Gradual Numbers and their Application to Fuzzy Interval Analysis

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***Abstract*— We introduce a new way of looking at fuzzy intervals. Instead of considering them as fuzzy sets, we see them as crisp sets of entities we call gradual (real) numbers. They are a gradual extension of real numbers, not of intervals. Such a concept is apparently missing in fuzzy set theory. Gradual numbers basically have the same algebraic properties as real numbers, but they are functions. A fuzzy interval is then viewed as a pair of fuzzy thresholds, which are monotonic gradual real numbers. This view enable interval analysis to be directly extended to fuzzy intervals, without resorting to α-cuts, in agreement with Zadeh's extension principle. Several results show that interval analysis methods can be directly adapted to fuzzy interval computation where end points of intervals are changed into left and right fuzzy bounds. Our approach is illustrated on two known problems: computing fuzzy weighted averages, and determining fuzzy floats and latest starting times in activity network scheduling.**

I. Introduction

The extension principle of Zadeh [1] has been mainly applied to computing with so-called "fuzzy numbers" [2]. Fuzzy arithmetic was developed in order to perform addition, subtraction multiplication and division of fuzzy numbers [3], [4], [5], [6]. This has too often led to consider that functions of fuzzy interval arguments involving arithmetic operations could be computed using fuzzy arithmetic operations, thus neglecting interactions between variables and yielding incorrect results. However since calculations with fuzzy intervals extend interval analysis [7], interval analysis methods should be more widely known and used in this context. Several authors noticed this fact and proposed to apply interval analysis to all (in practice a selection of) cuts of the fuzzy intervals, owing to Nguyen [8] showing that, under mild assumptions, the cut of the resulting fuzzy set can be computed from the cuts of the fuzzy operands. Works by Dong et al. (the vertex method) [9], Yang et Al. [10], Antonsson [11], Anile [12], Hanss [13] propose effective computational methods to this end.

In this area, a fuzzy set of numbers is often called a "fuzzy number" (for instance a triangular fuzzy number). This terminology is questionable because it has recurrently led some authors to try to equip such fuzzy numbers with the same algebraic structure as the set of numbers (a recent example is the proposal by Kolesnik et al. [14] and Kosinski et al [15]). Fuzzy numbers generalize intervals, not numbers, and thus only inherit the properties of intervals. In particular, there is no genuine notion of opposite of a fuzzy number such that the sum of a fuzzy number and its opposite is zero. Beyond the terminological difficulty (which we can live with), a pending question remains, namely what is a genuine fuzzy number, that would be to a number what a fuzzy set is to a set? This paper offers a response to this question, by proposing a rigorous definition of a so-called *gradual real number* (or *gradual number* for short). It is a generalization of "fuzzy real numbers" proposed by fuzzy set mathematicians in the seventies like Hutton [16], and further actively studied in fuzzy topology (Rodabaugh, Höhle, Lowen, especially [17], [18], [19]). Gradual numbers display fuzziness (the property of being gradual rather than sharp), but they are not sets, they are rather "fuzzy elements" [20] or "gradual elements" and do not account for incomplete information (contrary to intervals). A gradual number is not a fuzzy set, nor does it have a membership function. Instead, it is defined by a function from the unit interval to the real line. Interestingly, for such gradual numbers, genuine opposite exists and the sum of a gradual number and its opposite is zero. The notion of "gradual element" did not receive lot af attention in fuzzy set theory, even if they can be found in the literature other than purely mathematical (for instance the fuzzy integers of Rocacher [21] are gradual elements on the sets of natural integers).

Beyond the philosophical attractiveness of this concept, it has potentially useful properties for simplifying fuzzy interval analysis. It leads to considering a fuzzy interval as a crisp interval limited by a pair of (genuine) gradual real numbers. A fuzzy interval is then an interval of gradual real numbers bounded by two profiles respectively obtained by the increasing and decreasing parts of the membership function of the fuzzy interval. Selecting a gradual real number in a fuzzy interval comes down to picking an element in each alpha-cut. This view enables interval analysis to be directly applied to performing fuzzy interval analysis, yielding exact closed form expressions of the results when functions involved are locally monotonic. It also suggests a method for evaluating the potential increase of computational complexity when going from interval analysis to fuzzy interval analysis, in terms of number of kinks in the resulting membership functions.

The paper is organized as follows; Section II recalls basic notions of interval analysis. Section III does the same for fuzzy intervals and motivates the new approach. Section IV introduces the notion of gradual real numbers contained in a fuzzy interval. Section V applies standard interval analysis to fuzzy interval computation. Since fuzzy bounds of fuzzy intervals can be formally handled just like crisp bounds of intervals, the theoretical complexity of fuzzy interval analysis is the same as standard interval analysis. However the resulting membership functions may have kinks, due to the fact that the set of gradual numbers is not totally ordered. Their determination increases the practical complexity of computing with gradual real numbers. Section VI-B addresses this issue and sheds some light on their algebraic structure. These results are illustrated by several examples in section VII: fuzzy weighted average, latest starting times and floats of activities in fuzzy scheduling.

II. A Refresher On Classical Interval Computation

The goal of interval computations is to find the minimum and the maximum of a function when the different possible values of its arguments range in intervals. Formally the basic problem is: given a n-place real function f from $\mathbb{R}^n$ to $\mathbb{R}$, depending on inputs $(x_1, \cdots, x_n)$ and given n intervals $[x_i^-, x_i^+], i = 1, \ldots, n$, find the range of the variable $y = f(x_1, \cdots, x_n)$ when the x_i's lie in the intervals $[x_i^-, x_i^+]$ [7]. When the function is continuous this range is an interval.

Given such n intervals $[x_i^-, x_i^+]$, we call real *configuration* an $n-tuple$ of values in the set $\mathcal{X} = \times_i [x_i^-, x_i^+]$. Among configurations of $\mathcal{X}$, the *extreme* ones, obtained by selecting for each value one interval end, define the set $\mathcal{H} = \times_i \{x_i^-, x_i^+\}$. An element $\omega \in \mathcal{H}$ has the form $\omega = (x_1^{\epsilon_1}, \cdots, x_n^{\epsilon_n})$, with $\epsilon_i \in \{+, -\}$. The notion of configuration was proposed by Buckley for the fuzzy scheduling problem [22], but extreme configurations are also called poles in the literature [10].

Under suitable monotonicity assumptions, the maximum of f over $\mathcal{X}$ is attained on an extreme configuration, that is, in $\mathcal{H}$, or on one of its subset $\mathcal{C} \subseteq \mathcal{H}$.

Definition 1 (increasing/decreasing function): An n-place function f is said to be *increasing with respect to* x_i (respectively decreasing) if for all $n-1$-tuples $(a_1, a_2, \cdots, a_{i-1}, a_{i+1}, \cdots, a_n) \in \mathbb{R}^{n-1}$ the restriction of f from $\mathbb{R}$ to $\mathbb{R}$ $f(a_1, a_2, \cdots, a_{i-1}, x_i, a_{i+1}, \cdots, a_n)$ is increasing (respectively decreasing).

Definition 2 (monotonic function): f is said to be *monotonic with respect to each* x_i if for each variable x_i, f is either increasing or decreasing according to x_i.

Definition 3 (locally monotonic function): f is said to be *locally monotonic with respect to each* x_i if for each variable x_i, for each n-tuple $(a_1, a_2, \cdots, a_{i-1}, a_{i+1}, \cdots, a_n) \in \mathbb{R}^{n-1}$ the restriction $f(a_1, a_2, \cdots, a_{i-1}, x_i, a_{i+1}, \cdots, a_n)$ of f is monotonic.

In the latter definition, we should note that f can be increasing for one tuple and decreasing for another. Therefore, a locally monotonic function with respect to each argument is not monotonic in the usual sense of Definition 2. A differentiable function is locally monotonic with respect to x_i if the sign of its partial derivative $\frac{\partial f}{\partial x_i}$ does not depend on x_i.

We can now state a well-known proposition [7]:

Proposition 1: Let $x = (x_1, x_2, \cdots, x_n)$ be a tuple of n variables such that $x_i \in [x_i^-, x_i^+]$, and $y = f(x_1, \cdots, x_n) = [y^-, y^+]$. If f is locally monotonic with respect to each argument, then $y^- = \min_{\omega \in \mathcal{H}} \{f(\omega)\}$ and $y^+ = \max_{\omega \in \mathcal{H}} \{f(\omega)\}$

For example, the multiplication function defined by $mult(x, y) = xy$ is locally monotonic with respect to x and y: for $\frac{\partial mult}{\partial x}(x, y) = y$, so for a fixed y, the sign of $\frac{\partial mult}{\partial x}$ is constant. Note that the sign of $\frac{\partial mult}{\partial x}$ depends of the choice of the fixed y, and so $mult$ is not monotonic. Now with the help of Proposition 1, we can apply the well-known vertex method of interval arithmetics [7] (page 12) to product: $[a, b] \cdot [c, d] = [\min(ac, ad, bc, bd), \max(ac, ad, bc, bd)]$

Another known result decreases the number of fuzzy configurations useful for the computation of a function f when stronger monotony conditions hold:

Proposition 2: Under the assumption of Proposition 1, if f is locally monotonic with respect to each argument, and $\forall j \in E_1$, f is increasing according to x_j and $\forall j \in E_2$, f is decreasing according to x_j, then

$$y^- = \min_{\omega \in \mathcal{H}} \left\{ f(\omega) | \begin{array}{l} \forall j \in E_1, \omega_j = x_j^- \\ \forall j \in E_2, \omega_j = x_j^+ \end{array} \right\}$$

and $$y^+ = \max_{\omega \in \mathcal{H}} \left\{ f(\omega) | \begin{array}{l} \forall j \in E_1, \omega_j = x_j^+ \\ \forall j \in E_2, \omega_j = x_j^- \end{array} \right\}$$

