Parity index of binary words and powers of prime words

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Abstract

Let f be a binary word and let $\mathcal{F}_d(f)$ be the set of words of length d which do not contain f as a factor (alias words that avoid the pattern f). A word is called even/odd if it contains an even/odd number of 1s. The parity index of f (of dimension d) is introduced as the difference between the number of even words and the number of odd words in $\mathcal{F}_d(f)$. A word f is called prime if every nontrivial suffix of f is different from the prefix of f of the same length. It is proved that if f is a power of a prime word, then the absolute value of the parity index of f is at most 1. We conjecture that no other word has this property and prove the conjecture for words $0^r1^s0^t$, $r,s,t \geq 1$. The conjecture has also been verified by computer for all words f of length at most 10 and all $d \leq 31$.

Keywords: binary words, combinatorics on words, words avoiding a pattern, parity index, generalized Fibonacci cubes.

AMS Subject Classification (2010): 05A05, 68R15, 68W32.

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1 Introduction

Elements of $B = \{0,1\}$ are called *bits* and an element of B^d is a *binary word* of length d. Since all words considered here are binary, we will simply speak about *words*. A word $u \in B^d$ will be written in the coordinate form as $u = u_1 u_2 \ldots u_d$. A word f is a *factor* of a word x if f appears as a sequence of $|f|$ consecutive bits of x. A word u is called *f-free* if it does not contain f as a factor. For a word f and positive integer d, let

$$\mathcal{F}_d(f) = \{u \in B^d \mid u \text{ is } f\text{-free}\}.$$

The product notation will mean concatenation, for example, 1^r is the word of length r with all bits equal 1. A word b is a *power* of a word c if $b = c^k$ for some $k \geq 1$. A word is called *even* if it contains an even number of 1s and *odd* otherwise.

Suppose that f is a word and d is a positive integer. Then the *generalized Fibonacci cube*, $Q_d(f)$, is the graph obtained from the d-dimensional cube Q_d by removing all vertices that contain f as a factor. In other words, $V(Q_d(f)) = \mathcal{F}_d(f)$, two vertices being adjacent if they differ in exactly one bit. These graphs were studied for the first time in [3], but special cases were extensively studied earlier. The most notable special case is formed by Fibonacci cubes $\Gamma_d = Q_d(11)$, $d \geq 1$, see the survey [4]. The special case of $Q_d(1^s)$ was introduced in [2] (under the same name of generalized cubes) and further investigated in [6, 9].

The definition of the generalized Fibonacci cubes naturally leads to different problems on words. The most fundamental problem is to determine the order of these graphs. This problem was studied earlier under the notion of words avoiding a pattern. Calling f a pattern, then the number of words avoiding f is just the number of f-free words. Baccherini, Merlini and Sprugnoli [1] were interested in the number of f-free words that contain prescribed numbers of 0s and 1s and established that they are closely related to proper Riordan arrays. This work was extended in [7].

Another natural problem about generalized Fibonacci cubes is when they embed isometrically into hypercubes. This question naturally leads to the concept of the so called good and bad words. A word f is said to be *d-good* if for any f-free words u and v of length d, v can be obtained from u by complementing one by one the bits of u on which u and v differ, such that all intermediate words are f-free. Then f is *good* if it is d-good for any $d \geq 1$. The main result of [5] asserts that about eight percent of all words are good.

Our principal motivation for the present paper is a result of [6] asserting that each $Q_d(1^r)$ contains a hamiltonian path. This in particular implies that the bipartition of $Q_d(1^r)$ is balanced. (By the way, it is not difficult to see that every generalized

Fibonacci cube is connected.) Clearly, the bipartition sets of $Q_d(f)$ are formed by even and odd words, respectively. Hence, for a set of words X, let $e(X)$ and $o(X)$ be the number of even and odd words in X, respectively. Let in addition $\Delta(X) = e(X) - o(X)$, in particular write $\Delta(x) = \Delta(\{x\})$ for a word x. That is, $\Delta(x) = 1$ if x is even and $\Delta(x) = -1$ if x is odd. Then we define the *parity index* of f of dimension d as

$$\mathrm{PI}_d(f) = \Delta(\mathcal{F}_d(f)) .$$

Using this notation, a necessary condition for $Q_d(f)$ to contain a hamiltonian path is that $|\mathrm{PI}_d(f)| \le 1$.

