A Proposed Method for Reliability Analysis in Higher Dimension

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Abstract— **In this paper, a new method is proposed to evaluate the reliability of stochastic mechanical systems. This technique is based on the combination of the probabilistic transformation methods for multiple random variables (MPTM) and the finite element method (FEM). The transformation technique evaluates the Probability Density Function (PDF) of the system response by the use of the Jacobian of the inverse mechanical function. This approach has the advantage of giving directly the whole density function of the response - in a closed form -, which is very helpful in reliability analysis.**

Index Terms— **Probabilistic method, Finite Element Method, Reliability Analysis, FORM, Monte-Carlo simulation.**

I. Introduction

Many mechanical applications require the consideration of stochastic properties of materials, geometry and loads. The basic representation of uncertain parameters in the underlying models is obtained by introducing random variables or fields. Different kinds of analyses accounting for uncertainties can be carried out. The second moment analysis aims to characterize the means and variances of response quantities (displacements, strain and stress components, etc.) in terms of the input variable moments. The perturbation method [1] and the weighted integral method [2] belong to this category. On the other hand, the reliability methods focus on the calculation of the probability of failure associated with a given limit-state function. First and second order reliability methods (FORM / SORM) and various simulation methods are commonly used in reliability analysis [3]. To account for the spatial variability of uncertain quantities (e.g. material properties), a characterization in terms of a random field is usually employed. Through a process of discretization, it is possible to represent the random field by a vector of random variables. One of the methods mentioned above may then be used to carry out second-moment or reliability analyses. The spectral stochastic finite element method (SSFEM) proposed by Ghanem and Spanos [4] is an approach well suited to analyses involving random fields. It is based on two types of discretization of the system of stochastic partial differential equations governing the problem under consideration: one in the spatial domain and one in the probabilistic domain. The response (e.g. the random vector of

nodal displacements) is cast as a series expansion in terms of standard normal variables. This can be interpreted as an 'intrinsic' representation of the random response, from which quantities such as statistical moments can be computed by post-processing, either analytically or by sampling.

The aim of this paper is to generalize the probabilistic transformation methods (PTM), introduced recently by the authors [5] for one-dimensional. This new method is presented in order to evaluate the stochastic mechanical response. The method is based on the combination of the probabilistic transformation methods (PTM) for multiple random variables (e.g. Young's modulus and load) and the deterministic finite element method (FEM). The transformation technique evaluates the Probability Density Function (PDF) of the system response by multiplying the input PDF with the Jacobean of the inverse mechanical function.

II. Reliability Analysis

The reliability methods aim at evaluates the probability of failure of structural systems subjected to randomness. The design of structures and the prediction of their good functioning lead to the verification of a certain number of rules resulting from the knowledge of physical and mechanical experience of designers and constructors. These rules traduce the necessity to limit the loading effects such as stresses and displacements. Each rule represents an elementary event and the occurrence of several events leads to a failure scenario. The objective is then to evaluate the failure probability corresponding to the occurrence of critical failure modes. In addition to the vector of deterministic variables **x** used in the system control and optimization, the uncertainties are modeled by a vector of stochastic physical variables **Y** affecting the failure scenario. The knowledge of these variables is not, at best, more than statistical information and we admit a representation in the form of random variables. For a given design rule, the basic random variables are defined by their probability distribution associated with some expected parameters; the vector of random variables is noted herein **Y** whose realizations are written **y**. The safety is the state where the structure is able to fulfill all the functioning requirements: mechanical and service ability, for which it is designed. To evaluate the failure probability with respect to a chosen failure scenario, a limit state function $G(x, y)$ is defined by the

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condition of good functioning of the structure. The limit between the *state of failure* $G(x,y) \le 0$ and the *state of safety* $G(x,y) > 0$ is known as the *limit state surface* $G(x,y) = 0$ (figure 1). The failure probability is then calculated by:

$$P_f = \Pr[G(x) \le 0] = \int_{G(x)\le 0} f_X(x)dx_1 ... dx_n \qquad (1)$$

where P_f is the failure probability, $f_X(\mathbf{x})$ is the joint density function of the random variables **X** and Pr[.] is the probability operator. The evaluation of integral (1) is not easy, because it represents a very small quantity and all the necessary information for the joint density function are not available. For these reasons, the *First and the Second Order Reliability Methods FORM/SORM* has been developed. They are based on the reliability index concept, followed by an estimation of the failure probability. The invariant reliability index β was introduced by working in the space of standard independent Gaussian variables instead of the space of physical variables. The transformation from the physical variables **y** to the normalized variables **u** is given by:

$$u = T(x,y) \text{ and } y = T^{-1}(x,u) \qquad (2)$$

This transformation *T*(.) is called the *probabilistic transformation*. In this standard space, the limit state function takes the form:

$$H(u) \equiv G(x) = 0 \qquad (3)$$

For practical engineering, equation (3) gives sufficiently accurate estimation of the failure probability.

