Mixed Type Higher Order Symmetric Duality Over Cones

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***Abstract*—In this paper, a new mixed type higher-order symmetric duality in scalar programming over cone is formulated. The weak, strong and converse duality theorems are proved for these programs under η -invexity/ η -pseudoinvexity assumptions. Self duality also discussed. As a special case of our duality relation, we give some known duality results. Our results generalize these existing dual formulations.**

***Index Terms*—Higher-order symmetric duality, duality theorems, higher-order invexity/generalized invexity.**

I. INTRODUCTION

Mangasarian [1] introduced the concept of second and higher-order duality for nonlinear problems. The study of higherorder duality is significant due to the computational advantage over the first order duality as it provides tighter bounds for the value of the objective function when approximations are used. Mond and Zhang [2] obtained duality results for various higher-order dual problems under higher-order invexity assumptions, Chen [3] presented Mond-Weir type higher order symmetric duality for scalar and multiobjective nondifferentiable programming problem under F-convexity while Mishra and Rueda [4] generalized Mangasarian [5] and Mond-Weir [6] type higher-order duality to higher-order type I functions. Ahmad et al. [6] discussed higher-order duality in nondifferentiable Multiobjective Programming. Recently, Ahmad [7] Unified higher-order duality in nondifferentiable multiobjective programming involving cones.

Xu [8] formulated two mixed type duals in multiobjective programming and also proved duality theorems. Ahmad and Husain [9] studied mixed symmetric multiobjective dual programs and obtained duality results under K-preinvexity and K-pseudoinvexity assumptions. Chandra et al. [10] and Yang et al. [11] discussed a mixed symmetric dual formulation for a nonlinear programming problem and for a class of nondifferentiable nonlinear programming problems, respectively. Later on, Ahmad [12] formulated mixed type symmetric dual in multiobjective programming problems ignoring nonnegativity restrictions of Bector et al. [13].

In this paper, a new mixed type higher-order symmetric duality over cone in multiobjective programming is formulated. The weak, strong and converse duality theorems are proved for these programs under η -invexity/ η -pseudoinvexity assumptions. Self duality also discussed. As a special case of our duality relation, we give some known duality results. Special cases are discussed to show that this study extends some of the known results in [14], [15] and [4]

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II. PREREQUISITES

For $N=\{1, 2, 3, ..., n\}$ and $M=\{1, 2, 3, ..., m\}$, let $J_1 \subset N,\ K_1 \subset M$ and $J_2 = N \setminus J_1$ and $K_2 = M \setminus K_1$. Let $|J_1|$ denote the number of elements in the set J_1. The other numbers $|J_2|$, $|K_1|$ and $|K_2|$ are defined similarly. Notice that if $J_1 = \varnothing$, then $J_2 = N$, that is $|J_1| = 0$ and $|J_2| = n$. Hence, $R^{|J_1|}$ is zero dimensional Euclidean space and $R^{|J_2|}$ is n-dimensional Euclidean space. It is clear that any $x \in R^n$ can be written as $x = \{x^1, x^2\},\ x^1 \in R^{|J_1|},\ x^2 \in R^{|J_2|}$. Similarly, any $y \in R^m$ can be written as $y = \{y^1, y^2\},\ y^1 \in R^{|K_1|},\ y^2 \in R^{|K_2|}$.

We consider the following programming problem :

(P) Minimize $F(x)$,

$$-g(x) \in Q, \quad x \in S$$

where $S \subseteq R^{n+m}$ and $F: S \to R$ and Q is a closed convex cone.

The following convention for vector inequalities will be used: If a, b $\in R^n$, then

$$a \geqq b \Leftrightarrow a_i \geqq b_i, i = 1, 2, ..., n;$$
$$a \geq b \Leftrightarrow a \geqq b \text{ and } a \neq b;$$
$$a > b \Leftrightarrow a_i > b_i, i = 1,2,...,n.$$

Definition 2.1 A function $\phi: S \mapsto R$ is said to be higher-order invex at $u \in S$ with respect to $\eta: S \times S \mapsto S$ and $h: S \times S \mapsto R,$ if for all $(x, p) \in S \times S,$

$$\phi(x) - \phi(u) - h(u, p) + p^T \nabla_p h(u, p) \geqq \eta^T(x,u)\{\nabla_x \phi(u) + \nabla_p h(u, p)\}.$$

Definition 2.2 A function $\phi: S \mapsto R$ is said to be higher-order pseudoinvex at $u \in R^n$ with respect to $\eta: S \times S \mapsto S$ and $h: S \times S \mapsto R,$ if for all

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$$(x,p)\in S\times S,\ \begin{array}{l}\eta^T(x,u)\{\nabla_x\phi(u)+\nabla_p h(u,p)\}\geqq 0\\ \Rightarrow \phi(x)-\phi(u)-h(u,p)+p^T\nabla_p h(u,p)\geqq 0.\end{array}$$

Unless otherwise stated, C_1, C_2, C_3 and C_4 represent closed convex cones in $R^{|J_1|}$, $R^{|J_2|}$, $R^{|K_1|}$ and $R^{|K_2|}$, respectively, with non-empty interiors and $C_i^*, i=1,2,3,4$ is its polar cones and $S_1 \subset R^n$ and $S_2 \subset R^m$ are open sets such that $C_1 \times C_2 \subset S_1 \times S_2$.

III. Higher-order Mixed Type Symmetric Duality

We consider the following pair of higher order symmetric duals and establish weak, strong and converse duality theorems.

