FIBONACCI CONVOLUTION SEQUENCES

V. E. HOGGATT, JR., and MARJORIE BICKNELL-JOHNSON
San Jose State University, San Jose, California 95129

The Fibonacci convolution sequences $\left\{F_n^{(r)}\right\}$ which arise from convolutions of the Fibonacci sequence $\{1, 1, 2, 3, 5, 8, \cdots, F_n, \cdots\}$ lead to some new Fibonacci identities, limit theorems, and determinant identities.

1. THE FIBONACCI CONVOLUTION SEQUENCES

Let the r^{th} Fibonacci convolution sequence be denoted $\left\{F_n^{(r)}\right\}$; note that $F_n^{(0)} = F_n$, the n^{th} Fibonacci number. Then

(1.1) $$F_n^{(1)} = \sum_{i=0}^{n} F_{n-i}F_i$$

(1.2) $$F_n^{(r)} = \sum_{i=0}^{n} F_{n-i}^{(r-1)}F_i .$$

However, there are some easier methods of calculation.

Let the Fibonacci polynomials $F_n(x)$ be defined by

(1.3) $$F_{n+2}(x) = xF_{n+1}(x) + F_n(x), \qquad F_0(x) = 0, \quad F_1(x) = 1 .$$

Then, since $F_n(1) = F_n$, the recursion relation for the Fibonacci numbers, $F_{n+2} = F_{n+1} + F_n$, follows immediately by taking $x = 1$. In a similar manner we may write recursion relations for $\left\{F_n^{(r)}\right\}$.

From (1.3), taking the first derivative we have

$$F'_{n+2}(x) = xF'_{n+1}(x) + F'_n(x) + F_{n+1}(x).$$

Since $F'_n(1) = F_n^{(1)}$, taking $x = 1$ gives us the recursion relation for $\left\{F_n^{(1)}\right\}$,

(1.4) $$F_{n+2}^{(1)} = F_{n+1}^{(1)} + F_n^{(1)} + F_{n+1} .$$

Since the generating function for the Fibonacci polynomials is

(1.5) $$\frac{Y}{1 - xY - Y^2} = \sum_{n=1}^{\infty} F_n(x)Y^n ,$$

while the generating function for the Fibonacci convolution sequences is

(1.6) $$\left(\frac{x}{1 - x - x^2}\right)^{r+1} = \sum_{n=1}^{\infty} F_n^{(r)}x^n ,$$

it is easy to see that

(1.7) $$F_n^{(r)} = F_n^{(r)}(1)/r! ,$$

where $F_n^{(r)}(x)$ is the r^{th} derivative of the Fibonacci polynomial $F_n(x)$. Thus we can write

(1.8) $$F_{n+2}^{(r+1)} = F_{n+1}^{(r+1)} + F_n^{(r+1)} + F_{n+1}^{(r)} ,$$

which enables us to make the following table with a minimum of effort.

We can extend our sequences for negative subscripts to write

(1.9) $$F_{-n}^{(r)} = (-1)^{n+1}F_n^{(r)} ,$$

n	F_n	$F_n^{(1)}$	$F_n^{(2)}$	$F_n^{(3)}$	$F_n^{(4)}$	$\cdots$
0	0	0	0	0	0	$\cdots$
1	1	0	0	0	0	$\cdots$
2	1	1	0	0	0	$\cdots$
3	2	2	1	0	0	$\cdots$
4	3	5	3	1	0	$\cdots$
5	5	10	9	4	1	$\cdots$
6	8	20	22	14	5	$\cdots$
7	13	38	51	40	20	$\cdots$
8	21	71	111	105	65	$\cdots$
9	34	130	233	256	190	$\cdots$
10	55	235	474	594	511	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$

where we note that $\left\{F_n^{(r)}\right\}$ has $2r+1$ zeros, and $F_{r+1}^{(r)} = 1$, $F_{r+2}^{(r)} = r$.