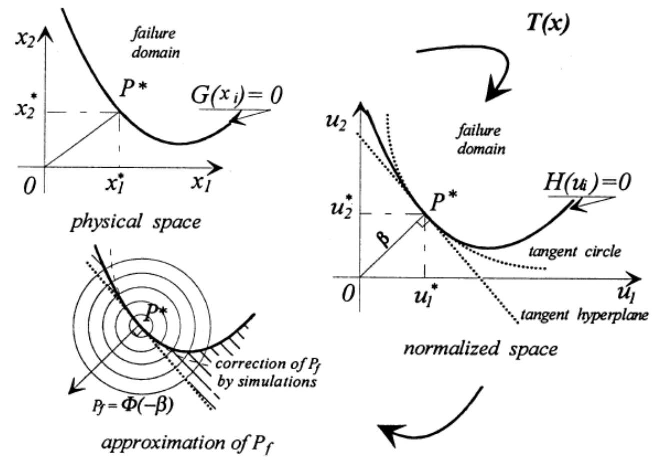

Fig. 1: Reliability analysis methodology

A. Approximate reliability methods (Form & Sorm)

The First Order Reliability Method (FORM) uses the closest point on the limit state function H(u)=0 to the origin in the standard normal space as a measure of the reliability. This point is called design point u^* and its beta value $\beta = \|u^*\|$ specifies the reliability level; the failure probabilities $P_f \approx \Phi(-\beta)$, where $\Phi(.)$ denotes the standard cumulative distribution function.

Second Order Reliability Methods (SORM) approximate the limit state functions by an incomplete second order polynomial, which assumes in a rotated space the simple form

$$H(u) \approx \tilde{H}(u) = \beta - u_1 + \frac{1}{2}\sum_{i=2}^{n} k_i u_i^2 \qquad (4)$$

The form is incomplete since the terms $\{(\beta - u_1)u_i\}_{i=2}^{n}$, leading to a complete second order approximation around the design point, are missing. An exact result in form of a one dimensional integral has been derived in [6] for the incomplete representation [6] and an asymptotic result, sufficiently accurate for large β value, has been developed [7].

B. Importance Sampling

Importance Sampling has been one of the most prevalent approaches in the context of Simulation based methods for the estimation of structural reliability [8]. The underlying concept is to draw samples of the vector of random parameters X from a distribution $f_X(x)$ which is concentrated in the 'important region' of the random parameter space.

C. Line Sampling

An alternative, quite suitable for large dimension X and efficient in context of FEA, is the following approach denoted as "line sampling". It requires an important direction as starting point, defining the direction along the limit state G(x,y) will be determined. Each point (x,y) in the standard point is decomposed into the one dimensional space X [9].

D. Some factors of comparison

The procedures for estimating the failure probability developed over the last twenty years like FORM/SORM, Importance Sampling, and all its variants, lack robustness or computational efficiency as the number of random variables tends to infinity. However, the robust and simple straight forward Line Sampling procedures, is able to overcome most difficulties encountered in traditional procedures. Summarizing the arguments, the following can be concluded:

- FORM provides a point estimate, subject to linearization errors, without confidence. Moreover, it requires the evaluation of the design point, which becomes difficult in high dimensions for nonlinear limit state functions in the normalized space. The efforts to compute the design point grows proportional with the number of variables.
- SORM requires in addition to the design point, the main curvatures which cannot be obtained in a feasible manner for high dimensions. The procedure implies that the domain close to the design point is

the important domain which is not the case for high dimensions.

- Importance Sampling is more robust and accurate than FORM / SORM, but not competitive to Line Sampling, because it is generally impossible to sample according to the optimal sampling density. The approach has difficulties to deal with multiple failure domains if they are not well separated.
- Line Sampling is capable to take advantage of simple flat limit states in standard normal space and samples in the most important domain, without assuming a linear or quadratic limit state surface, or requiring a design point computation.