Primal Problem (MHPC):

Minimize $L(x,y,p)=$

$$f_1(x^1,y^1)+f_2(x^2,y^2)+h_1(x^1,y^1,p^1)+h_2(x^2,y^2,p^2)$$
$$-(p^1)^T\nabla_{p^1}h_1(x^1,y^1,p^1)-(p^2)^T\nabla_{p^2}h_2(x^2,y^2,p^2)$$
$$-(y^2)^T[\nabla_{y^2}f(x^2,y^2)+\nabla_{p^2}h_2(x^2,y^2,p^2)]$$

Subject to

$$\nabla_{y^1}f(x^1,y^1)+\nabla_{p^1}h_1(x^1,y^1,p^1)\in C_3^*, \tag{3.1}$$
$$\nabla_{y^2}f(x^2,y^2)+\nabla_{p^2}h_2(x^2,y^2,p^2)\in C_4^*, \tag{3.2}$$
$$(y^1)^T[\nabla_{y^1}f(x^1,y^1)+\nabla_{p^1}h_1(x^1,y^1,p^1)]\geqq 0, \tag{3.3}$$
$$(p^1)^T[\nabla_{y^1}f(x^1,y^1)+\nabla_{p^1}h_1(x^1,y^1,p^1)]\geqq 0, \tag{3.4}$$
$$(p^2)^T[\nabla_{y^2}f(x^2,y^2)+\nabla_{p^2}h_2(x^2,y^2,p^2)]\geqq 0, \tag{3.5}$$
$$x^1\in C_2, x^2\in C_2, y^2\geqq 0, \tag{3.6}$$

Dual Problem (MHDC):

Minimize $L(x,y,p)=$

$$f_1(u^1,v^1)+f_2(u^2,v^2)+g_1(u^1,v^1,r^1)+g_2(u^2,v^2,r^2)$$
$$-(r^1)^T\nabla_{r^1}g_1(u^1,v^1,r^1)-(r^2)^T\nabla_{r^2}g_2(u^2,v^2,r^2)$$
$$-(u^2)^T[\nabla_{u^2}f^2(u^2,v^2)+\nabla_{r^2}g_2(u^2,v^2,r^2)]$$

Subject to

$$\nabla_{u^1}f_1(u^1,v^1)+\nabla_{r^1}g_1(u^1,v^1,r^1)\in C_1^*, \tag{3.7}$$
$$\nabla_{u^2}f^2(u^2,v^2)+\nabla_{r^2}g_2(u^2,v^2,r^2)\in C_2^*, \tag{3.8}$$
$$(u^1)^T[\nabla_{u^1}f_1(u^1,v^1)+\nabla_{r^1}g_1(u^1,v^1,r^1)]\leqq 0, \tag{3.9}$$
$$(r^1)^T[\nabla_{u^1}f_1(u^1,v^1)+\nabla_{r^1}g_1(u^1,v^1,r^1)]\leqq 0, \tag{3.10}$$
$$(r^2)^T[\nabla_{u^2}f^2(u^2,v^2)+\nabla_{r^2}g_2(u^2,v^2,r^2)]\leqq 0, \tag{3.11}$$
$$v^1\in C_3, v^2\in C_4, u^2\geqq 0, \tag{3.12}$$

where

(i) $f^1: R^{|J_1|}\times R^{|K_1|}\to R$,

(ii) $f^2: R^{|J_2|}\times R^{|K_2|}\to R$,

(iii) $g^1: R^{|J_1|}\times R^{|K_1|}\to R$,

(iv) $g^2: R^{|J_2|}\times R^{|K_2|}\to R$,

(v) $h^1: R^{|J_1|}\times R^{|K_1|}\times R^{|K_1|}\to R,$

(vi) $h^2: R^{|J_2|}\times R^{|K_2|}\times R^{|K_2|}\to R$, are twice differentiable functions, respectively,

(vii) $p^1\in R^{|K_1|}$, $p^2\in R^{|K_2|}$, $r^1\in R^{|J_1|}$ and $r^2\in R^{|J_2|}$.

IV. Duality Theorems

Theorem 4.1 (Weak Duality).

Let $(x^1,x^2,y^1,y^2,p^1,p^2)$ be feasible for (PP) and $(u^1,u^2,v^1,v^2,r^1,r^2)$ be feasible for (DP). Suppose that

(i) $f^1(.,v^1)$ is higher-order pseudo-invex at u^1 with respect to η_1 and $g^1(u^1,v^1,r^1)$,

(ii) $-f^1(x^1,.)$ is higher-order pseudo-invex at y^1 with respect to η_2 and $-h^1(x^1,y^1,p^1)$,

(iii) $f^2(.,v^2)$ is higher-order invex at u^2 with respect to η_3 and $g^2(u^2,v^2,r^2)$,

(iv) $-f^2(x^2,.)$ is higher-order invex at y^2 with respect to η_4 and $-h^2(x^2,y^2,p^2)$,

(v) $\eta_1(x^1,u^1)+u^1+r^1\in C_1$,

(vi) $\eta_2(v^1,y^1)+y^1+p^1\in C_3$,

(vii) $\eta_3(x^2,u^2)+u^2+r^2\in C_2$,

(viii) $\eta_4(v^2,y^2)+y^2+p^2\in C_4$.

Then

$$L(x^1,x^2,y^1,y^2,p^1,p^2)\geqq M(u^1,u^2,v^1,v^2,r^1,r^2). \tag{4.1}$$

Proof: From hypothesis (vii), (viii) and equations (3.2) and (3.8), we get

$$\eta_3(x^2,u^2)[\nabla_{x^2}f^2(u^2,v^2)+\nabla_{r^2}g_2(u^2,v^2,r^2)]$$
$$+u^2[\nabla_{x^2}f^2(u^2,v^2)+\nabla_{r^2}g_2(u^2,v^2,r^2)]$$
$$\geqq -r^2[\nabla_{x^2}f^2(u^2,v^2)+\nabla_{r^2}g_2(u^2,v^2,r^2)],$$

$$\eta_4(v^2,y^2)[\nabla_{y^2}f(x^2,y^2)+\nabla_{p^2}h_2(x^2,y^2,p^2)]$$
$$+y^2[\nabla_{y^2}f(x^2,y^2)+\nabla_{p^2}h_2(x^2,y^2,p^2)]$$
$$\geqq -p^2[\nabla_{y^2}f(x^2,y^2)+\nabla_{p^2}h_2(x^2,y^2,p^2)].$$

