COUPLED THIRD–ORDER RECURRENCES

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ABSTRACT

There are eight systems of coupled third–order recurrence relations. We find the solutions of all eight systems in terms of the Tribonacci numbers.

In the recently published book [1], we find the following statement
"We can construct 8 different schemes of generalised Tribonacci sequences in the case of two sequences ... An open problem is the construction of an explicit formula for each of the schemes given... ". (See also [2] and [3].)

Well, no longer! The object of this note is to give the explicit solution for each of the 8 schemes. We give the solutions in terms of the bilateral sequence $\{T_n\}$ defined by

$$T_0 = 1,\ T_1 = 1,\ T_2 = 2 \ \text{ and for all } n, \quad T_{n+3} = T_{n+2} + T_{n+1} + T_n.$$

Thus

$$\{T_n\} = \{\ \cdots\ , 4, 1, -3, 2, 0, -1, 1, 0, 0, 1, 1, 2, 4, 7, 13, 24, 44, 81,\ \cdots\ \}.$$

Of course, it is possible to give an explicit formula for T_n. Thus, if α is the real number satisfying

$$\alpha^3 = \alpha^2 + \alpha + 1$$

and θ is given by

$$\cos\theta = \frac{(1-\alpha)\sqrt{\alpha}}{2},$$

then for all n,

$$T_n = \frac{1}{22}(5 + \alpha + 2\alpha^2)\alpha^n + \frac{1}{11}\left(5\cos n\theta + \frac{1}{\sqrt{\alpha}}\cos{(n+1)\theta} + \frac{2}{\alpha}\cos{(n+2)\theta}\right)\frac{1}{\sqrt{\alpha}^n}.$$

The 8 schemes and their solutions are as follows. In every case the solution can be proved by induction.

$$\begin{aligned} a_{n+3} &= a_{n+2} + a_{n+1} + a_n, \\ b_{n+3} &= b_{n+2} + b_{n+1} + b_n. \end{aligned} \tag{1}$$

$$\begin{aligned} a_n &= T_{n-3}a_0 + (T_{n-1} - T_{n-2})a_1 + T_{n-2}a_2, \\ b_n &= T_{n-3}b_0 + (T_{n-1} - T_{n-2})b_1 + T_{n-2}b_2. \end{aligned}$$

$$\begin{aligned} a_{n+3} &= a_{n+2} + a_{n+1} + b_n, \\ b_{n+3} &= b_{n+2} + b_{n+1} + a_n. \end{aligned} \tag{2}$$

$$\begin{aligned} a_n = &\left(\frac{1}{2}T_{n-3} - \frac{2n-3-(-1)^n}{8}\right)a_0 + \left(\frac{1}{2}T_{n-1} - \frac{1}{2}T_{n-2} + \frac{1-(-1)^n}{4}\right)a_1 \\ &+ \left(\frac{1}{2}T_{n-2} + \frac{2n-1+(-1)^n}{8}\right)a_2 + \left(\frac{1}{2}T_{n-3} + \frac{2n-3-(-1)^n}{8}\right)b_0 \\ &+ \left(\frac{1}{2}T_{n-1} - \frac{1}{2}T_{n-2} - \frac{1-(-1)^n}{4}\right)b_1 + \left(\frac{1}{2}T_{n-2} - \frac{2n-1+(-1)^n}{8}\right)b_2, \end{aligned}$$

$$\begin{aligned} b_n = &\left(\frac{1}{2}T_{n-3} + \frac{2n-3-(-1)^n}{8}\right)a_0 + \left(\frac{1}{2}T_{n-1} - \frac{1}{2}T_{n-2} - \frac{1-(-1)^n}{4}\right)a_1 \\ &+ \left(\frac{1}{2}T_{n-2} - \frac{2n-1+(-1)^n}{8}\right)a_2 + \left(\frac{1}{2}T_{n-3} - \frac{2n-3-(-1)^n}{8}\right)b_0 \\ &+ \left(\frac{1}{2}T_{n-1} - \frac{1}{2}T_{n-2} + \frac{1-(-1)^n}{4}\right)b_1 + \left(\frac{1}{2}T_{n-2} + \frac{2n-1+(-1)^n}{8}\right)b_2. \end{aligned}$$