III. TRANSFORMATION METHOD

The theory of the Probabilistic Transformation Method or PTM is based on the following theorem [10]:

Theorem (Single input-single output system): Suppose that *X* is a random variable with PDF (probability density function) $f(x)$ and $A \subset \Re$ is the one–dimensional space where $f(x) > 0$. Consider the random variable (function of x) $Y = u(X)$, where $y = u(x)$ defines a one-to-one transformation that maps the set A onto a set $B \subset \Re$ so that the equation $y = u(x)$ can be uniquely solved for *x* in terms of *y*, say $x = u^{-1}(y)$. Then, the PDF of *Y* is:

$$g(y) = f\left[u^{-1}(y)\right]|J|, \qquad y \in B,$$

where, $|J| = \dfrac{dx}{dy} = \dfrac{du^{-1}(y)}{dy}$ is the Jacobean of the transformation.

The mathematical condition for this theorem is that the transformation must be one-to-one. Problems frequently arise when we wish to find the probability distribution of the random function *Y=U(X)* when *X* is a continuous random variable and the transformation is not one-to-one. So the two major problems or limitations of PTM are:

1. the transformation function (U) should be bijective
2. the determinant of Jacobean should be not null

To avoid these limitations of PTM, a $n \times n$ system can be defined such as $y_1 = u(x_1, x_2, ..., x_n)$:

$$\begin{cases} U : IR^n \to IR^n \\ Y \to Y = U(X) \end{cases} \quad with \quad U(X) = \begin{cases} y_1 = u(x_1, x_2, ..., x_n) \\ y_i = x_i \;\; i = 2, ..., n \end{cases} \tag{5}$$

The above function U is reversible if and only if the determinant of the Jacobean is not null. Indeed

$$|J| = \begin{vmatrix} \dfrac{\partial u}{\partial x_1} & . & . & . & \dfrac{\partial u}{\partial x_n} \\ 0 & 1 & 0 & . & 0 \\ . & 0 & 1 & . & . \\ . & . & . & 1 & 0 \\ 0 & . & . & 0 & 1 \end{vmatrix} = \frac{\partial u}{\partial x_1} \neq 0$$

IV. PROPOSED TECHNIQUE: MPTM-FEM

The proposed technique is a combination of the deterministic finite element method and the random variable transformation technique for multiple variables (MPTM) using the previous theorem. In this technique, the differential equation is solved firstly by using the deterministic theory of finite element (a) which yields to accurate nodal solutions. These solutions are then used to obtain the **"exact"** PDF using the random variable transformation (b) between the input random variables and the output variable. The accuracy of the solution is increased when increasing the number of elements in the FEM as usual. The organization chart of this technique is shown in figure 2 and the algorithm is shown in figure 3.

a) the FEM leads to:
$[K]U = F, i.e. \, U = K^{-1}F = SF$ (where S is the flexibility matrix).

b) Using PTM technique $f_U = \left|\dfrac{\partial S}{\partial U}\right| . f_S . f_F$