In the next section we introduce prime words and prove that if f is a power of a prime word then $|\mathrm{PI}_d(f)| \le 1$ holds for any d. In Section 3 we consider the parity index of the words $0^r1^s0^t$ and prove that for any d large enough, $|\mathrm{PI}_d(0^r1^s0^t)| \ge 2$. For the special case of 0^r10^r a more precise result is obtained, in particular it is noted that $\{|PI_d(010)|\}_{d\ge 3}$ is the so-called Padovan sequence. In the final section we pose a conjecture that powers of prime words are the only words with the property $|\mathrm{PI}_d(f)| \le 1$ for any d and verify the conjecture for all words of length ≤ 10 and for all $d \le 31$.

2 Powers of prime words

A word f of length d is *prime* if for any k, $1 \le k \le d-1$, the suffix of f of length k is different from the prefix of f of the same length. In particular, words 0 and 1 are prime, and if $d \ge 2$, then the first bit and the last bit of a prime word are different. For instance, 001101 is a prime word which easily follows from the fact that the factor 00 appears only at its beginning. On the other hand the word 01101011 is not prime as it starts and ends with 011.

For a word f of length ℓ let $\mathcal{S}_d(f) = B^d \setminus \mathcal{F}_d(f)$, that is,

$$\mathcal{S}_d(f) = \{b = b_1b_2\dots b_d \mid b \text{ contains factor } f\} .$$

For $i = 1, 2, \dots, d-\ell+1$ let in addition

$$\mathcal{S}_d^{(i)}(f) = \{b = b_1b_2\cdots b_d \mid b \in S, b_ib_{i+1}\cdots b_{i+\ell-1} = f\} .$$

Then $\mathcal{S}_d(f) = \bigcup_{i=1}^{d-\ell+1} \mathcal{S}_d^{(i)}(f)$.

By $\binom{X}{k}$ we denote the set of all k-subsets of the set X.

Lemma 2.1 *Let f be a word of length ℓ. Then*

$$\Delta(\mathcal{S}_d(f)) = \sum_{k=1}^{d-\ell+1} (-1)^{k-1} \sum_{I \subseteq \binom{\mathbb{N}_{d-\ell+1}}{k}} \Delta\left(\cap_{i\in I}\mathcal{S}_d^{(i)}(f)\right) .$$

Proof. Let χ_A be the characteristic function of a set A:

$$\chi_A(x) = \begin{cases} 1; & x \in A, \\ 0; & \text{otherwise}. \end{cases}$$

Since $\mathcal{S}_d(f) = \bigcup_{i=1}^{d-\ell+1} \mathcal{S}_d^{(i)}(f)$, the inclusion and exclusion principle implies that for every $x \in \mathcal{S}_d(f)$,

$$\sum_{k=1}^{d-\ell+1} (-1)^{k-1} \sum_{I \subseteq \binom{\mathbb{N}_{d-\ell+1}}{k}} \chi_{\cap_{i\in I}\mathcal{S}_d^{(i)}(f)}(x) = 1 .$$

Therefore,

$$\begin{aligned} \Delta(\mathcal{S}_d(f)) &= \sum_{x\in\mathcal{S}_d(f)} \Delta(x) \\ &= \sum_{x\in\mathcal{S}_d(f)} \Delta(x) \left(\sum_{k=1}^{d-\ell+1} (-1)^{k-1} \sum_{I \subseteq \binom{\mathbb{N}_{d-\ell+1}}{k}} \chi_{\cap_{i\in I}\mathcal{S}_d^{(i)}(f)}(x) \right) \\ &= \sum_{k=1}^{d-\ell+1} (-1)^{k-1} \sum_{I \subseteq \binom{\mathbb{N}_{d-\ell+1}}{k}} \sum_{x\in\mathcal{S}_d(f)} \Delta(x) \chi_{\cap_{i\in I}\mathcal{S}_d^{(i)}(f)}(x) \\ &= \sum_{k=1}^{d-\ell+1} (-1)^{k-1} \sum_{I \subseteq \binom{\mathbb{N}_{d-\ell+1}}{k}} \Delta(\cap_{i\in I}\mathcal{S}_d^{(i)}(f)) . \end{aligned}$$