Equation (1.9) can be established for $r = 1$ quite easily by induction. Assume that (1.9) holds for 1, 2, 3, $\cdots$, r, and for $r+1$ for $n = 1, 2, \cdots, k$. Then by (1.8)

$$F_{k+1}^{(r+1)} = F_k^{(r+1)} + F_{k-1}^{(r+1)} + F_k^{(r)} = (-1)^{k+1}F_{-k}^{(r+1)} + (-1)^k F_{-k+1}^{(r+1)} + (-1)^{k+1}F_{-k}^{(r)}$$

$$= (-1)^{k+2}[F_{-k+1}^{(r+1)} - F_{-k}^{(r+1)} - F_{-k}^{(r)}] = (-1)^{k+2}F_{-k-1}\,,$$

which is equivalent to (1.9) for $n = k+1$, finishing a proof by induction.

Returning to (1.6), recall that the recurrence relation for $\left\{F_n^{(1)}\right\}$ has auxiliary polynomial $(x^2 - x - 1)^2$, whose roots are, of course, $\alpha, \alpha, \beta, \beta$, where $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2$. Then,

$$F_n^{(1)} = (A + Bn)\alpha^n + (C + Dn)\beta^n \tag{1.10}$$

for some constants A, B, C and D due to the repeated roots. Since the Fibonacci numbers are a linear combination of the same roots,

$$F_n^{(1)} = (A^* + B^*n)F_{n+1} + (C^* + D^*n)F_{n-1} \tag{1.11}$$

for some constants A^*, B^*, C^*, and D^*. By letting n = 0, 1, 2, 3 and solving the resulting system of equations, one finds $A^* = -1/5$, $B^* = C^* = D^* = 1/5$, resulting in

$$5F_n^{(1)} = (n-1)F_{n+1} + (n+1)F_{n-1}\,, \tag{1.12}$$

which leads easily to

$$F_n^{(1)} = (nL_n - F_n)/5 \tag{1.13}$$

where L_n is the n^{th} Lucas number.

Returning again to the auxiliary polynomial for $\left\{F_n^{(1)}\right\}$, since $(x^2 - x - 1)^2 = x^4 - 2x^3 - x^2 + 2x + 1$, we can write

$$F_{n+4}^{(1)} = 2F_{n+3}^{(1)} + F_{n+2}^{(1)} - 2F_{n+1}^{(1)} - F_n^{(1)} \tag{1.14}$$

2. SPECIAL LIMITING RATIOS

It is well known that

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \alpha = \frac{1+\sqrt{5}}{2}\,. \tag{2.1}$$

We extend this property of the Fibonacci numbers to the Fibonacci convolution sequences. First, (1.10) gives us

$$F_n^{(1)} = (A+Bn)\alpha^n + (C+Dn)\beta^n$$

for some constants A, B, C and D. Thus one concludes

$$\lim_{n\to\infty} \frac{F_{n+1}^{(1)}}{F_n^{(1)}} = \lim_{n\to\infty} \frac{[A+B(n+1)]\,\alpha + [C+D(n+1)]\,\beta](\beta/\alpha)^n}{A+Bn+(C+Dn)(\beta/\alpha)^n} = \alpha\ .$$

Clearly, this holds for any $\left\{F_n^{(r)}\right\}$ since, by examining the auxiliary polynomial,

(2.2) $$F_n^{(r)} = p_r(n)\alpha^n + q_r(n)\beta^n,$$

where $p_r(n)$ and $q_r(n)$ are polynomials in n of degree r. Then, we have

(2.3) $$\lim_{n\to\infty}\frac{F_{n+1}^{(r)}}{F_n^{(r)}} = \lim_{n\to\infty}\frac{p_r(n+1)\alpha^{n+1}+q_r(n+1)\beta^{n+1}}{p_r(n)\alpha^n+q_r(n)\beta^n} = \lim_{n\to\infty}\frac{p_r(n+1)}{p_r(n)}\ \alpha = \alpha .$$

While it is not necessary to be able to write $p_r(n)$ and $q_r(n)$ to establish (2.3), it would be interesting to find a recurrence for these polynomials.