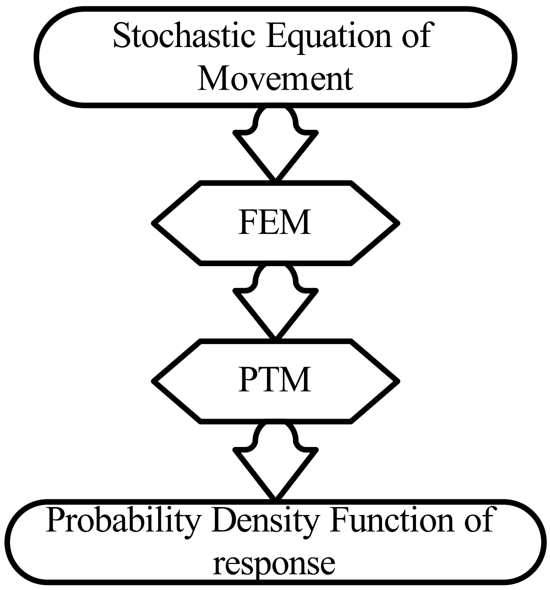

Fig. 2: Organizaton chart

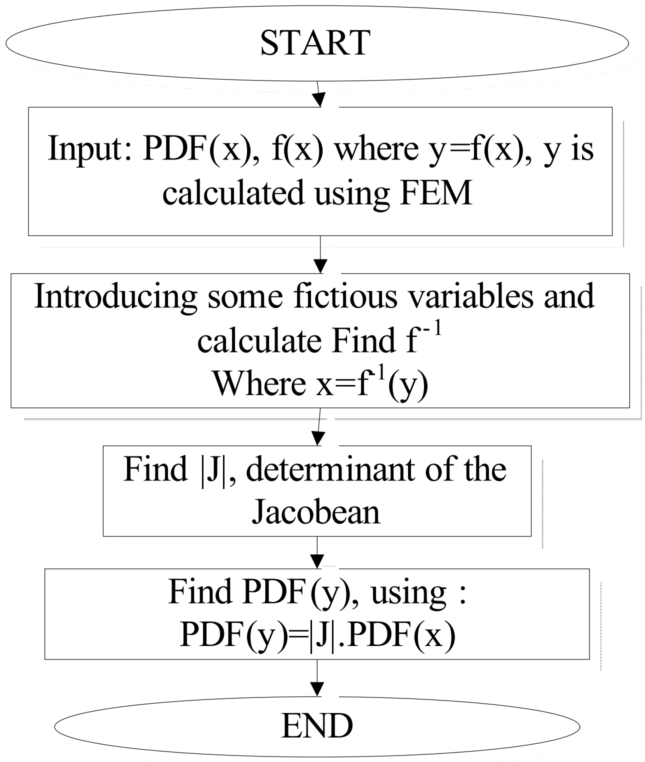

Fig. 3: algorithm of MPTM-FEM

V. Application

In this example, a 6-bar truss structure with random Young's modulus *E* and random load *F*. It is required to find the p.d.f of the response of node 1 (Figure 4), which is a function of two input random variables (load and Young's modulus).

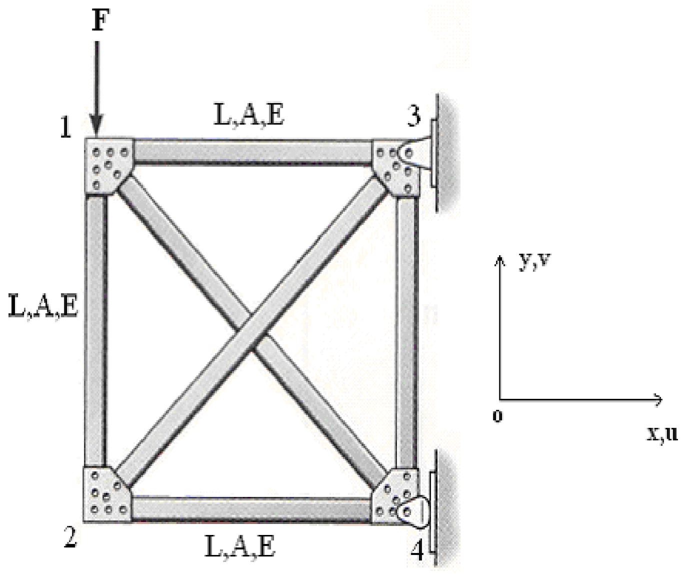

Fig. 4: 6-bar truss structure

To use the MPTM-FEM technique, we must have the number of output random variables equal to the number of input random variables. So, we should introduce a fictitious random output, which is an arbitrary function of the random inputs. The choice of this function doesn't affect the p.d.f of the solution process (response).

The MPTM-FEM is applied according to the following steps:

Models the truss using FEM, to obtain the displacement of the first and the second nodes. For the truss bars, the stiffness matrices are given by:

Bar 3-1

$$\frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} u_3 \\ u_1 \end{bmatrix}, \begin{bmatrix} F_{x3} \\ F_{x1} \end{bmatrix}$$

Bar 4-2

$$\frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} u_4 \\ u_2 \end{bmatrix}, \begin{bmatrix} F_{x4} \\ F_{x2} \end{bmatrix}$$

Bar 4-3

$$\frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} v_4 \\ v_3 \end{bmatrix}, \begin{bmatrix} F_{y4} \\ F_{y3} \end{bmatrix}$$

Bar 2-1

$$\frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} v_4 \\ v_3 \end{bmatrix}, \begin{bmatrix} F_{y2} \\ F_{y1} \end{bmatrix}$$

Bar 3-2

$$\frac{EA}{4L}\begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}\begin{bmatrix} u_3 \\ v_3 \\ u_2 \\ v_2 \end{bmatrix}, \begin{bmatrix} F_{x3} \\ F_{y3} \\ F_{x2} \\ F_{y2} \end{bmatrix}$$