This proposition is the basis of the (FWA) Algorithm [9] which computes the fuzzy weighted average using cuts of fuzzy intervals. Consider the function $fwa(w_1, \cdots, w_n, x_1, \cdots, x_n) = \frac{\sum_{i=1}^n w_i x_i}{\sum_{i=1}^n w_i}$ on the domain $(\mathbb{R}^+)^n \times \mathbb{R}^n$ ($\forall i \in [1..n]$, $w_i \in \mathbb{R}^+$ and $x_i \in \mathbb{R}$). This function corresponds to the weighted average of an n-tuple of values, with positive weights. $fwa(.)$ is locally monotonic with respect to each variable. Moreover, on this domain, we know that $fwa(.)$ is increasing according each x_i. Suppose, we must compute the range of $fwa(.)$ when the x_i's lie in the intervals $[x_i^-, x_i^+]$ and the w_i's lie in the intervals $[w_i^-, w_i^+]$. Proposition 2 tells us that only the partial configuration $(x_1^-, x_2^-, \cdots, x_n^-)$ (resp. $(x_1^+, x_2^+, \cdots, x_n^+)$) need to be considered when minimizing (resp. maximizing) $fwa(.)$, but this is not enough for the weights. In fact, existing results about this function show that only a linear complexity underlies the required search for useful configurations [23].

The range of functions such as the interval-valued average *cannot* be obtained by means of interval arithmetic operations like

- Interval addition:$[a, b] + [c, d] = [a + c, b + d]$.
- Interval subtraction: $[a, b] - [c, d] = [a - d, b - c]$.
- Positive interval product: $[a, b] \cdot [c, d] = [a \cdot c, b \cdot d]$ for $a > 0, c > 0$.
- Positive interval quotient : $\frac{[a,b]}{[c,d]} = [\frac{a}{d}, \frac{b}{c}]$ for $a > 0, c > 0$.

This is especialy true if the same variable appears several times in the expression of a function. Indeed, the notation $[a, b] - [a, b]$, $[a, b] \cdot [a, b]$ can be ambiguous. More specifically, what does $f([a, b], [a, b])$ mean? It may mean two things:

1) $\{f(x, x) | x \in [a, b]\}$
2) $\{f(x, y) | x, y \in [a, b]\}$

Indeed if $[a, b]$ restricts the value of both parameters x and y, it is false that $x = y$ follows. Generally the two results differ. For instance consider the product of the interval $[-1, +1]$ by itself:

1) $\{x \cdot x | x \in [-1, +1]\} = \{x^2 | x \in [-1, +1]\} = [0, 1]$
2) $\{x \cdot y | x \in [-1, +1]\} = [-1, 1]$

In order to remain consistent with the above interval artihmetic, $f([a, b], [a, b])$ must be interpreted as in the second expression, for the sake of clarity. Then $[a,b]^2$ differs from $[a,b]\cdot[a,b]$, the former being the range of x^2, the latter being the range of $x \cdot y$.

III. A REFRESHER ON FUZZY INTERVALS

Modeling possible values of quantities by means of real intervals accounts for some incomplete knowledge in a very simple way. However the expressive power of intervals is limited: if an interval is too narrow, the quantity it represents may lie outside it. If it is too large, the obtained results will be uninformative. But we can be more refined by modeling incomplete knowledge about a parameter x_i by means of a fuzzy interval X_i. In this paper, we shall generalize the notion of configuration to fuzzy interval problems, and we give counterparts to the previous propositions.

Fuzzy intervals are defined as follow [24]:

Definition 4: A fuzzy interval M, defined by its membership function $\mu_M(.)$ is a fuzzy subset of the real line such that, if $\forall(x,y,z) \in \mathbb{R}^3$ $z \in [x,y]$ then $\mu_M(z) \geq \min(\mu_M(x), \mu_M(y))$.

M is said to be normal iff $\exists y \in \mathbb{R}$ such that $\mu_M(y) = 1$. In this paper we only work with normal fuzzy intervals and with upper semi-continuous (USC) membership functions, in such a way that the α-cut of a fuzzy interval ($M_\alpha = \{x|\mu_M(x) \geq \alpha > 0\}$) is a closed interval.

A fuzzy interval M is then such that

- its core is a closed interval $[m_1^-, m_1^+]$, actually the 1-cut of M;
- its support is an open interval $(m_0^-, m_0^+) = \{x|\mu_M(x) > 0\}$;
- μ_M is non-decreasing on $(-\infty, m_1^-]$
- μ_M is non-increasing on $[m_1^+, +\infty)$

According the extension principle of Zadeh [1], [2], given fuzzy intervals $(X_1, \cdots, X_n)$, the fuzzy set $f(X_1, \cdots, X_n)$, image of $(X_1, \cdots, X_n)$ by f is defined by

$$\mu_{f(X_1,\cdots,X_n)}(z) = \sup_{(x_1,\cdots,x_n):z=f(x_1,\cdots,x_n)} \min_{i=1\ldots n} \mu_{X_i}(x_i).$$

In many cases (for instance, if f is continuous) the decomposition by α-cuts can be used to compute the function $f(X_1, \cdots, X_n)$ by means of standard interval calculations, due to a result of Nguyen [8]:

$$[f(X_1, \cdots, X_n)]_\alpha = f([X_1]_\alpha, \cdots, [X_n]_\alpha).$$

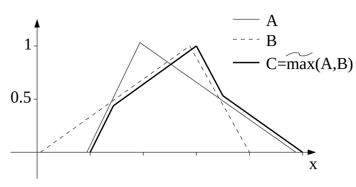

Fig. 1. Possibility distribution of two triangular fuzzy intervals A and B and their maximum C

For example, let A and B be the fuzzy intervals on Figure 1. Let C be the maximum of A and B ($C = \widetilde{\max}(A,B)$). According the extension principle of Zadeh [1], C is defined at level α by $C_\alpha = \overline{\max}(A_\alpha, B_\alpha)$, where $\overline{\max}$ is the operator maximum on classical intervals ($\overline{\max}([a,b],[c,d]) = [\max(a,c), \max(b,d)]$). Most results on fuzzy interval analysis deal with fuzzy arithmetic operations since Mizumoto and Tanaka in 1976 [3] (see also Nahmias 1978 [4], Dubois and Prade 1978 [5] and the books by Kaufmann and Gupta [6], Mares [25]). Just like for intervals, the use of fuzzy arithmetic operations to compute polynomial functions, averages, etc., with fuzzy intervals does not yield the exact fuzzy range of $f(X_1, \cdots, X_n)$. More recent books move away from fuzzy arithmetics and use interval analysis optimization methods on a finite sample of α-cuts (Hanss, 2004 [13]). This is also the idea behind the fuzzy vertex method [9].

This process has drawbacks: it computes only a discrete approximation of Y, and for each α-cut, the interval algorithm has to be completely executed (actually, sometimes only partially as explained in [13]). This method can be heavy if the interval analysis step is difficult, and seldom provides a closed form result.

The second idea is to use arithmetics on L-R fuzzy intervals (Dubois and Prade 1978 [5]). This method can give a closed form result for the basic arithmetic operations (addition, subtraction, multiplication and division of positive numbers,[26]), but may be very inaccurate if L-R approximations of the results are used.

In this paper, we try to envisage fuzzy interval analysis without resorting to α-cuts, and provide (at least in the theory) exact results for all possibility degrees. We propose an approach to the fuzzy interval computation problem, based on a particular representation of fuzzy intervals, as a crisp intervals of entities we call gradual real numbers. We represent a fuzzy interval by two fuzzy bounds just like a classical interval can be presented as a pair of reals, representing the two bounds of the interval. This representation enables techniques of interval analysis, for instance the vertex method, to be directly extended to computations on fuzzy intervals, under different monotonicity assumptions on the function, using the set of what we call fuzzy configurations.

The aim of this paper is to formalize rigorously this approach for a large class of functions. To proceed, we need to define a new concept which was apparently overlooked in fuzzy set theory: the gradual real numbers.

IV. GRADUAL NUMBERS VERSUS FUZZY INTERVALS

In the literature, fuzzy intervals are often called fuzzy numbers, especially if their cores reduce to a point (for instance, triangular fuzzy numbers). But such fuzzy numbers also generalize intervals, not numbers. So the calculus of fuzzy triangular numbers is an extension of interval analysis. Fuzzy arithmetics inherit algebraic properties of interval arithmetics, not of numbers. For instance the addition of fuzzy numbers does not lead to a group operation. It even appears impossible to equip the set of fuzzy intervals under addition with a group structure, as no inverse exists for intervals. So the term "fuzzy number" is misleading, even if the core of M is reduced to a singleton. From this discussion, we see that we should not call fuzzy sets of the real line whose cuts are intervals "fuzzy numbers", but fuzzy intervals. The rest of this section is devoted to give a precise definition of what could be called a gradual number, with a view to look at fuzzy intervals as crisp intervals on the set of gradual real numbers. In fact this notion has not been studied in fuzzy set theory under a specific name, even if such entities can be encountered in previous works, as seen later.