It is not difficult to show that

(2.4) $$\lim_{n\to\infty}\frac{F_n}{F_n^{(1)}} = 0$$

and that

(2.5) $$\lim_{n\to\infty}\frac{F_n^{(r^*)}}{F_n^{(r)}} = 0,\quad r^* < r\ .$$

We also find α^2 as a value for a special limiting ratio. We define

(2.6) $$W_n^{(r)} = F_{n+1}^{(r)}F_{n-1}^{(r)} - [F_n^{(r)}]^2\ .$$

For $r = 0$, the Fibonacci numbers themselves, $W_n^{(0)} = (-1)^n$, but when $r \geq 1$, $W_n^{(r)}$ is not a constant. However, we have the surprising limiting ratio,

(2.7) $$\lim_{n\to\infty}\frac{W_{n+1}^{(r)}}{W_n^{(r)}} = \alpha^2,\quad r \geq 1.$$

To establish (2.7), we use (2.2) to calculate $W_n^{(r)}$ as

$$\begin{aligned} W_n^{(r)} &= [p_r(n+1)\alpha^{n+1} + q_r(n+1)\beta^{n+1}][p_r(n-1)\alpha^{n-1} + q_r(n-1)\beta^{n-1}] - [p_r(n)\alpha^n + q_r(n)\beta^n]^2 \\ &= [p_r(n+1)p_r(n-1)\alpha^{2n} + q_r(n+1)q_r(n-1)\beta^{2n} + p_r(n+1)q_r(n-1)\alpha^{n+1}\beta^{n-1} \\ &\quad + p_r(n-1)q_r(n+1)\alpha^{n-1}\beta^{n+1}] - [p_r^2(n)\alpha^{2n} + 2p_r(n)q_r(n)\alpha^n\beta^n + q_r^2(n)\beta^{2n}] \\ &= [p_r(n+1)p_r(n-1) - p_r^2(n)]\,\alpha^{2n} + [q_r(n+1)q_r(n-1) - q_r^2(n)]\,\beta^{2n} + R_r(n), \end{aligned}$$

where $R_r(n)$ is a polynomial in n of degree $2r$, but each term contains a factor of α^s or β^t, where s, t are at most two, since $\alpha\beta = -1$. Then, if $p_r(n+1)p_r(n-1) - p_r^2(n) \neq 0$, we find that

$$\lim_{n\to\infty}\frac{W_{n+1}^{(r)}}{W_n^{(r)}} = \frac{F_{n+2}^{(r)}F_n^{(r)} - [F_{n+1}^{(r)}]^2}{F_{n+1}^{(r)}F_{n-1}^{(r)} - [F_n^{(r)}]^2} = \alpha^2\ .$$

Please note that for the Fibonacci numbers themselves, it is indeed true that $p = -q = 1/(\alpha - \beta)$ and

$$p(n+1)p(n-1) - p^2(n) \equiv 0.$$

That there are no other polynomials such that $p(n+1)p(n-1) - p^2(n) \equiv 0$ is proved by considering

$$F_n^{(r)} = p_r(n)\alpha^n + q_r(n)\beta^n\ ,$$

where $p_r(n)$ is a polynomial of degree at most r. Consider

$$P(n) = p_r(n+1)p_r(n-1) - p_r^2(n)$$

which is a polynomial of degree at most $2r$. Thus, $P(n) \neq 0$ for more than $2r$ values of n. Clearly, then, for all large enough n, $P(n) \neq 0$.