□

Theorem 2.2 *Let f be a power of a prime word. Then $|\mathrm{PI}_d(f)| \le 1$ for any $d \ge 1$.*

Proof. Let $d \ge 1$. Suppose first that f is a prime word. When $d < \ell$, we have $\mathcal{F}_d(f) = B^d$ and if $d = \ell$, then $\mathcal{F}_d(f)$ contains all but the word f. Hence we may assume in the rest that $d > \ell$. Since $e(B^d) = o(B^d)$, we have $e(\mathcal{F}_d(f)) + e(\mathcal{S}_d(f)) = o(\mathcal{F}_d(f)) + o(\mathcal{S}_d(f))$. Hence $\mathrm{PI}_d(f) = \Delta(\mathcal{F}_d(f)) = -\Delta(\mathcal{S}_d(f))$. It thus suffices to prove that $|\Delta(\mathcal{S}_d(f))| \le 1$.

We first note that $\Delta(\mathcal{S}_d^{(i)}(f)) = 0$. Indeed, the first $i-1$ bits and the last $d-\ell-i+1$ bits of the words from $\mathcal{S}_d^{(i)}(f)$ are arbitrary, hence $\mathcal{S}_d^{(i)}(f)$ contains $2^{d-\ell-1}$ even words and the same number of odd words. Consider now $X = \cap_{i\in I}\mathcal{S}_d^{(i)}(f)$ where $I = \{i_1, i_2, \ldots, i_k\}$ and $i_1 < i_2 < \cdots < i_k$. Because f is a prime word, $X = \emptyset$ as soon as for some index j, $i_{j+1} - i_j < \ell$. Moreover, by the same argument as the one used for $\Delta(\mathcal{S}_d^{(i)}(f))$, $\Delta(X) = 0$ as soon as for some index j, $i_{j+1} - i_j > \ell$. Hence $\Delta(X)$ can be nonzero only when $k\ell = d$ and $i_j = (j-1)\ell + 1$ for each $1 \le j \le k$. Therefore, applying

Lemma 2.1,

$$\Delta(\mathcal{S}_d(f)) = \begin{cases} 0; & \ell \nmid d, \\ -1; & \ell | d,\ k \text{ odd},\ f \text{ contains odd number of 1s}, \\ 1; & \text{otherwise}. \end{cases}$$

The proof is complete for a prime word f.

Assume now that $f = (f')^r$, where f' is a prime word and $r \geq 2$. Let $|f'| = \ell'$. The proof continues similarly as in the case when f was prime. The only difference is that now $X = \varnothing$ as soon as for some index j, the difference $i_{j+1} - i_j$ is not a multiple of ℓ' and so $\Delta(X)$ can be nonzero only when d is a multiple of ℓ'. □

3 Non-prime words

In this section we study the parity index of words consisting of three blocks, that is, of words $0^r1^s0^t$, $r, s, t \geq 1$. Clearly, none of these words is prime. In our main result (Theorem 3.2) we prove that for no such word f, $|\mathrm{PI}_d(f)| \leq 1$ holds for all d. Before that we separately give a more precise result for the special case of 0^r10^r. The obtained results in particular imply that $Q_d(0^r1^s0^t)$ does not contain a hamiltonian path as soon as d is large enough.

Theorem 3.1 *Let $r \geq 1$. Then*

$$|\mathrm{PI}_d(0^r10^r)| = \begin{cases} 0; & d \leq 2r, 2r+2 \leq d \leq 3r+1\,, \\ 1; & d = 2r+1, 3r+2 \leq d \leq 4r+3\,. \end{cases}$$

Moreover, for any $d \geq 4r+4$, $|\mathrm{PI}_d(0^r10^r)| \geq 2$.

Proof. Suppose first that $d \leq 2r$. Then $\mathcal{F}_d(0^r10^r) = B^d$ and hence $\mathrm{PI}_d(0^r10^r) = 0$. Since $\mathcal{F}_{2r+1}(0^r10^r) = B^d \setminus \{0^r10^r\}$ we have $\mathrm{PI}_{2r+1}(0^r10^r) = 1$.

