 1, j-i=0, 1, j-i=
1,(1-(-1)^j)/2, 0);
K:=Matrix(n,n,k):
for r from 0 to n-2 do
print(r,K):
for j from 2 to n-r do
K[1,j]:=K[2,1]*K[1,j]+K[1,1]*K[2,j]:
od:
K:=DeleteRow(DeleteColumn(Matrix(n-r,n-r,K),1),2):
od:
print(r,eval(K)):
end proc:with(LinearAlgebra):
permanent(n);
```

Appendix C. The following Maple procedure calculates the numbers of perfect matchings of bipartite graph $G(L_n)$ given in Theorem 2.5.

```
restart:
with(LinearAlgebra):
permanent:=proc(n)
local i,j,r,s,t,u,S,T,U,L;
s:=(i,j)->piecewise(abs(j-i)=1,1,j-i=2,1,0);
t:=(i,j)->piecewise(i=1 and j=1,1,i=2 and j=2,1,0);
u:=(i,j)->piecewise(i=3 and j=5,1,0);
S:=Matrix(n,n,s):
T:=Matrix(n,n,t):
U:=Matrix(n,n,u):
L:=S+T-U:
for r from 0 to n-2 do
print(r,L):
for j from 2 to n-r do
L[1,j]:=L[2,1]*L[1,j]+L[1,1]*L[2,j]:
od:
L:=DeleteRow(DeleteColumn(Matrix(n-r,n-r,L),1),2):
od:
print(r,eval(L)):
end proc:with(LinearAlgebra):
permanent(n);
```

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