3. DETERMINANT IDENTITIES FOR THE FIBONACCI CONVOLUTION SEQUENCES

Several interesting determinant identities can be found for the Fibonacci convolution sequences. First, we examine a class of unit determinants. Let

$$(3.1) \qquad D_n = \begin{vmatrix} F_{n+3}^{(1)} & F_{n+2}^{(1)} & F_{n+1}^{(1)} & F_n^{(1)} \\ F_{n+2}^{(1)} & F_{n+1}^{(1)} & F_n^{(1)} & F_{n-1}^{(1)} \\ F_{n+1}^{(1)} & F_n^{(1)} & F_{n-1}^{(1)} & F_{n-2}^{(1)} \\ F_n^{(1)} & F_{n-1}^{(1)} & F_{n-2}^{(1)} & F_{n-3}^{(1)} \end{vmatrix}$$

Then it is easily proved that $D_n = 1$ by using (1.14), since replacing the fourth column with a linear combination of the present columns gives us the negative of the first column of D_{n+1}. That is, since

$$-F_{n+4}^{(1)} = -2F_{n+3}^{(1)} - F_{n+2}^{(1)} + 2F_{n+1}^{(1)} + F_n^{(1)},$$

$$D_n = \begin{vmatrix} F_{n+3}^{(1)} & F_{n+2}^{(1)} & F_{n+1}^{(1)} & -F_{n+4}^{(1)} \\ F_{n+2}^{(1)} & F_{n+1}^{(1)} & F_n^{(1)} & -F_{n+3}^{(1)} \\ F_{n+1}^{(1)} & F_n^{(1)} & F_{n-1}^{(1)} & -F_{n+2}^{(1)} \\ F_n^{(1)} & F_{n-1}^{(1)} & F_{n-2}^{(1)} & -F_{n+1}^{(1)} \end{vmatrix}$$

so that $D_n = D_{n+1}$ after making appropriate column exchanges. Lastly, since $D_1 = 1$, $D_n = 1$ for all n.

Now, let $D_n^{(r)}$ be the determinant of order $(2r+2)$ with successive members of the sequence $\{F_n^{(r)}\}$ written along its rows and columns in decreasing order such that $F_n^{(r)}$ appears everywhere along the minor diagonal. Since $\{F_n^{(r)}\}$ has an auxiliary polynomial of degree $(2r+2)$, $F_{n+2r+2}^{(r)}$ is a linear combination of

$$F_{n+2r+1}^{(r)},\ F_{n+2r}^{(r)},\ F_{n+2r-1}^{(r)},\ \cdots,\ F_{n+1}^{(r)},\ F_n^{(r)},$$

so that $D_n^{(r)} = \pm D_{n+1}^{(r)}$ after $(2r+1)$ appropriate column exchanges. The auxiliary polynomial $(x^2 - x - 1)^{r+1}$ has a positive constant term when r is odd, making the last column the negative of the first column of $D_{n+1}^{(r)}$, so that

$$D_n^{(r)} = (-1)^{2r+1}(-1)D_{n+1}^{(r)} = D_{n+1}^{(r)}, \quad r \text{ odd;}$$

but, for r even, a negative constant term makes the last column equal the first column of $D_{n+1}^{(r)}$, and

$$D_n^{(r)} = (-1)^{2r+1}D_{n+1}^{(r)} = -D_{n+1}^{(r)}, \quad r \text{ even.}$$

We need only to evaluate $D_n^{(r)}$ for one value of n, then. Now, $F_n^{(r)} = 0$ for $n = 0, \pm1, \pm2, \cdots, \pm r$, and $F_{r+1}^{(r)} = 1$. Thus, $D_{r+1}^{(r)} = (-1)^{r+1}$ since ones appear on the minor diagonal there with zeroes everywhere below. Then, $D_n^{(r)} = 1$ when r is odd, and $D_n^{(r)} = (-1)^n$ when r is even, which can be combined to

$$(3.2) \qquad D_n^{(r)} = (-1)^{n(r+1)}.$$

The special case $r = 0$ is the well known formula, $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$.