Let $d \geq 2r+2$. Recall that $-\mathrm{PI}_d(0^r10^r) = \Delta(\mathcal{S}_d(f)) = \sum_{b \in \mathcal{S}_d(f)} \Delta(b)$. By Lemma 2.1,

$$-\mathrm{PI}_d(f) = \sum_{k=1}^{d-\ell+1} (-1)^{k-1} \sum_{I \subseteq \binom{\mathbb{N}_{d-\ell+1}}{k}} \Delta(\cap_{i \in I} \mathcal{S}_d^{(i)}(f))\,.$$

Suppose that for a set $X = \cap_{i \in I} \mathcal{S}_d^{(i)}(f)$ there exists an index i such that if $w \in X$ then also $w + e_i \in X$. Then $\Delta(X) = 0$. It follows that $\Delta(X) \neq 0$ if and only if there exist $k \geq 0$ and $r \leq r_j \leq 2r$ for all $1 \leq j \leq k$, such that

$$X = \{0^r10^{r_1}10^{r_2}1 \cdots 0^{r_k}10^r\}\,.$$

Moreover, in that case $\Delta(X) = \Delta(0^r10^{r_1}10^{r_2}1 \cdots 0^{r_k}10^r) = (-1)^{k+1}$.

Hence let $k \geq 0$ and $r \leq r_j \leq 2r$, $1 \leq j \leq k$, and set $b = 0^r 1 0^{r_1} 1 0^{r_2} 1 \cdots 0^{r_k} 1 0^r$. Let $v = b_{r_1+2} b_{t_1+3} \cdots b_d$ be the word obtained from b by omitting the first $r_1 + 1$ bits, so that $v \in B^{d-r_1-1}$. Since b has one more bit of 1 than v does, $\Delta(v) = -\Delta(b)$.

Note that v starts with $0^r 1$. Then

$$v \in \bigcap_{j \in J} S^{(j)}_{d-r_1-1}(f) \qquad \text{if and only if} \qquad b \in \mathcal{S}^{(1)}_d(f) \bigcap \left(\cap_{j \in J} \mathcal{S}^{(j+r_1+1)}_d(f) \right).$$

Now we can compute as follows:

$$\begin{aligned}
\mathrm{PI}_d(0^r 1 0^r) &= -\Delta(\mathcal{S}_d(f)) = -\sum_{b \in \mathcal{S}_d(f)} \Delta(b) \\
&= -\left(\sum_{\substack{b \in \mathcal{S}_d(f) \\ r_1 = r}} \Delta(b) + \sum_{\substack{b \in \mathcal{S}_d(f) \\ r_1 = r+1}} \Delta(b) + \cdots + \sum_{\substack{b \in \mathcal{S}_d(f) \\ r_1 = 2r}} \Delta(b) \right) \\
&= -\Bigg(\sum_{v \in S_{d-r-1}(f)} -\Delta_{d-r-1}(v) + \sum_{v \in S_{d-r-2}(f)} -\Delta_{d-r-2}(v) + \cdots \\
&\qquad + \sum_{v \in S_{d-2r-1}(f)} -\Delta_{d-2r-1}(v) \Bigg) \\
&= \sum_{v \in S_{d-r-1}(f)} \Delta_{d-r-1}(v) + \sum_{v \in S_{d-r-2}(f)} \Delta_{d-r-2}(v) + \cdots \\
&\qquad + \sum_{v \in S_{d-2r-1}(f)} \Delta_{d-2r-1}(v) \\
&= \sum_{r \leq r_1 \leq 2r} \sum_{v \in S_{d-r_1-1}(f)} \Delta_{d-r_1-1}(v) \\
&= \sum_{r \leq r_1 \leq 2r} \Delta_{d-r_1-1} \left(\bigcup_{v \in S_{d-r_1-1}(f)} v \right) \\
&= \sum_{r \leq r_1 \leq 2r} \Delta_{d-r_1-1}(S_{d-r_1-1}(f)) \\
&= -\sum_{r \leq r_1 \leq 2r} \mathrm{PI}_{d-r_1-1}(0^r 1 0^r). \qquad (1)
\end{aligned}$$

It follows that $|\mathrm{PI}_d(0^r 1 0^r)| = \left| \sum_{r \leq r_1 \leq 2r} \mathrm{PI}_{d-r_1-1}(0^r 1 0^r) \right|$.