Which on using equations (3.5) and (3.11) implies that

$$\begin{aligned}&\eta_3(x^2,u^2)[\nabla_{x^2} f^2(u^2,v^2)+\nabla_{r^2} g_2(u^2,v^2,r^2)]\\&+u^2[\nabla_{x^2} f^2(u^2,v^2)+\nabla_{r^2} g_2(u^2,v^2,r^2)]\geqq 0,\end{aligned} \quad (4.2)$$

$$\begin{aligned}&\eta_4(v^2,y^2)[\nabla_{y^2} f(x^2,y^2)+\nabla_{p^2} h_2(x^2,y^2,p^2)]\\&+y^2[\nabla_{y^2} f(x^2,y^2)+\nabla_{p^2} h_2(x^2,y^2,p^2)]\geqq 0.\end{aligned} \quad (4.3)$$

Now from hypothesis (iii) and (iv), we have

$$\begin{aligned}&f^2(x^2,v^2)-f^2(u^2,v^2)-h^2(x^2,v^2,p^2)\\&+(p^2)^T\nabla_{p^2}h^2(x^2,v^2,p^2)+h^2(u^2,v^2,p^2)\\&-(p^2)^T\nabla_{p^2}h^2(u^2,v^2,p^2)\\&\geqq\eta_3(x^2,u^2)[\nabla_{x^2} f^2(u^2,v^2)+\nabla_{r^2} g_2(u^2,v^2,r^2)],\end{aligned}$$

and

$$\begin{aligned}&f^2(x^2,y^2)-f^2(x^2,v^2)-h^2(x^2,y^2,p^2)\\&+(p^2)^T\nabla_{p^2}h^2(x^2,y^2,p^2)+h^2(x^2,v^2,p^2)\\&-(p^2)^T\nabla_{p^2}h^2(x^2,v^2,p^2)\\&\geqq\eta_4(v^2,y^2)[\nabla_{y^2} f(x^2,y^2)+\nabla_{p^2} h_2(x^2,y^2,p^2)],\end{aligned}$$

which along with equations (4.2) and (4.3), we obtain

$$\begin{aligned}&f^2(x^2,v^2)-f^2(u^2,v^2)-h^2(x^2,v^2,p^2)\\&+(p^2)^T\nabla_{p^2}h^2(x^2,v^2,p^2)+h^2(u^2,v^2,p^2)\\&-(p^2)^T\nabla_{p^2}h^2(u^2,v^2,p^2)\\&\geqq -u^2[\nabla_{x^2} f^2(u^2,v^2)+\nabla_{r^2} g_2(u^2,v^2,r^2)],\end{aligned}$$

$$\begin{aligned}&f^2(x^2,y^2)-f^2(x^2,v^2)-h^2(x^2,y^2,p^2)\\&+(p^2)^T\nabla_{p^2}h^2(x^2,y^2,p^2)+h^2(x^2,v^2,p^2)\\&-(p^2)^T\nabla_{p^2}h^2(x^2,v^2,p^2)\\&\geqq y^2[\nabla_{y^2} f(x^2,y^2)+\nabla_{p^2} h_2(x^2,y^2,p^2)].\end{aligned}$$

Now adding the above two equation, we get

$$\begin{aligned}&f^2(x^2,y^2)-h^2(x^2,y^2,p^2)+(p^2)^T\nabla_{p^2}h^2(x^2,y^2,p^2)\\&-y^2[\nabla_{y^2} f(x^2,y^2)+\nabla_{p^2} h_2(x^2,y^2,p^2)]\\&\geqq f^2(u^2,v^2)-h^2(u^2,v^2,p^2)+(p^2)^T\nabla_{p^2}h^2(u^2,v^2,p^2)\\&-u^2[\nabla_{x^2} f^2(u^2,v^2)+\nabla_{r^2} g_2(u^2,v^2,r^2)].\end{aligned} \quad (4.4)$$

Similarly, from hypothesis (i), (ii) and equations (3.1), (3.7), we get

$$\begin{aligned}&\eta_1(x^1,u^1)[\nabla_{u^1} f_1(u^1,v^1)+\nabla_{r^1} g_1(u^1,v^1,r^1)]\\&\geqq -(u^1+r^1)[\nabla_{u^1} f_1(u^1,v^1)+\nabla_{r^1} g_1(u^1,v^1,r^1)],\end{aligned}$$

and

$$\begin{aligned}&\eta_2(v^1,y^1)[\nabla_{y^1} f(x^1,y^1)+\nabla_{p^1} h_1(x^1,y^1,p^1)]\\&\geqq -(y^1+p^1)[\nabla_{y^1} f(x^1,y^1)+\nabla_{p^1} h_1(x^1,y^1,p^1)].\end{aligned}$$

Now inequalities (3.3), (3.4), (3.9) and (3.10) gives

$$\eta_1(x^1,u^1)[\nabla_{u^1} f_1(u^1,v^1)+\nabla_{r^1} g_1(u^1,v^1,r^1)]\geqq 0,$$

and

$$\eta_2(v^1,y^1)[\nabla_{y^1} f(x^1,y^1)+\nabla_{p^1} h_1(x^1,y^1,p^1)]\geqq 0,$$

which by hypothesis (i) and (ii) implies

$$\begin{aligned}&f_1(x^1,v^1)-f_1(u^1,v^1)-h^1(x^1,v^1,p^1)\\&+(p^1)^T\nabla_{p^1}h^1(x^1,v^1,p^1)\\&+h^1(u^1,v^1,p^1)-(p^1)^T\nabla_{p^1}h^1(u^1,v^1,p^1)\geqq 0,\end{aligned}$$

and

$$\begin{aligned}&f_1(x^1,y^1)-f_1(x^1,v^1)-h^1(x^1,y^1,p^1)\\&+(p^1)^T\nabla_{p^1}h^1(x^1,y^1,p^1)\\&+h^1(x^1,v^1,p^1)-(p^1)^T\nabla_{p^1}h^1(x^1,v^1,p^1)\geqq 0.\end{aligned}$$