A second proof of (3.2) is instructive. Returning to (3.1), apply (1.8) as

$$(3.3) \qquad F_{n+1}^{(r)} = F_{n+2}^{(r+1)} - F_{n+1}^{(r+1)} - F_n^{(r+1)},$$

taking $r = 0$. Subtracting pairs of columns and then pairs of rows gives

$$D_n = \begin{vmatrix} F_{n+2} & F_{n+1} & F_{n+1}^{(1)} & F_n^{(1)} \\ F_{n+1} & F_n & F_n^{(1)} & F_{n-1}^{(1)} \\ F_n & F_{n-1} & F_{n-1}^{(1)} & F_{n-2}^{(1)} \\ F_{n-1} & F_{n-2} & F_{n-2}^{(1)} & F_{n-3}^{(1)} \end{vmatrix} = \begin{vmatrix} 0 & 0 & F_n & F_{n-1} \\ 0 & 0 & F_{n-1} & F_{n-2} \\ F_n & F_{n-1} & F_{n-1}^{(1)} & F_{n-2}^{(1)} \\ F_{n-1} & F_{n-2} & F_{n-2}^{(1)} & F_{n-3}^{(1)} \end{vmatrix} .$$

Thus,

$$D_n = (F_nF_{n-2} - F_{n-1}^2)^2 = 1.$$

Notice that this proof can be generalized, and after sufficient subtractions, one always makes a block of zeroes in the upper left, with two smaller determinants of the same form in the lower left and upper right, so that $D_n^{(r)}$ is always a product of smaller known determinants $D_n^{(r^*)}$, $r^* < r$, making a proof by induction possible. Each higher order determinant requires more subtractions of pairs of rows and columns, but careful counting of subscripts leads one to

$$D_n^{(r)} = \begin{cases} [D_n^{(r/2)}]\cdot[D_n^{((r-2)/2)}], & r \text{ even}; \\ [D_n^{((r-1)/2)}]^2, & r \text{ odd}; \end{cases} \tag{3.4}$$

which again gives us (3.2).

The process of subtraction of pairs of columns and rows can also be applied to determinants of odd order. For example,

$$D_n^* = \begin{vmatrix} F_{n+2}^{(1)} & F_{n+1}^{(1)} & F_n^{(1)} \\ F_{n+1}^{(1)} & F_n^{(1)} & F_{n-1}^{(1)} \\ F_n^{(1)} & F_{n-1}^{(1)} & F_{n-2}^{(1)} \end{vmatrix} = \begin{vmatrix} 0 & F_n & F_{n-1} \\ F_n & F_n^{(1)} & F_{n-1}^{(1)} \\ F_{n-1} & F_{n-1}^{(1)} & F_{n-2}^{(1)} \end{vmatrix} .$$

Then, by applying (1.13) and known Fibonacci and Lucas identities, one can evaluate D_n^*. The algebra, however, is long and inelegant. One obtains, after patience,

$$D_n^* = (-1)^{n+1}F_n^{(1)} . \tag{3.5}$$

However, D_n^* can also be written out from the form given above on the right, so that

$$D_n^* = (-1)^{n+1}F_n^{(1)} = 2F_nF_{n-1}F_{n-1}^{(1)} - F_{n-1}^{(2)}F_n^{(1)} - F_n^2F_{n-1}^{(1)}$$

$$[(-1)^{n-1} + F_{n-1}^2]F_n^{(1)} = 2F_nF_{n-1}[F_n^{(1)} - F_{n-2}^{(1)} - F_{n-1}] - F_n^2F_{n-2}^{(1)}$$

$$[(F_{n-1}F_n - F_{n-1}^2) + F_{n-1}^2 - 2F_nF_{n-1}]F_n^{(1)} = (-2F_nF_{n-1} - F_n^2)F_{n-2}^{(1)} - 2F_nF_{n-1}^2$$

$$-F_nL_{n-2}F_n^{(1)} = -F_nL_nF_{n-2}^{(1)} - 2F_nF_{n-1}^2$$

by applying known Fibonacci identities. Finally, dividing by $-F_n$, $n \neq 0$ and rearranging, we have