A. Another Kind of Fuzzy Number: Gradual Numbers

The origin of the confusion regarding the term "fuzzy number" is that since fuzzy numbers usually refer to a generalized interval, a fuzzy number is not for a number what a fuzzy set is to a set. The idea of fuzziness is to move from Boolean, all-or-nothing concepts to gradual ones, introducing intermediate values. So membership to a fuzzy set is gradual. However one often reads that the essence of fuzzy sets is to account for (subjective) uncertainty. The reason why this statement is correct is however not due to fuzziness, but to the use of sets as a tool for representing incomplete knowledge. We argue that in a fuzzy interval, it is the interval that models incomplete knowledge (we know that some parameter lies between two bounds), not the fuzziness per se. Intervals model uncertainty in a Boolean way: a value in the interval is possible; a value outside is impossible. What fuzziness brings is to make the boundaries of the interval softer, thus making uncertainty gradual.

Obviously the boundaries of an interval are simple, precise real numbers. There is no reason why fuzzy boundaries of fuzzy intervals should be fuzzy sets. They are gradual, but they should not convey any idea of incomplete information. They are examples of what we try to understand as fuzzy real numbers. Let μ_M^- (resp. μ_M^+) be the nondecreasing (non-increasing) part of the membership function of a fuzzy interval M. They are functions from the real line to $[0,1]$ respectively defined on $[m_0^-, m_1^-]$ and $[m_1^+, m_0^+]$. Suppose these functions are injective (that is, μ_M^- is increasing, and μ_M^+ is decreasing), and let $(\mu_M^-)^{-1}, (\mu_M^+)^{-1}$ be their inverse functions. It is well-known that for any monotonically increasing continuous function $f(x,y)$, the fuzzy interval $f(M,N)$ obtained by the extension principle has a membership function $\mu_{f(M,N)}$ such that (Dubois and Prade [27], [26]):

$$(\mu^-_{f(M,N)})^{-1}(\alpha) = f((\mu_M^-)^{-1}(\alpha), (\mu_M^-)^{-1}(\alpha)) \quad (1)$$
$$(\mu^+_{f(M,N)})^{-1}(\alpha) = f((\mu_M^+)^{-1}(\alpha), (\mu_M^+)^{-1}(\alpha)) \quad (2)$$

This result extends the interval analysis result $f([a,b],[c,d]) = [f(a,c), f(b,d)]$ for increasing functions to fuzzy intervals: the left (resp. right) fuzzy boundary of $f(M,N)$ is straightforwardly obtained from the left (resp. right) fuzzy boundaries of M and N. It is obvious that what plays the role of the boundaries a and b of the interval $[a,b]$ are the functions from $(0,1]$ to the real line $(\mu_M^-)^{-1}$ and $(\mu_M^+)^{-1}$. They are the gradual real numbers we are looking for.

Definition 5 (Gradual real number): A gradual real number (or gradual number for short) $\tilde{r}$ is defined by an assignment function $A_{\tilde{r}}$ from $(0,1]$ (the unit interval minus 0) to the reals.[1] Functions $(\mu_M^-)^{-1}$ and $(\mu_M^+)^{-1}$ are special cases of this definition. A real number r is then viewed as an assignment function $A_{\tilde{r}}$ such that $A_{\tilde{r}}(\alpha) = r$, $1 \geq \alpha > 0$. A gradual real number can be understood as a real value parametrized by α: To each value $\alpha \in (0,1]$ it assigns a real number $r_\alpha = A_{\tilde{r}}(\alpha)$. In particular, the values $(\mu_M^-)^{-1}(\alpha)$ and $(\mu_M^+)^{-1}(\alpha)$ are the endpoints of the α-cut of M (hence the exclusion of the membership value 0). Since $(\mu_M^-)^{-1}$ is increasing and $(\mu_M^+)^{-1}$ is decreasing, we do not make any specific monotonicity assumptions for assignment functions.

Since we consider mappings from $(0,1]$ to the real line and no monotonicity is required, an assignment function is not necessarily one-to-one. It may be that $A_{\tilde{r}}(\alpha) = A_{\tilde{r}}(\beta)$ when $\alpha \neq \beta$. So, an assignment function may not correspond to the inverse of any membership function, nor be interpreted as a fuzzy set. Note that a fuzzy interval M can be interpreted as an interval made fuzzy in the sense that the α-cut mapping $\alpha \longrightarrow M_\alpha$ is an assignment function from $(0,1]$ to the set of intervals. Some simple properties of these gradual numbers are as follows:

- Using assignment functions, the sum of gradual numbers $\tilde{r}$ and $\tilde{s}$ is simply defined by summing their assignment functions. It is $\tilde{r} + \tilde{s}$ such that $\forall \alpha \in (0,1]$,
$$A_{\tilde{r}+\tilde{s}}(\alpha) = A_{\tilde{r}}(\alpha) + A_{\tilde{s}}(\alpha).$$
- The set of the gradual real numbers with the addition operation forms a commutative group with identity $\tilde{0}$ ($\mathcal{A}_{\tilde{0}}(\lambda) = 0,\ \forall \lambda \in (0,1]$). Indeed the gradual real number $\tilde{r}$ has an inverse $-\tilde{r}$ under the addition: $A_{-\tilde{r}}(\alpha) = -A_{\tilde{r}}(\alpha)$, and $\tilde{r} + (-\tilde{r}) = \tilde{0}$.
- From the group structure we naturally define the subtraction operation by: $A_{\tilde{r}-\tilde{s}}(\alpha) = A_{\tilde{r}}(\alpha) + A_{-\tilde{s}}(\alpha) = A_{\tilde{r}}(\alpha) - A_{\tilde{s}}(\alpha)$.
- The product and the quotient of fuzzy numbers can be defined likewise (up to caution when dividing by a gradual real number $\tilde{r}$ such that $A_{\tilde{r}}(\alpha) = 0$ for some α).

Most algebraic properties of real numbers are preserved for gradual real numbers, contrary to the case of fuzzy intervals. For the latter, the lack of inverses is because they inherit the properties of interval calculations. However gradual real numbers are not totally ordered. A partial order on gradual real numbers can be defined as follows

Definition 6: A gradual real number $\tilde{r}$ is greater than a gradual real number $\tilde{s}$ (written $\tilde{r} \geq \tilde{s}$) if and only if $\forall \alpha \in (0,1], A_{\tilde{r}}(\alpha) \geq A_{-\tilde{s}}(\alpha)$.

When assignment functions are the inverse of probabilistic cumulative distributions, this definition coincides with stochastic dominance.

B. Related works

Actually, a special case of gradual number called "fuzzy (real) number" was used by mathematicians in the late 1970s and the 1980s, starting with Hutton [5].

[1]We use the term "gradual" number so as to avoid confusions with usual fuzzy numbers (triangular fuzzy intervals) or fuzzy real numbers, as used in fuzzy topology.

Definition 7: A fuzzy real number r is a decreasing membership function μ_r such that $\lim_{x\to-\infty}\mu_r(x) = 1$ and $\lim_{x\to+\infty}\mu_r(x) = 0$.

A real number r is then viewed as a step-function μ_r with membership 1 for $x \leq r$ and 0 for $x > r$. There is indeed a one-to-one mapping between such sets and reals[2]. For fuzzy real numbers, the transition from one side of r to the other is made gradual. Often, this notion takes the form of a decreasing mapping from the reals to the unit interval or a suitable lattice (Grantner et al. [28]), or a probability distribution function (Lowen [19]); variants of such a fuzzy reals were also studied by Rodabaugh [17] and Höhle [18]. The sum of two fuzzy Hutton real numbers $\tilde{r}$ and $\tilde{s}$ has been defined such that: $\mu_{\tilde{r}+\tilde{s}}(x+y) = \alpha$, for the unique x and y such that $\mu_{\tilde{r}}(x) = \mu_{\tilde{s}}(y) = \alpha$. If monotonic decreasing, the assignment function $A_{\tilde{r}}$ is the converse of the membership function of a fuzzy real number a la Hutton. However, under Definition 7, such fuzzy real numbers have no inverse for addition, because the decreasing property of such inverse can not be ensured. Then the subtraction can not be easily introduced in this setting. Worse, even if there is a bijection between real numbers and their associated step-function, results of a subtraction will differ according to whether this step-function is viewed as an assignment function or the characteristic function of a set. Consider a real number r

- If r is equated to the assignment function $A_r(\alpha) = r\ \forall\alpha$, the fact that $r - r = 0$ is recovered using the difference of gradual real numbers.
- If r is equated to the characteristic function of the semi-open interval $(-\infty, r]$, then it is clear that $(-\infty, r] - (-\infty, r] = (-\infty, r] + (-r, \infty]$ is the whole set of reals. It means that μ_r is not the encoding of a real number. In particular $-r$ is encoded by the characteristic function of $(-\infty, -r]$ in the Hutton setting, not by the interval $(-r, \infty] = -(-\infty, r]$.

Definition 5 is a direct way of defining the set of gradual real numbers, but we can construct this set differently, using the classical approach of set theory, starting with integers. Fuzzy natural integers have been recently defined by Rocacher [29] as the cardinality of a finite fuzzy subset. In many previous publications, the fuzzy cardinality of a fuzzy set is itself viewed as a fuzzy set of natural integers [30], [31], [32]. But this is misleading because cardinality maps finite sets to natural integers, not to subsets thereof. A fuzzy set (insofar as it accounts for gradual membership, not uncertain two-valued membership) is precise, because its membership function is precisely defined. Hence its cardinality is well-known even if parameterized by the membership degrees expressing tolerance levels on membership. Given a fuzzy subset F of a finite set S, the fuzzy cardinality of F is a fuzzy natural integer $|F|$ with assignment function $A_{|F|}$ defined by $A_{|F|}(\alpha) = |F_\alpha|$. It is an injective (and non-increasing) function from $(0, 1]$ to the natural integers, hence its inverse can be (mis)interpreted as a membership function. Rocacher [21] then constructs a relative fuzzy integer as the difference between two fuzzy integers and then construct the set of fuzzy rational numbers just as rationals are constructed from relative integers. Gradual real numbers can then be viewed as limits of fuzzy rational numbers.