$$\begin{aligned} a_{n+3} &= a_{n+2} + b_{n+1} + a_n, \\ b_{n+3} &= b_{n+2} + a_{n+1} + b_n. \end{aligned} \tag{3}$$

$$\begin{aligned} a_n = &\left(\frac{1}{2}T_{n-3} + \frac{1+\cos\frac{n\pi}{2}+\cos\frac{(n+1)\pi}{2}}{4}\right)a_0 + \left(\frac{1}{2}T_{n-1} - \frac{1}{2}T_{n-2} - \frac{\cos\frac{(n+1)\pi}{2}}{2}\right)a_1 \\ &+ \left(\frac{1}{2}T_{n-2} + \frac{1-\cos\frac{n\pi}{2}+\cos\frac{(n+1)\pi}{2}}{4}\right)a_2 + \left(\frac{1}{2}T_{n-3} - \frac{1+\cos\frac{n\pi}{2}+\cos\frac{(n+1)\pi}{2}}{4}\right)b_0 \\ &+ \left(\frac{1}{2}T_{n-1} - \frac{1}{2}T_{n-2} + \frac{\cos\frac{(n+1)\pi}{2}}{2}\right)b_1 + \left(\frac{1}{2}T_{n-2} - \frac{1-\cos\frac{n\pi}{2}+\cos\frac{(n+1)\pi}{2}}{4}\right)b_2, \end{aligned}$$

$$\begin{aligned} b_n = &\left(\frac{1}{2}T_{n-3} - \frac{1+\cos\frac{n\pi}{2}+\cos\frac{(n+1)\pi}{2}}{4}\right)a_0 + \left(\frac{1}{2}T_{n-1} - \frac{1}{2}T_{n-2} + \frac{\cos\frac{(n+1)\pi}{2}}{2}\right)a_1 \\ &+ \left(\frac{1}{2}T_{n-2} - \frac{1-\cos\frac{n\pi}{2}+\cos\frac{(n+1)\pi}{2}}{4}\right)a_2 + \left(\frac{1}{2}T_{n-3} + \frac{1+\cos\frac{n\pi}{2}+\cos\frac{(n+1)\pi}{2}}{4}\right)b_0 \end{aligned}$$

$$+\left(\frac{1}{2}T_{n-1}-\frac{1}{2}T_{n-2}-\frac{\cos\frac{(n+1)\pi}{2}}{2}\right)b_1+\left(\frac{1}{2}T_{n-2}+\frac{1-\cos\frac{n\pi}{2}+\cos\frac{(n+1)\pi}{2}}{4}\right)b_2,$$

$$\begin{aligned} a_{n+3} &= a_{n+2}+b_{n+1}+b_n, \\ b_{n+3} &= b_{n+2}+a_{n+1}+a_n. \end{aligned} \tag{4}$$

$$\begin{aligned} a_n &= \left(\frac{1}{2}T_{n-3}+\frac{1}{2}(-1)^nT_{-n}\right)a_0 \\ &+\left(\frac{1}{2}T_{n-1}-\frac{1}{2}T_{n-2}-\frac{1}{2}(-1)^nT_{-n-2}-\frac{1}{2}(-1)^nT_{-n-1}\right)a_1 \\ &+\left(\frac{1}{2}T_{n-2}+\frac{1}{2}(-1)^nT_{-n-1}\right)a_2+\left(\frac{1}{2}T_{n-3}-\frac{1}{2}(-1)^nT_{-n}\right)b_0 \\ &+\left(\frac{1}{2}T_{n-1}-\frac{1}{2}T_{n-2}+\frac{1}{2}(-1)^nT_{-n-2}+\frac{1}{2}(-1)^nT_{-n-1}\right)b_1 \\ &+\left(\frac{1}{2}T_{n-2}-\frac{1}{2}(-1)^nT_{-n-1}\right)b_2, \end{aligned}$$