$$L_{n-2}F_n^{(1)} - L_nF_{n-2}^{(1)} = 2F_{n-1}^2, \tag{3.6}$$

which we compare with the known

$$L_{n-2}F_n - L_nF_{n-2} = 2(-1)^n .$$

If we let $D_n^{*(r)}$ denote the determinant of order $(2r+1)$ which has successive members of the sequence $\{F_n^{(r)}\}$ written along its rows and columns in decreasing order such that $\{F_n^{(r)}\}$ appears everywhere along the minor diagonal, we conjecture that

$$D_n^{*(r)} = (-1)^{r(n+1)}F_n^{(r)} . \tag{3.7}$$

Equation (3.7) has been proved for $r = 1$ above, and $r = 0$ is trivial. When $r = 2$, it is possible to prove (3.7) by using the identity

(3.8) $$F_n^{(2)} = [(5n^2 - 2)F_n - 3nL_n]/50$$

as well as (1.13). The algebra, however, is horrendous. The identity (3.8) can be derived by solving for the constants A, B, C, D, E, and F in

$$F_n^{(2)} = (A + Bn + Cn^2)F_n + (D + En + Fn^2)L_n$$

which arises since $\{F_n^{(2)}\}$ has auxiliary polynomial $(x^2 - x - 1)^3$, whose roots are α, α, α and β, β, β.

Two other determinant identities follow without proof.

$$\begin{vmatrix} F_{n+2}^{(1)} & F_{n+1}^{(1)} & F_{n-1}^{(1)} \\ F_{n+1}^{(1)} & F_n^{(1)} & F_{n-2}^{(1)} \\ F_n^{(1)} & F_{n-1}^{(1)} & F_{n-3}^{(1)} \end{vmatrix} = (-1)^n [F_{n-5}^{(1)} + 2F_{n-4}]$$

$$\begin{vmatrix} F_{n+2}^{(1)} & F_n^{(1)} & F_{n-1}^{(1)} \\ F_{n+1}^{(1)} & F_{n-1}^{(1)} & F_{n-2}^{(1)} \\ F_n^{(1)} & F_{n-2}^{(1)} & F_{n-3}^{(1)} \end{vmatrix} = (-1)^n [F_{n-2}^{(1)} - F_{n-2}]$$

****

TWO RECURSION RELATIONS FOR $F(F(n))$

EDWARD A. PARBERRY
Well's College, Aurora, New York 13026

Some time ago, in [1], the question of the existence of a recursion relation for the sequence of Fibonacci numbers with Fibonacci numbers for subscripts was raised. In the present article we give a 6$^{\text{th}}$ order non-linear recursion for $f(n) = F(F(n))$.

Proposition. Let $f(n) = F(F(n))$, where $F(n)$ is the n^{th} Fibonacci number, then

$$f(n) = (5f(n-2)^2 + (-1)^{F(n+1)})f(n-3) + (-1)^{F(n)}(f(n-3) - (-1)^{F(n+1)}f(n-6))f(n-2)/f(n-5).$$

Remark. Identity (1) below is given in [2], and identity (2) is proved similarly. Note also that $a \equiv b$ (mod 3) implies that

$$(-1)^{F(a)} = (-1)^{F(b)} = (-1)^{L(a)} = (-1)^{L(b)},$$

which is used frequently.

(1) $$F(a+b) = F(a)L(b) - (-1)^b F(a-b)$$

(2) $$5F(a)F(b) = L(a+b) - (-1)^a L(b-a).$$

Proof of Proposition. In (1), let $a = F(n-2)$, $b = F(n-1)$ to obtain

$$\begin{aligned} f(n) &= f(n-2)L(F(n-1)) - (-1)^{F(n-1)}F(-F(n-3)) \\ &= f(n-2)L(F(n-1)) - (-1)^{F(n-1)}(-1)^{F(n-3)+1}f(n-3) \\ &= f(n-2)L(F(n-1)) + (-1)^{F(n+1)}f(n-3). \end{aligned}$$

[Continued on page 139.]