Similarly one may argue that the fuzzy probability of a fuzzy event F is a gradual number in $[0, 1]$, assigning to each membership grade α the probability $P(F_\alpha)$. Gradual numbers naturally appear in the same way when computing Hausdorff distances between fuzzy intervals M and N (Dubois and Prade [33]), assigning to each membership grade α the distance $d(M_\alpha, N_\alpha)$. This assignment function is usually non-monotonic.

Gradual numbers also appear in some defuzzification methods like the old proposal by Yager [34]: Given a fuzzy interval M he proposed, for ranking purposes, to substitute a real number $r(M)$ defined by

$$r(M) = \int_0^1 \frac{infM_\alpha + \sup M_\alpha}{2} d\alpha$$

It is clear that the function assigning to each membership grade α the number $m_\alpha = \frac{infM_\alpha + \sup M_\alpha}{2}$ is the assignment function of a gradual number which can be viewed as the gradual midpoint of the fuzzy interval M.

With this analysis in mind, it is clear that a notion such as "defuzzification", usually understood as changing a fuzzy interval into a number, is also somewhat misleading. Stripping a fuzzy set from its fuzziness, a crisp set results, not a single element. Similarly, defuzzifying a fuzzy interval should give an interval. The idea of interval-valued defuzzification has been studied by Dubois and Prade [35] in terms of imprecise probabilities, and more recently by Ralescu [36], for instance, as the Aumann integral of the α-cut mapping. If a single element is to be obtained in the end, it can be so by picking a default value after the set-valued defuzzification, thus getting rid of the incompleteness of the information. The two steps can be exchanged by picking a gradual number in the fuzzy interval, thus getting an entity that is fuzzy but no longer incomplete, and defuzzifying it in a second step. This is what Yager method does.

C. Fuzzy Intervals as Crisp Intervals of Gradual Numbers

Using the notion of gradual number, we can describe a fuzzy interval M by an ordered pair of gradual numbers $(\tilde{m}^-, \tilde{m}^+)$. $\tilde{m}^-$ is called the fuzzy lower bound (or left profile) of M and $\tilde{m}^+$ the fuzzy upper bound (or right profile). To ensure the well known form of a fuzzy interval in Definition 4, several properties of $\tilde{m}^-$ and $\tilde{m}^+$ must hold:

- the domains of $\mathcal{A}_{\tilde{m}^-}$ and $\mathcal{A}_{\tilde{m}^+}$ must be $(0, 1]$
- $\mathcal{A}_{\tilde{m}^-}$ need to be increasing
- $\mathcal{A}_{\tilde{m}^+}$ need to be decreasing
- $\tilde{m}^-$ and $\tilde{m}^+$ must be well ordered ($\mathcal{A}_{\tilde{m}^-} \leq \mathcal{A}_{\tilde{m}^+}$).

Such a pair of gradual numbers intuitively describe a fuzzy interval with membership function:

$$\mu_M(x) = \begin{cases} \sup\{\lambda | \mathcal{A}_{\tilde{m}^-}(\lambda) \leq x\} & \text{if } x \in \mathcal{A}_{\tilde{m}^-}((0,1]) \\ 1 & \text{if } \mathcal{A}_{\tilde{m}^-}(1) \leq x \leq \mathcal{A}_{\tilde{m}^+}(1) \\ \sup\{\lambda | \mathcal{A}_{\tilde{m}^+}(\lambda) \geq x\} & \text{if } x \in \mathcal{A}_{\tilde{m}^+}((0,1]) \\ 0 & \text{otherwise} \end{cases}$$

[2]In fact, mathematicians consider equivalence classes of such membership functions; members of the same class differ only for discontinuity points.

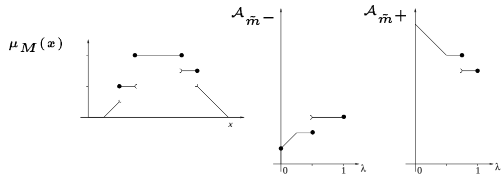

Fig. 2. Examples of left and right profiles of a non continuous fuzzy interval

Note that in the case that $\mathcal{A}_{\tilde{m}^-}$ and $\mathcal{A}_{\tilde{m}^+}$ have inverses, the last equation becomes:

$$\mu_M(x) = \begin{cases} (\mathcal{A}_{\tilde{m}^-})^{-1}(x) & \text{if } x \in \mathcal{A}_{\tilde{m}^-}((0,1]) \\ 1 & \text{if } \mathcal{A}_{\tilde{m}^-}(1) \le x \le \mathcal{A}_{\tilde{m}^+}(1) \\ (\mathcal{A}_{\tilde{m}^+})^{-1}(x) & \text{if } x \in \mathcal{A}_{\tilde{m}^+}((0,1]) \\ 0 & \text{otherwise} \end{cases}$$

Generally, we will use the words "left (resp. right) profile", to speak about the fuzzy lower (resp. higher) bound of a fuzzy interval, and the expression "gradual number" to deal with any other kind of gradual element of the real line. However, we should keep in mind that a fuzzy bound, or a "profile", is a gradual number.

An USC fuzzy interval can be entirely defined by its left profile and its right profile (this is not the case for all fuzzy sets, since two profiles can only define the membership function of a fuzzy interval, the α-cut defining bounds of intervals): a fuzzy interval M with an upper semi continuous membership function, can be described by the pair of gradual reals $(\tilde{m}^-, \tilde{m}^+)$ defined by

$$\begin{array}{rrcl} \mathcal{A}_{\tilde{m}^-}: & (0,1] & \longrightarrow & \mathbb{R} \\ & \lambda & \longmapsto & \mathcal{A}_{\tilde{m}^-}(\lambda) = inf\{x|\mu_M(x) \ge \lambda\} \end{array} \tag{3}$$

$$\begin{array}{rrcl} \mathcal{A}_{\tilde{m}^+}: & (0,1] & \longrightarrow & \mathbb{R} \\ & \lambda & \longmapsto & \mathcal{A}_{\tilde{m}^+}(\lambda) = \sup\{x|\mu_M(x) \ge \lambda\} \end{array} \tag{4}$$

So the gradual real $\tilde{m}^-$ represents the left profile of interval M, and $\tilde{m}^+$ represents the right profile of M. Just as $[a,b]$ stands for the set $\{r : a \le r \le b\}$, a fuzzy interval M defined by an ordered pair of gradual numbers $(\tilde{m}^-, \tilde{m}^+)$, with

$$\mathcal{A}_{\tilde{m}^-} = (\mu_{M^-})^{-1}, \mathcal{A}_{\tilde{m}^+} = (\mu_{M^+})^{-1}$$

It stands for a crisp interval of gradual numbers $\{\tilde{r}|(\mu_{M^-})^{-1} \le \mathcal{A}_{\tilde{r}} \le (\mu_{M^+})^{-1}\}$. So we can write $\tilde{r} \in M$ whenever $(\mu_{M^-})^{-1} \le \mathcal{A}_{\tilde{r}} \le (\mu_{M^+})^{-1}$ to denote the (crisp) membership of a fuzzy real number in a fuzzy interval.

Note that the profiles of M (Figure 2) respect Equations (3) and (4), and therefore, the membership function of M can be exactly recovered from $\tilde{m}^-$ and $\tilde{m}^+$. Note that the left and right profiles of an USC fuzzy interval are both left-continuous. The left profile is a lower semi-continuous increasing function (in the wide sense), and on the contrary, the right profile is an USC decreasing function.

The definition of a fuzzy interval as a pair of gradual numbers is akin to the so-called graded numbers of Herencia [37]. This author also considers fuzzy numbers as mappings from the unit interval to the set of real intervals, instead of the usual USC mapping from the reals to the unit interval. However, gradual reals are more general here because they are not necessarily monotonic. Only monotonic gradual reals are useful to define fuzzy intervals, but, as shown in the sequel, computations with fuzzy intervals may lead to non-monotonic gradual reals as intermediary results.

V. Applying Interval Analysis to Fuzzy Intervals

Now that a fuzzy interval can be viewed as a crisp interval in the set of gradual numbers, interval analysis can be directly applied to fuzzy intervals. Given a n-place real function f, its domain can be straightforwardly extended to gradual numbers, composing f and assignment functions viewed as fuzzy configurations. Left and right profiles of fuzzy intervals can be viewed as inducing fuzzy extreme configurations. On this basis, most results of interval analysis extend to fuzzy intervals, and also computation methods like the vertex method can be used with no increase in complexity in terms of number of fuzzy configurations. A preliminary version of results in this section was presented at the 2004 IEEE conference on Fuzzy Systems [38]. For the sake of simplicity, we will not tell a gradual number $\tilde{r}$ from its assignement function $\mathcal{A}_{\tilde{r}}$, and we denote $\tilde{r}(\lambda)$ the value of the assignement function $\mathcal{A}_{\tilde{r}}(\lambda)$.

A. Fuzzy Configurations

In the classical interval computation theory, one way to get the result of a computation on intervals was to find a *small* set of extreme configurations, a set on which we are sure that the function will reach its lower and upper bounds on the considered domain. An extreme configuration is a tuple of reals representing the lower or the upper bound of the interval containing each input of the function. To extend this notion to fuzzy interval computation, we need to define the concept of fuzzy extreme configuration, which is a tuple of gradual numbers.