As the values $\mathrm{PI}_{d-r_1-1}(0^r 1 0^r)$ have the same sign for all $r \leq r_1 \leq 2r$, from Equation (1) we get

$$|\mathrm{PI}_d(0^r 1 0^r)| = \sum_{r \leq r_1 \leq 2r} |\mathrm{PI}_{d-r_1-1}(0^r 1 0^r)|. \qquad (2)$$

Set $a_d = |\mathrm{PI}_d(0^r 1 0^r)|$ for all d. We already know that $a_d = 0$ for all $d \leq 2r$ and that $a_{2r+1} = 1$. Let $d \geq 2r + 2$. If $d \leq 3r + 1$ and there is a word $b \in S_d(f)$, then there is an index i such that if $w \in X$ then also $w + e_i \in X$ and hence $\Delta(X) = 0$.

Assume $d \geq 3r+2$. When $3r+2 \leq d \leq 4r+2$, $\mathrm{PI}_d(0^r10^r) = \Delta(0^r10^{d-2r-2}10^r) = 1$ or -1 and hence $a_d = 1$. When $d = 4r+3$, $a_d = a_{2r+2} + \cdots + a_{3r+2} = 1$. Let $d = 4r+3+u$ for some $u \geq 1$. Then by Equation (2), $a_{4r+3+u} = a_{2r+2+u} + \cdots + a_{3r+2+u}$. If $u \leq r$, then $2r+2+u \leq 3r+2 < 3r+2+u \leq 4r+2$ and therefore $a_d \geq 2$. Assume $u \geq r+1$. Let $u' = u-r$. Then $d = 5r+3+u'$ for $u' \geq 1$. By Equation (2), $a_{5r+3+u'} = a_{3r+2+u'} + \cdots + a_{4r+2+u'}$. If $u' \leq r$, then $3r+2 \leq 3r+2+u' < 4r+3 \leq 4r+2+u'$ and therefore $a_d \geq 2$. Assume $u' \geq r+1$. Then let $u'' = u'-r$. Then $d = 6r+3+u''$ for $u'' \geq 1$. By Equation (2), $a_{6r+3+u''} = a_{4r+2+u''} + \cdots + a_{5r+2+u''}$. As $3r+2 \leq 4r+2+u'' < 5r+2+u''$, $a_d \geq 2$. Thus when $d \geq 4r+4$, $a_d \geq 2$. □

The special case of Theorem 3.1 when $r = 1$ deserves a special attention. In that case,

$$|\mathrm{PI}_d(010)| = |\mathrm{PI}_{d-2}(010)| + |\mathrm{PI}_{d-3}(010)|$$

with initial conditions $|\mathrm{PI}_3(010)| = 1$, $|\mathrm{PI}_4(010)| = 0$, $|\mathrm{PI}_5(010)| = 1$ which is the Padovan sequence, see sequence A000931 from [8].

Theorem 3.2 *Let $r,s,t \geq 1$. Let z be the integer such that $(z-1)t+2 \leq r+s \leq zt+1$. Then*

$$|\mathrm{PI}_d(0^r1^s0^t)| \begin{cases} = 0; & d < r+s+t, \\ & y(r+s+t) < d < (y+1)(r+s)+t \quad \text{for } 1 \leq y \leq z\,, \\ \geq 1; & d = r+s+t, \\ & (y+1)(r+s)+t \leq d \leq (y+1)(r+s+t) \quad \text{for } 1 \leq y \leq z, \\ & d = (z+1)(r+s+t)+1\,. \end{cases}$$

Moreover, for any $d \geq (z+1)(r+s+t)+2$, $|\mathrm{PI}_d(0^r1^s0^t)| \geq 2$.

Proof. Since $\mathrm{PI}_d(0^r1^s0^t) = \mathrm{PI}_d(0^t1^s0^r)$, it suffices to prove the result for words $0^r1^s0^t$ with $r \geq t$. By the same argument as in the proof of Theorem 3.1, $\Delta(X) \neq 0$ if and only if there exist $k \geq 0$ and $r \leq r_j \leq r+t$ for all $1 \leq j \leq k$ such that

$$X = \{0^r1^s0^{r_1}1^s0^{r_2}1^s \cdots 0^{r_k}1^s0^t\}$$

where $\Delta(X) = (-1)^{(k+1)s}$. Also

$$|\mathrm{PI}_d(0^r1^s0^t)| = \sum_{r \leq r_1 \leq r+t} |\mathrm{PI}_{d-r_1-s}(0^r1^s0^t)|\,. \tag{3}$$