Bar 4-1

$$\frac{EA}{4L}\begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix}\begin{bmatrix} u_4 \\ v_4 \\ u_1 \\ v_1 \end{bmatrix}, \begin{bmatrix} F_{x4} \\ F_{y4} \\ F_{x1} \\ F_{y1} \end{bmatrix}$$

The Assembly Theorem leads, to the global equilibrium system:

$$\frac{EA}{4L}\begin{bmatrix} 5 & -1 & -1 & 1 \\ 1 & 5 & 1 & -1 \\ 0 & 0 & 5 & -1 \\ 0 & -4 & -1 & 5 \end{bmatrix}\begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} F_{x1} \\ F_{y1} \\ F_{x2} \\ F_{y2} \end{bmatrix}$$

Where the solution gives:

$$u_1 = \frac{6FL}{11EA},\ v_1 = -\frac{30FL}{11EA},\ u_2 = -\frac{5FL}{11EA},$$

$$v_2 = -\frac{25FL}{11EA}$$

the displacement of the node 1: $u_1 = \frac{6FL}{11EA}$, where L is the bar length L=10m, A is the cross-section A=0.0015m^2, E is the Young's modulus and F is the load. F and E are physically two independent random variables that have some probability distributions.

Using the MPTM-FEM, we can find the p.d.f of the response u_1 if E and F are random variables. In this problem we have two inputs (E,F) and one output (u_1). So, we introduce a fictitious random variable, say $Z=E$, as the 2nd output. The transformation equations between inputs and outputs becomes:

$$\begin{cases} u_1 = \frac{6FL}{11EA} \\ Z = E \end{cases}$$

where the domain of E,F will be completely defined by the distributions of E and F.

The Jacobean of the transformation is

$$J = \begin{vmatrix} \partial E/\partial u_1 & \partial E/\partial Z \\ \partial F/\partial u_1 & \partial F/\partial Z \end{vmatrix} = \frac{11ZA}{6L}$$

Using the PTM technique, we can get:

$$f_{u_1,Z}(u_1,Z) = f_{E,F}[E,F].|J|$$

In our problem, from the independency between the material property and the load we can simply say:

$$f_{E,F}(E,F) = f_E(E).f_F(F)$$

Finally, the p.d.f of the response is computed from the following integral:

$$f_{u_1}(u_1) = \int_{D_{u_1,z}} f_{u_1,Z}(u_1,z).dz \quad ,$$

which is the exact p.d.f of u_1.

Now, we let us consider the example where F and E are uniformly distributed in the ranges $[10,30]$ and $[10^8, 3.10^8]$ respectively.

In this case, the joint probability density function of F and E is:

$$f_{E,F}(E,F) = \begin{cases} \frac{10^{-8}}{40}, & if \;\; 10 \le F \le 30, \;\; 10^8 \le E \le 3.10^8 \\ 0.0, & Otherwise \end{cases}$$

The joint distribution for (u_1, Z) is:

$$f_{u_1,Z}(u_1,Z) = \begin{cases} \frac{11ZA}{240.10^8 L}, \; if & \frac{60L}{11ZA} \le u_1 \le \frac{180L}{11ZA}, \\ & 10^8 \le Z \le 3.10^8 \\ 0.0 \;, & Otherwise. \end{cases}$$

The p.d.f of the first node u_1 is:

$$f_{u_1}(u_1) = \begin{cases} \frac{1}{8\times3636\times10^9}\left[\left(\frac{30\times3636}{u_1}\right)^2 - 10^{16}\right] & ; \frac{3636}{10^7} \le u_1 \le \frac{10908}{10^7} \\ \frac{1}{8\times3636\times10^9}\left[9.10^{16} - \left(\frac{10\times3636}{u_1}\right)^2\right] & ; \frac{3636}{3.10^7} \le u_1 \le \frac{3636}{10^7} \\ 0.0 \;\; ; & Otherwise. \end{cases}$$

Figure 5 shows the p.d.f of input and figure 6 shows the output variables. The result is verified by 10000 Monte-Carlo simulation.