Definition 8: Let $x = (x_1, x_2, \cdots, x_n)$ be a tuple of n independent variables, restricted by the fuzzy intervals $X_1, \cdots, X_n$ viewed as pairs of left and right profiles $(\tilde{x}_i^-, \tilde{x}_i^+)$. A fuzzy configuration Ω is a n-tuple of gradual numbers $(\tilde{r}_1, \tilde{r}_2, \cdots, \tilde{r}_n)$ such that $\forall i = 1 \ldots n$, $\tilde{x}_i^- \le \tilde{r}_i \le \tilde{x}_i^+$. A *fuzzy extreme configuration* Ω is an n-tuple of left or right profiles: $\Omega = (\tilde{x}_1^{\epsilon_1}, \tilde{x}_2^{\epsilon_2}, \cdots, \tilde{x}_n^{\epsilon_n})$, where $\epsilon_i \in \{+,-\}$. We denote $\widetilde{\mathcal{H}}$ the set of all fuzzy extreme configurations: $\widetilde{\mathcal{H}} = \times_i\{\tilde{x}_i^-, \tilde{x}_i^+\}$ $(|\widetilde{\mathcal{H}}| = 2^n)$

We denote Ω_i the i^{th} gradual number of configuration Ω. For any $\Omega \in \widetilde{\mathcal{H}}$, let $\Omega(\lambda) = (\Omega_1(\lambda), \Omega_2(\lambda), \cdots, \Omega_n(\lambda)) \in \mathbb{R}^n$ denote the classical configuration obtained at level λ. $\Omega(\lambda)$ is a vertex of the hyper-rectangle $\times_i[X_i]_\lambda$.

B. Main Results

We have defined notions of profiles and configurations for fuzzy intervals. We can now extend the range of real functions to gradual arguments.

Definition 9: Let f be a function of arity n. Let us denote $\dot{f}$ the extension of f applicable to gradual numbers: for any n-tuple of gradual reals $\Omega = (\tilde{r}_1, \tilde{r}_2, \cdots, \tilde{r}_n)$, $\dot{f}(\Omega)$ is the gradual real defined as follows: $\forall\lambda \in [0,1]$

$$\begin{aligned}\mathcal{A}_{\dot{f}(\Omega)}(\lambda) &= f(\Omega(\lambda))\\ &= f(\tilde{r}_1(\lambda), \tilde{r}_2(\lambda), \cdots, \tilde{r}_n(\lambda))\end{aligned}$$

Now, the computation of a n-place function f on fuzzy intervals $X_1, \cdots, X_n$ can then be seen as a standard interval computation of the form $f(X_1, \cdots, X_n) = \{\dot{f}(\tilde{r}_1, \cdots, \tilde{r}_n) | \tilde{r}_1 \in X_1, \cdots, \tilde{r}_n \in X_n)\}$.

Let us define a set $\xi \subseteq \{-, +\}^n$ such that for all intervals $\mathcal{X} = \times_i [x_i^-, x_i^+]$, ξ defines a set of configurations $\mathcal{H}_{\mathcal{X},\xi}$ as follows: $\mathcal{H}_{\mathcal{X},\xi} = \{(x_1^{\epsilon_1}, \cdots, x_n^{\epsilon_n}) | (\epsilon_1, \cdots, \epsilon_n) \in \xi\}$. If there are n fuzzy intervals $X_1, \cdots, X_n$, ξ also defines a set of fuzzy configurations: $\widetilde{\mathcal{H}}_\xi = \{(\tilde{x}_1^{\epsilon_1}, \cdots, \tilde{x}_n^{\epsilon_n}) | (\epsilon_1, \cdots, \epsilon_n) \in \xi\}$.

With these notations we can state the following theorem:

Theorem 1: Let $x = (x_1, x_2, \cdots, x_n)$ be a tuple of n independent variables, restricted by the fuzzy intervals $X_1, \cdots, X_n$, defined by their membership functions $\mu_{X_1}, \cdots \mu_{X_n}$ all USC. f is a function from $\mathbb{R}^n$ to $\mathbb{R}$, and $Y = [\tilde{y}^-, \tilde{y}^+]$ is the fuzzy set of the possible values of the variable $y = f(x)$.
If there is a set $\xi \subseteq \{(\epsilon_1, \cdots, \epsilon_n) | \epsilon_i \in \{-, +\}\}$, such that, for all α-cuts, f attains its maximum and minimum on $\mathcal{X}_\alpha = \times_i [X_i]_\alpha$ for a configuration in $\mathcal{H}_{\mathcal{X}_\alpha, \xi}$,
then $\tilde{y}^+ = \dot{\max}_{\Omega \in \widetilde{\mathcal{H}}_\xi} \{\dot{f}(\Omega)\}$
and $\tilde{y}^- = \dot{\min}_{\Omega \in \widetilde{\mathcal{H}}_\xi} \{\dot{f}(\Omega)\}$

Proof: Let $\alpha \in [0, 1]$ be a possibility degree. By definition of the right profile, we know that:

$$\begin{aligned}\tilde{y}^+(\alpha) &= \max\{y | \mu_Y(y) \geq Y^\alpha\}\\ &= \max\{y | y = f(x_1, \cdots, x_n), x_i \in X_i^\alpha\}\end{aligned}$$

And then, under the hypothesis of the theorem, we can write:

$$\tilde{y}^+(\alpha) = \max\{y | y = f(x_1^{\epsilon_1}, \cdots, x_n^{\epsilon_n}), (\epsilon_1, \cdots, \epsilon_n) \in \xi\},$$

which exactly means $\tilde{y}^+(\alpha) = \dot{\max}_{\Omega \in \widetilde{\mathcal{H}}_\xi} (\dot{f}(\Omega)(\alpha)\}$
This equation is true for all $\alpha \in [0, 1]$, therefore, we can conclude that $\tilde{y}^+ = \dot{\max}_{\Omega \in \widetilde{\mathcal{H}}_\xi} \{\dot{f}(\Omega)\}$ $\square$

Note that $\dot{f}(\Omega)$ ($\Omega \in \widetilde{\mathcal{H}}_\xi$) can be a non monotonic gradual number. However, the final result (after the $\dot{\max}$ operation) is a well-formed profile: increasing for $\tilde{y}^-$ and decreasing for $\tilde{y}^+$. This is due the fact that the gradual number approach provides the same result as the extension principle, and that α-cuts are nested.

As in the interval case, we can state two corollaries based on the monotony of f:

Corollary 1: Under the assumption of Theorem 1, if f is locally monotonic with respect to each argument,
then $\tilde{y}^- = \dot{\min}_{\Omega \in \widetilde{\mathcal{H}}} (\dot{f}(\Omega))$
and $\tilde{y}^+ = \dot{\max}_{\Omega \in \widetilde{\mathcal{H}}} (\dot{f}(\Omega))$
Proof: This is Theorem 1, where $\xi = \{(\epsilon_1, \cdots, \epsilon_n) | \epsilon_i \in \{-, +\}\}$ $\square$

Corollary 2: Under the assumption of Theorem 1, if f is locally monotonic with respect to each argument, and $\forall j \in E_1$, f is increasing according to x_j and $\forall j \in E_2$, f is decreasing according to x_j, then

$$\tilde{y}^- = \dot{\min}_{\Omega \in \widetilde{\mathcal{H}}} \left\{ \dot{f}(\Omega) | \begin{array}{l} \forall j \in E_1, \Omega_j = \tilde{x}_j^- \\ \forall j \in E_2, \Omega_j = \tilde{x}_j^+ \end{array} \right\}$$

and $$\tilde{y}^+ = \dot{\max}_{\Omega \in \widetilde{\mathcal{H}}} \left\{ \dot{f}(\Omega) | \begin{array}{l} \forall j \in E_1, \Omega_j = \tilde{x}_j^+ \\ \forall j \in E_2, \Omega_j = \tilde{x}_j^- \end{array} \right\}$$

Proof: Obvious with Theorem 1 and Definition of profiles. $\square$

Corollary 1 and 2 are the counterpart of Proposition 1 and 2 for fuzzy intervals. This last corollary was in fact known for strictly increasing functions [24]. Not all problems of interval computation deal with locally monotonic functions. For example, a n–place differentiable function f may have extrema on internal points x^* where partial derivatives of f equal to 0. The proposed configuration enumeration method can then be adapted by adding such constant profiles $\tilde{x}^*$ to the set of fuzzy configurations. For an example of fuzzy interval analysis based on a non-locally monotonic function, see [39] where the computation of a fuzzy empirical variance is proposed.

In the remainder of this paper, we will not tell f from its extension $\dot{f}$ applicable to profiles.

C. Why this Approach is Useful in Practice

The above setting looks elaborate, but many of its applications are really simple. Let us imagine some computations on piecewise linear fuzzy intervals (such fuzzy sets are not hard to implement [40] [41]). The profiles of such fuzzy sets are obviously piecewise linear and can be implemented in the same way. Some operations on such profiles preserve the piecewise linearity property: for example the maximum, minimum, addition, subtraction. Moreover, for addition and subtraction, no new break-points are generated, and for the minimum or maximum of two piecewise linear fuzzy intervals, the number of break-points may double in the worst case.

Besides our setting enables analytical representations of the results if the membership functions of the fuzzy intervals are defined analytically, thus extending the expressive power of parameterized representations such as L-R fuzzy numbers. This is because the results of fuzzy interval computations is much easier to express in closed form in terms of assignment functions than in terms of membership functions.

Let us see some simple examples illustrating the above claims.

1) Multiplication: A very simple application of Theorem 1 is the multiplication of two fuzzy intervals overlapping 0. This computation can be done analytically with the usual L-R representation in the case of non-negative fuzzy intervals of the same shape (Dubois and Prade [26]), but our result can be applied to the multiplication of any number of fuzzy intervals whose left and right profiles are analytically defined.