Set $a_d = |\mathrm{PI}_d(0^r1^s0^t)|$ for all d. We already know that $a_d = 0$ for all $d < r+s+t$ and that $a_{r+s+t} = 1$. Let $d \geq r+s+t+1$. In the first part of the proof, we prove the theorem for $d \leq (z+1)r+(z+1)s+(z+1)t$ by induction on y for $1 \leq y \leq z$. Then we prove the theorem for $d \geq (z+1)r+(z+1)s+(z+1)t+1$. The idea of the proof is as follows. From the first part of the proof, we notice that for each $y \geq 1$, $a_d = 0$ for $r + s - (y-1)t - 1$ consecutive numbers of d and then $a_d \geq 1$ for the next $yt + 1$ consecutive numbers of d. As y increases, $r + s - (y-1)t - 1$ decreases to zero and $yt + 1$ increases. While, by Equation (3), $a_d = a_{d-r-s-t} + \cdots + a_{d-r-s}$, which is a sum of $t + 1$ consecutive numbers, where $t + 1$ is a constant for given $0^r 1^s 0^t$. Therefore for large enough d, $a_d \geq 2$.

By a similar argument as in the proof of Theorem 3.1, $a_d = 0$ if $d < 2r + 2s + t$ and $a_d = 1$ if $2r + 2s + t \leq d \leq 2r + 2s + 2t$. Thus the statement is true for $y = 1$. Let $y \geq 2$. Suppose the statement is true for all $1 \leq y_0 < y$. Let $d = yr + ys + yt + u$ for some $u \geq 1$. Then by Equation (3), $a_{yr+ys+yt+u} = a_{(y-1)r+(y-1)s+(y-1)t+u} + \cdots + a_{(y-1)r+(y-1)s+yt+u}$. When $d < (y+1)r + (y+1)s + t$, i.e., $u < r + s - (y-1)t$, $(y-1)r + (y-1)s + (y-1)t < (y-1)r + (y-1)s + (y-1)t + u < (y-1)r + (y-1)s + yt + u < yr + ys + t$ and hence by the induction assumption, $a_d = 0$. When $d = (y+1)r + (y+1)s + t$, i.e., $u = r + s - (y-1)t$, $(y-1)r + (y-1)s + yt + u = yr + ys + t$ and hence by the induction assumption, $a_d \geq a_{yr+ys+t} \geq 1$. Assume $d \geq (y+1)r + (y+1)s + t + 1$, i.e., $u \geq r + s - (y-1)t + 1$. Let $u' = u - r - s + (y-1)t$. Then $d = (y+1)r + (y+1)s + t + u'$ where $u' \geq 1$. By Equation (3), $a_d = a_{yr+ys+u'} + \cdots + a_{yr+ys+t+u'}$. If $d \leq (y+1)r + (y+1)s + (y+1)t$, i.e., $u' \leq yt$, then $yr + ys + u' \leq yr + ys + yt$. Considering that $yr + ys + t < yr + ys + t + u'$, $a_d \geq a_{yr+ys+t} + a_{yr+ys+t+1}$ or $a_d \geq a_{yr+ys+u'} + a_{yr+ys+u'+1}$ depending on whether $yr + ys + u' < yr + ys + t$ or not. Therefore by the induction assumption, $a_d \geq 1$. Thus the theorem is proved for all $d \leq (z+1)r + (z+1)s + (z+1)t$.

Assume $d \geq (z+1)r + (z+1)s + (z+1)t + 1 \geq (z+2)r + (z+2)s + t$. Let $d = (z+2)r + (z+2)s + t + u''$ where $u'' \geq 0$. Then by Equation (3), $a_d = a_{(z+2)r+(z+2)s+t+u''} = a_{(z+1)r+(z+1)s+u''} + \cdots + a_{(z+1)r+(z+1)s+t+u''}$. Note that $(z+1)r + (z+1)s + t + u'' \geq (z+1)r + (z+1)s + t$. Assume $d = (z+2)r + (z+2)s + t$, i.e., $u'' = 0$. Then $(z+1)r + (z+1)s + t + u'' = (z+1)r + (z+1)s + t$. If $r + s > zt$, then $(z+1)r + (z+1)s > zr + zs + zt$ and hence $a_d = a_{(z+1)r+(z+1)s+t} \geq 1$. If $r + s \leq zt$, then $(z+1)r + (z+1)s \leq zr + zs + zt$ and hence $a_d \geq a_{zr+zs+zt} + a_{(z+1)r+(z+1)s+t} \geq 2$.