Adding the above two inequalities, we get

$$\begin{aligned}&f_1(x^1,y^1)-h^1(x^1,y^1,p^1)+(p^1)^T\nabla_{p^1}h^1(x^1,y^1,p^1)\\&\geqq f_1(u^1,v^1)-h^1(u^1,v^1,p^1)+(p^1)^T\nabla_{p^1}h^1(u^1,v^1,p^1).\end{aligned} \quad (4.5)$$

Combining inequalities (4.4) and (4.5), we have

$$L(x^1,x^2,y^1,y^2,p^1,p^2)\geqq M(u^1,u^2,v^1,v^2,r^1,r^2).$$

Thus the results holds.

Theorem 4.2 (Strong Duality).

Let $(\bar{x}^1,\bar{x}^2,\bar{y}^1,\bar{y}^2,\bar{p}^1,\bar{p}^2)$ be an optimal solution of (MHPC). Suppose that

(i) $\nabla_{p^1p^1}h^1(\bar{x}^1,\bar{y}^1,\bar{p}^1)$ is positive or negative definite and $\nabla_{p^2p^2}h^2(\bar{x}^2,\bar{y}^2,\bar{p}^2)$ is negative definite,

(ii) $\nabla^1_y f^1(\bar{x}^1,\bar{y}^1)+\nabla^1_p h^1(\bar{x}^1,\bar{y}^1,\bar{p}^1)\neq 0$ and $\nabla^2_y f^2(\bar{x}^2,\bar{y}^2)+\nabla^2_p h^2(\bar{x}^2,\bar{y}^2,\bar{p}^2)\neq 0$,

(iii)

$$(\bar{p}^1)^T[\nabla^1_y f^1(\bar{x}^1,\bar{y}^1)+\nabla^1_p h^1(\bar{x}^1,\bar{y}^1,\bar{p}^1)]=0\Rightarrow \bar{p}^1=0$$

and

$$y^2[\nabla_{y^2} h^2(\bar{x}^2,\bar{y}^2,\bar{p}^2)-\nabla_{p^2} h^2(\bar{x}^2,\bar{y}^2,\bar{p}^2)$$
$$+\nabla_{y^2y^2} f^2(\bar{x}^2,\bar{y}^2)\bar{p}^2]=0 \Rightarrow \bar{p}^2=0,$$

(iv) $h^1(\bar{x}^1,\bar{y}^1,0)=g^1(\bar{x}^1,\bar{y}^1,0)$,
$\nabla_{x^1} h^1(\bar{x}^1,\bar{y}^1,0)=\nabla_{r^1} g^1(\bar{x}^1,\bar{y}^1,0)$,
$\nabla_{y^1} h^1(\bar{x}^1,\bar{y}^1,0)=\nabla_{p^1} h^1(\bar{x}^1,\bar{y}^1,0)$ and
$h^2(\bar{x}^2,\bar{y}^2,0)=g^2(\bar{x}^2,\bar{y}^2,0)$,
$\nabla_{x^2} h^2(\bar{x}^2,\bar{y}^2,0)=\nabla_{r^2} g^2(\bar{x}^2,\bar{y}^2,0)$.

Then

(I) $(\bar{x}^1,\bar{x}^2,\bar{y}^1,\bar{y}^2,\bar{r}^1=0,\bar{r}^2=0)$ is feasible for (MHDC) and

(II)
$$L(\bar{x}^1,\bar{x}^2,\bar{y}^1,\bar{y}^2,\bar{p}^1,\bar{p}^2)=M(\bar{x}^1,\bar{x}^2,\bar{y}^1,\bar{y}^2,\bar{r}^1,\bar{r}^2).$$

Furthermore, if the hypothesis of Theorem (4.1) are satisfied for all feasible solutions of (MHPC) and (MHDC), then $(\bar{x}^1,\bar{x}^2,\bar{y}^1,\bar{y}^2,\bar{r}^1=0,\bar{r}^2=0)$ is an optimal solution for (MHDC).

Proof: Since $(\bar{x}^1,\bar{x}^2,\bar{y}^1,\bar{y}^2,\bar{p}^1,\bar{p}^2)$ is a n optimal solution of (MHPC), by the Fritz John necessary optimality conditions [1], there exist $\alpha,\gamma,\delta^1,\delta^2\in R$, $\beta^1\in C_3$, $\beta^2\in C_4$, and $\mu^1\in C_1$, $\mu^2\in C_2$ such that the following conditions are satisfied at $(\bar{x}^1,\bar{x}^2,\bar{y}^1,\bar{y}^2,\bar{p}^1,\bar{p}^2)$:

$$[\alpha[\nabla_{x^1} f^1(\bar{x}^1,\bar{y}^1)+\nabla_{x^1} h^1(\bar{x}^1,\bar{y}^1,\bar{p}^1)$$
$$-\nabla_{p^1x^1} h^1(\bar{x}^1,\bar{y}^1,\bar{p}^1)\bar{p}^1]+[\nabla_{y^1x^1} f^1(\bar{x}^1,\bar{y}^1)$$
$$+\nabla_{p^1x^1} h^1(\bar{x}^1,\bar{y}^1,\bar{p}^1)](\beta^1-\gamma\bar{y}^1-\delta^1\bar{p}^1)-\mu^1](x^1-\bar{x}^1)$$
$$\geqq 0,\ forall,\ x^1\in C_1, \tag{4.6}$$