The multiplication $C = A \cdot B$ of two fuzzy intervals A and B can then be done by Corollary 1. Indeed, the function

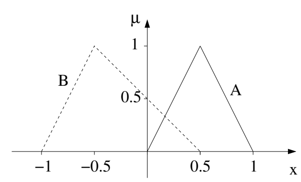

Fig. 3. Possibility distribution of two fuzzy intervals A and B

$mult(x, y) = x \cdot y$ is locally monotonic on $\mathbb{R}^2$. Then we can conclude that the following equations are valid:
$\tilde{c}^- = \min(\tilde{a}^- \cdot \tilde{b}^-, \tilde{a}^+ \cdot \tilde{b}^-, \tilde{a}^- \cdot \tilde{b}^+, \tilde{a}^+ \cdot \tilde{b}^+)$
$\tilde{c}^+ = \max(\tilde{a}^- \cdot \tilde{b}^-, \tilde{a}^+ \cdot \tilde{b}^-, \tilde{a}^- \cdot \tilde{b}^+, \tilde{a}^+ \cdot \tilde{b}^+)$
It extends a well-known formula of interval arithmetics [7] pointed out in section III. Note that for any two real functions

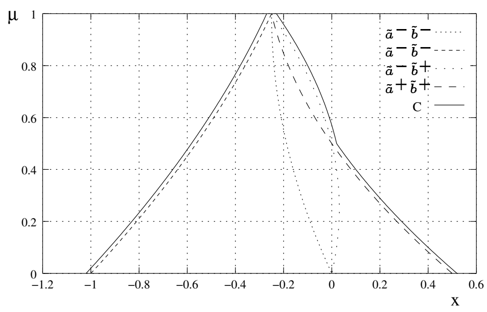

Fig. 4. $C = A \cdot B$

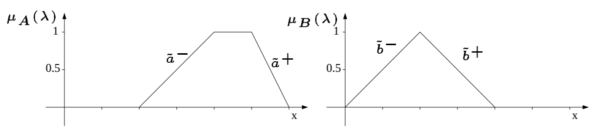

Fig. 5. Membership function of the fuzzy intervals A and B

Φ, Ψ, $\min(\Phi, \Psi) = \Phi$ *or* Ψ may not hold. It means that several extreme fuzzy configurations may be involved in the final results, depending on the considered membership level.

For example, consider the two fuzzy intervals A and B, defined by Figure 3.

The profiles of A and B are defined as follows:
$\tilde{a}^-(\lambda) = \frac{\lambda}{2}$, $\tilde{a}^+(\lambda) = 1 - \frac{\lambda}{2}$,
$\tilde{b}^-(\lambda) = \frac{\lambda}{2} - 1$, $\tilde{b}^+(\lambda) = \frac{1}{2} - \lambda$
Then we get: $(\tilde{a}^- \cdot \tilde{b}^-)(\lambda) = \frac{\lambda}{2} \cdot (\frac{\lambda}{2} - 1)$
$(\tilde{a}^+ \cdot \tilde{b}^-)(\lambda) = (1 - \frac{\lambda}{2}) \cdot (\frac{\lambda}{2} - 1)$
$(\tilde{a}^- \cdot \tilde{b}^+)(\lambda) = \frac{\lambda}{2} \cdot (\frac{1}{2} - \lambda)$
$(\tilde{a}^- \cdot \tilde{b}^-)(\lambda) = (1 - \frac{\lambda}{2}) \cdot (\frac{1}{2} - \lambda)$

The computed profile and the result $C = A \cdot B$ are shown on Figure 4. Note that $\tilde{a}^-\tilde{c}^+$ is a non monotonic gradual number, obtained as partial result. The above calculations are in the style of graded numbers [37] but some profiles obtained as partial results are not monotonic. They cannot be interpreted as membership functions. Yet, the resulting right and left profiles define the membership function of a genuine fuzzy interval. However, since assignment functions obtained in the end may intersect, kinks may appear in the resulting membership functions as patent on Figure 4.

2) Coping with multiple copies of a variable: Let h be the function defined in the form $h(x, y) = \max(x, y) - y$. h is locally monotonic with respect to each argument. Let x and y range in the fuzzy intervals A and B, defined by the membership functions of Figure 5.

The set of fuzzy extreme configurations is
$\widetilde{\mathcal{H}}_1 = \{(\tilde{a}^-, \tilde{b}^-), (\tilde{a}^+, \tilde{b}^-), (\tilde{a}^-, \tilde{b}^+), (\tilde{a}^+, \tilde{b}^+)\}$.
According to Corollary 1, we can apply h on each element of $\widetilde{\mathcal{H}}_1$ (Figure 6), put all the results of these computations on the same graph, and compute their fuzzy hull (Figure 7). Note that we again get a non-monotonic profile on Figure 6 as partial result: an example of non monotonic profile is $\max(\tilde{a}^-, \tilde{b}^+)$.

In the expression, y appears twice. One might be tempted to compute $\max(A, B) - B$ using fuzzy arithmetics. But it would give a wrong result counting the uncertainty of B twice. In

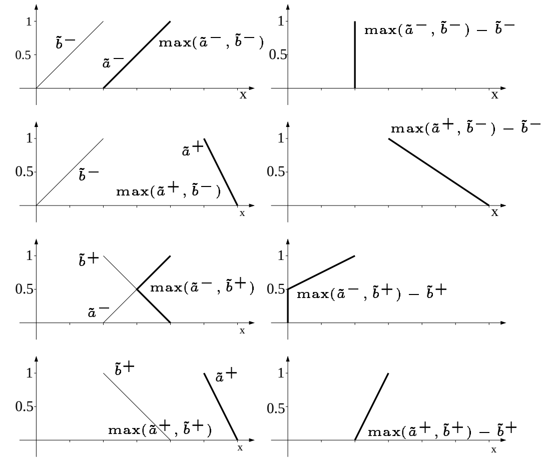

Fig. 6. Details of the computation of h applied on $\hat{H}_1$

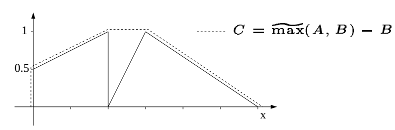

Fig. 7. Superposition of the result of h applied on $\hat{H}_1$

fact, the function h is increasing with x, and decreasing with y since: $h(x, y) = \max(x - y, 0)$. The right result is thus $\max(A - B, 0)$, more precise than $\max(A, B) - B$. However computing $\max(\tilde{r}, \tilde{s}) - \tilde{s}$ for two fuzzy real numbers $\tilde{r}, \tilde{s}$ yields the same result as computing $\max(\tilde{r} - \tilde{s}, 0)$, since gradual numbers convey no uncertainty. So, in the above computation scheme based on profiles and interval analysis, using $\max(x - y, 0)$ or $\max(x, y) - y$ is immaterial.

Yet if we notice the simplified form where variables appear only once, the computation would have been easier: Corollary 2 recommends to use only configuration on $\widetilde{\mathcal{H}}_2 = \{(\tilde{a}^+, \tilde{b}^-), (\tilde{a}^+, \tilde{b}^-)\}$. Therefore only the second and the third case on Figure 6 would have been useful to determine C completely. However, it is not always possible to rewrite the function in such a way that each variable appears once only. For instance, consider the function $g(x, y, z) = x + y + z - x \cdot y \cdot z$ on the real line. It cannot be obviously factorized at all so as to let x, y and z appear only once. It is clearly locally, not globally, monotonic. Hence fuzzy arithmetic cannot be applied, but the same computation scheme as above for the multiplication of general fuzzy intervals can provide exact analytical results.

VI. The Added Complexity of Working with Fuzzy Intervals

In Section IV-C, we (re-)defined what is classically called a fuzzy interval, using gradual numbers. These fuzzy intervals are defined by two monotonic gradual numbers, the first assignement function being increasing, and the second decreasing. Now, the monotonicity assumption of this definition can be relaxed.

From the set of all gradual numbers $\mathbb{G}$, one can define an *interval of gradual numbers*, or gradual interval for short, by

an ordered pair of gradual numbers $\tilde{r}^- \leq \tilde{r}^+$, without any monotonic restriction on them as $[\tilde{r}^-, \tilde{r}^+] = \{\tilde{r} | \tilde{r}^- \leq \tilde{r} \leq \tilde{r}^+\}$. Let us note $\mathbb{I}(\mathbb{G})$ the set of all gradual intervals.
$\mathbb{I}(\mathbb{G}) = \{[\tilde{r}^-, \tilde{r}^+] | \tilde{r}^- \in \mathbb{G}, \tilde{r}^+ \in \mathbb{G}, \tilde{r}^- \leq \tilde{r}^+\}$.
This set is the natural way to define an interval structure on a partially ordered set. Of course the obtained interval is then not a fuzzy interval in the sense used in the literature: the α-cuts are no longer nested. If one remembers that a fuzzy interval can be viewed as a nested random set (considering upper and lower probabilities induced by the α-cutmapping from the Lebesgue measure on the unit intervals [35]), a gradual interval can be the basis of of a concise representation of a random interval.

A. An Algebraic Setting for Fuzzy Intervals

The aim of this subsection is to investigate the status of gradual numbers in the scope of traditional algebraic theory. For example, a large class of graph problems, non linear in the usual sense, may appear linear in the appropriate algebraic structure [42], [43], [44]. Path of maximal capacity, shortest path, longest path problems can be treated as linear problems in an idempotent semi-ring $((\mathbb{R}, \max, +)$ for the longest path problem). These structures justify the use of the same algorithm for solving several different graph problems such as algorithms of *Jacobi*, *Gauss-Seidel*, *Gauss*...