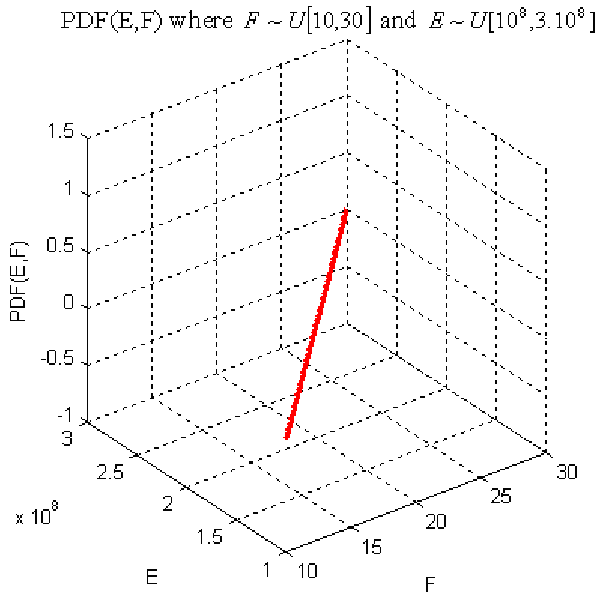

Fig. 5: PDF of input E and F

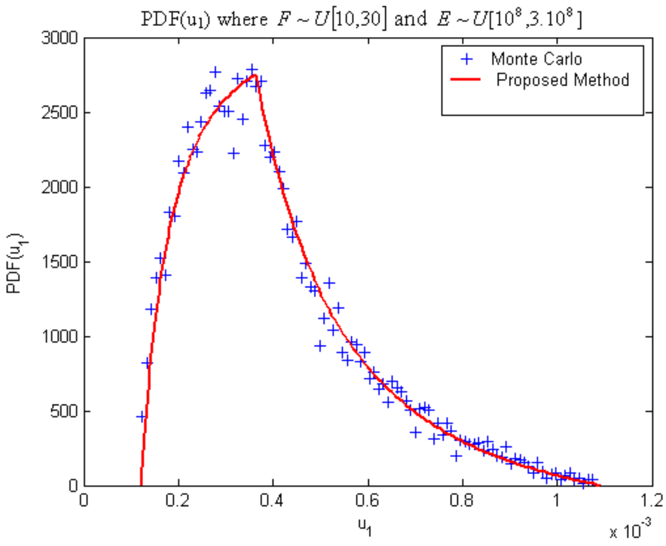

Fig. 6: PDF of output u_1

Reliability Analysis

Let us suppose the limit displacement is $u_{1l} = 6.10^{-5}$ mm it is required to find the failure probability $P_f = P(u_1 \geq u_{1l}) = \int_{u_{1l}}^{\infty} f_{u_1}(u_1)$.

$$P_f = \int_{0.00006}^{\infty} f_{u_1}(u_1) =$$

$$\int_{0.00006}^{0.00109} \frac{1}{8\times 3636\times 10^9}\left[\left(\frac{30\times 3636}{u_1}\right)^2 - 10^{16}\right] du_1 =$$

$$\frac{11}{50} = 0.22$$

Table 1: comparison of P_f with Monte Carlo simulation

	Proposed Method	Monte Carlo simulation(10000)
P_f	0.22	0.21758

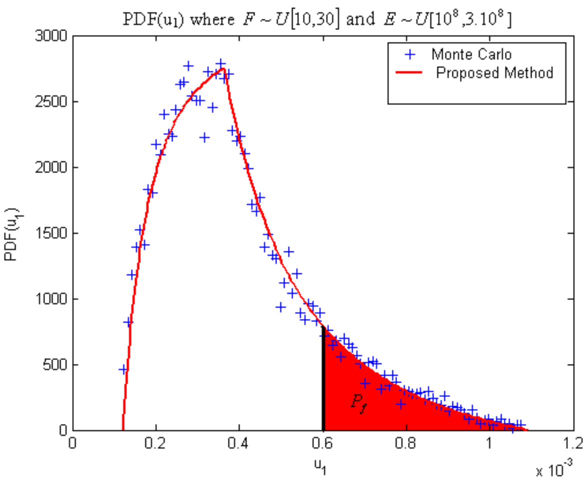

Fig. 7: probability of failure (red region)

VI. Conclusion

In this paper, the reliability analysis of mechanical systems with parameter uncertainties has been considered. The method is based on the combination of the probabilistic transformation method and the deterministic finite element method. The application is given for a simple 6-bar truss, where the PTM-FEM gives the closed form expression of the response density function. To proof the performance of the proposed method, the result is compared with 10000 Monte Carlo simulations.

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