Let $d > (z+2)r + (z+2)s + t$, i.e., $u'' > 0$. Then $(z+1)r + (z+1)s + t + u'' > (z+1)r + (z+1)s + t$. First assume $d < (z+2)r + (z+2)s + (z+2)t$, i.e., $u'' < (z+1)t$. Then $(z+1)r + (z+1)s + u'' < (z+1)r + (z+1)s + (z+1)t$. Therefore $a_d \geq a_{(z+1)r+(z+1)s+t} + a_{(z+1)r+(z+1)s+t+1}$ or $a_d \geq a_{(z+1)r+(z+1)s+u''} + a_{(z+1)r+(z+1)s+u''+1}$ depending on whether $(z+1)r + (z+1)s + u'' < (z+1)r + (z+1)s + t$ or not. Thus $a_d \geq 2$ in any case. Second assume $d = (z+2)r + (z+2)s + (z+2)t$, i.e., $u'' = (z+1)t$. Then $(z+1)r + (z+1)s + u'' = (z+1)r + (z+1)s + (z+1)t$ and hence $a_d \geq a_{(z+1)r+(z+1)s+u''} + a_{(z+1)r+(z+1)s+u''+1} \geq 2$ considering that $(z+1)r + (z+1)s + (z+1)t < (z+1)r + (z+1)s + u'' + 1 < (z+2)r + (z+2)s + (z+2)t$. Finally assume $d > (z+2)r + (z+2)s + (z+2)t$, i.e., $u'' > (z+1)t$. Suppose there is $u'' > (z+1)t$ such that $a_d \le 1$. Let u''_0 be the smallest such an integer and $d_0 = (z+2)r + (z+2)s + t + u''_0$. Then $a_{d_0} = a_{(z+1)r+(z+1)s+u''_0} + \cdots + a_{(z+1)r+(z+1)s+t+u''_0}$. Since $(z+1)r+(z+1)s+u''_0 > (z+1)r+(z+1)s+(z+1)t$ and $(z+1)r+(z+1)s+t+u''_0 < d_0$, $a_{d_0} \ge 2$, which is a contradiction. Thus the statement is true for all d. □

4 Computer evidence and conjecture

Using computer we obtained the parity index for all words f of length at most 10 and all $d \le 31$. Since $Q_d(f)$ is isomorphic to $Q_d(\overline{f})$, where $\overline{f}$ is the binary complement of f, we have restricted the computation to words f that contain not more 1s than 0s. From the same reason reversed words need not to be considered. In Table 1 all words f of length at most 8 and with $|\mathrm{PI}_d(f)| \le 1$ for $d \le 31$ are collected.

length	f
3	001
4	0001, 0011, 0101
5	00001, 00011, 00101
6	000001, 000011, 000101, 000111 001001, 001011, 001101, 010101
7	0000001, 0000011, 0000101, 0000111 0001001, 0001011, 0001101, 0010011 0010101, 0011101
8	00000001, 00000011, 00000101, 00000111 00001001, 00001011, 00001101, 00001111 00010001, 00010011, 00010101, 00010111 00011001, 00011011, 00011101, 00100011 00100101, 00101011, 00101101, 00110011 00110101, 00111101, 01010101

Table 1: List of words f with $|f| \le 8$ and $|\mathrm{PI}_d(f)| \le 1$ for $d \le 31$

It can be checked that every word from the table is a power of a prime word. Moreover, the same was verified also for the obtained words of length 9 and 10 (not given in the table). Based on this experiment and Theorems 2.2 and 3.2 we pose:

Conjecture 4.1 *Let f be a word such that $|\mathrm{PI}_d(f)| \le 1$ holds for any d. Then f is a power of a prime word.*