$$\begin{aligned} b_n &= \left(\frac{1}{2}T_{n-3}-\frac{1}{2}(-1)^nT_{-n}\right)a_0 \\ &+\left(\frac{1}{2}T_{n-1}-\frac{1}{2}T_{n-2}+\frac{1}{2}(-1)^nT_{-n-2}+\frac{1}{2}(-1)^nT_{-n-1}\right)a_1 \\ &+\left(\frac{1}{2}T_{n-2}-\frac{1}{2}(-1)^nT_{-n-1}\right)a_2+\left(\frac{1}{2}T_{n-3}+\frac{1}{2}(-1)^nT_{-n}\right)b_0 \\ &+\left(\frac{1}{2}T_{n-1}-\frac{1}{2}T_{n-2}-\frac{1}{2}(-1)^nT_{-n-2}-\frac{1}{2}(-1)^nT_{-n-1}\right)b_1 \\ &+\left(\frac{1}{2}T_{n-2}+\frac{1}{2}(-1)^nT_{-n-1}\right)b_2, \end{aligned}$$

$$\begin{aligned} a_{n+3} &= b_{n+2}+a_{n+1}+a_n, \\ b_{n+3} &= a_{n+2}+b_{n+1}+b_n. \end{aligned} \tag{5}$$

$$\begin{aligned} a_n = &\left(\frac{1}{2}T_{n-3}-(-1)^n\frac{2n-3-(-1)^n}{8}\right)a_0+\left(\frac{1}{2}T_{n-1}-\frac{1}{2}T_{n-2}+\frac{1-(-1)^n}{4}\right)a_1 \\ &+\left(\frac{1}{2}T_{n-2}+(-1)^n\frac{2n-1+(-1)^n}{8}\right)a_2+\left(\frac{1}{2}T_{n-3}+(-1)^n\frac{2n-3-(-1)^n}{8}\right)b_0 \end{aligned}$$

$$+\left(\frac{1}{2}T_{n-1}-\frac{1}{2}T_{n-2}-\frac{1-(-1)^n}{4}\right)b_1+\left(\frac{1}{2}T_{n-2}-(-1)^n\frac{2n-1+(-1)^n}{8}\right)b_2,$$

$$\begin{aligned}b_n&=\left(\frac{1}{2}T_{n-3}+(-1)^n\frac{2n-3-(-1)^n}{8}\right)a_0+\left(\frac{1}{2}T_{n-1}-\frac{1}{2}T_{n-2}-\frac{1-(-1)^n}{4}\right)a_1\\&+\left(\frac{1}{2}T_{n-2}-(-1)^n\frac{2n-1+(-1)^n}{8}\right)a_2+\left(\frac{1}{2}T_{n-3}-(-1)^n\frac{2n-3-(-1)^n}{8}\right)b_0\\&+\left(\frac{1}{2}T_{n-1}-\frac{1}{2}T_{n-2}+\frac{1-(-1)^n}{4}\right)b_1+\left(\frac{1}{2}T_{n-2}+(-1)^n\frac{2n-1+(-1)^n}{8}\right)b_2,\end{aligned}$$

$$\begin{aligned}a_{n+3}&=b_{n+2}+a_{n+1}+b_n,\\b_{n+3}&=a_{n+2}+b_{n+1}+a_n.\end{aligned}\tag{6}$$

$$\begin{aligned}a_n&=\frac{1+(-1)^n}{2}T_{n-3}a_0+\frac{1-(-1)^n}{2}(T_{n-1}-T_{n-2})a_1+\frac{1+(-1)^n}{2}T_{n-2}a_2\\&+\frac{1-(-1)^n}{2}T_{n-3}b_0+\frac{1+(-1)^n}{2}(T_{n-1}-T_{n-2})b_1+\frac{1-(-1)^n}{2}T_{n-2}b_2,\end{aligned}$$

$$\begin{aligned}b_n&=\frac{1-(-1)^n}{2}T_{n-3}a_0+\frac{1+(-1)^n}{2}(T_{n-1}-T_{n-2})a_1+\frac{1-(-1)^n}{2}T_{n-2}a_2\\&+\frac{1+(-1)^n}{2}T_{n-3}b_0+\frac{1-(-1)^n}{2}(T_{n-1}-T_{n-2})b_1+\frac{1+(-1)^n}{2}T_{n-2}b_2,\end{aligned}$$

$$\begin{aligned}a_{n+3}&=b_{n+2}+b_{n+1}+a_n,\\b_{n+3}&=a_{n+2}+a_{n+1}+b_n.\end{aligned}\tag{7}$$