$$\alpha[\nabla_{x^2} f^2(\bar{x}^2,\bar{y}^2)+\nabla_{x^2} h^2(\bar{x}^2,\bar{y}^2,\bar{p}^2)]$$
$$+\{\nabla_{p^2x^2} h^2(\bar{x}^2,\bar{y}^2,\bar{p}^2)\}(\beta^2-\alpha^2\bar{y}^2-\alpha^2\bar{p}^2-\delta^2\bar{p}^2)$$
$$+\{\nabla_{y^2x^2} f^2(\bar{x}^2,\bar{y}^2)\}(\beta^2-\alpha^2\bar{y}^2-\delta^2\bar{p}^2)$$
$$-\mu^2](x^2-\bar{x}^2)\geqq 0,\ forall,\ x^2\in C_2, \tag{4.7}$$

$$\alpha[\nabla_y^1 f^1(\bar{x}^1,\bar{y}^1)+\nabla_y^1 h^1(\bar{x}^1,\bar{y}^1,\bar{p}^1)$$
$$-\nabla_{p^1y^1} h^1(\bar{x}^1,\bar{y}^1,\bar{p}^1)\bar{p}^1]+(\nabla_{y^1y^1} f^1(\bar{x}^1,\bar{y}^1)$$
$$+\nabla_{p^1y^1} h^1(\bar{x}^1,\bar{y}^1,\bar{p}^1))(\beta^1-\gamma\bar{y}^1-\delta^1\bar{p}^1)$$
$$-\gamma[\nabla_y^1 f^1(\bar{x}^1,\bar{y}^1)+\nabla_p^1 h^1(\bar{x}^1,\bar{y}^1,\bar{p}^1)]\geqq 0,$$
$$forall,\ y^1\in R^{|K_1|}, \tag{4.8}$$

$$\{\nabla_{p^2y^2} h^2(\bar{x}^2,\bar{y}^2,\bar{p}^2)\}(\beta^2-\alpha^2\bar{y}^2-\alpha^2\bar{p}^2-\delta^2\bar{p}^2)$$
$$+\alpha^2[\nabla_{y^2} h^2(\bar{x}^2,\bar{y}^2,\bar{p}^2)-\nabla_{p^2} h^2(\bar{x}^2,\bar{y}^2,\bar{p}^2)]$$
$$+\{\nabla_{y^2y^2} f^2(\bar{x}^2,\bar{y}^2)\}(\beta^2-\alpha^2\bar{y}^2-\delta^2\bar{p}^2)$$
$$-\xi^2\geqq 0,\ forall,\ y^2\in R^{|K_2|}, \tag{4.9}$$

$$\{\nabla_{p^1p^1} h^1(\bar{x}^1,\bar{y}^1,\bar{p}^1)\}(\beta^1-\alpha\bar{p}^1-\gamma\bar{y}^1-\delta^1\bar{p}^1)$$
$$-\delta^1[\nabla_y^1 f^1(\bar{x}^1,\bar{y}^1)+\nabla_p^1 h^1(\bar{x}^1,\bar{y}^1,\bar{p}^1)]=0, \tag{4.10}$$

$$\{\nabla_{p^2p^2} h^2(\bar{x}^2,\bar{y}^2,\bar{p}^2)\}(\beta^2-\alpha\bar{y}^2-\alpha\bar{p}^2-\delta^2\bar{p}^2)$$
$$-\delta^2[\nabla_{y^2} f^2(\bar{x}^2,\bar{y}^2)+\nabla_{p^2} h^2(\bar{x}^2,\bar{y}^2,\bar{p}^2)]=0, \tag{4.11}$$

$$\beta^1[\nabla_{y^1} f^1(\bar{x}^1,\bar{y}^1)+\nabla_p^1 h^1(\bar{x}^1,\bar{y}^1,\bar{p}^1)]=0, \tag{4.12}$$

$$\beta^2[\nabla_{y^2} f^2(\bar{x}^2,\bar{y}^2)+\nabla_{p^2} h^2(\bar{x}^2,\bar{y}^2,\bar{p}^2)]=0, \tag{4.13}$$

$$\gamma\bar{y}^1[\nabla_{y^1} f^1(\bar{x}^1,\bar{y}^1)+\nabla_p^1 h^1(\bar{x}^1,\bar{y}^1,\bar{p}^1)]=0, \tag{4.14}$$

$$\delta^1\bar{p}^1[\nabla_{y^1} f^1(\bar{x}^1,\bar{y}^1)+\nabla_p^1 h^1(\bar{x}^1,\bar{y}^1,\bar{p}^1)]=0, \tag{4.15}$$

$$\delta^2\bar{p}^2[\nabla_{y^2} f^2(\bar{x}^2,\bar{y}^2)+\nabla_{p^2} h^2(\bar{x}^2,\bar{y}^2,\bar{p}^2)]=0,$$

$$\mu^1\bar{x}^1=0, \tag{4.17}$$

$$\mu^2\bar{x}^2=0, \tag{4.18}$$

$$\xi^2\bar{y}^2=0, \tag{4.19}$$

$$(\alpha,\beta^1,\beta^2,\gamma,\delta^1,\delta^2,\mu^1,\mu^2,\xi^2)\neq 0, \tag{4.20}$$

$$(\alpha,\beta^1,\beta^2,\gamma,\delta^1,\delta^2,\mu^1,\mu^2,\xi^2)\geqq 0, \tag{4.21}$$

Premultiplying equations (4.10), (4.11) by $(\beta^1-\alpha\bar{p}^1-\gamma\bar{y}^1-\delta^1\bar{p}^1)$, $(\beta^2-\alpha\bar{p}^2-\alpha\bar{y}^2-\delta^2\bar{p}^2)$, respectively and then using equations (4.12)-(4.16), we get

$$(\beta^1-\alpha\bar{p}^1-\gamma\bar{y}^1-\delta^1\bar{p}^1)^T$$
$$\nabla_{p^1p^1} h(\bar{x}^1,\bar{y}^1,\bar{p}^1)(\beta^1-\alpha\bar{p}^1-\gamma\bar{y}^1-\delta^1\bar{p}^1)=0,$$