We recall that a set S equipped with the algebraic operations addition (noted $\oplus$) and multiplication (noted $\odot$) is a semi-ring if:

- the addition $\oplus$ is closed: $\forall x, y \in S,\ x \oplus y \in S$
- the multiplication $\odot$ is closed: $\forall x, y \in S,\ x \odot y \in S$
- the addition $\oplus$ is associative: $\forall x, y, z \in S,\ x \oplus (y \oplus z) = (x \oplus y) \oplus z$
- the multiplication $\odot$ is associative: $\forall x, y, z \in S,\ x \odot (y \odot z) = (x \odot y) \odot z$
- the addition is commutative: $\forall x, y \in S,\ x \oplus y = y \oplus x$
- the multiplication $\odot$ is distributive with respect to the addition $\oplus$: $\forall x, y, z \in S,\ x \odot (y \oplus z) = (x \odot y) \oplus (x \odot z)$ and $(x \oplus y) \odot z = (x \odot z) \oplus (y \odot z)$

$(S, \oplus, \odot)$ is said idempotent if for all $x \in S$, $x \oplus x = x$.

We can show that $(\mathbb{G}, \max, +)$ is an idempotent semi–ring on gradual numbers. Now, from work of Litvinov et al. [44], we can directly deduce that $(\mathbb{I}(\mathbb{G}), \max, +)$ form an idempotent semi–ring on gradual intervals. Which means that we can use the standard tools to solve the longest path problems in graph with arcs weigthed by gradual intervals, without requesting further study. Since the traditional set of fuzzy intervals is a subset of gradual intervals $\mathbb{I}(\mathbb{G})$, the method is still valid to deal with classical fuzzy intervals, so we can solve longest path problems with fuzzy weigthed arcs with the traditional forward recursion for example, which correspond to the Jacobi Algorithm (this result was already proved by Prade in [45], but not in an algebraic way). Note that from our study, we know that we can also use Gauss or Gauss–Seidel Algorithm to obtain the same result with a better theoretical time complexity. The same reasoning is still valid for the maximal capacity path problem in the semi–ring $(\mathbb{I}(\mathbb{G}), \max, \min)$, and the shortest path problem in $(\mathbb{I}(\mathbb{G}), \min, +)$.

Of course, we can prove that fuzzy intervals with operations $\tilde{\max}$ and $+$ is an idempotent semi–ring, without resorting to gradual numbers. However, our approach sheds new light on the nature of fuzzy intervals.

There is one crucial difference between the semi-ring structure of gradual numbers and the semi-ring structure of $((\mathbb{R}, \max, +)$: the former is not totally ordered. The ordering of gradual numbers induced by the semi-ring addition is partial (it is the one in Definition 6). So even if we can use standard algorithms allowed by the algebraic structure, we need more effort to store the result of calculation of $\max(\tilde{r}, \tilde{s})$ as it can be neither $\tilde{r}$ nor $\tilde{s}$. We must store levels λ in $[0, 1]$ for which $A_{\max(\tilde{r}, \tilde{s})}(\lambda) = A_{\tilde{r}}(\lambda)$ and levels for which $A_{\max(\tilde{r}, \tilde{s})}(\lambda) = A_{\tilde{s}}(\lambda)$. In the simplest situation there may be a threshold λ above which one result obtains and under which the other result obtains. This is materialized by the presence of a kink in the resulting assignment function.

B. Handling the Kinks

In theory, one gets the impression that the computational complexity of moving from interval analysis to fuzzy interval analysis does not increase for locally monotonic functions, since we are substituting fuzzy extreme configurations to extreme configurations, and there is the same number of each. Yet a computational limitation of this fuzzy interval analysis method is due to the kinks that appear during the computation. The fact that kinks appear in the result does not come from the method, but from the nature of the exact result of the problem. However, the complexity of the representation of a fuzzy interval (or a gradual number) depends for a large part of the number of kinks in their membership (assignment) function: Suppose only piecewise linear fuzzy intervals are used. Any fuzzy bound of such an interval can be easily modeled by the list of its kinks, and so the size of the data structure is proportional to the number of kinks of the profile. We may obtain some very complex gradual numbers as intermediary results of a computation even if the final result is simple. Let us see these problems on small examples.

Let A_i be a triangular fuzzy interval defined by: $A_i = (n - i, n + 2^i, n + 2^i)$ where $i \in (N)$ and $I = (a, b, c)$ represents the fuzzy triangular interval with core b and support $[a, c]$. Figure 8 shows the representation of fuzzy intervals A_i for $n = 3, i = 1, 2, 3$.

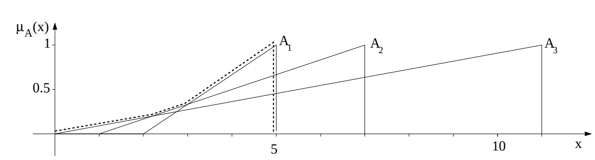

Fig. 8. Possibility distribution of the triangular fuzzy intervals A_1, A_2 and A_3

Now let M_n be the minimum of the n fuzzy intervals A_i: $M_n = \min(A_1, \cdots, A_n)$. M_n has $n - 1$ kinks on its left profile. So we can state that the maximal number of kinks that

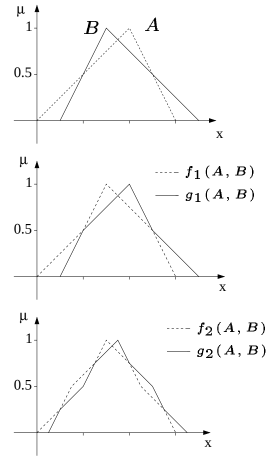

Fig. 9. $f_i(A,B)$ and $g_i(A,B)$ for $i=1,2$

can appear in the computation of the minimum of n triangular fuzzy intervals is in $O(n)$ (we can show that it is not possible to have more than $n-1$ kinks).

Now, suppose that we want to compute the minimum of $B=(-1,0,0)$ and the n intervals A_i. The result is of course $\min(B, A_1, \cdots, A_n) = B$, but the profile method may lead to compute many different gradual numbers, left profile of M_n included. So the execution of the computation may be very long for certain kind of functions or arguments, even if the result is trivial.

For more complex functions, the number of kinks can be really huge: Suppose the sequence of functions defined by[3]
$f_1(x,y) = \min(x, y-2^{-i})$
$g_1(x,y) = \max(x, y-2^{-i})$
$f_i(x,y) = \min(f_{i-1}(x,y), g_{i-1}(x,y) - 2^{-i})$
$g_i(x,y) = \max(f_{i-1}(x,y), g_{i-1}(x,y) - 2^{-i})$
and the two triangular fuzzy intervals $A=(0,2,3)$ and $B=(1,2,4)$. Figure 9 shows $f_i(A,B)$ for $i=1,2$. In this example left and right profiles of $f_i(A,B)$ and $g_i(A,B)$ have 2^i-1 kinks, which in fact corresponds to 2^i-1 $\tilde{\min}$ or $\tilde{\max}$ operations ($f_2(A,B) = \tilde{\min}(\tilde{\min}(A, B-2^{-1}), \tilde{\max}(A, B-2^{-1}) - 2^{-2})$).

So we can see on this example that it is very difficult to provide an upper bound of the number of kinks that can appear in the final result. However, such kinks are intrinsic to the problem when they belong to the result. Managing kinks is more problematic when they only appear in partial results of the computation and thus should be avoided.

In fact beyond the artificial character of the above examples, and the issue of kinks proper, the added complexity of working with fuzzy intervals stems from the situation when the useful configurations depend on a comparison test between gradual numbers. As they are not totally ordered, the result of such a comparison is that several configurations must be tried, each in a different subrange of the unit interval. This is what happens when computing the fuzzy variance of a set of fuzzy intervals [39].

[3]We thank Przemyslaw Kobylanski from the Institute of Mathematics and Computer Science (Wroclaw University of Technology, Poland) who gives us this example.

VII. Some Applications

The previous computation method can be easily applied to classical problems, like the fuzzy weighted average. Besides the motivation for developing the above framework first came from the scheduling problem under uncertainty, in the form first posed by Buckley [22]. These applications are briefly discussed below. Other problems like the computation of a fuzzy variance are also more difficult because the underlying function is not locally monotonic.

A. Fuzzy Weighted Average

The fuzzy weighted average (FWA) problem is as follows: how to obtain the fuzzy range of the function $fwa(w_1, \cdots, w_n, x_1, \cdots, x_n) = \frac{\sum_{i=1}^n w_i x_i}{\sum_{i=1}^n w_i}$, given n fuzzy intervals X_i restricting ill-known evaluations x_i, and n positive fuzzy intervals W_i restricting weights w_i. This kind of problem is typical of some decision-making procedures under multiple criteria.

The (FWA) Algorithm [9] decomposes the problem into M interval problems (corresponding to M α-cuts). Then for each α-cut, it computes $fwa(w_1, \cdots, w_n, x_1, \cdots, x_n)$ on each vertex of the hyper-rectangle $[W_1]_\alpha \times \ldots [W_n]_\alpha \times [X_1]_\alpha \times [X_n]_\alpha$, where $[Z]_\alpha$ is the α-cut of Z at level α. The maximal (respectively minimal) possible value of z at possibility level α is then the greatest (respectively the lowest) computed value. This is due the following result:

Proposition 3: The function $fwa(.)$ is locally monotonic with respect to each argument (according to Definition 3) and increasing with respect to the $x_i's$.