$$\begin{aligned}a_n&=\left(\frac{1}{2}T_{n-3}+\frac{1}{2}T_{-n}\right)a_0+\left(\frac{1}{2}T_{n-1}-\frac{1}{2}T_{n-2}+\frac{1}{2}T_{-n-2}+\frac{1}{2}T_{-n-1}\right)a_1\\&+\left(\frac{1}{2}T_{n-2}+\frac{1}{2}T_{-n-1}\right)a_2+\left(\frac{1}{2}T_{n-3}-\frac{1}{2}T_{-n}\right)b_0\\&+\left(\frac{1}{2}T_{n-1}-\frac{1}{2}T_{n-2}-\frac{1}{2}T_{-n-2}-\frac{1}{2}T_{-n-1}\right)b_1+\left(\frac{1}{2}T_{n-2}-\frac{1}{2}T_{-n-1}\right)b_2.\end{aligned}$$

$$b_n=\left(\frac{1}{2}T_{n-3}-\frac{1}{2}T_{-n}\right)a_0+\left(\frac{1}{2}T_{n-1}-\frac{1}{2}T_{n-2}-\frac{1}{2}T_{-n-2}-\frac{1}{2}T_{-n-1}\right)a_1$$

$$+\left(\frac{1}{2}T_{n-2}-\frac{1}{2}T_{-n-1}\right)a_2+\left(\frac{1}{2}T_{n-3}+\frac{1}{2}T_{-n}\right)b_0$$
$$+\left(\frac{1}{2}T_{n-1}-\frac{1}{2}T_{n-2}+\frac{1}{2}T_{-n-2}+\frac{1}{2}T_{-n-1}\right)b_1+\left(\frac{1}{2}T_{n-2}+\frac{1}{2}T_{-n-1}\right)b_2.$$

$$\begin{aligned}a_{n+3}&=b_{n+2}+b_{n+1}+b_n,\\ b_{n+3}&=a_{n+2}+a_{n+1}+a_n.\end{aligned}\tag{8}$$

$$\begin{aligned}a_n=&\left(\frac{1}{2}T_{n-3}+\frac{(-1)^n+\cos\frac{n\pi}{2}-\cos\frac{(n+1)\pi}{2}}{4}\right)a_0\\
&+\left(\frac{1}{2}T_{n-1}-\frac{1}{2}T_{n-2}-\frac{\cos\frac{(n+1)\pi}{2}}{2}\right)a_1\\
&+\left(\frac{1}{2}T_{n-2}+\frac{(-1)^n-\cos\frac{n\pi}{2}-\cos\frac{(n+1)\pi}{2}}{4}\right)a_2\\
&+\left(\frac{1}{2}T_{n-3}-\frac{(-1)^n+\cos\frac{n\pi}{2}-\cos\frac{(n+1)\pi}{2}}{4}\right)b_0\\
&+\left(\frac{1}{2}T_{n-1}-\frac{1}{2}T_{n-2}+\frac{\cos\frac{(n+1)\pi}{2}}{2}\right)b_1\\
&+\left(\frac{1}{2}T_{n-2}-\frac{(-1)^n-\cos\frac{n\pi}{2}-\cos\frac{(n+1)\pi}{2}}{4}\right)b_2,\end{aligned}$$

$$\begin{aligned}b_n=&\left(\frac{1}{2}T_{n-3}-\frac{(-1)^n+\cos\frac{n\pi}{2}-\cos\frac{(n+1)\pi}{2}}{4}\right)a_0\\
&+\left(\frac{1}{2}T_{n-1}-\frac{1}{2}T_{n-2}+\frac{\cos\frac{(n+1)\pi}{2}}{2}\right)a_1\\
&+\left(\frac{1}{2}T_{n-2}-\frac{(-1)^n-\cos\frac{n\pi}{2}-\cos\frac{(n+1)\pi}{2}}{4}\right)a_2\\
&+\left(\frac{1}{2}T_{n-3}+\frac{(-1)^n+\cos\frac{n\pi}{2}-\cos\frac{(n+1)\pi}{2}}{4}\right)b_0\\
&+\left(\frac{1}{2}T_{n-1}-\frac{1}{2}T_{n-2}-\frac{\cos\frac{(n+1)\pi}{2}}{2}\right)b_1\end{aligned}$$

$$+\left(\frac{1}{2}T_{n-2}+\frac{(-1)^n-\cos\frac{n\pi}{2}-\cos\frac{(n+1)\pi}{2}}{4}\right)b_2.$$

REFERENCES

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