And

$$(\beta^2-\alpha\bar{p}^2-\alpha\bar{y}^2-\delta^2\bar{p}^2)\nabla_{p^2p^2} h(\bar{x}^2,\bar{y}^2,\bar{p}^2)$$
$$(\beta^2-\alpha\bar{p}^2-\alpha\bar{y}^2-\delta^2\bar{p}^2)$$
$$=-\alpha\delta\bar{y}^2[\nabla_{y^2} f^2(\bar{x}^2,\bar{y}^2)+\nabla_{p^2} h^2(\bar{x}^2,\bar{y}^2,\bar{p}^2)].$$

Using hypothesis (i), we get

$$\beta^1=\alpha\bar{p}^1+\gamma\bar{y}^1+\delta^1\bar{p}^1. \tag{4.22}$$

Further using inequality (3.1), (3.6) and (4.21), we obtain

$$(\beta^2-\alpha\bar{p}^2-\alpha\bar{y}^2-\delta^2\bar{p}^2)\nabla_{p^2p^2} h(\bar{x}^2,\bar{y}^2,\bar{p}^2)$$
$$(\beta^2-\alpha\bar{p}^2-\alpha\bar{y}^2-\delta^2\bar{p}^2)\geqq 0,$$

which on using hypothesis (i)

$$\beta^2 = \alpha\bar{p}^2 + \alpha\bar{y}^2 + \delta^2\bar{p}^2. \tag{4.23}$$

From equations (4.10) and (4.11), and hypothesis (ii), we obtain

$$\delta^1 = 0, \tag{4.24}$$

And

$$\delta^2 = 0. \tag{4.25}$$

Now suppose, $\alpha = 0$. Then equations (4.23), with (4.9) gives $\xi^2 = 0$ and, with (4.25) implies $\beta^2 = 0$ also equations (4.6), (4.22) implies $\mu^1 = 0$ and equations (4.7), (4.23) and (4.25) implies $\mu^2 = 0$. From equation (4.8) and hypothesis (ii) yield $\gamma = 0$, which along with equation (4.22), (4.24) reduces $\beta^1 = 0$. Thus $(\alpha, \beta^1, \beta^2, \gamma, \delta^1, \delta^2, \mu^1, \mu^2) = 0$, a contradiction to (4.20).

$$\text{Hence } \alpha = 0. \tag{4.26}$$

Using equations (4.12), (4.14) and (4.15), we have

$$(\beta^1 - \gamma\bar{y}^1 - \delta^1\bar{p}^1)^T[\nabla_y^1 f^1(\bar{x}^1, \bar{y}^1) + \nabla_{p^1} h(\bar{x}^1, \bar{y}^1, \bar{p}^1)] = 0,$$

Now equation (4.22), gives

$$\alpha\bar{p}^{1^T}[\nabla_y^1 f^1(\bar{x}^1, \bar{y}^1) + \nabla_{p^1} h(\bar{x}^1, \bar{y}^1, \bar{p}^1)] = 0. \tag{4.27}$$

which along with hypothesis (iii) yield

$$\bar{p}^1 = 0. \tag{4.28}$$

Further, from equation (4.9) and (4.23), we get

$$\alpha^2[\nabla_{y^2} h^2(\bar{x}^2, \bar{y}^2, \bar{p}^2) - \nabla_{p^2} h^2(\bar{x}^2, \bar{y}^2, \bar{p}^2) + \nabla_{y^2 y^2} f^2(\bar{x}^2, \bar{y}^2)\bar{p}^2] - \xi^2 = 0. \tag{4.29}$$

Now from hypothesis (iii), we obtain

$$\bar{p}^2 = 0. \tag{4.30}$$

Therefore equation (4.22) and (4.23) reduce to

$$\beta^1 = \gamma\bar{y}^1, \tag{4.31}$$

And

$$\beta^2 = \alpha\bar{y}^2. \tag{4.32}$$

Also, it follows from equations (4.8), (4.22), (4.28) and hypothesis (ii) and (iv) that

$$\alpha = \gamma > 0. \tag{4.33}$$

So equation (4.31) implies

$$\bar{y}^1 = \frac{\beta^1}{\gamma} \geq 0. \tag{4.34}$$

Moreover, equation (4.6), (4.7) along with (4.22), (4.23), (4.30) and hypothesis (iv) yields

$$\alpha[\nabla_{x^1} f^1(\bar{x}^1, \bar{y}^1) + \nabla_{r^1} h^1(\bar{x}^1, \bar{y}^1, \bar{p}^1) - \mu^1](x^1 - \bar{x}^1) \geqq 0, \ forall\ x^1 \in C_1, \tag{4.35}$$

$$\alpha[\nabla_{x^2} f^2(\bar{x}^2, \bar{y}^2) + \nabla_{r^2} h^2(\bar{x}^2, \bar{y}^2, \bar{p}^2) - \mu^2](x^2 - \bar{x}^2) \geqq 0, \ forall\ x^2 \in C_2, \tag{4.36}$$

Let $x^1 \in C_1$, then $\bar{x}^1 + x^1 \in C_1$ and then above inequality implies

$$\alpha x^{1T}[\nabla_{x^2} f^2(\bar{x}^2, \bar{y}^2) + \nabla_{r^2} h^2(\bar{x}^2, \bar{y}^2, \bar{p}^2) - \mu^2] \geqq 0, \ forall\ x^1 \in C_1,$$

Further by using equation (4.17) the above inequality also be rewritten as

$$\bar{x}^{1T}[\nabla_{x^1} f^1(\bar{x}^1, \bar{y}^1) + \nabla_{r^1} h^1(\bar{x}^1, \bar{y}^1, \bar{p}^1)] \geqq \mu^1\bar{x}^1 = 0, \ forall\ x^1 \in C_1.$$

Therefore

$$\nabla_{x^1} f^1(\bar{x}^1, \bar{y}^1) + \nabla_{r^1} h^1(\bar{x}^1, \bar{y}^1, \bar{p}^1) \in C_1^*, \tag{4.37}$$

Similarly, we also obtain that

$$\nabla_{x^2} f^2(\bar{x}^2, \bar{y}^2) + \nabla_{r^2} h^2(\bar{x}^2, \bar{y}^2, \bar{p}^2) \in C_2^*. \tag{4.38}$$

Thus $(\bar{x}^1, \bar{x}^2, \bar{y}^1, \bar{y}^2, \bar{r}^1 = 0, \bar{r}^2 = 0)$ satisfies the dual constraints (3.7)-(3.12), i.e, it is an feasible solution of $(MHDC)$.