Proof: $\frac{\partial fwa}{\partial x_j}(w_1, \cdots, w_n, x_1, \cdots, x_n) = \frac{w_j}{\sum_{i=1}^n w_i} > 0$
Besides: $\frac{\partial fwa}{\partial w_j}(w_1, \cdots, w_n, x_1, \cdots, x_n) = \frac{x_j}{\sum_{i=1}^n w_i} - \frac{\sum_{i=1}^n w_i x_i}{(\sum_{i=1}^n w_i)^2}$
$= \frac{\sum_{i=1}^{j-1} w_i(x_j - x_i) + \sum_{i=j+1}^n w_i(x_j - x_i)}{(\sum_{i=1}^n w_i)^2}$
It follows that the sign of $\frac{\partial fwa}{\partial w_j}$ does not depend on w_j, and so $fwa(.)$ is locally monotonic with respect to w_j. □

Applying Corollary 1 to the fuzzy weighted average yields to a closed form representation of the exact FWA, with a time complexity in $O(2^n)$. On the contrary, the classical (FWA) Algorithm gives the exact value of the average only for a restricted number (say M) of possibility degrees (the rest of the result is approximated) with complexity $O(M * 2^n)$.

Polynomial algorithms have been later on developed for the real interval problem based on fractional linear programming [46], [47]. But a fuzzy configuration-based approach with linear complexity can be used to extend the interval-valued one of Lee and Park [23]. Since $fwa(.)$ is increasing with respect to the $x_i's$, the case where all x_i are precisely defined suffices to solve the problem. The sign of $\frac{\partial fwa}{\partial w_j}$ depends on the relative magnitude of the $x_i's$. Suppose the x_i are ordered such that for all $j < i$, $x_j \leq x_i$. With this order, there exists $k \in [1, n-1]$ such that for all $i \leq k$, $fwa(.)$ is decreasing with respect to w_i, and for all $i > k$, $fwa(.)$ is increasing with respect to w_i. So we can apply Theorem 1 to the set of n fuzzy configurations obtained by letting $\xi = \{(-,+,\cdots,+), (-,-,+,\cdots,+), \cdots, (-,\cdots,-,+)\}$, and compute the analytical closed form of the fuzzy weighted average as

$(fwa(W_1, \cdots, W_n, x_1, \cdots, x_n))^- =$
$$\min_{i=1,\cdots,n-1} \frac{\sum_{j=1}^{i}(W_j)^+ \cdot x_j + \sum_{j=i+1}^{n}(W_j)^- \cdot x_j}{\sum_{j=1}^{i}(W_j)^+ + \sum_{j=i+1}^{n}(W_j)^-}$$
$(fwa(W_1, \cdots, W_n, x_1, \cdots, x_n))^+ =$
$$\max_{i=1,\cdots,n-1} \frac{\sum_{j=1}^{i}(W_j)^- \cdot x_j + \sum_{j=i+1}^{n}(W_j)^+ \cdot x_j}{\sum_{j=1}^{i}(W_j)^- + \sum_{j=i+1}^{n}(W_j)^+}$$

B. Project Scheduling Problem

A project scheduling problem can be defined by a set of tasks (or activities) which represents the different parts of a project, and a set of precedence constraints expressing that some tasks cannot start before others are completed. When there are no resource constraints, we can represent an activity network as a directed acyclic graph where n nodes represent tasks, and arcs represent precedence constraints. In this context, the goal of a project manager is generally to minimize the makespan of the project. For a general description of project scheduling problems, the reader should refer to [48].

Three quantities are computed for each task of the project (they allow to identify the critical tasks):

- the *earliest starting date* e_i of a task i is the date before which we cannot start the task without violation of a precedence constraint.
- The *latest starting date* l_i is the date after which we cannot start the task without delaying the end of the project.
- The *float* $f_i = l_i - e_i$ is the difference between the latest starting date and the earliest starting date.

A task is then critical iff its float is null.

We note $P_{i,j}$ the set of all paths from task i to task j, and $W(p_{i,j})$ the length of path $p_{i,j} \in P_{i,j}$. The earliest starting date e_i is actually the length of the longest path from the starting task (noted 1) to task i:

$$e_i = \max\{W(p) | p \in P_{1,i}\}. \qquad (5)$$

The latest starting date l_i is the length of the longest path from the starting task to the ending task (noted n) minus the longest path from task i to the ending task.

$$l_i = \max\{W(p) | p \in P_{1,n}\} - \max\{W(p) | p \in P_{i,n}\}. \qquad (6)$$

These two quantities are computed by the PERT Algorithm based on recursion equations which only use min, max, $+$ and $-$ operators:

$$e_i = \ = \ \max\{e_j + d_j | j \in pred(i)\} \qquad (7)$$
$$l_i = \ = \ \min\{l_j - d_j | j \in succ(i)\} \qquad (8)$$

where d_j is the duration of the task j, $pred(i)$ is the set of tasks preceding i, $succ(i)$ is the set of tasks following i.

In *project scheduling problems under uncertainty* (on fuzzy PERT) a task duration can be modeled by an interval, crisp or fuzzy.

Let D_i be the fuzzy interval representing the possible valuations of the duration of task i. E_i, L_i and F_i are respectively the fuzzy earliest starting date, latest starting date, and float of i, to be computed.

For each task i, the three functions $e_i(.)$, $l_i(.)$ and $f_i(.)$ depend on n variables representing ill-known durations (if there are n tasks in the problem).

First, the earliest starting date $e_i(.)$ is increasing according to each argument. So applying Corollary 2, we directly get two recursive expressions that compute the fuzzy earlier starting date:
$\tilde{e}_i^- = \max\{\tilde{e}_j^- + \tilde{d}_j^- | j \in pred(i)\}$
$\tilde{e}_i^+ = \max\{\tilde{e}_j^+ + \tilde{d}_j^+ | j \in pred(i)\}$.
This result has been known and used for a long time (see for instance Prade [45]). Nevertheless the recursive approach can no longer be used for latest starting times [49].

For obtaining the fuzzy intervals L_i and F_i containing the latest starting date and the float of a task i, it is easy to see that the functions $l_i(.)$ and $f_i(.)$ are locally monotonic with respect to each argument. In the pure interval case, a set of configurations can be pointed out where the bounds of the quantities are attained. Namely, we can find a subset of variables according to which $l_i(.)$ and $f_i(.)$ are increasing [50]: $l_i(x_1, \cdots, x_n)$ is increasing with respect to all x_j such that j is not a successor of task i, and $f_i(x_1, \cdots, x_n)$ is increasing with respect to all x_j such that i is neither a successor nor a predecessor of i. Therefore we can apply Corollary 2. No more fuzzy configurations are necessary in the fuzzy case than in the interval case. It yields an exponential brute force method to come up with fuzzy latest starting times of operations.

The fuzzy interval containing the float F_i of task i can no longer be obtained by subtracting the fuzzy earliest starting time from the fuzzy latest starting time, as the corresponding quantities are interactive. Fuzzy floats must be computed separately from a suitable enumeration of fuzzy configurations applying Corollary 2 again, on the formal expressions of f_i in terms of $W(p)$. Building on the results in [50], a more efficient algorithm was recently developed (the Path-Algorithm) for the interval-valued problem [51], in which, the computations of $l_i(.)$ and $f_i(.)$ can be done on a *small* set of configurations. This set is the basis of the application of Theorem 1 in the fuzzy version of the problem. The minimum of $l_i(.)$ is attained on a configuration $\omega = (x_1^{\epsilon_1}, \cdots, x_n^{\epsilon_n})$ where the set of task assigned to their maximum $(\{i | \epsilon_i = +\})$ is exactly a path from task i to the ending task n. The maximum of $l_i(.)$ and $f_i(.)$ and the minimum of $f_i(.)$ is attained on a configuration $\omega = (x_1^{\epsilon_1}, \cdots, x_n^{\epsilon_n})$ where the set of tasks assigned to their maximum $(\{i | \epsilon_i = +\})$ is exactly a path from the starting task 1 to the ending task n. With Theorem 1, we obtain the exact fuzzy profiles of the latest starting dates and floats with the same time complexity as in the interval case. Actually as shown in the case of pure intervals [52], [53], this complexity is polynomial for the latest starting dates and is exponential for the floats. The path algorithm is generally exponential, even if more efficient than a brute force configuration enumeration. It can be adapted to fuzzy intervals in a natural way. Adapting the less computationally demanding techniques proposed in [53] is left for further research.

VIII. Conclusion

In many situations involving computations with fuzzy intervals, fuzzy arithmetics cannot be legitimately used. A new approach for computing with fuzzy quantities is proposed, based on a careful distinction between gradual numbers that are precise parameterized values, and fuzzy intervals, that express incomplete knowledge about an ill-known value. This paper lays bare the possibility of viewing fuzzy intervals as crisp intervals of gradual numbers, bounded by two monotonic gradual numbers acting as fuzzy bounds. In such a setting, standard interval analysis techniques can be directly extended to fuzzy intervals, and the obtained results are in agreement with the extension principle. They are more precise than what is obtained via fuzzy arithmetics. The proposed approach handles the presence of multiple copies of the same variable in the expression of a function. It is especially interesting in the case of locally monotonic functions (functions which reach their maximum and minimum values on the bounds of input intervals). Analytical closed form expressions can then often be obtained when fuzzy inputs are represented by suitable parameterized membership functions, thus extending L-R fuzzy interval arithmetics. Many potential applications to aggregation operations in decision-making, handling imprecise data in operations research, and statistics (like extracting useful information from of random fuzzy data) can be envisaged.

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