A possible approach to the conjecture would be to prove that if f is not a power of a prime word, then the sequence $\{|\mathrm{PI}_d(f)|\}_d$ satisfies a certain recurrence relation from which we can deduce the behavior of the sequence. For instance, one can establish the recurrent formula

$$|\mathrm{PI}_d(01110)| = |\mathrm{PI}_{d-4}(01110)| + |\mathrm{PI}_{d-5}(01110)|,$$

with initial conditions $|\mathrm{PI}_5(01110)| = 1$, $|\mathrm{PI}_6(01110)| = |\mathrm{PI}_7(01110)| = |\mathrm{PI}_8(01110)| = 0$ and $|\mathrm{PI}_9(01110)| = 1$. Similarly, either by applying Equation (3) or by a tedious case analysis yields, one can get:

$$\begin{aligned}|\mathrm{PI}_d(000001000)| &= |\mathrm{PI}_{d-6}(000001000)| + |\mathrm{PI}_{d-7}(000001000)| \\ &\quad + |\mathrm{PI}_{d-8}(000001000)| + |\mathrm{PI}_{d-9}(000001000)|,\end{aligned}$$

with initial conditions $|\mathrm{PI}_9(000001000)| = 1$, $|\mathrm{PI}_{10}(000001000)| = |\mathrm{PI}_{11}(000001000)| = |\mathrm{PI}_{12}(000001000)| = |\mathrm{PI}_{13}(000001000)| = |\mathrm{PI}_{14}(000001000)| = 0$, $|\mathrm{PI}_{15}(000001000)| = |\mathrm{PI}_{16}(000001000)| = |\mathrm{PI}_{17}(000001000)| = 1$.

In Fig. 1 the values of $|\mathrm{PI}_d(01110)|$ and $|\mathrm{PI}_d(000001000)|$ for $5 \le d \le 55$ are plotted. Note that the sequence $|PI_d(f)|$ does not need to be monotone, but it seems that starting from some large enough dimension the sequence is strictly increasing.

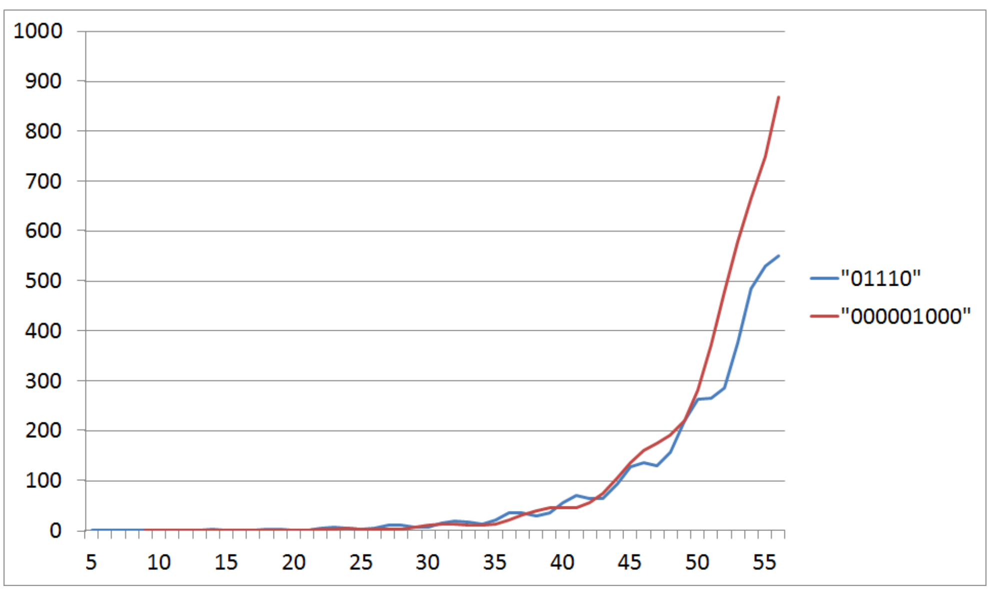

Figure 1: Values of $|\mathrm{PI}_d(f)|$ for $f = 01110$ and $f = 000001000$

Acknowledgements

This work was supported by the research Grants 174010 and 174033 of Serbian Ministry of Education and Science, by the research grant P1-0297 of Ministry of Higher Education, Science and Technology Slovenia, and by Basic Science Research Program through the National Research Foundation of Korea funded by the Ministry of Education, Science and Technology grant 2011-0025319.

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