Also, using hypothesis (iv) we get the values of the objective functions of $(MHPC)$ and $(MHDC)$ at $(\bar{x}^1, \bar{x}^2, \bar{y}^1, \bar{y}^2, \bar{p}^1 = 0, \bar{p}^2 = 0)$ and $(\bar{x}^1, \bar{x}^2, \bar{y}^1, \bar{y}^2, \bar{r}^1 = 0, \bar{r}^2 = 0)$ are equal. Using Weak duality it easily shown that

$(\bar{x}^1, \bar{x}^2, \bar{y}^1, \bar{y}^2, \bar{r}^1 = 0, \bar{r}^2 = 0)$ and $(\bar{x}^1, \bar{x}^2, \bar{y}^1, \bar{y}^2, \bar{p}^1 = 0, \bar{p}^2 = 0)$ are an optimal solutions for $(MHPC)$ and $(MHDC)$, respectively.

Theorem 4.3 (Converse Duality).

Let $(\bar{u}^1, \bar{u}^2, \bar{v}^1, \bar{v}^2, \bar{r}^1, \bar{r}^2)$ be an optimal solution of $(MHDC)$. Suppose that

(i) $\nabla_{r^1 r^1} g^1(\bar{u}^1, \bar{v}^1, \bar{r}^1)$ is positive or negative definite and $\nabla_{r^2 r^2} g^2(\bar{u}^2, \bar{v}^2, \bar{r}^2)$ is negative definite,

(ii) $\nabla_u^1 f^1(\bar{u}^1, \bar{v}^1) + \nabla_r^1 g^1(\bar{u}^1, \bar{v}^1, \bar{r}^1) \neq 0$ and

$$\nabla_y^2 f^2(\bar{x}^2, \bar{y}^2) + \nabla_p^2 h^2(\bar{x}^2, \bar{y}^2, \bar{p}^2) \neq 0,$$

(iii)

$$(\bar{r}^1)^T [\nabla_u^1 f^1(\bar{u}^1, \bar{v}^1) + \nabla_r^1 g^1(\bar{u}^1, \bar{v}^1, \bar{r}^1)] = 0 \Rightarrow \bar{r}^1 = 0$$

and

$$\nabla_{u^2} g^2(\bar{u}^2, \bar{v}^2, \bar{r}^2) - \nabla_{r^2} g^2(\bar{u}^2, \bar{v}^2, \bar{r}^2) + \nabla_{u^2 u^2} f^2(\bar{u}^2, \bar{v}^2)\bar{r}^2 = 0 \Rightarrow \bar{r}^2 = 0,$$

(iv) $g^1(\bar{u}^1, \bar{v}^1, 0) = g^1(\bar{u}^1, \bar{v}^1, 0)$, $\nabla_{u^1} g^1(\bar{u}^1, \bar{v}^1, 0) = \nabla_{r^1} g^1(\bar{u}^1, \bar{v}^1, 0)$, $\nabla_{v^1} g^1(\bar{u}^1, \bar{v}^1, 0) = \nabla_{r^1} g^1(\bar{u}^1, \bar{v}^1, 0)$ and $g^2(\bar{u}^2, \bar{v}^2, 0) = h^2(\bar{u}^2, \bar{v}^2, 0)$, $\nabla_{u^2} g^2(\bar{u}^2, \bar{v}^2, 0) = \nabla_{r^2} h^2(\bar{u}^2, \bar{v}^2, 0)$.

Then

(I) $(\bar{u}^1, \bar{u}^2, \bar{v}^1, \bar{v}^2, \bar{r}^1 = 0, \bar{r}^2 = 0)$ is feasible for $(MHPC)$ and

(II)

$$L(\bar{u}^1, \bar{u}^2, \bar{v}^1, \bar{v}^2, \bar{p}^1, \bar{p}^2) = M(\bar{u}^1, \bar{u}^2, \bar{v}^1, \bar{v}^2, \bar{r}^1, \bar{r}^2).$$

Furthermore, if the hypothesis of Theorem 4.1 are satisfied for all feasible solutions of $(MHPC)$ and $(MHDC)$, then $(\bar{x}^1, \bar{x}^2, \bar{y}^1, \bar{y}^2, \bar{r}^1 = 0, \bar{r}^2 = 0)$ is an optimal solution for $(MHPC)$.

Proof. Follows on the line of Theorem 4.2.

Theorem 4.4 (Self Duality).

A primal (dual) problem having equivalent dual (primal) formulation is said to be self-dual, that is, if the dual can be recast in the form of the primal. In general, $(MHPC)$ and $(MHDC)$ are not self-duals without some added restrictions on f ; g; and h. If we assume $f^1 : R^{|J_1|} \times R^{|K_1|} \to R$, $f^2 : R^{|J_2|} \times R^{|K_2|} \to R$, $g^1 : R^{|J_1|} \times R^{|K_1|} \to R$, $g^2 : R^{|J_2|} \times R^{|K_2|} \to R$, $h^1 : R^{|J_1|} \times R^{|K_1|} \times R^{|K_1|} \to R$, $h^2 : R^{|J_2|} \times R^{|K_2|} \times R^{|K_2|} \to R$, to be skew symmetric, that is

$$f^i(u^1, v^1) = -f^i(u^1, v^1), i = 1,2.,$$

and

$$g^i(u^1, v^1, r^1) = -g^i(u^1, v^1, r^1), i = 1,2,$$

Then we shall show that $(MHPC)$ and $(MHDC)$ are self-duals. By recasting the dual problem $(MHDC)$ as a minimization problem, we have

Minimize $M(u, v, r) =$

$$\begin{aligned}-\{ & f_1(u^1, v^1) + f_2(u^2, v^2) + g_1(u^1, v^1, r^1) + g_2(u^2, v^2, r^2) \\ & -(r^1)^T \nabla_{r^1} g_1(u^1, v^1, r^1) - (r^2)^T \nabla_{r^2} g_2(u^2, v^2, r^2) \\ & -(u^2)^T [\nabla_{u^2} f^2(u^2, v^2) + \nabla_{r^2} g_2(u^2, v^2, r^2)]\}\end{aligned}$$

Subject to

$$\nabla_{u^1} f_1(u^1, v^1) + \nabla_{r^1} g_1(u^1, v^1, r^1) \geqq 0,$$
$$\nabla_{u^2} f^2(u^2, v^2) + \nabla_{r^2} g_2(u^2, v^2, r^2) \geqq 0,$$

$$(u^1)^T [\nabla_{u^1} f_1(u^1, v^1) + \nabla_{r^1} g_1(u^1, v^1, r^1)] \leqq 0,$$
$$(r^1)^T [\nabla_{u^1} f_1(u^1, v^1) + \nabla_{r^1} g_1(u^1, v^1, r^1)] \leqq 0,$$
$$(r^2)^T [\nabla_{u^2} f^2(u^2, v^2) + \nabla_{r^2} g_2(u^2, v^2, r^2)] \leqq 0,$$
$$v^1, v^2, u^2 \geqq 0,$$

As f, g and h are skew symmetric, i.e.,

$$\nabla_{u^1} f_1(u^1, v^1) = -\nabla_{u^1} f_1(v^1, u^1),$$
$$\nabla_{u^1} f_2(u^2, v^2) = -\nabla_{u^1} f_2(v^2, u^2),$$
$$\nabla_{u^1} g_1(u^1, v^1, r^1) = -\nabla_{u^1} g_1(v^1, u^1, r^1), \text{ and}$$
$$\nabla_{u^1} g_2(u^2, v^2, r^2) = -\nabla_{u^1} g_2(v^2, u^2, r^2),$$

Then the above problem becomes :

Minimize $M(u, v, r) =$

$$\begin{aligned} & f_1(v^1, u^1) + f_2(v^2, u^2) + g_1(v^1, u^1, r^1) + g_2(v^2, u^2, r^2) \\ & -(r^1)^T \nabla_{r^1} g_1(v^1, u^1, r^1) - (r^2)^T \nabla_{r^2} g_2(v^2, u^2, r^2) \\ & -(u^2)^T [\nabla_{u^2} f^2(v^2, u^2) + \nabla_{r^2} g_2(v^2, u^2, r^2)]\end{aligned}$$

Subject to

$$\nabla_{u^1} f_1(v^1, u^1) + \nabla_{r^1} g_1(v^1, u^1, r^1) \leqq 0,$$
$$\nabla_{u^2} f^2(v^2, u^2) + \nabla_{r^2} g_2(v^2, u^2, r^2) \leqq 0,$$
$$(u^1)^T [\nabla_{u^1} f_1(v^1, u^1) + \nabla_{r^1} g_1(v^1, u^1, r^1)] \geqq 0,$$
$$(r^1)^T [\nabla_{u^1} f_1(v^1, u^1) + \nabla_{r^1} g_1(v^1, u^1, r^1)] \geqq 0,$$
$$(r^2)^T [\nabla_{u^2} f^2(v^2, u^2) + \nabla_{r^2} g_2(v^2, u^2, r^2)] \geqq 0,$$
$$v^1, v^2, u^2 \geqq 0,$$

Which shows that $M(u,v,p)$ is identical to $L(x,y,p)$, that is, the objective and the constraint functions are identical. Thus, the problem $L(x,y,p)$ becomes self-dual.

It is obvious that the feasibility of $(x^1,x^2,y^1,y^2,p^1,p^2)$ for $L(x,y,p)$ implies the feasibility of $(y^1,y^2,x^1,x^2,p^1,p^2)$ for $(MHPC)$ implies the feasibility of for $(MHDC)$ and vice versa.

V. Special Case

In this section, we consider some special cases of our problems by choosing particular forms of the closed convex sets C_1 and C_2 . In all these cases,

$$h(x,y,p)=(1/2)p^T\nabla_{yy}f(x,y)p$$

$$and\ \ g(u,v,r)=(1/2)r^T\nabla_{xx}f(u,v)r,$$

(a) If $|K_1|=0$, $|J_1|=0$, $p=0$ and $r=0$, then $(MHPC)$ and $(MHDC)$ reduce to the programs studied in Chandra and Kumar [14].

(b) If $|K_1|=0$, $|J_1|=0$, $C_1=R_+^n$ and $C_2=R_+^m$, then after removing inequalities (3.5), (3.11), our programs reduce to the problems considered in Mishra [4].

(c) If $|K_1|=0$, $|J_1|=0$, $C_1=R_+^n$ and $C_2=R_+^m$, then the then after removing inequalities (3.5), (3.11), programs reduce to the second-order symmetric dual programs of Gulati et al. [15].

VI. Conclusion

A pair of mixed symmetric dual programs has been formulated by considering the optimization under the assumptions of η-invexity and η-pseudoinvexity. It may be noted that the symmetric duality between $(MHPC)$ and $(MHDC)$ can be utilized to establish non differentiable mixed symmetric duality in integer over cone and other related programming problems.

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